საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 2, №4, 2008
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 2, no. 4, 2008

Mathematics

On Parametrization of Compact Wavelet Matrices

Lasha Ephremidze, Edem Lagvilava

A. Razmadze Mathematical Institute, Tbilisi

(Presented by Academy Member V. Kokilashvili)

ABSTRACT. We give an efficient complete parametrization of wavelet matrices of rank *m*, genus *g*+1, and degree *g*, which are naturally identified with corresponding polynomial paraunitary matrix-functions. The parametrization depends on Wiener-Hopf factorization of unitary matrix-functions with constant determinant given in the unit circle. This method allows us to construct in real time the coefficients of wavelet matrices from the above class. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *wavelet matrices, paraunitary matrix-functions, Wiener-Hopf factorization.*

A wavelet matrix $\mathrm{A}=\left(a_j^r\right)$ of rank m consists of m rows of possibly infinite vectors

$$\mathrm{A}=\begin{pmatrix} \cdots & a_{-1}^0 & a_0^0 & a_1^0 & a_2^0 & \cdots \\ \cdots & a_{-1}^1 & a_0^1 & a_1^1 & a_2^1 & \cdots \\ & \vdots & \vdots & & & \\ \cdots & a_{-1}^{m-1} & a_0^{m-1} & a_1^{m-1} & a_2^{m-1} & \cdots \end{pmatrix}, \tag{1}$$

$a_j^r \in C$, satisfying the following two conditions.

(i) *Quadratic condition:*

$$\sum_j a_{j+ml}^r \overline{a}_{j+mn}^s = m\delta^{rs}\delta_{nl}; \tag{2}$$

(ii) *Linear condition:*

$$\sum_{j=-\infty}^{\infty} a_j^r = m\delta^{r,0}, \tag{3}$$

where δ stands for the Kronecker symbol.

In this paper we assume that (1) is compact, i.e. only the finite number of its entries is different from 0. Therefore, the series in (2) and (3) are only formally infinite, and no problem of convergence appears.

The quadratic condition (2) asserts that the rows of a wavelet matrix $a^r := \left(a_j^r\right)_{j=\overline{-\infty,\infty}}$ have length equal to $\sqrt{m}$ and that they are pairwise orthogonal when shifted by an arbitrary multiple of *m*. The first row a^0 is called the *scaling vector or low-pass filter*, while the remaining rows a^r, $0<r<m$, are called the *wavelet vectors or high-pass filters*. In signal processing applications, the linear constraint (3) implies that a constant signal emerges from the first subband of the *multirate filter bank* (1).

Associate to each wavelet matrix A the matrix function $\mathbf{A}(z)$ as follows: let A_k, $k \in Z$, be submatrices of A of size $m \times m$ defined by $A_k = \left(a^r_{km+s}\right)$, $0 \le s$, $r \le m-1$, in other words, (1) is expressed in terms of block matrices in the form

$$\mathrm{A} = \cdots, A_{-1}, A_0, A_1, A_2, \cdots,$$

and assume

$$\mathrm{A}(z) = \sum_{k=-\infty}^{\infty} A_k z^k . \tag{4}$$

Obviously, there is one-to-one correspondence between matrices (1) and formal series expansions (4), and, for a compact matrix, the corresponding matrix-function is a Laurent polynomial.

It can be verified that the quadratic and the linear constraints on A are equivalent, respectively, to the following two conditions on $\mathbf{A}(z)$:

$$\mathbf{A}(z)=\mathbf{A}^*\left(z^{-1}\right) = mI , \tag{5}$$

where $\mathrm{A}^*\left(z^{-1}\right) := \sum_{k=-\infty}^{\infty} A_k^* z^{-k}$ is the adjoint of $\mathbf{A}(z)$, and

$$\sum_{j=1}^{m} A_{ij}(1) = m\delta_{i,1} , \ 1 \le i \le m , \tag{6}$$

where $\mathbf{A}(z)=(A_{ij}(z))_{i,j=1}^{m}$. The condition (5) means that $\mathbf{A}$ is a *paraunitary matrix-function*.

If U is a unitary matrix of size $m \times m$, $U \in U(m)$, and $\mathbf{A}(z)$ satisfies (5), then $U\mathbf{A}(z)$ satisfies (5) as well. Furthermore, for each paraunitary matrix-function $\mathbf{A}(z)$, there exists and one can explicitly construct a unitary matrix U, such that $U\mathbf{A}(z)$ satisfies the linear condition (6) as well. If U and U' is two such matrices, then

$$U' = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} U ,$$

where $V \in \mathbf{U}(m\text{-}1)$. Thus the construction of paraunitary matrix-functions is decisive for construction of wavelet matrices.

It is said that a wavelet matrix (1) has the *rank m* and the *genus g*, $\mathrm{A} \in WM(m, g; C)$, if the corresponding matrix-function $\mathbf{A}(z)$ has a form

$$\mathrm{A}(z) = \sum_{k=0}^{g-1} A_k z^k . \tag{7}$$

It can be easily shown (see [1, p. 58] that the determinant of a paraunitary matrix-function $\mathbf{A}(z)$ is a monomial in z, that is, there is a nonnegative integer d, called the degree of $\mathbf{A}(z)$, such that

$$\det \mathbf{A}(z) = cz^d .$$

Generically, (7) has degree g-1, although in specific degenerated cases, it can be larger or smaller than g-1.

The relation between compact wavelet matrices and compactly supported wavelet systems as orthonormal functions in $L^2(R)$ is well-known (see [1], Ch. 5).

Theorem ([2], [3], for rank 2; [1, pp. 87, 91], *for rank m>2). Let*

$$A \in WM(m, g; C)$$

be a wavelet matrix and consider the functional difference equation

$$\phi(x) = \sum_{k=0}^{mg-1} a_k^0 \phi(mx - k) \tag{8}$$

called the scaling equation associated with A. Then, there exists a unique $\phi \in L_2(R)$, *called the scaling function, which solves* (8) *and satisfies*

$$\int_R \phi(x)dx = 1 \quad \textit{and} \quad \operatorname{supp}\phi \subset \left[0, (g-1)\left(\frac{m}{m-1}\right) + 1\right].$$

Furthermore, if we define wavelet functions (associated with A) by the formula

$$\psi^r(x)=\sum_{k=0}^{mg-1} a_k^r \phi(mx-k), \quad 1\le r<m ,$$

and consider the collection of functions

$$\phi_{jk}(x)=m^{j/2}\phi\left(m^j x-k\right) \quad j,k\in Z ,$$

$$\psi_{jk}^r(x)=m^{j/2}\psi\left(m^j x-k\right) \quad 1\le r<m ; \qquad j,k\in Z ,$$

called the wavelet system **W**[A] (associated with wavelet matrix A), then there exists an L_2-convergent expansion for each $f\in L_2$:

$$f(x)=\sum_{k=-\infty}^{\infty} c_k\phi_{0k}(x)+\sum_{r=1}^{m-1}\sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty} c_{jk}^r\psi_{jk}^r(x), \tag{9}$$

where the coefficients are given by

$$c_k=\int_R f(x)\overline{\phi_{0k}(x)}\,dx.,$$

$$c_{jk}^r=\int_R f(x)\overline{\psi_{jk}^r(x)}\,dx .$$

Remark. For most wavelet matrices A, the wavelet system **W**[A] is a complete orthonormal system and hence an orthonormal basis for $L_2(R)$, which would imply the above theorem. However, for some wavelet matrices, the system W[A] is not orthonormal, and yet (9) is always true, which means that **W**[A] is a *tight frame*.

Independently of the above-mentioned connection between the wavelet matrices and associated wavelet systems, the former can be directly used in various discrete signal processing applications. Namely, the following theorem is one of the key links between the mathematical theory of wavelets and its practical applications.

Theorem (wavelet matrix expansion [1, p. 80]). *Let*

$$f:Z\to C$$

be an arbitrary function (discrete signal) and let

$$A=\left(a_k^r\right)\in WM(m,g;C)$$

be a wavelet matrix of rank m and genus g. Then f has a unique wavelet matrix expansion

$$f(n)=\sum_{r=0}^{m-1}\sum_{k=-\infty}^{\infty} c_k^r a_{mk+n}^r ,$$

where

$$c_k^r=\frac{1}{m}\sum_{n=-\infty}^{\infty} f(n)\overline{a}_{mk+n}^r .$$

The wavelet matrix expansion is locally finite; that is, for given *n*, only finitely many terms of the series are different from 0.

From the above said the theoretical and practical importance is evident of deeper understanding of the internal structure of paraunitary matrix-functions, which would allow to construct efficiently a wide class of such matrices. To date, the only way of classification of paraunitary matrix functions was via the following factorization theorem. This theorem resembles the factorization of polynomials of degree *d* according to their *d* roots and highest coefficients.

For a unit column vector $v\in C^m$, $v^*v=1$, let

$$V(z):=I-vv^*+vv^*z . \tag{10}$$

Obviously, (10) is a polynomial matrix function of order 1. It can be shown that *V*(*z*) is the paraunitary matrix-function of degree 1 (see [1, p. 59]) and it is called *primitive*.

Theorem (Paraunitary Matrix Factorization, [1, p. 60]). *A paraunitary matrix-function (7) of degree d, where $A_{g-1}\neq 0$ can be factorized as*

$$\mathbf{A}(z)=V_1(z)V_2(z)\cdots V_d(z)U ,$$

where $V_j(z)$, $j=1,2,\dots,d$, are primitive paraunitary matrix-functions and U is a (constant) unitary matrix.

We propose an absolutely new way of parametrization of paraunitary matrix-functions of rank m, genus g+1 and order g, which depends on Wiener-Hopf factorization of unitary matrix-functions (with constant determinant) given on the unit circle in the complex plane. Actually this method was developed in [4], [5] and it allows to construct efficiently matrix-functions of the above type (consequently, to prepare the coefficients of the whole class of compactly supported wavelets) in real time.

Let

$$\mathrm{A(z)}=\begin{pmatrix} A_{1,1}(z) & A_{1,2}(z) & \cdots & A_{1,m}(z) \\ A_{2,1}(z) & A_{2,2}(z) & \cdots & A_{2,m}(z) \\ \vdots & \vdots & \vdots & \vdots \\ A_{m,1}(z) & A_{m,2}(z) & \cdots & A_{m,m}(z) \end{pmatrix},$$

$$A_{rj}(z)=\sum_{k=0}^{g}\alpha_k^{rj}\ ,\ \ 1\le r,j\le m\ ,$$

$$\mathbf{A}(z)\mathrm{A}^*(z^{-1})=I\ ,$$

$$\det\ \mathbf{A}(z)=cz^g\ ,\ \ |c|=1\ .$$

We first convert the matrix-function **A**(z) into a unitary (on the unit circle) matrix-function U(z) by dividing any row of **A**(z) (say, the last row, to be specific) by z^g :

$$\mathrm{U(z)}:=\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m-1,1} & A_{m-1,2} & \cdots & A_{m-1,m} \\ z^{-g}A_{m,1} & z^{-g}A_{m,2} & \cdots & z^{-g}A_{m,m} \end{pmatrix}.$$

Then we have

$$U(z)=U^*(z)=I\ ,\qquad \text{for } |z|=1\ ,$$

$$\det\ U(z)=c\ ,\qquad |c|=1\ ,$$

$$U_{rj}\in L_g^+\ ,\qquad 1\le r<m\ ;\ \ 1\le j\le m\ ,$$

$$U_{m,j}\in L_g^-\ ,\qquad 1\le j\le m\ ,$$

where

$$L_g^+=\left\{f: f(z)=\sum_{k=0}^{g}c_k z^k\right\},$$

$$L_g^-=\left\{f: f(z)=\sum_{k=0}^{g}c_k z^{-k}\right\}.$$

Thus, the two theorems below give a simple and transparent way of one-to-one parametrization of paraunitary matrix-functions of rank m, genus g+1 and degree g. To compare this with the above presented factorization theorem from the simplicity point of view, the proposed parametrization resembles the classification of polynomials of degree d according to their d+1 coefficients.

Theorem 1. (see [5], p. 22) *For each* $m\times m$ *matrix-function F(z) of form*

$$F=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \varphi_1 & \varphi_2 & \varphi_3 & \cdots & \varphi_{m-1} & 1 \end{pmatrix}, \tag{11}$$

where

$$\varphi_j \in L_g^-, \quad 1 \le j \le m-1, \tag{12}$$

there exists a unitary matrix-function U(z) (unique up to a constant unitary right multiplier) of form

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1m} \\ u_{21} & u_{22} & \cdots & u_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1} & u_{m-1,2} & \cdots & u_{m-1,m} \\ u_{m1}^* & u_{m2}^* & \cdots & u_{mm}^* \end{pmatrix}, \tag{13}$$

where

$$u_{kj} \in L_g^+, \quad 1 \le k, j \le m, \tag{14}$$

$$\det U(z) = Const, \tag{15}$$

$$\sum_{j=1}^{m} \left| u_{mj}(0) \right| > 0, \tag{16}$$

such that

$$F(z)U(z) \in L_g^+. \tag{17}$$

The condition (16) means that $z=0$ is not a common zero of the polynomials $u_{mj}(z)$, $j=1,2,\ldots,m$. This can be assumed without loss of generality in the following theorem as well.

Theorem 2. (see [4]) *For each unitary matrix-function U(z) of form (13)-(16) there exists a unique F(z) of form (11), (12) such that (17) holds.*

Observe that if we denote by $F_{-}(z)$ the matrix function of type (11) where each φ_j is replaced by $-\varphi_j$, $j=1,2,\ldots,m-1$, then $F_{-}(z) = (F(z))^{-1}$, so that the equation

$$U(z) = F_{-}(z) \cdot F(z)U(z)$$

gives the Wiener-Hopf right factorization of $U(z)$.

Finally, we should mention that less than 1 sc computer time is required to compute coefficients of matrix-function (13) in Theorem 1 whenever coefficients of functions φ_j, $j=1,2,\ldots,m-1$, are selected in (11) for such large dimensions as m=30 and g=50. This speed of calculations opens the possibility to choose the optimal wavelet matrix for a specific problem, which is the most important step in practical applications, by total selection.

მათემატიკა

კომპაქტური ვეივლეტ-მატრიცების პარამეტრიზაციის შესახებ

ლ. ეფრემიძე, ე. ლაგვილავა

ა. რაზმაძის მათემატიკის ინსტიტუტი, თბილისი

(წარმოდგენილია აკადემიის წევრის ვ. კოკილაშვილის მიერ)

მოცემულია ეფექტური სრული პარამეტრიზაცია ვეივლეტ-მატრიცების რანგით m, რიგით $g+1$ და ხარისხით g, რომლებიც ბუნებრივი გზით გაიგივებულია შესაბამისი რიგის პოლინომიალურ პარაუნიტარულ მატრიც-ფუნქციებთან. მოცემული პარამეტრიზაცია ეყრდნობა ერთეულოვან წრეწირზე განსაზღვრული მუდმივი დეტერმინანტის მქონე პოლინომიალური უნიტარული მატრიც-ფუნქციების ვინერ-ჰოპფის ფაქტორიზაციას. ეს მეთოდი საშუალებას იძლევა რეალურ დროში აგებულ იქნას ვეივლეტ-მატრიცის კოეფიციენტები ზემოაღწერილი კლასიდან.

REFERENCES

1. *I. Daubechies* (1988), Commmun. Pure Appl. Math., **41**: 909-996.
2. *L. Ephremidze, G. Janashia, E. Lagvilava* (2007), Proceedings of the SICE Annual Conference, Sept. 17-20, Kagawa University, Japan.
3. *G. Janashia, E. Lagvilava* (1997), Georgian Math. J., **4**: 439-442.
4. *W. M. Lawton* (1990), J. Math. Phys., **31**(8): 1898-1901.
5. *H. L. Resnikoff, R. O. Wells* (1998), Wavelet Analysis, Berlin: Springer-Verlag.

Received July, 2008