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# THE QUANTUM GROUP AND HARPER EQUATION ON A HONEYCOMB LATTICE

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Abstract. The tight-binding model of quantum particle on a honeycomb lattice is investigated in the presence of homogeneous magnetic field. The one-particle Hamiltonian is expressed in terms of the generators of the quantum group $U_q(sl_2)$. The corresponding Harper equation is rewritten as a system of two coupled functional equations in the complex plane. The system is shown to exhibit certain symmetry that allows one to resolve the entanglement, and the basic single equation determining the eigenvalues and eigenstates is obtained. Equations specifying the roots of eigenstates in the complex plane are found.

In 1994, Wiegmann and Zabrodin pointed out (see [1]) that the tight-binding Hamiltonian on a square lattice is closely related to the quantum group $U_q(sl_2)$ (for mathematical treatment, see [2]). Here we develop a similar approach and reveal the novel symmetry for a honeycomb lattice (see [4]).

We study the tight-binding model on a honeycomb lattice with the nearest neighboring hoppings only. In the presence of a homogeneous magnetic field, the Hamiltonian under consideration has the form

$$H = \sum_{n,\boldsymbol{r}} \left[ e^{-i\gamma_n(\boldsymbol{r})} c_B^\dagger(\boldsymbol{r}+\boldsymbol{\delta}_n) c_A(\boldsymbol{r}) \right] + h.c., \tag{1}$$

where the sum with respect to $\boldsymbol{r}$ is implied over the sites $\boldsymbol{r} = j_1\boldsymbol{a}_1 + j_2\boldsymbol{a}_2$. Here $c_A^\dagger(\boldsymbol{r})$ $(c_A(\boldsymbol{r}))$ and $c_B^\dagger(\boldsymbol{r}+\boldsymbol{\delta}_n)$ $(c_B(\boldsymbol{r}+\boldsymbol{\delta}_n))$ are the particle creation (annihilation) operators on the site $\boldsymbol{r}$ of the sublattice $A$ and on the site $\boldsymbol{r}+\boldsymbol{\delta}_n$ of the sublattice $B$, respectively.

The magnetic field $\mathcal{B}$ is included in the Hamiltonian via the Peierls phases

$$\gamma_n(\boldsymbol{r}) = \frac{e}{\hbar} \int\limits_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{\delta}_n} \boldsymbol{\mathcal{A}}\, d\boldsymbol{l}, \tag{2}$$

where the vector-potential is taken in the Landau gauge $\boldsymbol{\mathcal{A}} = (-\mathcal{B}y, 0)$.

We consider

$$\frac{\Phi}{\Phi_0} = \frac{\nu}{N}, \tag{3}$$

where $\nu$ and $N$ are coprime integers, and concentrate on the odd values of $N$.

Then the Hamiltonian (1) can be presented as follows:

$$H = \int\limits_{\mathtt{MBZ}} \Psi^\dagger(\boldsymbol{k}) \mathcal{H}(\boldsymbol{k}) \Psi(\boldsymbol{k})\, d\boldsymbol{k}, \tag{4a}$$

$$\mathcal{H}(\boldsymbol{k}) = \begin{pmatrix} 0 & \mathbb{I} + X^-(\boldsymbol{k}) \\ \mathbb{I} + X^+(\boldsymbol{k}) & 0 \end{pmatrix}, \tag{4b}$$

where $\Psi(\boldsymbol{k})$ is a $(2N)$-component column and the integration covers the magnetic Brillouin zone, which is the $N$'th part of the first Brillouin zone.

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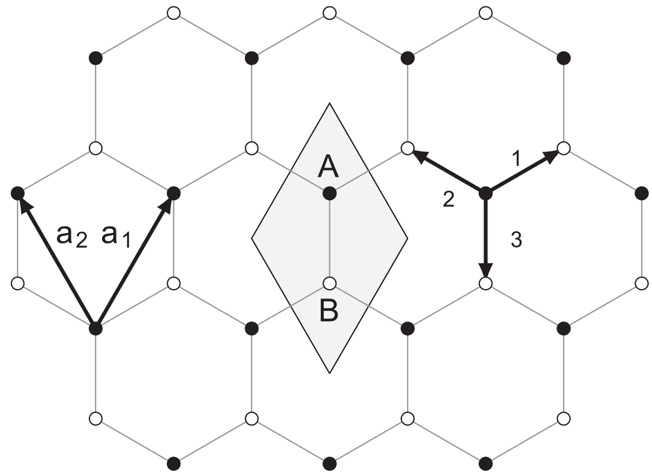


Fig. 1. Honeycomb lattice (left) consists of two Bravais sublattices $A$ and $B$. The $A$-sites are located at $\boldsymbol{r} = j_1\boldsymbol{a}_1 + j_2\boldsymbol{a}_2$, where $\boldsymbol{a}_{1,2} = \frac{1}{2}(\pm 1, \sqrt{3})a$, and $j_{1,2}$ are integers. The three $B$-sites nearest to a given $A$-site are located at $\boldsymbol{r} + \boldsymbol{\delta}_{1,2,3}$.

The $(N \times N)$-matrices $X^{\pm}(\boldsymbol{k})$ are given by

$$X^{+}(\boldsymbol{k}) = e^{-i\boldsymbol{k}\boldsymbol{a}_1}\beta^{\dagger}Q + e^{-i\boldsymbol{k}\boldsymbol{a}_2}Q\beta, \tag{5a}$$

$$X^{-}(\boldsymbol{k}) = e^{+i\boldsymbol{k}\boldsymbol{a}_1}Q^{\dagger}\beta + e^{+i\boldsymbol{k}\boldsymbol{a}_2}\beta^{\dagger}Q^{\dagger}, \tag{5b}$$

where

$$\beta = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{6}$$

and

$$Q = \mathrm{diag}\left(q^1, q^2, \ldots, q^N\right) \tag{7}$$

with

$$q = e^{+i\pi(\nu/N)}. \tag{8}$$

The operators $X^{\pm}$ satisfy the following relations of the quantum group $U_q(sl_2)$:

$$\left[X^{+}, X^{-}\right] = i^2(q - q^{-1})(K - K^{-1}), \tag{9a}$$

$$KX^{\pm}K^{-1} = q^{\pm 2}X^{\pm}, \tag{9b}$$

where

$$K(\boldsymbol{k}) = qe^{+i\boldsymbol{k}(\boldsymbol{a}_1 - \boldsymbol{a}_2)}Q\beta Q^{\dagger}\beta, \tag{10a}$$

$$K^{-1}(\boldsymbol{k}) = q^{-1}e^{-i\boldsymbol{k}(\boldsymbol{a}_1 - \boldsymbol{a}_2)}\beta^{\dagger}Q\beta^{\dagger}Q^{\dagger}. \tag{10b}$$

For every complex number $q \neq \pm 1$, the quantum group $U_q(sl_2)$ possesses the highest weight representations similar to those of ordinary $sl_2$. Furthermore, when $q$ is the root of unity, the so-called cyclic representation also exists.

We study the eigenvalue equation

$$\begin{pmatrix} 0 & \mathbb{I}+X^-(\boldsymbol{k}) \\ \mathbb{I}+X^+(\boldsymbol{k}) & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = E \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \tag{11}$$

which in the functional representation has the form

$$\xi(z) + e^{+\frac{i}{2}k_x a} q^{+\frac{1}{2}} z\xi(qz) + e^{-\frac{i}{2}k_x a} q^{-\frac{1}{2}} z\xi(q^{-1}z) = E\zeta(z), \tag{12a}$$

$$\zeta(z) + e^{+\frac{i}{2}k_x a} q^{-\frac{1}{2}} z^{-1}\zeta(qz) + e^{-\frac{i}{2}k_x a} q^{+\frac{1}{2}} z^{-1}\zeta(q^{-1}z) = E\xi(z). \tag{12b}$$

If any pair $(\xi, \zeta)$ is the eigenvector corresponding to the eigenvalue $E = \lambda$, then the pair $(\xi, -\zeta)$ is the eigenvector corresponding to the eigenvalue $E = -\lambda$.

The main disadvantage of (12) is the entanglement of $\xi(z)$ and $\zeta(z)$. This difficulty has been overcome in [3] by "squaring up" Eqs. (11). Due to the "square up" trick, some portion of the information encoded in (12) is lost. Here we propose an essentially different approach allowing us to avoid this drawback (disadvantage) of the "square up" procedure.

The key point of our consideration is the observation that the system (12) is invariant under the following transformation:

$$\xi(z) \longrightarrow (iz)^\omega \zeta(-z^{-1}), \tag{13a}$$

$$\zeta(z) \longrightarrow (iz)^\omega \xi(-z^{-1}), \tag{13b}$$

where the parameter $\omega$ is defined by $q^\omega e^{+ik_x a} = -1$. This symmetry allows us to express solutions of (12) as

$$\begin{pmatrix} \xi(z) \\ \zeta(z) \end{pmatrix} = \begin{pmatrix} f(z) \\ \pm(iz)^\omega f(-z^{-1}) \end{pmatrix}, \tag{14}$$

where $f(z)$ satisfies the equation

$$f(z) + e^{+\frac{i}{2}k_x a} q^{+\frac{1}{2}} z f(qz) + e^{-\frac{i}{2}k_x a} q^{-\frac{1}{2}} z f(q^{-1}z) = \lambda (iz)^\omega f(-z^{-1}). \tag{15}$$

The signs $\pm$ in (14) correspond to $E = \pm\lambda$, respectively.

Setting $e^{+\frac{i}{2}k_x a} = iq^{\frac{1}{2}}$, we reduce Eq. (15) to the form

$$f(z) + iqzf(qz) - iq^{-1}zf(q^{-1}z) = \lambda z^{2J} f(-z^{-1}), \tag{16}$$

which admits polynomial solutions and generates $N$ eigenvalues $\lambda_1, \ldots, \lambda_N$.

Since $f(z)$ is a polynomial, we can write

$$f(z) = \prod_{j=1}^{2J} (z - z_j), \tag{17}$$

where $z_1, z_2, \ldots, z_{2J}$ are the zeros of $f(z)$.

Substituting (17) into (16) we find the following result:

$$iz_n = \prod_{j=1}^{2J} \frac{1 + qz_n z_j}{1 + z_n z_j} - \prod_{j=1}^{2J} \frac{1 + q^{-1} z_n z_j}{1 + z_n z_j}, \tag{18}$$

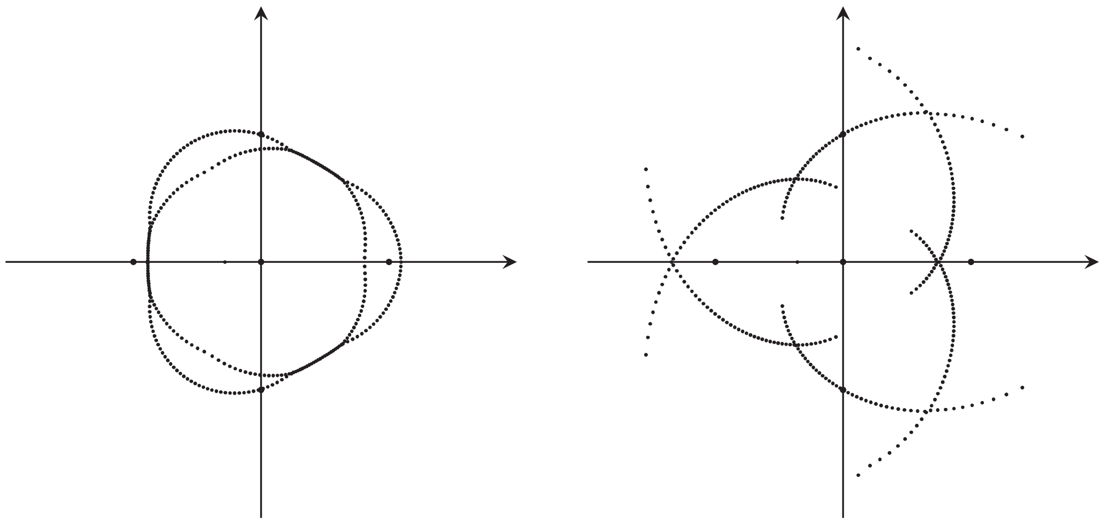

Fig. 2. Typical distribution of the zeroes $z_n$ on the complex plane for $\dfrac{\Phi}{\Phi_0} = \dfrac{204}{307} \approx \dfrac{2}{3}$.

which is the honeycomb analog of the Bethe ansatz equation obtained in [1] for a square lattice,

$$\lambda = \prod_{j=1}^{2J} z_j, \tag{19}$$

$$\lambda^2 = 1 + i\bigl(q - q^{-1}\bigr) \sum_{j=1}^{2J} z_j, \tag{20}$$

$$\sum_{j=1}^{2J} \Bigl(z_j + \frac{1}{z_j}\Bigr) = iq + \frac{1}{iq}. \tag{21}$$

Thus, we have derived the relation among an eigenvalue and the roots of the corresponding polynomial. The set of roots is determined by (18). Once the roots of a polynomial are known, the corresponding eigenvalue can be calculated using any of (19) and (20). Relation (21) represents the "sum rule."

Note that the aforementioned invariance of (12) may be interpreted in terms of special conformal transformation

$$z \longrightarrow w = w(z) = -\frac{1}{z}. \tag{22}$$

Under this conformal map a quasi-primary field $\Phi(z)$ transforms as follows:

$$\Phi(z) \longrightarrow \Phi'(w) = \Bigl(\frac{dw}{dz}\Bigr)^{-h} \Phi(z), \tag{23}$$

where $h$ is the corresponding conformal weight.

Employing the results of numerical calculations, we present distributions of the zeroes on the complex plane (see Fig. **??**).

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