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REMARK ON OUTER ANALYTIC MATRIX-FUNCTIONS

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Abstract. We present an analytic proof of a Helson-Lowdenslager theorem which characterizes the outer matrix-functions as those with outer determinant.

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1. Introduction

Wiener's matrix spectral factorization theorem asserts that (see [5], [3]) *if*

$$S(t)=\begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1n}(t)\\ s_{21}(t) & s_{22}(t) & \cdots & s_{2n}(t)\\ \vdots & \vdots & \vdots & \vdots\\ s_{n1}(t) & s_{n2}(t) & \cdots & s_{nn}(t)\end{pmatrix},$$

$|t|=1$, $s_{ij}\in L_1(\mathbb{T})$, *is a positive definite matrix-function satisfying*

$$\log\det S(t)\in L_1(\mathbb{T}),$$

then there exists a unique (up to a constant right unitary multiplier) outer analytic $n\times n$ *matrix function* $S^+(z)$, $|z|<1$, *with entries from the Hardy space* H_2, *such that*

$$S(t)=S^+(t)\cdot\big(S^+(t)\big)^*\quad a.e.\ on\ \ \mathbb{T}.$$

The requirement on the spectral factor $S^+(z)$ to be *outer* is decisive for the uniqueness of spectral factorization and exactly such type of factorizations are important for applications.

Two different definitions of outer analytic matrix-functions follows (see §2 for notation): $S^+(z)\in H_2(n\times n)$ is called outer if

(i) $\det S^+(z)$ is outer;
(ii) $\operatorname{clos}\{S^+\cdot\mathcal{P}(n)\}=H_2(n)$.

As an ingredient of the proof of the matrix spectral factorization theorem,

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the equivalence of these two definitions was proved by Helson and Lowdenslager [3] using the methods of invariant subspaces (a nontrivial part of this equivalence, (i) $\Rightarrow$ (ii), is actually the generalization of Beurling theorem to the matrix case)

Recently authors [1] have provided the new proof of the matrix spectral factorization theorem using the methods of Complex Analysis. The uniqueness of the spectral factor (see [2]), as well as "analytic" definition (i) of outer matrix functions, was heavily used in this proof which made it shorter and transparent. In the present note we would like to demonstrate that the implication (i) $\Rightarrow$ (ii) can also be proved using "analytic" methods.

2. Notation

Let $L_p = L_p(\mathbb{T})$, $p > 0$, be the Lebesgue class of p-integrable (complex) functions defined on $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$, and $L_p(n \times n)$ be the set of $n \times n$ matrix-functions with entries from L_p. $L_p(n \times 1)$ is denoted by $L_p(n)$.

$H_p = H_p(\mathbb{D})$, $p > 0$, is the Hardy space of analytic functions (defined in the unit disk $\mathbb{D}$ of the complex plane),

$$H_p = \left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty \right\}$$

(see [4] for basic properties of the Hardy spaces) and $H_p(n \times n)$ is the set of $n \times n$ matrix-functions with entries from H_p. $H_p(n \times 1)$ is denoted by $H_p(n)$.

$L_p^+ = L_p^+(\mathbb{T})$ denotes the class of boundary functions of H_p. (Correspondingly are defined $L_p^+(n \times n)$ and $L_p^+(n)$.) It is well-known that there is a one-to-one correspondence between $H_p(\mathbb{D})$ and $L_p^+(\mathbb{T})$, $p > 0$, so that we naturally identify these classes and we can speak about the values of $f \in L_p^+(\mathbb{T})$ inside $\mathbb{D}$. Furthermore, we denote the boundary function of $f = f(z) \in H_p(\mathbb{D})$ by $f = f(t) \in L_p^+(\mathbb{T})$, i.e. $f(z)|_{z=t} =: f(t)$. For $p \geq 1$, $L_p^+(\mathbb{T})$ coincides with the class of functions from $L_p(\mathbb{T})$ whose Fourier coefficients with negative indices are equal to zero.

$f \in H_p$ is called *outer*, $f \in H_p^O$, if

$$f(z) = c \cdot \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \left| f(e^{i\theta}) \right| d\theta \right), \qquad |c| = 1.$$

If M is a matrix, then $\overline{M}$ denotes the matrix with conjugate entries and $M^* := \overline{M}^T$.

Let $\mathcal{P}$ be the set of polynomials $\mathcal{P} = \{\sum_{k=0}^m c_k z^k,\ c_k \in \mathbb{C},\ k = 1, 2, \ldots, m\}$. Correspondingly are defined $\mathcal{P}(n \times n)$ and $\mathcal{P}(n)$.

Under these notation, one can formulate an essence of the Beurling theorem as $f \in H_2^O \Rightarrow \mathrm{clos}\{f \cdot \mathcal{P}\}$ (= closer of $\{fP : P \in \mathcal{P}\}$)$= H_2$. An analytic proof of this theorem can be found in [4] (see Ch.IV, E) and we generalize this proof to the matrix case in §3.

We will make use of the following generalization of Smirnov's theorem (see [4], p. 109): *If* $f(z) = g(z)/h(z)$*, where* $g \in H_{p_1}$*,* $p_1 > 0$*, and* $h \in H^O_{p_2}$*,* $p_2 > 0$ *and* $f(t) \in L_p(\mathbb{T})$*,* $p > 0$*, then* $f(z) \in H_p$ *.*

3. A New Proof of the Helson-Lowdenslager Theorem

First we prove the following

Lemma. If $M(z) \in H_2(n \times n)$,

$$\det M(z) \in H^O_p\,, \qquad (p = 2/n) \tag{1}$$

$F(t) \in L_2(n)$, and $M(t)F(t) \in L^+_1(\mathbb{T})$, then

$$F(t) \in L^+_2(n)\,. \tag{2}$$

Proof. Since $M(t)F(t) =: \Phi(t) \in L^+_1(\mathbb{T})$, we have

$$F(t) = M^{-1}(t)\Phi(t) = \frac{1}{\det M(t)} \cdot \operatorname{Adj} M(t) \cdot \Phi(t).$$

Since (1) holds and $\operatorname{Adj} M(t) \cdot \Phi(t) \in L^+_p$ for some $p > 0$, we can apply the above mentioned generalization of Smirnov's theorem to conclude that the entries of $F(t)$ belongs to L^+_2. Thus (2) holds. □

Theorem. Let $M(z) \in H_2(n \times n)$ and $\det M(z) \in H^O_{2/n}$. Then

$$\operatorname{clos}\{M \cdot \mathcal{P}(n)\} = H_2(n).$$

Proof. Define the scalar product in $L_2(n)$ by the equation

$$\langle F, G\rangle = \sum_{j=1}^{n} \int_{\mathbb{T}} f_j(t)\overline{g_j(t)}\,dt = \int_{\mathbb{T}} G^*(t) \cdot F(t)\,dt$$

where $F = (f_1, f_2, \dots, f_n)^T$ and $G = (g_1, g_2, \dots, g_n)^T$.

Obviously $M \cdot \mathcal{P}(n) \subset H_2(n)$. Assume

$$\operatorname{clos}\{M \cdot \mathcal{P}(n)\} \neq H_2(n).$$

Then it follows from an elementary theory of Hilbert spaces that there exists $\Psi \in H_2(n)\setminus \operatorname{clos}\{M \cdot \mathcal{P}(n)\}$ such that $\Psi \perp \operatorname{clos}\{M \cdot \mathcal{P}(n)\}$. Hence $\langle MP, \Psi\rangle = \int_{\mathbb{T}} \Psi^*(t)M(t)P(t)\,dt = 0$ for each $P \in \mathcal{P}(n)$. Thus every entry of the matrix-function $M^T(t)\overline{\Psi(t)}$ has all Fourier coefficients with non-positive indices equal to 0. Consequently $M^T(t)\overline{\Psi(t)}t^{-1} \in L^+_1(n)$, and by virtue of the above lemma $\overline{\Psi(t)}t^{-1} \in L^+_2(n)$. Since we also have $\Psi(t) \in L^+_2(n)$, we conclude that $\Psi(t) = 0$, which is a contradiction. □

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