JOURNAL OF
FUNCTION SPACES AND APPLICATIONS
Volume 5, Number 3 (2007), 319–331

http://www.jfsa.net

The Riesz "rising sun" lemma for arbitrary Borel measures with some applications

Lasha Ephremidze, Nobuhiko Fujii and Yutaka Terasawa

(*Communicated by Vakhtang Kokilashvili*)

2000 Mathematics Subject Classification. Primary 42B25.

Keywords and phrases. "Rising sun" lemma, weak type inequality, exact constants.

Abstract. The Riesz "rising sun" lemma is proved for arbitrary locally finite Borel measures on the real line. The result is applied to study an attainability problem of the exact constant in a weak $(1,1)$ type inequality for the corresponding Hardy-Littlewood maximal operator.

1. Introduction

Let M_+ be the one-sided Hardy-Littlewood maximal operator on the real line

$$M_+f(x) = \sup_{b>x} \frac{1}{b-x} \int_x^b |f|\, dm, \qquad f \in L^1_{\text{loc}}(\mathbb{R}),$$

where m stands for the Lebesgue measure. The following equality

$$m\{M_+f > \lambda\} = \frac{1}{\lambda} \int_{\{M_+f>\lambda\}} |f|\, dm, \qquad \lambda > 0, \tag{1}$$

is well known and sometimes called the Riesz "rising sun lemma" (see [4]) since it can be readily obtained from the following lemma which usually carries this name (see [7], [8]):

Lemma. *Suppose G is a continuous function on $\mathbb{R}$. Let E be the set of points x such that $G(x+h) > G(x)$ for some $h = h_x > 0$. If (a,b) is a finite connected component of E, then $G(a) = G(b)$.*

The equation (1) is an important tool in studying various problems related with maximal functions ant there exist its several proofs (see [3], [8]) which do not depend on a geometric structure of the Lebesgue measure and can be generalized for absolutely continuous measures (see Lemma 1 in [1]). In this note we formulate and prove the lemma in the most general setting for which it remains true.

Throughout the paper let ν be a locally finite signed Borel measure and μ be a positive Borel measure on $\mathbb{R}$. We assume without lose of generality that $\mu(a,b) > 0$ for each open interval. Define the one sided maximal function $M_+^\mu \nu$ as

$$M_+^\mu \nu(x) = \sup_{b>x} \frac{\nu[x,b)}{\mu[x,b)}.$$

For a signed measure ν, let ν^+ and ν^- be respectively the positive and the negative part of ν. We assume that $\nu^-(\mathbb{R}) < \infty$.

Theorem 1. *If measures ν^- and μ are free of atoms, i.e. $\nu\{x\} \geq 0$ and $\mu\{x\} = 0$ for each $x \in \mathbb{R}$, and*

$$\lambda > \liminf_{x \to -\infty} M_+^\mu \nu(x), \tag{2}$$

then

$$\lambda \mu\{M_+^\mu \nu > \lambda\} = \nu\{M_+^\mu \nu > \lambda\}. \tag{3}$$

We construct the counterexamples where the equality (3) fails to hold when measures have inadmissable atoms or (2) is not satisfied. Nevertheless the one sided inequality $\lambda \mu\{M_+^\mu \nu > \lambda\} \leq \nu\{M_+^\mu \nu > \lambda\}$ is always true, i.e. the weak (1,1) type inequality holds for the operator M_+^μ with exact constant 1, and we show this fact separately.

Theorem 2. *Let ν and μ be as in the introduction. For each $\lambda > 0$, we have*

$$\mu\{M_+^\mu \nu > \lambda\} \leq \frac{1}{\lambda} \nu\{M_+^\mu \nu > \lambda\}. \tag{4}$$

We use the above results to answer the question considered below on the attainability of the exact constant in the weak type $(1,1)$ inequality for the two-sided Hardy-Littlewood maximal operator. Namely, let

$$Mf(x) = \sup_{a<x<b} \frac{1}{b-a} \int_a^b |f|\, dm, \qquad f \in L^1_{\text{loc}}(\mathbb{R}).$$

It is well known that the constant 2 is exact in the following weak type $(1,1)$ inequality

$$m\{Mf>\lambda\}\leq\frac{C}{\lambda}\|f\|_1,\qquad f\in L(\mathbb{R}),\ \lambda>0, \tag{5}$$

for the operator M, i.e., the inequality (5) holds when $C=2$ and fails to hold for some $f\in L(\mathbb{R})$ and $\lambda>0$ whenever $C<2$. The question arises whether the exact constant 2 can be achieved for some nontrivial integrable function, i.e. if there exist $f\in L(\mathbb{R})$ (except $f\equiv 0$) and $\lambda>0$ such that

$$m\{Mf>\lambda\}=\frac{2}{\lambda}\|f\|_1\ .$$

A similar problem of the attainability of the exact constants is considered in [6]. We give a negative answer to the above posed question in the general setting below.

It is well known that $C=2$ is also the exact constant in the weak (1,1) type inequality for the Hardy-Littlewood maximal function $M^\mu\nu$ corresponding to the measures ν and μ:

$$M^\mu\nu(x)=\sup_{x\in(a,b)}\frac{\nu(a,b)}{\mu(a,b)},$$

i.e.

$$\mu\{M^\mu\nu>\lambda\}\leq\frac{2}{\lambda}\nu(\mathbb{R}),\qquad \lambda>0, \tag{6}$$

and if ν is allowed to have atoms, say $\nu=\delta_{\{0\}}$, then for each $\lambda>0$

$$\mu\{M^\mu\nu>\lambda\}=\frac{2}{\lambda}\nu(\mathbb{R})=\frac{2}{\lambda}\nu\{M^\mu\nu>\lambda.\} \tag{7}$$

For the sake of completeness, we give the proof of a slightly improved version of (6) and show that the equality (7) cannot be achieved (except for the trivial case) if ν^+ is free of atoms.

Proposition 1. *For any* $\lambda>0$*, we have*

$$\mu\{M^\mu\nu>\lambda\}\leq\frac{2}{\lambda}\nu^+\{M^\mu\nu>\lambda\}. \tag{8}$$

Proposition 2. *Let* ν^+ *be a finite measure free of atoms and* $\lambda>0$*. If the equality*

$$\mu\{M^\mu\nu>\lambda\}=\frac{2}{\lambda}\nu^+\{M^\mu\nu>\lambda\} \tag{9}$$

holds, then both sides of (9) *are zero.*

2. The proof of the main result

Note that in general $\{M_+^\mu \nu > \lambda\}$ may not be open, but it is always open from the left, i.e. for each $x \in \{M_+^\mu \nu > \lambda\}$ there exists $\varepsilon > 0$ such that $(x-\varepsilon, x] \subset \{M_+^\mu \nu > \lambda\}$. Hence its representation as a disjoint union of connected components has the form

$$\{M_+^\mu \nu > \lambda\} = \cup_{n=1}^{\infty} (a_n, b_n\rangle, \tag{10}$$

where the angle “$\rangle$” indicates that b_n either belongs or does not belong to $(a_n, b_n\rangle$ (i.e. $(a_n, b_n\rangle = (a_n, b_n]$ or (a_n, b_n)).

Proof of Theorem 2. If $\{M_+^\mu \nu > \lambda\} = \emptyset$, then the case is trivial.

Assume $(a, b\rangle$ is a connected component of $\{M_+^\mu \nu > \lambda\}$ (not excluding the cases $a = -\infty$ or $b = \infty$) and prove that

$$\lambda\mu(a, b\rangle \le \nu(a, b\rangle. \tag{11}$$

Then obviously (4) follows, because of (10). We will prove that

$$\lambda\mu[x, b\rangle \le \nu[x, b\rangle \tag{12}$$

for each $x \in (a, b\rangle$ and one can get (11) by passing to the limit in (12) as x tends to a from the right. For each $x \in (a, b\rangle$, define

$$y_x := \sup\{y > x : \lambda\mu[x, y) < \nu[x, y)\} \tag{13}$$

(the latter set is not empty). The limiting argument shows that

$$\lambda\mu[x, y_x) \le \nu[x, y_x). \tag{14}$$

Note that

$$y_x \ge b. \tag{15}$$

Indeed, if $y_x < b$, then $y_x \in \{M_+^\mu \nu > \lambda\}$ and there exists $y > y_x$ such that

$$\lambda\mu[y_x, y) < \nu[y_x, y). \tag{16}$$

It follows from (14) and (16) that $\lambda\mu[x, y) < \nu[x, y)$, which contradicts the definition (13) of y_x being a supremum. Thus inequality (15) holds.

Now consider several cases. If $(a,b\rangle = (a,b)$, then $b \notin \{M_+^\mu \nu > \lambda\}$ and

$$\lambda\mu[b,y) \geq \nu[b,y) \quad \text{for each } y > b. \tag{17}$$

In addition, if $y_x = b$, then (12) is the same as (14) and if $y_x > b$, then $\lambda\mu[x,b) > \nu[x,b)$ together with (17) would imply that $\lambda\mu[x,y_x) > \nu[x,y_x)$ which contradicts (14). Thus (12) holds in this case.

If $(a,b\rangle = (a,b]$, then

$$\lambda\mu(b,y) \geq \nu(b,y) \quad \text{for each } y > b \tag{18}$$

and

$$\lambda\mu\{b\} < \nu\{b\}. \tag{19}$$

Indeed, there exists a sequence $\{b_n\}_{n=1}^{\infty}$ from $\mathbb{R}\backslash\{M_+^\mu \nu > \lambda\}$ which converges to b from the right and consequently $\lambda\mu[b_n,y) \geq \nu[b_n,y)$, $n = 1,2,\ldots$, holds, where one can pass to the limit to get (18). The inequality (19) follows from the fact that if $\lambda\mu\{b\} \geq \nu\{b\}$, then $\lambda\mu[b,y) \geq \nu[b,y)$ for each $y > b$, by (18), so that b would not belong to $\{M_+^\mu \nu > \lambda\}$.

Now if $y_x = b$, then (12) follows from (14) and (19) and if $y_x > b$, then $\lambda\mu[x,b] > \nu[x,b]$ together with (18) would imply that $\lambda\mu[x,y_x) > \nu[x,y_x)$ which contradicts (14). Thus (12) holds in this case as well. □

Proof of Theorem 1. Without lose of generality we can assume that $\lambda > 0$. Indeed, if $\mu(\mathbb{R}) = \infty$, then $\liminf_{x\to-\infty} M_+^\mu \nu(x) \geq \liminf_{x\to-\infty} \frac{\nu[x,\infty)}{\mu[x,\infty)} \geq 0$ and the positiveness of λ is justified by (2). If $\mu(\mathbb{R}) < \infty$ and $\lambda < 0$, then take $\lambda' > |\lambda|$ and consider the maximal function of measure $\nu + \lambda'\mu$ with respect to μ, i.e. $M_+^\mu(\nu + \lambda'\mu)$. It can be readily checked that $(\nu + \lambda'\mu)^-$ does not have atoms and $M_+^\mu(\nu + \lambda'\mu) = M_+^\mu \nu + \lambda'$, which implies that $\{M_+^\mu(\nu + \lambda'\mu) > \lambda + \lambda'\} = \{M_+^\mu \nu > \lambda\}$. If we now apply the theorem for positive $\lambda + \lambda'$, then we get $(\lambda + \lambda')\mu\{M_+^\mu \nu > \lambda\} = (\nu + \lambda'\mu)\{M_+^\mu \nu > \lambda\}$, which readily implies (3).

Let us now show that if $(a,b\rangle$ is a connected component of $\{M_+^\mu \nu > \lambda\}$, where $-\infty < a$, then

$$\lambda\mu(a,b\rangle \geq \nu(a,b\rangle. \tag{20}$$

Indeed, since $a \notin \{M_+^\mu \nu > \lambda\}$, we have $\lambda\mu[a,y) \geq \nu[a,y)$ for each $y > a$ and, letting y to tend to b from the right, if necessary, we get

$$\lambda\mu[a,b\rangle \geq \nu[a,b\rangle. \tag{21}$$

Since $\nu\{a\} \geq 0$ and $\mu\{a\} = 0$, (20) follows from (21).

Since it follows from the condition (2) that $-\infty < a_n$ for each n in the representation (10), we have $\lambda\mu(a_n, b_n\rangle \geq \nu(a_n, b_n\rangle$ by virtue of (20). Thus

$$\lambda\mu\{M_+^\mu\nu > \lambda\} = \lambda\mu\left(\cup_{n=1}^\infty (a_n, b_n\rangle\right) \geq \nu\left(\cup_{n=1}^\infty (a_n, b_n\rangle\right) = \nu\{M_+^\mu\nu > \lambda\}$$

which together with (4) implies (3). □

3. Counterexamples

The following example shows that in general ν^- cannot have an atom in Theorem 1.

Example 1. Let $\mu = m$ be the Lebesgue measure and ν be the measure concentrated at two points -1 and 0, where $\nu\{-1\} = -1$ and $\nu\{0\} = 1$. Then $M_+^\mu\nu(x) = 1/|x|$ on $(-1;0]$ and is equal to 0 outside $(-1;0]$. Thus $\{M_+^\mu\nu > \lambda\} = (-1;0]$ for each $\lambda \in (0;1)$ and $\lambda\mu\{M_+^\mu\nu > \lambda\} = \lambda \cdot 1 \neq 1 = \nu\{0\} = \nu\{M_+^\mu\nu > \lambda\}$.

The following example shows that μ cannot have an atom.

Example 2. Let μ_0 and ν be the measures concentrated at -1 and 0, respectively, where $\mu_0\{-1\} = 1$ and $\nu\{0\} = 1$ and μ be $m + \mu_0$. Then $M_+^\mu\nu(x) = 1/|x|$ on $(-1;0]$, $M_+^\mu\nu(x) = 1/(|x|+1)$ on $(-\infty;-1]$ and is equal to 0 on $(0,\infty)$. Thus $\{M_+^\mu\nu > \lambda\} = (-1;0]$ for each $\lambda \in (\frac{1}{2};1)$ and $\lambda\mu\{M_+^\mu\nu > \lambda\} = \lambda \cdot 1 \neq 1 = \nu\{0\} = \nu\{M_+^\mu\nu > \lambda\}$.

The following example shows the necessity of condition (3).

Example 3. Let ν be the same as in the preceding example and μ be a positive measure such that $\mu = m$ on $[0,\infty)$ and $\mu(-\infty,0) = 1$. Then $M_+^\mu\nu(x) = 0$ on $(0,\infty)$, $M_+^\mu\nu(x) \geq 1$ on $(-\infty,0]$ and $\lim\limits_{x\to-\infty} M_+^\mu\nu(x) = 1$. Thus, for each $\lambda < 1$, $\{M_+^\mu\nu > \lambda\} = (-\infty,0]$ and $\lambda\mu\{M_+^\mu\nu > \lambda\} = \lambda \cdot 1 \neq 1 = \nu\{0\} = \nu\{M_+^\mu\nu > \lambda\}$.

4. The attainability problem

Proof of Proposition 1 (cf. [2], Lemma I.4.4.). Inequality (8) will be proved if we show that

$$\mu(K) \leq \frac{2}{\lambda}\nu^+\{M^\mu\nu > \lambda\}$$

for each compact set $K \subset \{M^\mu\nu > \lambda\}$. Let $I_1, I_2, \ldots, I_n$ be a finite cover of K by intervals such that

$$\lambda\mu(I_j) < \nu(I_j), \quad j = 1, 2, \ldots, n. \tag{22}$$

Observe that (22) implies that

$$I_j \subset \{M^\mu \nu > \lambda\}, \quad j = 1, 2, \dots, n. \tag{23}$$

We say that a finite system of covering intervals $\{I_j\}_{j=1}^n$ is minimal if $\cup_{j=1}^n I_j \supset K$ and $K \not\subset \cup_{l \neq j=1}^n I_j$ for each $l = 1, 2, \dots, n$. Since from every system of covering intervals one can select a minimal subsystem, we can assume that $\{I_j\}_{j=1}^n$ is minimal.

Let $I_j = (a_j, b_j)$, $j = 1, 2, \dots, n$. It can be observed that $a_i \neq a_j$ and $b_i \neq b_j$ whenever $i \neq j$, since the system is minimal. We can assume that $a_1 < a_2 < \dots < a_n$. The minimality of the system also implies that $(a_j, b_j) \cap (a_{j+2}, b_{j+2}) = \emptyset$ for each $j \leq n-2$, since otherwise either (a_{j+1}, b_{j+1}) can be excluded from the system (kept K covered) if $b_{j+1} < b_{j+2}$ or (a_{j+2}, b_{j+2}) can be excluded from the system if $b_{j+1} > b_{j+2}$. Thus we have the two open sets consisting of disjoint intervals

$$E_1 = (a_1, b_1) \cup (a_3, b_3) \cup \dots \quad \text{and} \quad E_2 = (a_2, b_2) \cup (a_4, b_4) \cup \dots$$

such that $K \subset E_1 \cup E_2$ and $\lambda \mu(E_i) < \nu(E_i)$, $i = 1, 2$, by (22).

Since $\nu(E_i) \leq \nu^+(E_i) \leq \nu^+\{M^\mu \nu > \lambda\}$, $i = 1, 2$, by (23), we have

$$\lambda \mu(K) \leq \lambda(\mu(E_1) + \mu(E_2)) \leq \nu(E_1) + \nu(E_2) \leq 2\nu^+\{M^\mu \nu > \lambda\}. \qquad \square$$

By developing further the idea of the proof of Proposition 1, one can prove the following covering lemmas, which leads to the Lebesgue differentiation theorem for arbitrary measures. Although it is essential in the following two lemmas that we deal with the one-dimensional case, nevertheless we emphasize that the measure μ is arbitrary and may not satisfy the doubling condition.

Lemma 1. *Let $\{[a_j, b_j]\}_{j \in J}$ be a system of closed intervals that covers a measurable set $A \subset \mathbb{R}$ with $\mu(A) < \infty$. Then, for each $\varepsilon > 0$, there exists a disjoint subsystem $\{[a_j, b_j]\}_{j \in J_0}$, $J_0 \subset J$, such that*

$$\mu\left(\cup_{j \in J_0} [a_j, b_j]\right) > \frac{1}{2}\mu(A) - \varepsilon.$$

Proof. Clearly, one can associate to each point $x \in Q := (\cup_{j \in J}[a_j, b_j]) \setminus (\cup_{j \in J}(a_j, b_j))$ an open interval Δ_x with one endpoint at x such that $Q \cap \Delta_x = \emptyset$. If $a \leq b < c \leq d$, then $(a, \frac{a+c}{2}) \cap (\frac{b+d}{2}, d) = \emptyset$. So, if we denote by Δ'_x the interval with length $\frac{1}{2}|\Delta_x|$ and with the same left or right endpoint x, then the system of intervals $\{\Delta'_x\}_{x \in Q}$ will be disjoint. Thus Q is numerable.

The system of open intervals $\{(a_j, b_j)\}_{j\in J}$ covers $A\backslash Q$. Take an arbitrary ε and a compact set $K \subset (A\backslash Q)$ such that

$$\mu(K) > \mu(A\backslash Q) - \varepsilon. \tag{24}$$

We can cover K by a finite subsystem of intervals $\{(a_j, b_j)\}_{j\in J_1}$, i.e. J_1 is a finite subset of J and

$$K \subset \left(\cup_{j\in J_1}(a_j, b_j)\right), \tag{25}$$

and we can take another finite subsystem of closed intervals $\{[a_j, b_j]\}_{j\in J_2}$ which covers $A \cap Q$ up to a set of μ-measure less than ε,

$$\mu\left((A \cap Q)\backslash \cup_{j\in J_2} [a_j, b_j]\right) < \varepsilon. \tag{26}$$

It follows from (24),(25) and (26) that

$$\mu\left(A\backslash \cup_{j\in J_1\cup J_2} [a_j, b_j]\right) < 2\varepsilon. \tag{27}$$

Thus, we have found a finite system of closed intervals $\{[a_j, b_j]\}_{j\in J_3}$ which covers A up to a set of μ-measure 2ε.

The rest of the proof repeats the proof of Proposition 1. Assume $\cup_{j\in J_3}[a_j, b_j] = A_J$ and let $\{[a_j, b_j]\}_{j=1}^{n}$ be a minimal cover of A_J (with exactly the same definition as in the proof of Proposition 1). We can assume without loss of generality that $a_1 < a_2 < \ldots < a_n$. Then

$$E_1 = [a_1, b_1] \cup [a_3, b_3] \cup \ldots \quad \text{and} \quad E_2 = [a_2, b_2] \cup [a_4, b_4] \cup \ldots$$

consist of disjoint closed intervals and $A_J \subset E_1 \cup E_2$. Thus it follows from (27) that

$$\mu(E_i) \geq \frac{1}{2}\mu(A_J) \geq \frac{1}{2}(\mu(A) - 2\varepsilon)$$

either for $i = 1$ or 2 and the lemma follows. □

Using Lemma 2, one can prove the following lemma exactly in the same way as Theorem 2.8 is proved in [5].

Lemma 2. *Let $A \subset \mathbb{R}$ be a measurable set and $\{[a_j, b_j]\}_{j\in J}$ be a system of closed intervals such that for each $x \in A$ and $\varepsilon > 0$ there exist $j \in J$ such that $b_j - a_j < \varepsilon$ and $x \in [a_j, b_j]$. Then there is a disjoint subsystem $\{[a_j, b_j]\}_{j\in J_0}$ such that*

$$\mu\left(A\backslash \cup_{j\in J_0} [a_j, b_j]\right) = 0.$$

If $\nu = \nu_c + \nu_s^+ - \nu_s^-$ is the Lebesgue decomposition of ν with respect to μ, i.e. $\nu_c \ll \mu$, $\mu(E^\pm) = 0$ and $\nu_s^\pm(E \cap E^\pm) = \nu^\pm(E)$, then define

$$\frac{D\nu}{D\mu} := \begin{cases} +\infty & \text{on } E^+ \\ -\infty & \text{on } E^- \\ \dfrac{d\nu_c}{d\mu} & \text{otherwise} \end{cases} \tag{28}$$

where $\frac{d\nu_c}{d\mu}$ is the Radon-Nikodym derivative. Using Lemma 2, one can get the Lebesgue differentiation theorem for the pair of measures ν and μ in a standard way (see, e.g., the proof of Theorem 2.12 in [5]).

Lebesgue Differentiation Theorem. *For $(|\nu| + \mu)$-almost all $x \in \mathbb{R}$ the limits*

$$D_+^\mu \nu(x) = \lim_{b \to x+} \frac{\nu[x,b)}{\mu[x,b)} \quad \text{and} \quad D_-^\mu \nu(x) = \lim_{a \to x-} \frac{\nu(a,x]}{\mu(a,x]}$$

are equal to $\frac{D\nu}{D\mu}(x)$ defined by (28).

Proof of Proposition 2. Along with $M_+^\mu \nu$ consider the left side maximal function $M_-^\mu \nu(x) = \sup_{a<x} \frac{\nu(a,x]}{\mu(a,x]}$. Obviously, Theorems 1 and 2 hold for $M_-^\mu \nu$ too.

First consider the case where μ does not have atoms. Then $M^\mu \nu(x) > \lambda$ implies that either $M_+^\mu \nu(x) > \lambda$ or $M_-^\mu \nu(x) > \lambda$. At the same time, using the limiting argument, it is clear that $M_+^\mu \nu(x) > \lambda$ or $M_-^\mu \nu(x) > \lambda$ implies that $M^\mu \nu(x) > \lambda$. Thus

$$\{M^\mu \nu > \lambda\} = \{M_+^\mu \nu > \lambda\} \cup \{M_-^\mu \nu > \lambda\}.$$

By virtue of Theorem 2, we have

$$\begin{aligned} \mu\{M^\mu \nu > \lambda\} &\leq \mu\{M_+^\mu \nu > \lambda\} + \mu\{M_-^\mu \nu > \lambda\} \\ &\leq \frac{1}{\lambda}\nu\{M_+^\mu \nu > \lambda\} + \frac{1}{\lambda}\nu\{M_-^\mu \nu > \lambda\} \\ &\leq \frac{1}{\lambda}(\nu^+\{M_+^\mu \nu > \lambda\} + \nu^+\{M_-^\mu \nu > \lambda\}) \\ &\leq \frac{2}{\lambda}\nu^+\{M^\mu \nu > \lambda\}. \end{aligned}$$

If (9) holds then we should have all equalities instead of "$\leq$" in the above relations. This means that we should have

$$\mu\{M^\mu \nu > \lambda\} = \mu\{M_+^\mu \nu > \lambda\} + \mu\{M_-^\mu \nu > \lambda\}. \tag{29}$$

But since ν^+ does not have atoms, $\{M_+^\mu\nu > \lambda\}$ and $\{M_-^\mu\nu > \lambda\}$ are open sets and (29) means that $\{M_+^\mu\nu > \lambda\} \cap \{M_-^\mu\nu > \lambda\} = \emptyset$ since we have that the measure μ of each open interval is positive. By the Lebesgue differentiation theorem, $(|\nu|+\mu)$-almost all $x \in \{\frac{D\nu}{D\mu} > \lambda\}$ belong to the sets $\{M_+^\mu\nu > \lambda\}$ and $\{M_-^\mu\nu > \lambda\}$. Hence $(|\nu| + \mu)\{\frac{D\nu}{D\mu} > \lambda\} = 0$. Therefore $\nu_s^+ \equiv 0$ and $\frac{d\nu_c}{d\mu} \le \lambda$. Hence $\frac{\nu(a,b)}{\mu(a,b)} \le \lambda$ for each interval (a,b) so that $\{M^\mu\nu > \lambda\} = \emptyset$ and the theorem follows.

If μ contains atoms, we reduce the case to the non-atomic one by the following trick.

Let A be the set of atoms of μ. Since μ is locally finite, A is countable so that let

$$A = \{x_j\}_{j=1}^\infty \quad \text{and} \quad \mu\{x_j\} = \mu_j,\ j = 1, 2, \ldots.$$

Assume W.L.O.G. that $0 \notin A$ and define the maps

$$\Psi_-(x) = \begin{cases} x + \sum_{x_j \in (0,x)} \mu_j, & x > 0, \\ x - \sum_{x_j \in [x,0)} \mu_j, & x \le 0, \end{cases}$$

and

$$\Psi_+(x) = \begin{cases} x + \sum_{x_j \in (0,x]} \mu_j, & x \ge 0, \\ x - \sum_{x_j \in (x,0)} \mu_j, & x < 0, \end{cases}$$

(we assume $(a,b] = \emptyset = [a,b)$ when $a = b$). Obviously, $x < y \Rightarrow \Psi_\mp(x) < \Psi_\mp(y)$ and if $x \notin A$, then $\Psi_-(x) = \Psi_+(x) =: \Psi(x)$. The relation $x < y \Rightarrow \Psi(x) < \Psi(y)$ is naturally valid if we take any point from $[\Psi_-(x_j), \Psi_+(x_j)]$ in the role of $\Psi(x_j)$ for the atoms $x_j \in A$.

For arbitrary Borel measurable $E \subset \mathbb{R}$, define naturally the set

$$\Psi(E) := \Psi(E \backslash A) \cup (\cup_{x_j \in E}[\Psi_-(x_j), \Psi_+(x_j)]) = \cup_{x \in E}[\Psi_-(x), \Psi_+(x)]$$

and let $B = \Psi(\mathbb{R}\backslash A)$. The map Ψ is strictly increasing and hence is one-to-one on $\mathbb{R}\backslash A$. Thus $\Psi^{-1}(y)$ exists whenever $y \in B$ and we naturally assume that $\Psi^{-1}(y) = x_j$ whenever $y \in \Psi\{x_j\}$. We have

$$\text{(30)} \quad \mathbb{R} = B \cup (\cup_{j=1}^\infty[\Psi_-(x_j), \Psi_+(x_j)]) = B \cup \Psi(A) \;\text{ and }\; B \cap \Psi(A) = \emptyset.$$

Keeping these relations in mind, we can define the measures $\tilde{\mu}$ and $\tilde{\nu}$ on $\mathbb{R}$ by letting them to be measure-preserving on $\mathbb{R}\backslash A$, $\tilde{\mu}(E) = \mu(\Psi^{-1}(E))$ and $\tilde{\nu}(E) = \nu(\Psi^{-1}(E))$ as $E \subset B$, and also $\tilde{\mu} = dm$, $\tilde{\nu} = \nu\{x_j\}\delta_{\{x_j\}}$ on each $[\Psi_-(x_j), \Psi_+(x_j)] = \Psi\{x_j\}$. Note that $\tilde{\mu}$ has got free of atoms and, since $\tilde{\nu}\{x_j\} \le 0$,

$$\text{(31)} \qquad \tilde{\nu}(E) \le 0 \text{ for each } E \subset \cup_{j=1}^\infty[\Psi_-(x_j), \Psi_+(x_j)] = \Psi(A).$$

Observe that

$$\mu(a,b)=\tilde\mu(\Psi_+(a),\Psi_-(b)),\quad \nu(a,b)=\tilde\nu(\Psi_+(a),\Psi_-(b)) \text{ and } \nu^+(a,b)=\tilde\nu^+(\Psi_+(a),\Psi_-(b)) \tag{32}$$

for each $a<b$. Furthermore, for each Borel measurable $E\subset\mathbb{R}$,

$$\mu(E)=\tilde\mu(\Psi(E)) \text{ and } \nu^+(E)=\tilde\nu^+(\Psi(E)). \tag{33}$$

If $x\in(a,b)$ and $\frac{\nu(a,b)}{\mu(a,b)}>\lambda$, then $\Psi\{x\}\subset(\Psi_+(a),\Psi_-(b))$ and $\frac{\tilde\nu(\Psi_+(a),\Psi_-(b))}{\tilde\mu(\Psi_+(a),\Psi_-(b))}>\lambda$, by (32). Thus $x\in\{M^\mu\nu>\lambda\}$ implies that $\Psi(x)\in\{M^{\tilde\mu}\tilde\nu>\lambda\}$ if $x\notin A$ and $\Psi\{x\}=[\Psi_-(x_j),\Psi_+(x_j)]\subset\{M^{\tilde\mu}\tilde\nu>\lambda\}$ if $x\in A$. Consequently,

$$\Psi\{M^\mu\nu>\lambda\}\subset\{M^{\tilde\mu}\tilde\nu>\lambda\}. \tag{34}$$

Now we show that

$$y\in B=\Psi(\mathbb{R}\backslash A) \text{ and } M^{\tilde\mu}\tilde\nu(y)>\lambda \implies M^\mu\nu(\Psi^{-1}(y))>\lambda. \tag{35}$$

Indeed, let $(\alpha,\beta)\ni y$ and

$$\tilde\nu(\alpha,\beta)>\lambda\tilde\mu(\alpha,\beta). \tag{36}$$

Assume $\alpha_+=\Psi_+(x_j)$ when $\alpha\in[\Psi_-(x_j),\Psi_+(x_j)]=\Psi\{x_j\}$ for some j and $\alpha_+=\alpha$ otherwise ($\alpha\in B$ in this case). Similarly, $\beta_-=\Psi_-(x_j)$ when $\beta\in[\Psi_-(x_j),\Psi_+(x_j)]=\Psi\{x_j\}$ for some j and $\beta_-=\beta$ otherwise. Clearly, $\alpha\le\alpha_+<y<\beta_-\le\beta$ because $y\notin\cup_{j=1}^\infty[\Psi_-(x_j),\Psi_+(x_j)]$, and hence $\Psi^{-1}(\alpha_+)<\Psi^{-1}(y)<\Psi^{-1}(\beta_-)$. Since $\tilde\nu(\alpha,\alpha_+]\le 0$ and $\tilde\nu[\beta_-,\beta)\le 0$ (see (31)), it follows from (36) that $\tilde\nu(\alpha_+,\beta_-)>\lambda\tilde\mu(\alpha_+,\beta_-)$. Thus $\nu(\Psi^{-1}(\alpha_+),\Psi^{-1}(\beta_-))>\lambda\mu(\Psi^{-1}(\alpha_+),\Psi^{-1}(\beta_-))$ by (32), and $M^\mu\nu(\Psi^{-1}(y))>\lambda$.

It follows from (34), (35) and (30) that $\{M^{\tilde\mu}\tilde\nu>\lambda\}\backslash(\Psi\{M^\mu\nu>\lambda\})\subset\Psi(A)$. Hence

$$\tilde\nu^+\{M^{\tilde\mu}\tilde\nu>\lambda\}=\tilde\nu^+(\Psi\{M^\mu\nu>\lambda\}), \tag{37}$$

by (31). At the same time (34) implies that

$$\tilde\mu(\Psi\{M^\mu\nu>\lambda\})\le\tilde\mu\{M^{\tilde\mu}\tilde\nu>\lambda\}. \tag{38}$$

Since $\tilde{\mu}$ does not have atoms, we have that either

$$\tilde{\mu}\{M^{\tilde{\mu}}\tilde{\nu} > \lambda\} < \frac{2}{\lambda}\tilde{\nu}^{+}\{M^{\tilde{\mu}}\tilde{\nu} > \lambda\} \tag{39}$$

or the both sides of the inequality (39) are equal to 0. In the latter case it follows from (37) and (38) that $\tilde{\nu}^{+}(\Psi\{M^{\mu}\nu > \lambda\}) = 0 = \tilde{\mu}(\Psi\{M^{\mu}\nu > \lambda\})$. Consequently $\nu^{+}\{M^{\mu}\nu > \lambda\} = 0 = \mu\{M^{\mu}\nu > \lambda\}$, by (33).

If (39) holds, then it follows from (38), (39) and (37) that

$$\tilde{\mu}(\Psi\{M^{\mu}\nu > \lambda\}) < \frac{2}{\lambda}\tilde{\nu}^{+}(\Psi\{M^{\mu}\nu > \lambda\})$$

and consequently, by virtue of (33),

$$\mu\{M^{\mu}\nu > \lambda\} < \frac{2}{\lambda}\nu^{+}\{M^{\mu}\nu > \lambda\}. \qquad \square$$

Acknowledgements. The present paper arose from the authors' discussions at the Harmonic Analysis Seminar 2005 held at Keio University in Japan. We would like to express our gratitude to the organizers of this seminar for creating a warm and working atmosphere for the participants. The first two authors are also obliged to the Japanese Society of Promotion of Science for the financial support of this research.

References

[1] L. Ephremidze and R. Sato, *A weighted ergodic maximal equality for nonsingular semiflows*, Colloq. Math. 103 (2005), 207–213.

[2] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

[3] M. Guzman, *Differentiation of Integrals in* $\mathbb{R}^n$, Lecture Notes in Mathematics, 481, Springer-Verlag, 1975.

[4] A. A. Korenovskyy, A. K. Lerner and A. M. Stokolos, *On a multi-dimensional form of F. Riesz's "rising sun" lemma*, Proc. Amer. Math. Soc. 133 (2005), 1437–1440.

[5] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Univ. Press, Cambridge, 1995.

[6] A. D. Melas, *The best constant for the centered Hardy-Littlwood maximal inequality*, Ann. of Math. (2), 157 (2003), 647–688.

[7] E. M. Stein and R. Shakarchi, *Real Analysis*, Princeton University Press, 2005.

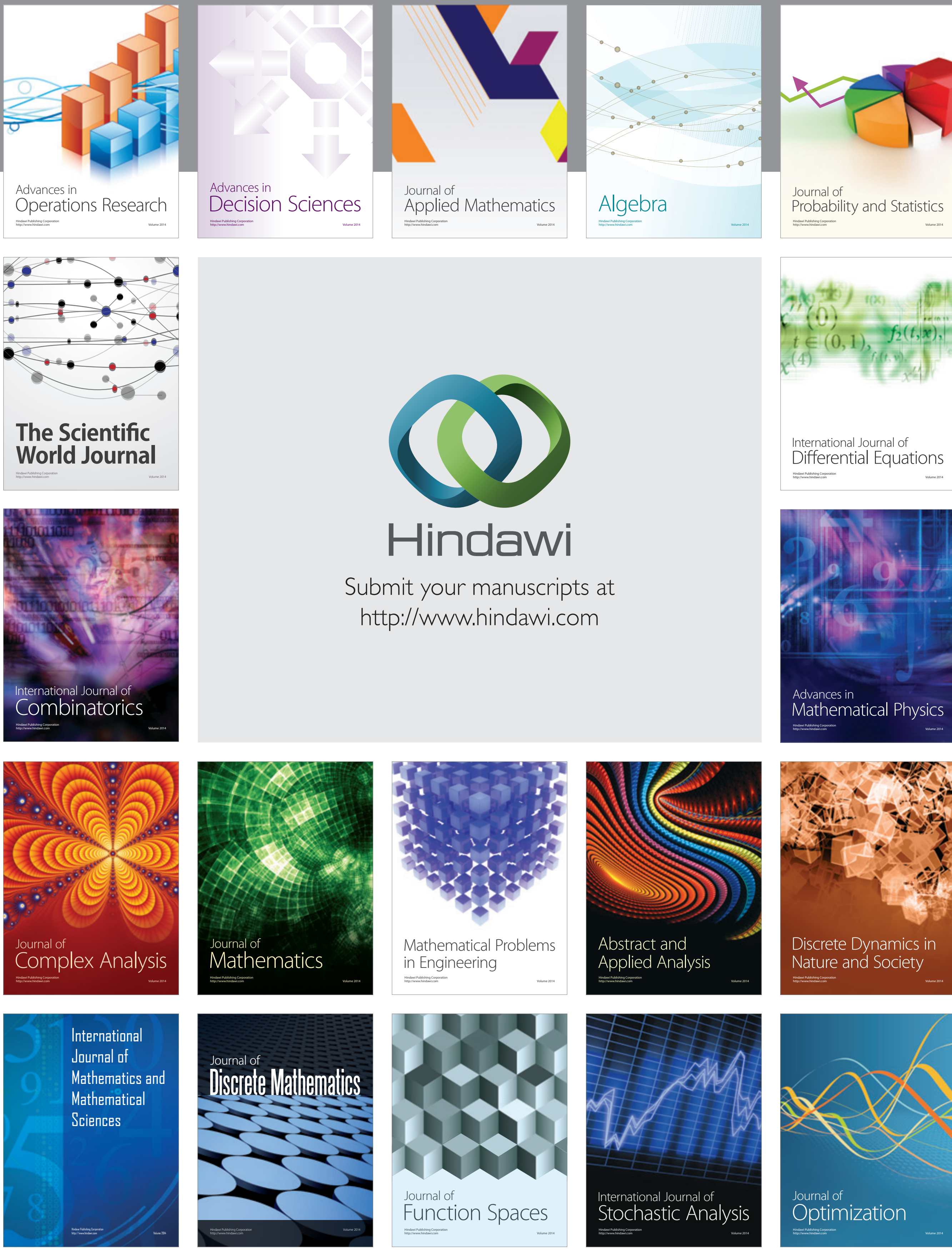
[8] A. Zygmund, *Trigonometrc Series*, Cambridge University Press, Cambridge, 1959.

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
Tbilisi 0193
Georgia
(*E-mail : 05k054@scc.u-tokai.ac.jp*)

Department of Mathematics
Tokai University
Shizuoka, 424-8610
Japan
(*E-mail : nfujii@scc.u-tokai.ac.jp*)

Department of Mathematics
Hokkaido University
Sapporo 060-0810
Japan
(*E-mail: yutaka@math.sci.hokudai.ac.jp*)

(*Received : July 2006*)



Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization