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THE JOHN-NIRENBERG INEQUALITY FOR ERGODIC SYSTEMS

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Abstract

The John-Nirenberg inequality is generalized to the ergodic case.

1. Introduction

Let $(X, \mathbb{S}, \mu)$ be a finite measure space, $\mu(X) < \infty$, and $T : X \to X$ be a measure-preserving ergodic transformation (see, e.g., [6] for definitions). For an integrable function $f : X \to \mathbb{R}$, $f \in L(X)$, the ergodic sharp maximal function is defined as

$$f^{\sharp}(x) = \sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} |f(T^k x) - E_n(f, x)|, \tag{1}$$

where $E_n(f, x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$, and the ergodic BMO norm of f is

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defined as (see [1])

$$\| f \|_{\mathrm{BMO}} = \operatorname{ess\,sup} f^{\sharp}. \tag{2}$$

In the present paper we generalize the classical John-Nirenberg theorem [5] to the ergodic case.

Theorem. *There exist universal constants* C_1 *and* C_2 *such that for any finite measure space* $(X, \mathbb{S}, \mu)$, *measure-preserving ergodic transformation* T *and* $f \in L(X)$, *we have*

$$\mu\{x \in X : | f(x) - E(f) | > \lambda\} \leq C_1 \mu(X) \exp\left(\frac{-\lambda C_2}{\| f \|_{\mathrm{BMO}}}\right), \tag{3}$$

where $E(f) = (1/\mu(X))\int_X f d\mu$ *and* $\lambda \geq 0$.

It is sufficient to take constants $C_1 = \sqrt{e}$ *and* $C_2 = 1/4e$.

Garsia [4] formulated and proved the John-Nirenberg inequality for martingales and Pitt [7] generalized this inequality for submartingales. We give a simple and transparent proof of inequality (3) depending on a new method of transferring results on the real line to the general ergodic setting developed in [2], [3]. For the sake of completeness, we give the proof of the discrete version of John-Nirenberg theorem as well.

2. The Discrete Case

In order to deal with the discrete case, we consider non-negative functions $h : \mathbb{N}_0 \to \mathbb{R}_+$ defined on the set of non-negative integers. Let $\mathcal{I}$ be the collection of all "intervals" in $\mathbb{N}_0$,

$$\mathcal{I} = \{I : I = I_{m,n} := \{m, m+1, \ldots, n-1\}, m < n, m, n \in \mathbb{N}_0\}.$$

For $I \in \mathcal{I}$, let $| I | = \mathrm{card}(I)$ denote the number of elements in I, and

$$E_I(h) = \frac{1}{| I |}\sum_{k \in I} h(k).$$

Suppose $\mathcal{I}_2$ is the set of all "intervals" $I \in \mathcal{I}$ for which $| I | = 2^p$ for some $p \in \mathbb{N}_0$.

The following lemma is the discrete version of the Calderón-Zygmund decomposition and can be proved in a similar way as its continuous analog.

Lemma 1. *Let* $g : I \to \mathbb{R}_+$, *where* $I \in \mathcal{I}_2$, *and* $\lambda \in (E_I(g), \max_{k \in I} g(k)]$. *Then there exist disjoint "intervals"* $I_i \subset I$, $i = 1, 2, \dots, n$, $I_i \cap I_j = \varnothing$ *for* $i \neq j$, *such that* $I_i \in \mathcal{I}_2$, $\{k \in I : g(k) \geq \lambda\} \subset \bigcup_{i=1}^{n} I_i$, *and*

$$\lambda \leq \frac{1}{|I_i|} \sum_{k \in I_i} g(k) < 2\lambda, \quad i = 1, 2, \dots, n.$$

The following lemma is a discrete analog of the John-Nirenberg theorem (see [8]).

Lemma 2. *For each* $h : \mathbb{N}_0 \to \mathbb{R}_+$, $I \in \mathcal{I}_2$, *and* $\lambda \geq 0$, *we have*

$$\operatorname{card}\{k \in I : |h(k) - E_I(h)| > \lambda\} \leq \sqrt{e}|I| \exp\left(-\frac{\lambda}{4e\|h\|_{\mathrm{BMO}}}\right), \tag{4}$$

where

$$\|h\|_{\mathrm{BMO}} = \sup_{I \in \mathcal{I}} \frac{1}{|I|} \sum_{k \in I} |h(k) - E_I(h)|.$$

Proof. It suffices to prove (4) for h with

$$\|h\|_{\mathrm{BMO}} = 1, \tag{5}$$

so we will assume this.

Using Lemma 1 for function $g(k) = |h(k) - E_I(h)|$, $k \in I$, and $\lambda = e$, the set $\{k \in I : |h(k) - E_I(h)| \geq e\}$ (whenever it is not empty) can be covered with disjoint "subintervals" $I_i \in \mathcal{I}_2$, $i = 1, 2, \dots, n$, such that

$$e \leq \frac{1}{|I_i|} \sum_{k \in I_i} |h(k) - E_I(h)| < 2e \quad \text{for each } i = 1, 2, \dots, n.$$

Hence

$$\sum_{i=1}^{n} |I_i| \leq \frac{1}{e} \sum_{i=1}^{n} \sum_{k \in I_i} |h(k) - E_I(h)| \leq \frac{1}{e} \sum_{k \in I} |h(k) - E_I(h)| \leq \frac{1}{e}|I| \tag{6}$$

(see (5)) and

$$2e > \frac{1}{|I_i|}\sum_{k\in I_i}|h(k) - E_I(h)| \geq |E_{I_i}(h) - E_I(h)| \quad \text{for each } i = 1, 2, ..., n. \tag{7}$$

We suppose that $I =: I_1^0$ is the "interval" of level 0 and "intervals" $I_i =: I_i^1$, $i = 1, 2, ..., n_1$, are of level 1, and we continue to construct "intervals" of the next levels $N = 2, 3, \ldots$. Namely, having disjoint "intervals" $I_i^N \in \mathcal{I}_2$, $i = 1, 2, ..., n_N$, which satisfy that each I_i^N is a subset of some I_j^{N-1}, and

$$\{k \in I : |h(k) - E_I(h)| \geq 2eN\} \subset \bigcup_{i=1}^{n_N} I_i^N, \tag{8}$$

$$\sum_{i=1}^{n_N} |I_i^N| \leq \frac{1}{e^N}|I| \tag{9}$$

and

$$|E_{I_i^N}(h) - E_I(h)| \leq 2eN \quad \text{for each } i = 1, 2, ..., n_N, \tag{10}$$

we use Lemma 1 for each I_i^N (whenever $\{k \in I_i^N : |h(k) - E_{I_i^N}(h)| \geq e\} \neq \emptyset$), the function $g(k) = |h(k) - E_{I_i^N}(h)|$, $k \in I_i^N$, and $\lambda = e$ to identify disjoint "subintervals" $I_{ij}^N \in \mathcal{I}_2$, $j = 1, 2, ..., n_{N_i}$, which satisfy

$$\{k \in I_i^N : |h(k) - E_{I_i^N}(h)| \geq e\} \subset \bigcup_{j=1}^{n_{N_i}} I_{ij}^N \tag{11}$$

and

$$e \leq \frac{1}{|I_{ij}^N|}\sum_{k\in I_{ij}^N} |h(k) - E_{I_i^N}(h)| < 2e \quad \text{for each } j = 1, 2, ..., n_{N_i}.$$

So, like (6) and (7)

$$\sum_{j=1}^{n_{N_i}} |I_{ij}^N| \leq \frac{1}{e} \sum_{j=1}^{n_{N_i}} \sum_{k \in I_{ij}^N} |h(k) - E_{I_i^N}(h)|$$

$$\leq \frac{1}{e} \sum_{k \in I_i^N} |h(k) - E_{I_i^N}(h)| \leq \frac{1}{e} |I_i^N| \tag{12}$$

and

$$2e > \frac{1}{|I_{ij}^N|} \sum_{k \in I_{ij}^N} |h(k) - E_{I_i^N}(h)| \geq |E_{I_{ij}^N}(h) - E_{I_i^N}(h)| \tag{13}$$

for each $j = 1, 2, \ldots, n_{N_i}$.

If we now reindex all the intervals I_{ij}^N, $i = 1, 2, \ldots, n_N$, $j = 1, 2, \ldots, n_{N_i}$, in arbitrary order and call them I_i^{N+1}, $i = 1, 2, \ldots, n_{N+1}$, then (12) and (9) imply that

$$\sum_{i=1}^{n_{N+1}} |I_i^{N+1}| \leq \frac{1}{e} \sum_{i=1}^{n_N} |I_i^N| \leq \frac{1}{e^{N+1}} |I|$$

and (13) and (10) imply that

$$|E_{I_i^{N+1}}(h) - E_I(h)| \leq 2e(N+1) \quad \text{for each } i = 1, 2, \ldots, n_{N+1}.$$

Thus, (9) and (10) hold whenever we change N by $N + 1$ in these inequalities. Now we wish to show that the same happens with relation (8) as well. Indeed, $|h(k) - E_I(h)| \geq 2e(N+1)$ implies that $k \in I_i^N$ for some $i \in \{1, 2, \ldots, n_N\}$ (by virtue of (8)) and taking into account (10) we can conclude that $|h(k) - E_{I_i^N}(h)| \geq 2e$. Hence, by virtue of (11), $k \in I_i^{N+1}$ for some $i \in \{1, 2, \ldots, n_{N+1}\}$. Thus (8) holds if we change N by $N + 1$.

We have shown that conditions (8)-(10) will be satisfied by the intervals of all level N in our construction process. Since I consists of finite number of points, this process will be finite, i.e., there will be such M that $\{k \in I_i^M : |h(k) - E_{I_i^M}(h)| > e\}$ will be empty for each $i \in \{1, 2, ..., n_M\}$. (Since each discrete "interval" consists at least one point, we can estimate from (9) that $M \le \log|I|$.)

Now we are ready to prove (4), which obviously holds whenever $\lambda \in (0, 2e)$.

If $\lambda \ge 2e$ is such that $\{k \in I : |h(k) - E_I(h)| \ge \lambda\} \ne \varnothing$, then there exists N such that $2eN \le \lambda < 2e(N+1)$ and (8) and (9) hold. Hence

$$\begin{aligned}\operatorname{card}\{k \in I : |h(k) - E_I(h)| > \lambda\} &\le \operatorname{card}\{k \in I : |h(k) - E_I(h)| > 2eN\} \\ &\le \sum_{i=1}^{n_N} |I_i^N| \le |I| \exp(-N) \\ &\le |I| \exp\left(-\frac{\lambda}{4e}\right).\end{aligned}$$

Thus (4) is proved. □

3. The Proof of Theorem

By the ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = E(f) \tag{14}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\{|f-E(f)|>\lambda\}}(T^k x) = \frac{1}{\mu(X)} \mu\{|f - E(f)| > \lambda\} \tag{15}$$

for a.a. $x \in X$.

For $x \in X$, let

$$h_x(k) = f(T^k x), \quad k \in \mathbb{N}_0. \tag{16}$$

Obviously, for each $k \in \mathbb{N}_0$, we have (see (1))

$$f^{\sharp}(T^k x) = \sup_{n>k} \frac{1}{n-k} \sum_{m=k}^{n-1} |f(T^m x) - E_{I_{k,n}}(h_x)|$$

and consequently $\| h_x \|_{\text{BMO}} = \sup_{k \in \mathbb{N}_0} f^{\sharp}(T^k x)$.

By virtue of definition (2), $\mu\{f^{\sharp} > \| f \|_{\text{BMO}}\} = 0$. Hence $\mu(U) = 0$, where $U = \bigcup_{k=0}^{\infty} T^{-k}\{f^{\sharp} > \| f \|_{\text{BMO}}\} = \{x \in X : f^{\sharp}(T^k x) > \| f \|_{\text{BMO}}$ for some $k \in \mathbb{N}_0\}$, and for each $x \in X \setminus U$ we have

$$\| h_x \|_{\text{BMO}} \le \| f \|_{\text{BMO}}. \tag{17}$$

It is sufficient to prove that

$$\mu\{| f - E(f) | > \lambda\} \le C_1 \mu(X) \exp\left(\frac{-(\lambda - \varepsilon) C_2}{\| f \|_{\text{BMO}}} \right) \tag{18}$$

for each $\varepsilon \in (0, \lambda)$, where $C_1 = \sqrt{e}$ and $C_2 = 1/4e$.

Fix any $x \in X$ for which (14), (15) and (17) hold (we can select such x since, as it was discussed, almost all points satisfy these conditions).

Let n be an arbitrary positive integer so large that (see (14), (16))

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} h_x(k) - E(f) \right| = | E_{I_{0,n}}(h_x) - E(f) | < \varepsilon, \tag{19}$$

where ε is the same as in (18). We can assume that $n = 2^p$ as well.

By virtue of (15), (16), (19), Lemma 2, and (17), we have

$$\mu\{| f - E(f) | > \lambda\}$$

$$= \lim_{n \to \infty} \frac{\mu(X)}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\{| f - E(f) | > \lambda\}}(T^k x)$$

$$= \lim_{n \to \infty} \frac{\mu(X)}{n} \text{card}\{k \in I_{0,n} : | f(T^k x) - E(f) | > \lambda\}$$

$$= \lim_{n \to \infty} \frac{\mu(X)}{n} \text{card}\{k \in I_{0,n} : | h_x(k) - E(f) | > \lambda\}$$

$$\leq \limsup_{n\to\infty} \frac{\mu(X)}{n} \operatorname{card}\{k \in I_{0,n} : |h_x(k) - E_{I_{0,n}}(h_x)| > \lambda - \varepsilon\}$$

$$\leq \limsup_{n\to\infty} \frac{\mu(X)}{n} C_1 |I_{0,n}| \exp\left(-\frac{(\lambda - \varepsilon)C_2}{\| h_x \|_{\mathrm{BMO}}}\right)$$

$$\leq C_1 \mu(X) \exp\left(\frac{-(\lambda - \varepsilon)C_2}{\| f \|_{\mathrm{BMO}}}\right).$$

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References

[1] R. R. Coifman and G. Weiss, Maximal functions and H^p spaces defined by ergodic transformations, Proc. Nat. Acad. Sci. U.S.A. 70 (1973), 1761-1763.

[2] L. Ephremidze, On the generalization of the Stein-Weiss theorem for the ergodic Hilbert transform, Studia Math. 155 (2003), 67-75.

[3] L. Ephremidze and R. Sato, On the generalization of the Riesz-Zygmund theorem for the ergodic Hilbert transform, Ergodic Theory Dynam. Systems 27 (2007), 113-122.

[4] A. M. Garsia, Martingale Inequalities: Seminar Notes on Recent Progress, Reading, Benjamin, 1973.

[5] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 (1961), 415-426.

[6] K. E. Petersen, Ergodic Theory, Cambridge Univ. Press, Cambridge, 1983.

[7] L. D. Pitt, A one sided John-Nirenberg inequality for submartingales, Probab. Theory Related Fields 34 (1976), 29-32.

[8] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, London Math. Soc. Student Texts 37, Cambridge Univ. Press, 1997.