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THE CONTINUITY AND THE LIMIT IN THE WIDE. THEIR CONNECTION WITH THE CONTINUITY AND LIMIT

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Abstract. The notions of a continuity and limit in the wide are introduced. It is proved that the continuity implies the continuity in the wide, without a converse statement, and the existence of a finite limit entails the existence of an equal limit in the wide, again without a converse statement.

§ 1. The Continuity in the Wide and Its Connection with the Continuity

1.1. Increment in the Wide. 1°. For the point $x=(x_1,\dots,x_n)$ from the real Euclidean n-dimensional space $\mathbb{R}^n$, by $\|x\|$ will be denoted any of the three equivalent norms $\|x\|_1=\max\limits_{1\le i\le n}|x_i|$, $\|x\|_2=\sum\limits_{i=1}^{n}|x_i|$, $\|x\|_3=(\sum\limits_{i=1}^{n}x_i^2)^{1/2}$.

A neighborhood $U(x^0)$ of the point $x^0=(x_1^0,\dots,x_n^0)\in\mathbb{R}^n$ is defined as a set of all points $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ for which $\|x-x^0\|<\delta$ for some $\delta>0$.

Along with the points $x=(x_1,\dots,x_n)$ and $x^0=(x_1^0,\dots,x_n^0)$, we introduce the following symbols of the points ([1])

$$x(x_k^0)=(x_1,\dots,x_{k-1},x_k^0,x_{k+1},\dots,x_n),$$
$$x^0(x_j)=(x_1^0,\dots,x_{j-1}^0,x_j,x_{j+1}^0,\dots,x_n^0).$$

By $x(x_k^0,x_j^0)$ is denoted a point obtained after simultaneous replacement of x_k and x_j by, respectively, x_k^0 and x_j^0 in $x=(x_1,\dots,x_n)$, $j\ne k$.

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Let a real finite function $f(x)$ be given in $U(x^0)$.

By $\Delta_{x^0} f(x)$ we denote an increment (called sometimes a complete increment) at the point x^0 for $f(x)$, i.e.,

$$\Delta_{x^0} f(x) = f(x) - f(x^0). \tag{1.1}$$

The continuity of the function $f(x)$ at the point x^0 is equivalent to the fulfilment of the equality

$$\lim_{x \to x^0} \Delta_{x^0} f(x) = 0. \tag{1.2}$$

Furthermore, by $\Delta_{x_k^0} f(x)$ is denoted a partial increment with respect to the variable x_k for $f(x)$ at x^0, i.e.,

$$\Delta_{x_k^0} f(x) = f(x^0(x_k)) - f(x^0). \tag{1.3}$$

The function $f(x)$ is called a partial continuous with respect to the variable x_k at the point x^0 if

$$\lim_{x \to x^0} \Delta_{x_k^0} f(x) = 0. \tag{1.4}$$

Unlike $\Delta_{x_k^0} f(x)$, the notion of a strong partial increment with respect to the variable x_k for $f(x)$ at the point x^0 is available ([2]):

$$\Delta_{[x_k^0]} f(x) = f(x) - f(x(x_k^0)). \tag{1.5}$$

The function $f(x)$ is strongly partial continuous with respect to the variable x_k at the point x^0 if ([2], [3])

$$\lim_{x \to x^0} \Delta_{[x_k^0]} f(x) = 0. \tag{1.6}$$

It is well known that the fulfilment of equality (1.4) for $k = 1, \dots, n$ is not sufficient for the function $f(x)$ to be continuous at the point x^0.

But the fulfilment of equality (1.6) for $k = 1, \dots, n$ is equivalent to the continuity of $f(x)$ at x^0. Moreover, the following theorem holds.

Theorem 1 ([2],[3]). *For the function $f(x)$ to be continuous at the point x^0, it is necessary and sufficient that equality* (1.6) *be fulfilled for $k = 1, \dots, n$.*

2°. The structure of the expression $\Delta_{x^0} f(x)$ indicates that all coordinates of the point x^0 receive simultaneously the increments $x_j - x_j^0$, $j = 1, \dots, n$.

Now we construct an expression which later will be called an increment in the wide.

It is clear from equality (1.5) that the expression $\Delta_{[x_k^0]} f(x)$ depends on variables $x_1, \dots, x_n$. Introduce the function $\lambda(x_1, \dots, x_n) = \Delta_{[x_k^0]} f(x)$ and for the function $\lambda(x_1, \dots, x_n)$ write equality (1.5) with respect to some variable x_j with $j \neq k$. As a result, we obtain a new function $\mu(x_1, \dots, x_n) =$

$\Delta_{[x_j^0]}\lambda(x)$ depending again on variables $x_1, \dots, x_n$. For the function $\mu(x_1, \dots, x_n)$ we write equality (1.5) for some variable x_e where $e \neq k$ and $e \neq j$.

We continue this procedure until all variables $x_1, \dots, x_n$ are exhausted. A final result will be called an increment in the wide for the function $f(x)$ at the point x^0, and we denote it by the symbol $\Delta^n_{[x^0]} f(x)$.

An important property of the procedure is the fact that the final result does not depend on the order of forming, by formula (1.5), of strong partial increments for the required functions.

On this basis we introduce the following

Definition 1. In the wide, an increment at the point x^0 for the finite in $U(x^0)$ function $f(x)$, $x = (x_1, \dots, x_n)$, will be called the value

$$\Delta^n_{[x^0]} f(x) = \Delta_{[x_1^0]}\big(\Delta_{[x_2^0]}\big(\cdots\big(\Delta_{[x_n^0]} f(x)\big)\cdots\big)\big), \tag{1.7}$$

where (see (1.5))

$$\Delta_{[x_j^0]} F(x) = F(x) - F\big(x(x_j^0)\big). \tag{1.8}$$

Thus, to obtain $\Delta^n_{[x^0]} f(x)$ it is necessary instead of $F(x)$ and j in (1.8) to take successively $f(x)$ and $j = n$, $\Delta_{[x_n^0]} f(x)$ and $j = n-1$, etc., and finally, $\Delta_{[x_2^0]}(\cdots(\Delta_{[x_n^0]} f(x))\cdots)$ and $j = 1$.

3°. Case $n = 2$. If a finite function of two variables $\varphi(x_1, x_2)$ is defined in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$, then a strong partial increment with respect to the variable x_1 at x^0 for $\varphi(x_1, x_2)$ is given by the equality

$$\Delta_{[x_1^0]}\varphi(x_1, x_2) = \varphi(x_1, x_2) - \varphi(x_1^0, x_2), \tag{1.9}$$

and a strong partial increment with respect to the variable x_2 is obtained by the equality

$$\Delta_{[x_2^0]}\varphi(x_1, x_2) = \varphi(x_1, x_2) - \varphi(x_1, x_2^0). \tag{1.10}$$

Therefore the increment in the wide at the point x^0 for $\varphi(x_1, x_2)$ is equal to

$$\Delta^2_{[x^0]}\varphi(x_1, x_2) = \\ = \varphi(x_1, x_2) - \varphi(x_1^0, x_2) - \varphi(x_1, x_2^0) + \varphi(x_1^0, x_2^0). \tag{1.11}$$

1.2. The Continuity in the Wide.

Definition 2 ([3]). A finite function $f(x)$ given in $U(x^0)$ is in the wide continuous at the point x^0 if

$$\lim_{x \to x^0} \Delta^n_{[x^0]} f(x) = 0. \tag{1.12}$$

In the sequel, it is advisable to formulate the continuity in the wide for the functions of two variables in the form of the following two equivalent equalities.

The function $\varphi(x_1,x_2)$, finite in the neighborhood of the point $x^0 = (x_1^0,x_2^0)$, is in the wide continuous at the point x^0 if

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}\big[\varphi(x_1,x_2)-\varphi(x_1^0,x_2)-\varphi(x_1,x_2^0)+\varphi(x_1^0,x_2^0)\big]=0 \tag{1.13}$$

or, what is the same thing, if

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}\big[\varphi(x_1^0,x_2)+\varphi(x_1,x_2^0)-\varphi(x_1,x_2)\big]=\varphi(x_1^0,x_2^0). \tag{1.14}$$

1.3. The Increment in the Wide for the Sum. Given in $U(x^0)$ finite functions $f_1(x_1,\dots,x_n)$, …, $f_m(x_1,\dots,x_n)$, we have

$$\Delta^n_{[x^0]}\sum_{j=1}^m f_j(x)=\sum_{j=1}^m\Delta^n_{[x^0]}f_j(x),\quad x=(x_1,\dots,x_n)\in U(x^0). \tag{1.15}$$

1.4. Equality to Zero of an Increment in the Wide for a Sum of Functions of Special Type. Let in $U(x^0)$ be given finite functions of special types:

$$\psi_k(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n),\quad k=1,\dots,n, \tag{1.16}$$

each depending on variables in number $n-1$.

Consider a summary function, depending on $x=(x_1,\dots,x_n)$,

$$\psi(x)=\psi_1(x_2,x_3,\dots,x_n)+\psi_2(x_1,x_3,\dots,x_n)+ \\ +\dots+\psi_n(x_1,\dots,x_{n-1}). \tag{1.17}$$

Since the function ψ_k does not depend on the variable x_k, from (1.5) we obtain equalities

$$\Delta_{[x_k^0]}\psi_k(x)=0,\quad k=1,\dots,n. \tag{1.18}$$

Because of the fact that an order of succession in (1.7) is not of importance, from (1.18) it follows that

$$\Delta^n_{[x^0]}\psi_k(x)=0,\quad k=1,\dots,n. \tag{1.19}$$

Equalities (1.15) and (1.19) yield

$$\Delta^n_{[x^0]}\psi(x)=0. \tag{1.20}$$

Consequently, every finite in $U(x^0)$ function $\psi(x)$ of type (1.15) is in the wide continuous at every point $x^0=(x_1^0,\dots,x_n^0)$, even for discontinuous at x^0 functions $\psi_k(x)$, $k=1,\dots,n$.

1.5. The Sufficient Condition for the Continuity in the Wide.

Theorem 2 ([3]). *If the function $f(x)$ with respect to at least of one variable is strongly partial continuous at x^0, then the function $f(x)$ is continuous in the wide at the point x^0.*

The converse statement is invalid.

Proof. [1] Suppose that the function $f(x)$ is strongly partial continuous with respect to the variable x_j at the point x^0. Then the equality

$$\lim_{x\to x^0}\left[f(x)-f(x(x_j^0))\right]=0. \tag{1.21}$$

holds. Since $\Delta^n_{[x^0]}f(x)$ does not depend on the order of forming strong partial increments, we begin forming the right-hand side of equality (1.7) with $\Delta_{[x_j^0]}f(x)$. The next step is to form with respect to the variable x_e with $e\neq j$, a strong partial increment for $\Delta_{[x_j^0]}f(x)$ at x^0. We have

$$\begin{aligned}\Delta_{[x_e^0]}\big(\Delta_{[x_j^0]}f(x)\big)&=\Delta_{[x_j^0]}f(x)-\big(\Delta_{[x_j^0]}f(x)\big)_{x_e=x_e^0}=\\&=\left[f(x)-f(x(x_j^0))\right]-\left[f(x)-f(x(x_j^0))\right]_{x_e=x_e^0}=\\&=\left[f(x)-f(x(x_j^0))\right]-\left[f(x(x_e^0))-f(x(x_j^0,x_e^0))\right].\end{aligned}$$

Both differences in the square brackets tend to zero by equality (1.21), as $x\to x^0$.

As a result of finite number of steps, we obtain equality (1.12), i.e. $f(x)$ is in the wide continuous at x^0.

The fact that the converse statement is invalid follows from the reasoning at the end of Section 1.4. Indeed, for any finite functions $\alpha(x_1)$ and $\beta(x_2)$, not necessarily continuous, we have

$$\Delta^2_{[x^0]}w(x_1,x_2)=0,$$

where $x^0=(x_1^0,x_2^0)$ is an arbitrary point from $\mathbb{R}^2$, and

$$\omega(x_1,x_2)=\alpha(x_1)+\beta(x_2). \tag{1.22}$$

Hence $\omega(x_1,x_2)$ is in the wide continuous at every point from $\mathbb{R}^2$. Thus the theorem is proved. □

It should be noted here that the function $\omega(x_1,x_2)$ defined by equality (1.22) is continuous at some point $x^0=(x_1^0,x_2^0)$ if and only if $\alpha(x_1)$ is continuous at x_1^0 and $\beta(x_2)$ is continuous at x_2^0. This follows from Theorem 1 and from the fact that

$$\begin{aligned}\omega(x_1,x_2)-\omega(x_1^0,x_2)&=\alpha(x_1)-\alpha(x_1^0),\\\omega(x_1,x_2)-\omega(x_1,x_2^0)&=\beta(x_2)-\beta(x_2^0).\end{aligned}$$

Theorems 1 and 2 result in

[1] This proof of Theorem 2 differs from that suggested in [3].

Corollary 1 ([3]). *If the function $f(x)$ is continuous at the point x^0, then $f(x)$ is likewise continuous in the wide at the point x^0 . The converse statement is invalid.*

1.6. The Continuity of a Function of Two Variables is Equivalent both to the Separate Continuity and to the Continuity in the Wide. It is well-known that the separate partial continuity, i.e., the fulfilment of equality (1.4) for all $k = 1, \dots, n$ does not imply the continuity. Moreover, we have just convinced ourselves that the continuity in the wide is the property, far from the continuity.

Let us prove that these two properties simultaneously guarantee the continuity, and vice versa.

Indeed, we have obtained an answer to the question: what useful information carries the notion of the continuity in the wide?

Theorem 3. *A finite function $\varphi(x_1, x_2)$ defined in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$ is continuous at x^0, if and only if $\varphi(x_1, x_2)$ at x^0 is both separately partial continuous and continuous in the wide.*

Proof. If $\varphi(x_1, x_2)$ possesses at the point x^0 the two properties mentioned in the theorem, then its continuity at x^0 follows from the equality

$$\begin{aligned} &\varphi(x_1^0 + h, x_2^0 + k) - \varphi(x_1^0, x_2^0) = \\ &= \big[\varphi(x_1^0 + h,\, x_2^0 + k) - \varphi(x_1^0,\, x_2^0 + k) - \varphi(x_1^0 + h,\, x_2^0) + \varphi(x_1^0, x_2^0)\big] + \\ &\quad + \big[\varphi(x_1^0 + h,\, x_2^0) - \varphi(x_1^0,\, x_2^0)\big] + \big[\varphi(x_1^0,\, x_2^0 + k) - \varphi(x_1^0,\, x_2^0)\big]. \quad (1.23) \end{aligned}$$

It is obvious that the continuous at the point x^0 function $\varphi(x_1, x_2)$ possesses both properties mentioned in Theorem 3. Thus the theorem is proved. □

§ 2. The Limit in the Wide and Its Connection with the Limit

2.1. The Notion of the Limit in the Wide. In the classical analysis, the notion of the continuity is introduced on the basis of the notion of the limit. For better understanding of the notion of the continuity in the wide we can apply to the preceding notion of the limit in the wide.

Bearing this in mind, we shall take advantage of the notion of the continuity in the wide.

It is evident that the defined by equality (1.7) increment in the wide for $f(x)$ at the point x^0 contains the value $f(x^0)$.

We now replace in $\Delta^n_{[x^0]} f(x)$ the value $f(x^0)$ by a finite value B and denote it by the symbol

$$\Delta^n_{[x^0]} f(x)\big|_{f(x^0)=B} . \tag{2.1}$$

Introduce the following

Definition 3. A finite value B is called a limit in the wide for the function $f(x)$ at the point x^0 if the equality

$$\lim_{x\to x^0}\left(\Delta^n_{[x^0]}f(x)\big|_{f(x^0)=B}\right)=0 \tag{2.2}$$

holds.

Definition 2 can now be rephrased in the form of

Definition 4. The function $f(x)$ is in the wide continuous at the point x^0 if the value $f(x^0)$ is finite and $f(x^0)$ is in the wide limit for $f(x)$ at x^0.

Proceeding from equality (1.11), the finite value B is the limit in the wide at the point $x^0=(x_1^0,x_2^0)$ for the function $\varphi(x_1,x_2)$ if

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}\left[\varphi(x_1,x_2)-\varphi(x_1^0,x_2)-\varphi(x_1,x_2^0)+B\right]=0. \tag{2.3}$$

In general, the number L, finite or of fixed sign infinite, is the limit in the wide at the point (x_1^0,x_2^0) for the function $\varphi(x_1,x_2)$ if

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}\left[\varphi(x_1^0,x_2)+\varphi(x_1,x_2^0)-\varphi(x_1,x_2)\right]=L. \tag{2.4}$$

Proposition 1. *If the function $f(x)$ has at the point x^0 the finite limit in the wide, then this limit is unique.*

To see that this is so, we have to write equality (2.2) for finite values B and B_1 and then to consider their difference. As a result, we obtain $B-B_1=0$.

2.2. The Existence of a Limit in the Wide.

Theorem 4. *If the function $f(x)$ has a finite limit B at x^0, then B is likewise a limit in the wide for $f(x)$ at x^0.*

Proof. If the function $f(x)$ is continuous at the point x^0, then $f(x)$ is continuous in the wide at the point x^0, by Corollary 1. Therefore the finite value $f(x^0)$ is, by Definition 4, the limit in the wide for the function $f(x)$ at the point x^0.

Suppose now that the function $f(x)$, possessing at the point x^0 the finite limit B, is discontinuous at x^0.

If we introduce a new function $f^*(x)=f(x)$ for $x\neq x^0$ and $f^*(x^0)=B$, then $f^*(x)$ is continuous at x^0, and the inequality $f^*(x)\neq f(x)$ is fulfilled only for $x=x^0$. Therefore

$$\Delta^n_{[x^0]}f(x)-\Delta^n_{[x^0]}f^*(x)=f(x^0)-f^*(x^0)=f(x^0)-B,$$

from which we get

$$\Delta^n_{[x^0]}f(x)\big|_{f(x^0)=B}-\Delta^n_{[x^0]}f^*(x)=B-B=0. \tag{2.5}$$

But, according to Corollary 1, we have

$$\lim_{x \to x^0} \Delta^n_{[x^0]} f^*(x) = 0. \tag{2.6}$$

Equality (2.2) is now obtained from (2.5) and (2.6). Thus the theorem is proved. □

Proposition 2. *The existence at the point x^0 of a finite limit in the wide for the function $\psi(x_1, \dots, x_n)$, $n > 1$, does not imply that there exist a finite or of a fixed sign infinite limit for $\psi(x_1, \dots, x_n)$ at x^0.*

Proof. The function $\psi(x_1, x_2)$, considered in [2] on p. 26, possesses at every point $(x_1^0, 0)$ the following two properties: with respect to variable x_1, it is strongly partial continuous and has no a finite or of a fixed sign infinite limit.

According to Theorem 2 and Definition 4, the function $\psi(x_1, x_2)$ has at all points $(x_1^0, 0)$ equal to $\psi(x_1^0, 0) = 0$ limit in the wide.

As the second example, we take the function $\omega(x_1, x_2)$ defined by equality (1.22), which in the wide is continuous at every point (x_1^0, x_2^0) and therefore possesses at (x_1^0, x_2^0) a limit in the wide, equal to the value $\alpha(x_1^0) + \beta(x_2^0)$. On the other hand, it is not necessary for the function $\omega(x_1, x_2)$ to have a finite or of a fixed sign infinite limit at (x_1^0, x_2^0). Thus the proposition is proved. □

2.3. The Necessary and Sufficient Conditions for Functions of Two Variables to Have a Finite Limit. We are already aware that the existence of a finite limit does not follow from the existence of equal separated partial limits ([2], p. 23 and functions $\varphi(x_1, x_2)$, $g(x_1, x_2)$ on p. 24) and from the existence of a finite limit in the wide (see Proposition 2).

It is noteworthy that these two properties together result in the existence of a finite limit, and vice versa.

Thus we have found that useful information load which carries the notion of a limit in the wide.

Theorem 5. *A finite number A is a limit at the point $x^0 = (x_1^0, x_2^0)$ for the function $\varphi(x_1, x_2)$, if and only if $\varphi(x_1, x_2)$ has simultaneously at the point x^0 separated partial limits and a limit in the wide, equal to A.*

Proof. If A is a limit at the point x^0 for the function $\varphi(x_1, x_2)$, then for $\varphi(x_1, x_2)$ A is both a limit in the wide (see Theorem 4) and, obviously, separated partial limits at the point x^0.

The converse follows from the equality obtained as a result of substitution of the value $\varphi(x_1^0, x_2^0)$ by the number A in (1.23). Thus the theorem is proved. □

2.4. Application to the Continuity of Functions of One Variable. Using Theorem 5, we can obtain the necessary and sufficient condition for the continuity of functions of one variable. To this end, we take advantage of the notion of symmetric continuity.

The function $\lambda(t)$ prescribed in the interval $(t_0-\delta, t_0+\delta)$ is called symmetrically continuous at the point t_0 if the equality

$$\lim_{h\to 0}\big[\lambda(t_0+h)-\lambda(t_0-h)\big]=0 \tag{2.7}$$

is fulfilled.

There naturally arises the question: What property the function $\lambda(t)$ possesses at the point t_0 if the equality

$$\lim_{(h,k)\to(0,0)}\big[\lambda(t_0+h)-\lambda(t_0-k)\big]=0, \tag{2.8}$$

which is stronger than (2.7), is fulfilled.

Theorem 6. *In order for the function $\lambda(t)$ to be continuous at the point t_0, it is necessary and sufficient to have equality* (2.8).

Proof. For the particular case $A=0$, Theorem 5 shows that equality (2.8) takes place if and only if the function of two variables $\mu(h,k)=\lambda(t_0+h)-\lambda(t_0-k)$ possesses at the point t_0 zero separated partial limits, as well as zero limits in the wide.

Consequently, in order for the equality (2.8) to be fulfilled, it is necessary and sufficient that the following three equalities $\lim\limits_{h\to 0}\mu(h,0)=0$, $\lim\limits_{k\to 0}\mu(0,k)=0$ and $\lim\limits_{(h,k)\to(0,0)}[\mu(h,k)-\mu(0,k)-\mu(h,0)+0]=0$, or what is the same, $\lim\limits_{h\to 0}\lambda(t_0+h)=\lambda(t_0)$, $\lim\limits_{k\to 0}\lambda(t_0-k)=\lambda(t_0)$ and $\lim\limits_{(h,k)\to(0,0)}0=0$, be fulfilled.

Thus the fulfilment of equality (2.8) implies the continuity of the function $\lambda(t)$ at the point t_0.

The converse statement follows from the equality $\lambda(t_0+h)-\lambda(t_0-k)=[\lambda(t_0+h)-\lambda(t_0)]-[\lambda(t_0-k)-\lambda(t_0)]$. Thus the theorem is proved. □

For the continuous only at the point $t_0=0$ function $\nu(t)=t\cdot D(t)$, where $D(t)$ is everywhere discontinuous Dirichlet function, the above theorem results in the following equality for the function $D(t)$:

$$\lim_{(h,k)\to(0,0)}\big[hD(h)+kD(k)\big]=0. \tag{2.9}$$

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