Available online at www.sciencedirect.com

ScienceDirect

Transactions of A. Razmadze Mathematical Institute 171 (2017) 16–23

Transactions of A. Razmadze Mathematical Institute

www.elsevier.com/locate/trmi

Original article

On Robinson's Energy Delay Theorem

L. Ephremidze[a,b,*], W.H. Gerstacker[c], I. Spitkovsky[a]

[a] *New York University Abu Dhabi, United Arab Emirates*
[b] *A. Razmadze Mathematical Institute, Georgia*
[c] *Friedrich–Alexander University Erlangen-Nürnberg, Institute for Digital Communications, Germany*

Received 30 November 2016; accepted 25 December 2016
Available online 30 January 2017

Abstract

An elementary proof of Robinson's Energy Delay Theorem on minimum-phase functions is provided. The situation in which the energy conservation property holds for an infinite number of lags is fully described.

Keywords: Minimum-phase functions; Hardy spaces

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane and $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$ be its boundary. The set of all analytic in $\mathbb{D}$ functions is denoted by $\mathcal{A}(\mathbb{D})$. The Hardy space $H^2 = H^2(\mathbb{D})$ consists of all the functions $f \in \mathcal{A}(\mathbb{D})$ the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

of which satisfy the condition

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

In engineering, these functions are known as *z-transforms* (resp. *transfer functions*) of discrete-time *causal signals* (resp. *filter impulse responses*) with a finite energy. It is well known that the boundary values of $f \in H^2$ exist a.e.,

$$f_+(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta}) \quad \text{for a.a. } \theta \in [0, 2\pi), \tag{1}$$

* Corresponding author at: A. Razmadze Mathematical Institute, Georgia.
E-mail address: lasha@rmi.ge (L. Ephremidze).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2016.12.004

and $f_+ \in L^2(\mathbb{T})$, the Lebesgue space of square integrable functions on $\mathbb{T}$. Furthermore, $f_+ \in L^2_+(\mathbb{T}) := \{f \in L^2(\mathbb{T}) : c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\, d\theta = 0 \text{ for } n < 0\}$. Actually, there is a one-to-one correspondence between H^2 and $L^2_+(\mathbb{T})$, and therefore we may naturally identify these two classes.

For any function $f \in H^2$, the inequality

$$|f(0)| \le \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f_+(e^{i\theta})|\, d\theta\right) \tag{2}$$

holds (see, e.g., [1, Th. 17.17]). The extreme functions for which (2) turns into an equality are called *outer*. In engineering they are also known as *minimum-phase*, or *optimal*, functions. According to the original definition of outer functions by Beurling [2], they admit the representation

$$f(z) = c \cdot \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f_+(e^{i\theta})|\, d\theta\right), \tag{3}$$

where c is a unimodular constant. This representation easily implies that the equality holds in (2) for outer functions and it can be proved that the converse is also true. In particular, boundary values of the modulus of an outer function uniquely determine the function itself up to a constant multiple with absolute value 1.

The following property of minimum-phase functions, first observed by Robinson [3], plays an important role in several signal processing applications.

Theorem 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be H^2-functions satisfying*

$$|f_+(e^{i\theta})| = |g_+(e^{i\theta})| \text{ for a.e. } \theta. \tag{4}$$

If f is of minimum-phase, then for each N,

$$\sum_{n=0}^{N} |a_n|^2 \ge \sum_{n=0}^{N} |b_n|^2. \tag{5}$$

Robinson gave a physical interpretation to inequality (5) "that among all filters with the same gain, the outer filter makes the energy built-up as large as possible, and it does so for every positive time" [4] and found geological applications of minimum-phase waveforms. Consequently, the term *minimum-delay* [5, p. 211] functions is being used to describe optimal functions, and Theorem 1 is known as the Energy Delay Theorem within the geological community [6, p. 52].

Theorem 1 was further extended to the matrix polynomial case and used in MIMO communications in [7]. In [8], the theorem is formulated and proved for general operator valued functions in abstract Hilbert spaces.

In this paper, we provide a very short and simple proof of Theorem 1 based on classical facts from the theory of Hardy spaces. This is done in Section 3, while the modification of this proof fitting the matrix case is discussed in Section 4. In final Section 5, we treat the situation in which (5) turns into an equality for infinitely many values of N. The preliminary Section 2 contains some notation and known results, included for convenience of reference.

2. Notation

Let $L^p = L^p(\mathbb{T})$, $0 < p \le \infty$, be the Lebesgue space of p-integrable complex functions f with the norm $\|f\|_{L^p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta\right)^{\frac{1}{p}}$ for $p \ge 1$ (with the standard modification for $p = \infty$), and let $H^p = H^p(\mathbb{D})$, $0 < p \le \infty$, be the Hardy space

$$\left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta < \infty \right\}$$

with the norm $\|f\|_{H^p} = \sup_{r<1} \|f(re^{i\cdot})\|_{L^p}$ for $p \ge 1$ (H^∞ is the space of bounded analytic functions with the supremum norm). It is well known that boundary value function f_+ (see (1)) exists for every $f \in H^p$, $p > 0$, and belongs to L^p. Furthermore,

$$\|f\|_{H^p} = \|f_+\|_{L^p} \tag{6}$$

for every $p \geq 1$, and it follows from the standard Fourier series theory that

$$\left\| \sum_{n=0}^{\infty} a_n z^n \right\|_{H^2} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}. \tag{7}$$

Condition

$$\int_0^{2\pi} \log |f_+(e^{i\theta})| \, d\theta > -\infty \tag{8}$$

holds for every $f \in H^p$, and the function f is called outer if the representation (3) is valid. We have the equality (the optimality condition) in (2) if and only if f is outer (see [1, Th. 17.17]). One can check, using the Hölder inequality, that if f and g are outer functions from H^p and H^q, respectively, then the product fg is the outer function from $H^{pq/(p+q)}$.

A function $u \in \mathcal{A}(\mathbb{D})$ is called *inner* if $u \in H^\infty$ and

$$|u_+(e^{i\theta})| = 1 \quad \text{for a.a. } \theta \in [0, 2\pi). \tag{9}$$

If in addition $u(z) \neq 0$ for $z \in \mathbb{D}$, then it is called a *singular inner* function. Every $h \in H^p$ can be factorized as

$$h(z) = B(z)\mathcal{I}(z) f(z), \tag{10}$$

where $B(z) = z^m \prod_{n=1} \frac{|\omega_n|}{\omega_n} \frac{\omega_n - z}{1 - \overline{z}_n z}$ is a Blaschke product, $\mathcal{I}$ is a singular inner function and f is an outer function from H^p. (Observe that $|h_+| = |f_+|$ a.e.) In these terms, a function is outer if and only if the inner factor in factorization (10) is constant, i.e., without loss of generality, $B \equiv \mathcal{I} \equiv 1$.

These definitions and factorization (10) are classical in mathematical theory of Hardy spaces. However, engineers frequently discard the middle term in the factorization (10): a singular inner factor, having the form

$$\mathcal{I}(z) = \exp\left(-\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu_s(\theta) \right),$$

where μ_s is a singular measure on $[0, 2\pi)$, is trivial in case of rational f and thus not encountered in practice. So, they sometimes define a minimum-phase function $f \in H^2(\mathbb{D})$ by the condition $1/f \in \mathcal{A}(\mathbb{D})$ (i.e. $f(z) \neq 0$ for $z \in \mathbb{D}$). This definition can be used for rational functions, however, not for arbitrary analytic functions. As an example of a singular inner function $\mathcal{I}$ shows, the inequality in (2) might be strict in this case ($|\mathcal{I}(0)| < 1$, while $\int_0^{2\pi} \log |\mathcal{I}_+(e^{i\theta})| \, d\theta = 0$). So, the equality may not hold in (2) even if $f^{-1} \in \mathcal{A}(\mathbb{D})$, as it was incorrectly claimed in [9, p. 574].

We will make use of the following standard result from the theory of Hardy spaces (see [10, p. 109]).

Smirnov's Generalized Theorem: if $f = g/h$, where $g \in H^p$, $p > 0$, h is an outer function from H^q, $q > 0$, and $f_+ \in L^r$, $r > 0$, then $f \in H^r$.

For a positive integer N, let P_N be the projection operator on H^2 defined by

$$P_N \colon \sum_{n=0}^{\infty} a_n z^n \longmapsto \sum_{n=0}^{N} a_n z^n.$$

For $h(z) = \sum_{n=0}^{\infty} \gamma_n z^n \in \mathcal{A}(\mathbb{D})$, let $\operatorname{supp}(\hat{h}) = \{n \in \mathbb{N}_0 : \gamma_n \neq 0\}$.

Now we turn to matrices and matrix functions. For a given set X of scalars or scalar valued functions, let $X_{m \times n}$ stand for the set of $m \times n$ matrices with the entries from X. The elements of $L^p_{d \times d}$ (resp. $H^p_{d \times d}$) are assumed to be matrix functions with domain $\mathbb{T}$ (resp. $\mathbb{D}$) and range $\mathbb{C}_{d \times d}$, and of course $F_+ \in L^p_{d \times d}$ for $F \in H^p_{d \times d}$.

For $M \in \mathbb{C}_{d \times d}$, we consider the Frobenius norm of M:

$$\|M\|_2 = \left(\sum_{i=1}^{d} \sum_{j=1}^{d} |m_{ij}|^2 \right)^{1/2} = \left(\operatorname{Tr}(MM^*) \right)^{1/2},$$

where $M^* = \overline{M}^T$, and for $F \in H^p_{d\times d}$, we define

$$\|F\|_{H^2_{d\times d}} = \left(\sum_{i=1}^{d}\sum_{j=1}^{d}|f_{ij}|^2_{H^2}\right)^{1/2}.$$

Similarly, we define $\|F_+\|_{L^2_{d\times d}}$ for $F_+ \in L^2_{d\times d}$. By virtue of (6), we have

$$\|F\|_{H^2_{d\times d}} = \|F_+\|_{L^2_{d\times d}} \tag{11}$$

and, as in (7),

$$\left\|\sum_{n=0}^{\infty} A_n z^n\right\|_{H^2_{d\times d}} = \left(\sum_{n=0}^{\infty}\|A_n\|_2^2\right)^{1/2} \tag{12}$$

for any sequence of matrix coefficients $A_0, A_1, \dots$ from $\mathbb{C}_{d\times d}$.

A matrix function $F \in H^2_{d\times d}$ is called *outer* if $\det F$ is an outer function from $H^{2/d}$. This definition is equivalent to number of other definitions of outer matrix functions (see, e.g., [11]). On the other hand, a matrix function $U \in \mathcal{A}(\mathbb{D})_{d\times d}$ is called *inner* if $U \in H^\infty_{d\times d}$ and U_+ is unitary a.e.:

$$U_+(e^{i\theta})U_+^*(e^{i\theta}) = I_d \quad \text{for a.a. } \theta \in [0, 2\pi). \tag{13}$$

3. Proof of Theorem 1

According to (7), the statement of Theorem 1 is equivalent to

$$\|P_N(f)\|_{H^2} \geq \|P_N(g)\|_{H^2}, \quad N \in \mathbb{N}_0. \tag{14}$$

For any bounded analytic function $u \in H^\infty$, we have

$$P_N(uf) = P_N\big(u \cdot P_N(f)\big) \tag{15}$$

since $P_N\big(u \cdot P_N(f)\big) = P_N\big(u(f - (f - P_N(f)))\big) = P_N(uf) - P_N\big(u(f - P_N(f))\big) = P_N(uf)$. Here we utilized the fact that the kernel of P_N is the set of functions in H^2 having zero as its root of multiplicity at least N, and thus invariant under multiplication by u.

Since (4) holds, by virtue of Beurling factorization (10), there exists an inner function u such that $g = uf$. Therefore, taking into account (6), (9), and (15), we get

$$\|P_N(f)\|_{H^2} = \|uP_N(f)\|_{H^2} \geq \|P_N\big(uP_N(f)\big)\|_{H^2} = \|P_N(uf)\|_{H^2} = \|P_N(g)\|_{H^2}. \tag{16}$$

Thus (14) follows, and Theorem 1 is proved.

4. The matrix case

In this section we prove the following matrix version of Theorem 1.

Theorem 2. *Let $F(z) = \sum_{n=0}^{\infty} A_n z^n$, $A_n \in \mathbb{C}_{d\times d}$, and $G(z) = \sum_{n=0}^{\infty} B_n z^n$, $B_n \in \mathbb{C}_{d\times d}$, be matrix functions from $H^2_{d\times d}$ satisfying*

$$F_+(e^{i\theta})\big(F_+(e^{i\theta})\big)^* = G_+(e^{i\theta})\big(G_+(e^{i\theta})\big)^* \quad \textit{for a.a. } \theta \in [0, 2\pi). \tag{17}$$

If F is optimal, then for each $N \in \mathbb{N}_0$,

$$\sum_{n=0}^{N}\|A_n\|_2^2 \geq \sum_{n=0}^{N}\|B_n\|_2^2. \tag{18}$$

Proof. Let $\mathbb{P}_N$ be the projection operator on $H^2_{d\times d}$ defined by

$$\mathbb{P}_N : \sum_{n=0}^{\infty} A_n z^n \longmapsto \sum_{n=0}^{N} A_n z^n.$$

By virtue of (12), we have to prove that

$$\|\mathbb{P}_N(F)\|_{H^2_{d\times d}} \geq \|\mathbb{P}_N(G)\|_{H^2_{d\times d}}. \tag{19}$$

Let

$$U(z) = F^{-1}(z)G(z). \tag{20}$$

It follows from (17) that (13) holds. Therefore, $U_+ \in L^\infty_{d\times d}$. Since, in addition, $F^{-1}(z) = \frac{1}{\det F(z)} Cof\big(F(z)\big)$, where $\det F(z)$ is an outer function, by the generalized Smirnov's theorem (see Section 2), we have $U \in H^\infty_{d\times d}$. Consequently, (20) is an inner matrix function.

Exactly in the same manner as (15) was proved, we can show that

$$\mathbb{P}_N(FU) = \mathbb{P}_N\big(\mathbb{P}_N(F)U\big). \tag{21}$$

Since unitary transformations preserve standard Euclidian norm on the space $\mathbb{C}^d$, it follows from (13) that, for any $V \in \mathbb{C}^{1\times d}$,

$$\|V\|_2 = \|V \cdot U_+(e^{i\theta})\|_2 \quad \text{for a.a. } \theta \in [0, 2\pi). \tag{22}$$

Therefore, by virtue of (11) and (22),

$$\|X\|_{H^2_{d\times d}} = \|X_+\|_{L^2_{d\times d}} = \|X_+U_+\|_{L^2_{d\times d}} = \|XU\|_{H^2_{d\times d}} \tag{23}$$

for any $X \in H^2_{d\times d}$. It follows now from (23), (21), and (20) that

$$\begin{aligned}\|\mathbb{P}_N(F)\|_{H^2_{d\times d}} &= \|\mathbb{P}_N(F)\cdot U\|_{H^2_{d\times d}} \geq \|\mathbb{P}_N\big(\mathbb{P}_N(F)\cdot U\big)\|_{H^2_{d\times d}} \\ &= \|\mathbb{P}_N(FU)\|_{H^2_{d\times d}} = \|\mathbb{P}_N(G)\|_{H^2_{d\times d}}.\end{aligned}$$

Thus (19) is true, and Theorem 2 is proved. □

5. An energy conservation property

As was mentioned in the Introduction, in the setting of Theorem 1 it can happen that the equality is attained in (5) for some values of N even when g is not a constant multiple of f. The next proposition describes exactly when it is possible. Though not very explicit, it will become instrumental when characterizing the case of (5) turning into an equality for infinitely many values of N.

Proposition 1. *Let $f, g \in H^2$ satisfy (4), with f being an outer function. Then*

$$\sum_{n=0}^{N} |a_n|^2 = \sum_{n=0}^{N} |b_n|^2 \tag{24}$$

holds for some $N \in \mathbb{N}$ if and only if

$$g = uf, \tag{25}$$

where u is a finite Blaschke product,

$$u(z) = cz^{m_0} \prod_{j=1}^{m_1} \frac{z-\alpha_j}{1-\overline{\alpha_j}z}, \qquad |c| = 1,\ m_0, m_1 \in \mathbb{N}_0,\ 0 < |\alpha_j| < 1 \text{ for } j = 1, 2, \ldots, m_1, \tag{26}$$

the polynomial $P_N(f)$ has the degree

$$\deg(P_N(f)) \leq N - m_0 \tag{27}$$

and vanishes at $w_j := 1/\overline{\alpha_j}$, $j = 1, 2, \ldots, m_1$:

$$P_N(f)(w_j) = 0, \quad j = 1, 2, \ldots, m_1. \tag{28}$$

Proof. It follows from (4) that (25) holds for some inner function u.

The chain of relations in (16) reveals that the equality

$$\|P_N(f)\|_{H_2} = \|P_N(g)\|_{H_2} \tag{29}$$

holds if and only if

$$\|uP_N(f)\|_{H_2} = \|P_N\big(uP_N(f)\big)\|_{H_2}.$$

Therefore (24), which is equivalent to (29), holds if and only if

$$uP_N(f) \text{ is a polynomial with } \deg\big(uP_N(f)\big) \le N. \tag{30}$$

Under the conditions (26), (27), and (28) the relation (30) holds since

$$\prod_{j=1}^{m_1} \frac{z-\alpha_j}{1-\overline{\alpha_j}z} P_N(f) \text{ is a polynomial of the same degree as } P_N(f) \tag{31}$$

and

$$\deg\big(uP_N(f)\big) = m_0 + \deg\big(P_N(f)\big). \tag{32}$$

Thus *sufficiency* is proved.

If now (30) holds, then $u = uP_N(f)/P_N(f)$ is a rational function and, being inner, it has to be of the form (26).

Furthermore, the polynomial $P_N(f)$ should be divisible by $\prod_{j=1}^{m_1}(1-\overline{\alpha_j}z)$. Therefore (28) holds and (31) follows. This implies that (32) holds and (27) follows by virtue of (30), thus proving the *necessity*. □

Note that conditions (27), (28) imply the inequality $N \ge m_0 + m_1 =: m$. In particular, $N = 0$ only if $m_0 = m_1 = 0$, that is, g is a scalar multiple of f. This is of course in agreement with the extremal property of outer functions, and guarantees (in a trivial way) that (24) holds for all $N \in \mathbb{N}$, and thus infinitely many times. The next theorem describes all the cases in which the latter phenomenon occurs.

Theorem 3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=0}^{\infty} b_n z^n$ *be functions from* H^2 *satisfying* (4)*, with* f *being outer. The set* $\mathcal{N}$ *of those positive integers* N *for which* (24) *holds is infinite if and only if* (25)*,* (26) *hold and*

$$f = qh, \tag{33}$$

where

$$q(z) = \prod_{j=1}^{m_1}(z - w_j) \text{ with } w_j = 1/\overline{\alpha_j},\ j = 1, 2, \ldots, m_1, \tag{34}$$

and h *is an outer "lacunary" analytic function with infinitely many gaps in its Fourier spectrum* $\operatorname{supp}(\hat{h})$ *of length at least* $m = m_0 + m_1$*. Moreover,* $N \in \mathcal{N}$ *if and only if*

$$N - m + 1, \ldots, N \notin \operatorname{supp}(\hat{h}). \tag{35}$$

Proof. *Sufficiency.* Let g be defined by (25) and (26), and let (33) hold for the polynomial (34) of degree m_1 and an outer analytic function h satisfying (35) for some N. Then we have

$$P_N(f) = P_N(qh) = P_N\big(qP_N(h)\big) = q\sum_{n=0}^{N-m}\gamma_n z^n$$

due to (33), (15), and (35). Therefore,

$$\deg\big(P_N(f)\big) \le m_1 + N - m = N - m_0.$$

Hence $N \in \mathcal{N}$ by virtue of Proposition 1.

Necessity. By Proposition 1, g is given by (25), where the inner multiple (26), is such that (27), (28) hold for all $N \in \mathcal{N}$.

Labeling elements of $\mathcal{N}$ as an increasing sequence N_k, we thus have

$$P_{N_k}(f) = qh_k, \tag{36}$$

where polynomials h_k satisfy

$$\deg(h_k) \le N_k - m. \tag{37}$$

The function q is the same for all k as it is uniquely determined by (26).

Since $P_{N_k}(f) \to f$ in H_2 as $k \to \infty$, we have $qh_k \to f$. Therefore $(h_k)_+$ converges to f_+/q_+ in $L_2(\mathbb{T})$ (since $1/q_+$ is bounded on $\mathbb{T}$), and consequently h_k is convergent in H_2. Let h be the limit. Letting $k \to \infty$ in (36), we arrive at (33). Since f is outer, the function h is such as well.

Let now $N = N_k$ be an arbitrary element of $\mathcal{N}$. Because of (33) and (36), we have

$$f - P_N(f) = q(h - h_k).$$

Since $f - P_N(f)$ is divisible by z^{N+1} and 0 is not the root of q, we have $h - h_k = z^{N+1}\tilde{h}_k$ for some analytic function $\tilde{h}_k \in H^2$. Therefore $h = h_k + z^{N+1}\tilde{h}_k$ with $\deg(h_k) \le N - m$ (see (37)) and this implies that the coefficients with indices from $\{N - m + 1, N - m + 2, \dots, N\}$ are omitted in the power expansion of h. Thus (35) holds and the theorem is proved. □

Corollary 1. *Let $\{N_1, N_2, \dots\} \subset \mathbb{N}$ be any infinite set. Then there exist functions $f, g \in H_2$ where f is an outer function such that*

$$\sum_{n=0}^{N} |a_n|^2 = \sum_{n=0}^{N} |b_n|^2 \tag{38}$$

if and only if $N \in \{N_1, N_2, \dots\}$.

Proof. Let $q(z) = z - w$ with $|w| > 1$, and let $h(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ be an outer function from H_2 such that $\gamma_n = 0$ if and only if $n \in \{N_1, N_2, \dots\}$ (the outerness of h can be achieved, for example, by making sure that $|\gamma_0| > \sum_{n=1}^{\infty} |\gamma_n|$). Define $f = qh$ and $g(z) = (1 - \overline{w}z)h(z)$. Then it follows from the proof of the theorem that (38) holds if and only if $N \in \{N_1, N_2, \dots\}$. □

Acknowledgments

The authors would like to express their gratitude to Anthony Ephremides for indicating several practical applications of minimum-phase signals in communication. We are also thankful to Leonid Golinskii for attracting our attention to the problem tackled in Section 5.

References

[1] Walter Rudin, Real and Complex Analysis, third ed., McGraw-Hill Book Co., New York, 1987, MR 924157.
[2] Arne Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math. 81 (1948) 17. MR 0027954.
[3] Enders A. Robinson, Random Wavelets and Cybernetic Systems, in: Griffin's Statistical Monographs & Courses, vol. 9, Hafner Publishing Co., New York, 1962, MR 0149973.
[4] Enders A. Robinson, http://ethw.org/oral-history:enders_robinson#cite_note-13.
[5] T. Kailath, B. Hassibi, A.H. Sayed, Linear Estimation, in: Prentice-Hall Information and System Sciences Series, Prentice-Hall, Inc., Englewood Cliffs, N.J, 1999.
[6] Jon F. Claerbout, Fundamentals of Geophysical Data Processing, with Applications to Petroleum Prospecting, Blackwell Scientific Publications, 1985.
[7] W.H. Gerstacker, D.P. Taylor, On prefiltering for reduced-state equalization of MIMO channels, in: Proceedings of 5th ITG Conference on Source and Channel Coding, Erlangen, 2004, pp. 25–30.
[8] Ciprian Foias, Arthur Frazho, Israel Gohberg, Central intertwining lifting, maximum entropy and their permanence, Integral Equations Operator Theory 18 (2) (1994) 166–201. MR 1256095.
[9] Athanasios Papoulis, S. Unnikrishna Pillai, Probability, Random Variables, and Stochastic Processes, fourth ed., in: McGraw-Hill Series in Electrical and Computer Engineering, McGraw-Hill Book Co., New York, 2002.

[10] Paul Koosis, Introduction to H_p Spaces, in: London Mathematical Society Lecture Note Series, vol. 40, Cambridge University Press, Cambridge, New York, 1980. With an appendix on Wolff's proof of the corona theorem. MR 565451.

[11] L. Ephremidze, E. Lagvilava, Remark on outer analytic matrix-functions, Proc. A. Razmadze Math. Inst. 152 (2010) 29–32. MR 2663529.