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UNILATERAL IN VARIOUS SENSES: THE LIMIT, CONTINUITY, PARTIAL DERIVATIVE AND THE DIFFERENTIAL FOR FUNCTIONS OF TWO VARIABLES

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Abstract. For functions of two variables we introduce the notions of the unilateral in various senses: the limit, continuity, partial derivative and differential. Their connection with the limit, continuity, partial derivative and total differential is established.

For functions of one variable we introduce the notions of the unilateral limit, continuity and derivative. The naturalness here is motivated by the uniqueness of a partitioning of the neighborhood of a point. The unique method for such a partitioning is not available in the two-dimensional case: the two-dimensional interval can be divided into parts by different methods. It is of interest which of these partitionings is most suitable for the above-formulated problem.

The ways of solving these problems are hidden in the notions of partial continuities in the strong and angular as well as in those of partial derivatives in the strong and angular ([1]–[4]).

PART I

The Unilateral Limit and Continuity in Various Senses

1^0. The notions of the unilateral limit and unilateral continuity for a function of one variable are well known. They consist in the following: the existence of a limit on the right at the point t_0 for the function $\lambda(t)$, in symbols $\lambda(t_0+)$, implies that $\lambda(t)$ tend to $\lambda(t_0+)$ as t tends to t_0 by values

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of $t > t_0$ (the tending $t \to t_0$ by values of $t > t_0$ is usually written symbolically as $t \to t_0+$). A limit on the left for $\lambda(t)$ at the point t_0, denoted by $\lambda(t_0-)$, is defined by $\lambda(t) \to \lambda(t_0-)$ as $t \to t_0-$.

The function $\lambda(t)$ has the limit at the point t_0, if and only if $\lambda(t_0-) = \lambda(t_0+)$, and in this case their common value is the limit for $\lambda(t)$ at the point t_0.

The function $\lambda(t)$ is said to be continuous on the right at the point t_0 if the value $\lambda(t_0)$ is finite and $\lambda(t_0+) = \lambda(t_0)$. Similarly, $\lambda(t)$ is continuous on the left at the point t_0 if $\lambda(t_0-) = \lambda(t_0)$ for a finite $\lambda(t_0)$.

For the function $\lambda(t)$ to be continuous at the point t_0, it is necessary and sufficient that the function $\lambda(t)$ be continuous at the point t_0 on the right and on the left simultaneously.

Thus, if t_0 is an interior point for the interval (a, b) and the function $\lambda(t)$ is defined in (a, b) or in $(a, b)\backslash\{t_0\}$, then the unilateral limits at the point t_0 for the function $\lambda(t)$ are the limits at the point t_0 for the function $\lambda(t)$ along the two open, non-intersecting intervals whose union is $(a, b)\backslash\{t_0\}$. In particular, the limit on the right at the point $t_0 = 0$ implies the limit at the point $t_0 = 0$ by values of $t > 0$. Therefore the limit on the right and the continuity on the right at the point $t_0 = 0$ is called the $+$ limit and the $+$ continuity at the point $t_0 = 0$. This terminology will be retained for arbitrary point t_0.

2^0. Such a terminological change is caused by the author's intention to investigate analogous problems for functions of two variables. These problems remain unstudied so far.

The limit and the continuity at the origin of the coordinates along a suitable set which covers positive parts of the axes lying near the origin and has the origin as a limiting point, will be called the $+$ limit and the $+$ continuity at the origin. This terminology will be retained for any kind of points.

Recall some well-known but ineffective notions and facts concerning the functions of two variables denoted in symbols as $\pm$.

Thus the notions of the $+$ limit and $+$ continuity as well as of the $-$ limit and $-$ continuity at the given point with respect to individual variable can be easily extended to the functions of two variables.

Namely, let the function $\varphi(x)$, $x = (x_1, x_2)$ be defined in the neighborhood $U(x^0)$ or in the punctured neighborhood $U^0(x^0) = U(x^0)\backslash\{x^0\}$ of the point $x^0 = (x_1^0, x_2^0)$.

We introduce into consideration the functions ${}^1\varphi$ and ${}^2\varphi$ depending respectively on the variables x_1 and x_2 by the equalities

$$ {}^1\varphi(x_1) = \varphi(x_1, x_2^0), \quad {}^2\varphi(x_2) = \varphi(x_1^0, x_2). $$

The function ${}^i\varphi(x_i)$ of one variable is called the i-th partial function at the point x^0 of the function $\varphi(x)$, $i = 1, 2$. Obviously, ${}^1\varphi(x_1^0) = \varphi(x^0) = {}^2\varphi(x_2^0)$.

If the function ${}^{i}\varphi(x_i)$ has at the point x_i^0 the unilateral + limit

$$\lim_{x_i \to x_i^0+} {}^{i}\varphi(x_i),$$

then the function $\varphi(x)$ has at the point x^0 the partial + limit with respect to the variable x_i, and this is written as

$$\lim_{x_1 \to x_1^0+} \varphi(x_1, x_2^0) \quad \text{and} \quad \lim_{x_2 \to x_2^0+} \varphi(x_1^0, x_2).$$

If the limit

$$\lim_{x_i \to x_i^0+} {}^{i}\varphi(x_i)$$

is equal to ${}^{i}\varphi(x_i^0) = \varphi(x^0)$, then the function $\varphi(x)$ is called the partial + continuous at the point x^0 with respect to the variable x_i, $i = 1, 2$.

The partial $-$ limit and the partial $-$ continuity for the function $\varphi(x)$ at the point x^0 with respect to the variable x_i, $i = 1, 2$, are defined analogously.

The existence of equal between themselves partial $\pm$ limits both with respect to x_1 and to x_2 is, in general, insufficient for the existence of the limit at $x^0 = (x_1^0, x_2^0)$ for the function $\varphi(x)$, $x = (x_1, x_2)$.

Analogously, simultaneous $\pm$ continuities does not imply the continuity, in general.

Using the notions of partial continuities in the strong and angular senses ([1], [2], [4]), below we introduce the corresponding effective notions of the unilateral limit and continuity and establish their connections with the existence of the limit and continuity.

§ I.1. The Unilateral Limit and Continuity in the Strong

1^0. Let the function $f(x)$, $x = (x_1, x_2)$ be defined in the neighborhood $U(x^0)$ or in the punctured neighborhood $U^0(x^0) = U(x^0) \setminus \{x^0\}$ of the point $x^0(x_1^0, x_2^0)$.

We introduce the following sets:

$$A_1^+ = \big\{(x_1, x_2) \in U(x^0) : \ x_1 > x_1^0\big\}, \quad A_2^+ = \big\{(x_1^0, x_2) \in U(x^0) : \ x_2 > x_2^0\big\},$$
$$A_1^- = \big\{(x_1, x_2) \in U(x^0) : \ x_1 < x_1^0\big\}, \quad A_2^- = \big\{(x_1^0, x_2) \in U(x^0) : \ x_2 < x_2^0\big\},$$
$$A_{12}^+ = A_1^+ \cup A_2^+, \quad A_{12}^- = A_1^- \cup A_2^-.$$

It is obvious that

$$A_{12}^+ \cap A_{12}^- = \varnothing \quad \text{and} \quad A_{12}^+ \cup A_{12}^- = U^0(x^0). \tag{1}$$

Hence the punctured neighborhood is represented as the union of two non-intersecting sets, and the limit along each set will be called the + limit in the strong and the $-$ limit in the strong, according to the definitions below.

Definition I.1.1. We say that the function $f(x)$ has at the point x^0 the $+$ limit in the strong, in symbols $f(x^0[+])$, if there exists a limit which is finite or with a fixed sign infinite,

$$f(x^0[+]) = \lim_{\substack{x \to x^0 \\ x \in A_{12}^+}} f(x). \tag{2}$$

Analogously we define the $-$ limit in the strong,

$$f(x^0[-]) = \lim_{\substack{x \to x^0 \\ x \in A_{12}^-}} f(x). \tag{3}$$

If there exist the values $f(x^0[-])$ and $f(x^0[+])$, then we say that the function $f(x)$ has the $\pm$ limit in the strong at the point x^0.

From these definitions and equality (1) follows

Proposition I.1.1. *For the function $f(x)$ to have the limit at the point x^0, it is necessary and sufficient that there exist $f(x^0[-])$ and $f(x^0[+])$ together with the equality $f(x^0[-]) = f(x^0[+])$. When these assumptions are fulfilled, we have*

$$f(x^0[-]) = \lim_{x \to x_0} f(x) = f(x^0[+]). \tag{4}$$

2^0. Definition I.1.2. The function $f(x)$ is called the $+$ continuous in the strong at the point x^0 if the value $f(x^0)$ is finite and

$$f(x^0[+]) = f(x^0). \tag{5}$$

Analogously, the function $f(x)$ is called the $-$ continuous in the strong at the point x^0 if $f(x^0)$ is finite and

$$f(x^0[-]) = f(x^0). \tag{6}$$

The function $f(x)$ is called the $\pm$ continuous in the strong at the point x^0 if $f(x)$ at x^0 is both the $+$ continuous and the $-$ continuous.

The following proposition is valid:

Proposition I.1.2. *For the function $f(x)$ to be continuous at the point x^0, it is necessary and sufficient that $f(x)$ be $\pm$ continuous in the strong at the point x^0.*

It should be noted that Definitions I.1.1 and I.1.2 trace back to the well-known facts ([1], Corollary 2.1; [2], Statements 3) and 4) from Theorem 1.1).

§ I.2. Saltus in the Strong

1^0. Definition I.2.1. If $f(x^0[-])$ and $f(x^0[+])$ are finite for the function $f(x)$, then the value

$$\Omega(f,x^0) = \left|f(x^0[+]) - f(x^0[-])\right| \tag{1}$$

is called a saltus in the strong of the function $f(x)$ at the point x^0, and x^0 is called a point in the strong of the finite saltus of $f(x)$.

The proposition below is obvious.

Proposition I.2.1. *For the function $f(x)$ to have a finite limit at the point x^0, it is necessary and sufficient that the condition*

$$\Omega(f,x^0) = 0 \tag{2}$$

and in case this equality is fulfilled, we shall have

$$f(x^0[-]) = \lim_{x\to x^0} f(x) = f(x^0[+]) \tag{3}$$

2^0. If x^0 is a point of discontinuity of the function $f(x)$ (i.e., $f(x)$ is not continuous at x^0, and this means that the function $f(x)$ has no or has the limit which is different from $f(x^0)$, if $f(x^0)$ is finite. However, if the value $f(x^0)$ is infinite, the function $f(x)$ is likewise discontinuous at the point x^0) and equality (2) holds, then x^0 is called a removable in the strong point of discontinuity of the function $f(x)$: if the limit

$$\lim_{x\to x^0} f(x),$$

which is finite and equal in the strong to the $\pm$ limits of $f(x)$ at x^0, is recognized as the value of the function $f(x)$ at the point x^0, then a new function obtained as a result of such a correction will be continuous at the point x^0.

This procedure is called the correction in the strong for the continuity of the function $f(x)$ at the point x^0, and x^0 the remediable in the strong point of discontinuity of the function $f(x)$.

If there are finite $f(x^0[-])$ and $f(x^0[+])$, but $f(x^0[-]) \neq f(x^0[+])$ or, what is the same thing, if there is a bilateral inequality

$$0 < \Omega(f,x^0) < +\infty, \tag{4}$$

then x^0 is called the first kind point of discontinuity in the strong of the function $f(x)$.

If at least one out of the values $f(x^0[-])$ and $f(x^0[+])$ does not exist, or there exist the both values, but at least one is of fixed sign infinite, then x^0 is called the second kind point of discontinuity in the strong of the function $f(x)$.

§ I.3. The Angular Limit and the Angular Continuity

In Section I.1 we have considered the unilateral in the strong $\pm$ limits and $\pm$ continuities. Our consideration was based on the partitioning of the neighborhood of a point which in its turn was dictated by the notion of a separate partial continuity in the strong.

Along with this, the continuity is likewise equivalent to the separate partial continuity in the angular ([1], Theorem 3.1; [2], Theorem 1.1). This fact allows one to introduce the angular limit, angular continuity and unilateral angular limit, unilateral angular continuity.

Let the function $\varphi(x)$, $x = (x_1, x_2)$ be defined in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$.

1⁰. We begin with the introduction of the notion of angular limit with respect to the given variable.

Definition I.3.1. We say that the function $\varphi(x)$ has at the point x^0 the angular limit with respect to the variable x_1, in symbols $\varphi(x^0 \wedge (x_1))$, if for every constant $c > 0$ there exists an independent of c finite or of fixed sign infinite limit

$$\varphi(x^0 \wedge (x_1)) = \lim_{\substack{h_1 \to 0 \\ |h_2| \le c|h_1|}} \varphi(x_1^0 + h_1,\, x_2^0 + h_2). \tag{1}$$

Analogously, the function $\varphi(x)$ has at the point x^0 the angular limit with respect to the variable x_2, if for every constant $l > 0$ there exists an independent of l finite or of fixed sign infinite limit

$$\varphi(x^0 \wedge (x_2)) = \lim_{\substack{h_2 \to 0 \\ |h_1| \le l|h_2|}} \varphi(x_1^0 + h_1,\, x_2^0 + h_2). \tag{2}$$

If the values $\varphi(x^0 \wedge (x_1))$ and $\varphi(x^0 \wedge (x_2))$ exist, we say that the function $\varphi(x)$ has separated angular limits at the point x^0.

Theorem I.3.1. *The function $\varphi(x)$ has the limit at the point x^0 if and only if there exist at the point x^0 the equal between themselves separated angular limits for $\varphi(x)$. When these conditions are fulfilled, we have*

$$\varphi(x^0 \wedge (x_1)) = \lim_{x \to x^0} \varphi(x) = \varphi(x^0 \wedge (x_2)). \tag{3}$$

Proof. The existence of the limit at the point x^0 for the function $\varphi(x)$ implies that of the function $\varphi(x)$ along every subset with the limiting point at the point x^0. In particular, as such are the sets given in equalities (1) and (2) under the limit sign. Therefore $\lim\limits_{x \to x^0} \varphi(x)$ is equal to each of the limits (1) and (2).

If $\varphi(x^0 \wedge (x_1)) = \varphi(x^0 \wedge (x_2))$, then the function $\varphi(x)$ has at the point x^0 limits equal with respect to those two subsets which correspond to the particular cases $c = 1$ and $l = 1$. But the union of the two subsets results in the neighborhood of the point x^0. Thus the theorem is complete. □

2^0. Below we shall introduce the notion of an angular continuity with respect to the given variable. This notion differs from the earlier adopted by us notion of angular partial continuity with respect to the given variable ([1], [2]).

The matter is that the angular partial continuity is introduced with the help of the specific difference. The subtrahend of the difference is obtained by substituting the value of the given variable in the function under consideration. Moreover, the minuend is the value of the function at the point belonging to the angle, while the subtrahend is the value of the function at the point not belonging to the given angle.

Now we introduce the notion of the angular continuity with respect to the given variable. The notion will contain the values of the function only at the points which belong to the angle corresponding to the given variable.

Definition I.3.2. The function $\varphi(x)$ is said to be angular continuous at the point x^0 with respect to the variable x_1, if $\varphi(x^0)$ is a finite number and

$$\varphi\big(x^0 \wedge (x_1)\big) = \varphi(x^0). \tag{4}$$

Similarly, the function $\varphi(x)$ is said to be angular continuous at the point x^0 with respect to the variable x_2, if the value $\varphi(x^0)$ is finite and

$$\varphi\big(x^0 \wedge (x_2)\big) = \varphi(x^0). \tag{5}$$

The function $\varphi(x)$ is said to be separately angular continuous at the point x^0, if $\varphi(x)$ is angular continuous at the point x^0 with respect to each of variables x_1 and x_2 separately (the separately angular partial continuity can be found in [1] and [2]).

Theorem I.3.2. *For the function $\varphi(x)$ to be continuous at the point x^0, it is necessary and sufficient that $\varphi(x)$ be separately angular continuous at the point x^0.*

Proof. In case the function $\varphi(x)$ is continuous at the point x^0, the value $\varphi(x^0)$ is finite and $\lim_{x\to x^0}\varphi(x) = \varphi(x^0)$. The limits (1) and (2) are the particular cases on the left-hand side of that equality. Therefore

$$\varphi\big(x^0 \wedge (x_1)\big) = \varphi(x^0) = \varphi\big(x^0 \wedge (x_2)\big).$$

Hence the function $\varphi(x)$ is separately angular continuous at the point x^0.

Conversely, if each of the limits (1) and (2) is equal to the finite number $\varphi(x^0)$, then the limits for $\varphi(x)$ will likewise be equal to the value $\varphi(x^0)$ with respect to the sets which correspond to the cases $c = 1$ and $l = 1$.

The union of these sets yield the neighborhood of the point x^0. Hence the limit of the function $\varphi(x)$ at the point x^0 is equal to the finite value $\varphi(x^0)$, i.e., $\varphi(x)$ is continuous at the point x^0. Thus the theorem is complete. □

§ I.4. The Unilateral Angular Limit and Continuity

The angular $\pm$ limits at the point x^0 for the function $\varphi(x)$ with respect to the variable x_1, in symbols $\varphi(x^0\widehat{+}(x_1))$ and $\varphi(x^0\widehat{-}(x_1))$, will be defined below by equalities (6) and (7) if these limits exist and do not depend on the constants $a > 0$ and $b > 0$,

$$\varphi(x^0\widehat{+}(x_1)) = \lim_{\substack{h_1\to 0+\\ |h_2|\le ah_1}} \varphi(x_1^0 + h_1,\, x_2^0 + h_2), \tag{6}$$

$$\varphi(x^0\widehat{-}(x_1)) = \lim_{\substack{h_1\to 0-\\ |h_2|\le -bh_1}} \varphi(x_1^0 + h_1,\, x_2^0 + h_2). \tag{7}$$

We define the angular $\pm$ limits at the point x^0 for the function $\varphi(x)$ with respect to the variable x_2 by equalities (8) and (9) under similar assumptions for the constants $c > 0$ and $d > 0$,

$$\varphi(x^0\widehat{+}(x_2)) = \lim_{\substack{h_2\to 0+\\ h_2\ge c|h_1|}} \varphi(x_1^0 + h_1,\, x_2^0 + h_2), \tag{8}$$

$$\varphi(x^0\widehat{-}(x_2)) = \lim_{\substack{h_2\to 0-\\ h_2\le -d|h_1|}} \varphi(x_1^0 + h_1,\, x_2^0 + h_2). \tag{9}$$

It can be easily seen that the following propositions holds.

Proposition I.4.1. *The function $\varphi(x)$ has at the point x^0 the angular limit with respect to the variable x_1 if and only if there exist equal between themselves values $\varphi(x^0\widehat{-}(x_1))$ and $\varphi(x^0\widehat{+}(x_1))$. In this case we have*

$$\varphi(x^0\widehat{-}(x_1)) = \varphi(x^0 \wedge (x_1)) = \varphi(x^0\widehat{+}(x_1)). \tag{10}$$

Analogous proposition is obviously holds for the variable x_2 as well.

Theorem I.3.1 and Proposition I.4.1 imply

Corollary I.4.1. *The function $\varphi(x)$ has at the point x^0 the limit if and only if there exist equal between themselves values defined by equalities* (6)–(9)*. In case these conditions are fulfilled, the* $\lim_{x\to x^0}\varphi(x)$ *is equal to their common value.*

§ I.5. Saltus in the Angular

Definition I.5.1. If the function $\varphi(x)$ has finite $\varphi(x^0 \wedge (x_1))$ and $\varphi(x^0 \wedge (x_2))$, then the value

$$\omega(\varphi, x^0) = \big|\varphi(x^0 \wedge (x_1)) - \varphi(x^0 \wedge (x_2))\big| \tag{1}$$

is called a saltus in the angular of the function $\varphi(x)$ at the point x^0.

The following proposition is valid.

Proposition I.5.1. *The equality*

$$\omega(\varphi, x^0) = 0 \tag{2}$$

is the necessary and sufficient condition for the function $\varphi(x)$ to have the finite limit at the point x^0. If equality (2) is fulfilled, then the value $\varphi(x^0 \wedge (x_1)) = \varphi(x^0 \wedge (x_2))$ is the limit at the point x^0 of the function $\varphi(x)$.

Here we can also introduce the following notions: in the angular, a removable point of discontinuity of the function $\varphi(x)$; in the angular, the correction for the continuity of the function $\varphi(x)$ at the point x^0; in the angular, a correctable point of discontinuity of the function $\varphi(x)$; in the angular, point of the first kind discontinuity for the function $\varphi(x)$; in the angular, a point of the second kind discontinuity for the function $\varphi(x)$.

§ I.6. Comparison of Corrections in the Strong and in the Angular

Proposition I.6.1. *If the function $f(x)$ admits the correction in the strong for the continuity at the point x^0, then $f(x)$ admits also that in the angular for the continuity at the point x^0.*

The converse statement is valid as well.

Proof. Since $f(x)$ admits the correction in the strong for the continuity at the point x^0, equality (2) from §I.2 is fulfilled. Consequently, there exists the finite limit for the function $f(x)$ at the point x^0, by Proposition I.2.1. This implies that equality (2) from §I.5 is fulfilled, by Proposition I.5.1. Therefore the correctness in the angular of the function $f(x)$ is quite possible for the continuity at the point x^0. □

The converse statement is proved analogously. Thus the proposition is complete.

Now Proposition I.6.1 allows one to come to the following agreement: the function $f(x)$ is called correctable for the continuity at the point x^0 if $f(x)$ admits the correction in the strong or in the angular for the continuity at the point x^0.

Finally, if the function $f(x)$ has at the point x^0 the incorrectable, or what is the same thing, unremovable discontinuity, then x^0 is called the point of essential discontinuity of the function $f(x)$, and the function $f(x)$ is called essentially discontinuous at the point x^0.

PART II

The Unilateral Partial Derivatives in Various Senses and the Differentials

Preliminary Facts

1^0. Let the function $\psi(x)$, $x=(x_1,x_2)$, be specified in the neighborhood $U(x^0)$ of the point $x^0=(x_1^0,x_2^0)$. If the function ${}^i\psi(x_i)$ has at the point x_i^0 a derivative $({}^i\psi(x_i))'(x_i^0)$, then the latter is called a partial derivative with respect to the variable x_i at the point x^0 for the function $\psi(x)$ and is denoted by one of the symbols $\psi'_{x_i}(x^0)$, $\frac{\partial\psi}{\partial x_i}(x^0)$, $\partial x_i\psi(x^0)$.

Of these notations we choose $\partial x_i\psi(x^0)$ because below we shall consider the unilateral partial derivatives which require additional symbol $+$ or $-$.

If there exist $\partial x_1\psi(x^0)$ and $\partial x_2\psi(x^0)$, then we consider the gradient of the function $\psi(x)$ at the point x^0,

$$\operatorname{grad}\psi(x^0)=\big(\partial x_1\psi(x^0),\partial x_2\psi(x^0)\big).$$

2^0. It is quite possible that the function ${}^i\psi(x_i)$ has no derivative at the point x_i^0, i.e., there is no $\partial x_i\psi(x^0)$, but ${}^i\psi(x_i)$ has the $+$ derivative at the point x_i^0, in symbols $\partial x_i^+\psi(x^0)$, which is called the right partial derivative with respect to the variable x_i at the point x^0 for the function $\psi(x)$. Hence

$$\partial^+_{x_i}\psi(x^0)=\lim_{x_i\to x_i^0+}\frac{{}^i\psi(x_i)-{}^i\psi(x_i^0)}{x_i-x_i^0}=\lim_{x_i\to x_i^0+}\frac{{}^i\psi(x_i)-\psi(x^0)}{x_i-x_i^0}.$$

The left partial derivative with respect to the variable x_i at the point x^0 for the function $\psi(x)$ is defined analogously:

$$\partial^-_{x_i}\psi(x^0)=\lim_{x_i\to x_i^0-}\frac{{}^i\psi(x_i)-{}^i\psi(x_i^0)}{x_i-x_i^0}=\lim_{x_i\to x_i^0-}\frac{{}^i\psi(x_i)-\psi(x^0)}{x_i-x_i^0}.$$

It is evident that for the existence of $\partial x_i\psi(x^0)$ it is necessary and sufficient that the equal between themselves values $\partial^+x_i\psi(x^0)$ and $\partial^-x_i\psi(x^0)$ exist.

In case the values $\partial^+x_1\psi(x^0)$ and $\partial^+x_2\psi(x^0)$ exist, we introduce the $+$ gradient at the point x^0 for the function $\psi(x)$,

$${}^+\operatorname{grad}\psi(x^0)=\big(\partial^+x_1\psi(x^0),\ \partial^+x_2\psi(x^0)\big).$$

Similarly, if there exist $\partial^-x_1\psi(x^0)$ and $\partial^-x_2\psi(x^0)$, then we introduce the $-$ gradient at the point x^0 for the function $\psi(x)$ by the equality

$${}^-\operatorname{grad}\psi(x^0)=\big(\partial^-x_1\psi(x^0),\ \partial^-x_2\psi(x^0)\big).$$

For the validity of the correlations

$${}^-\operatorname{grad}\psi(x^0)=\operatorname{grad}\psi(x^0)={}^+\operatorname{grad}\psi(x^0)$$

the equalities of all components

$$\partial^-x_i\psi(x^0)=\partial x_i\psi(x^0)=\partial^+x_i\psi(x^0),\quad i=1,2$$

are necessary and sufficient.

The last equalities are, in general, insufficient for the existence of the angular or of the strong gradient at the point x^0 of the function $\psi(x)$, which are tightly connected with the function $\psi(x)$ ([2], [3]).

Below we shall introduce the unilateral partial $\pm$ derivatives both in the strong and in the angular senses. The necessary and sufficient conditions will be proved for the existence of the strong and angular gradients. However, the conditions for the existence of the angular gradient will at the same time be the conditions for the existence of a total differential.

§ II.1. The Unilateral in the Strong Partial Derivatives

The notions ([2], [3]) of partial derivatives in the strong with respect to the variables x_1 and x_2 at the point $x^0 = (x_1^0, x_2^0)$ for the function $\psi(x)$, $x = (x_1, x_2)$ allow one to introduce partial $\pm$ derivatives in the strong with respect to x_1 and x_2 at the point x^0 for the function $\psi(x)$:

$$\partial^+_{[x_1]}\psi(x^0) = \lim_{\substack{(h_1,h_2)\to(0,0)\\ h_1>0}} \frac{\psi(x_1^0+h_1,\, x_2^0+h_2) - \psi(x_1^0, x_2^0+h_2)}{h_1}, \tag{1}$$

$$\partial^-_{[x_1]}\psi(x^0) = \lim_{\substack{(h_1,h_2)\to(0,0)\\ h_1<0}} \frac{\psi(x_1^0+h_1,\, x_2^0+h_2) - \psi(x_1^0, x_2^0+h_2)}{h_1}, \tag{2}$$

$$\partial^+_{[x_1]}\psi(x^0) = \lim_{\substack{(h_1,h_2)\to(0,0)\\ h_2>0}} \frac{\psi(x_1^0+h_1,\, x_2^0+h_2) - \psi(x_1^0+h_1,\, x_2^0)}{h_2}, \tag{3}$$

$$\partial^-_{[x_2]}\psi(x^0) = \lim_{\substack{(h_1,h_2)\to(0,0)\\ h_2<0}} \frac{\psi(x_1^0+h_1,\, x_2^0+h_2) - \psi(x_1^0+h_1, x_2^0)}{h_2}. \tag{4}$$

It is clear that for the existence of $\partial_{[x_i]}\psi(x^0)$ it is necessary and sufficient that there exist equal between themselves $\partial^+_{[x_i]}\psi(x^0)$ and $\partial^-_{[x_i]}\psi(x^0)$, and if they are equal, we have

$$\partial^-_{[x_i]}\psi(x^0) = \partial_{[x_i]}\psi(x^0) = \partial^+_{[x_i]}\psi(x^0), \quad i = 1, 2. \tag{5}$$

Let us now introduce the strong $\pm$ gradients at the point x^0 for the function $\psi(x)$ by the equalities

$$^+\operatorname{str}\operatorname{grad}\psi(x^0) = \big(\partial^+_{[x_1]}\psi(x^0),\, \partial^+_{[x_2]}\psi(x^0)\big), \tag{6}$$

$$^-\operatorname{str}\operatorname{grad}\psi(x^0) = \big(\partial^-_{[x_1]}\psi(x^0),\, \partial^-_{[x_2]}\psi(x^0)\big), \tag{7}$$

which are connected with the strong gradient ([2],[3])

$$\operatorname{str}\operatorname{grad}\psi(x^0) = \big(\partial_{[x_1]}\psi(x^0),\, \partial_{[x_2]}\psi(x^0)\big) \tag{8}$$

as follows:

Theorem II.1.1. *For the existence of* $\operatorname{str}\operatorname{grad}\psi(x^0)$ *it is necessary and sufficient that there exist equal between themselves* ${}^{-}\operatorname{str}\operatorname{grad}\psi(x^0)$ *and* ${}^{+}\operatorname{str}\operatorname{grad}\psi(x^0)$*, and if they are equal, we have*

$$ {}^{-}\operatorname{str}\operatorname{grad}\psi(x^0) = \operatorname{str}\operatorname{grad}\psi(x^0) = {}^{+}\operatorname{str}\operatorname{grad}\psi(x^0). \tag{9} $$

Theorem II.1.2. *The existence of finite* $\partial^{-}_{[x_i]}\psi|(x^0)$ *and* $\partial^{+}_{[x_i]}\psi|(x^0)$ *implies that the strong symmetric partial derivative is finite with respect to the variable* x_i *at the point* x^0 *for the function* $\psi(x_1,x_2)$ ([5], [6])*, denoted by* $\partial^{(1)}_{[x_i]}\psi(x^0)$*, for which the equality*

$$ \partial^{(1)}_{[x_i]}\psi(x^0) = \frac{1}{2}\big[\partial^{-}_{[x_i]}\psi(x^0) + \partial^{+}_{[x_i]}\psi(x^0)\big], \quad i=1,2, \tag{10} $$

holds.

In addition, there exists the function for which the left-hand side of equality (10) *is finite, and the summands on the right-hand side of the same equality are infinite of opposite signs.*

Proof. The above-said will be checked for the variable x_1. Since in the equality (see [6], Definition 3)

$$ \partial^{(1)}_{[x_1]}\psi(x^0) = \lim_{(h_1,h_2)\to(0,0)} \frac{\psi(x_1^0+h_1,\, x_2^0+h_2) - \psi(x_1^0-h_1, x_2^0+h_2)}{2h_1}. $$

the relation appearing under the limit sign is the even function with respect to h_1, we can assume that $h_1>0$ and have

$$ \begin{aligned} \partial^{(1)}_{[x_1]}\psi(x^0) &= \frac{1}{2}\lim_{(h_1,h_2)\to(0,0)} \frac{\psi(x_1^0+h_1,\, x_2^0+h_2) - \psi(x_1^0, x_2^0+h_2)}{2h_1} + \\ &+ \frac{1}{2}\lim_{(h_1,h_2)\to(0,0)} \frac{\psi(x_1^0-h_1,\, x_2^0+h_2) - \psi(x_1^0, x_2^0+h_2)}{2h_1} = \\ &= \frac{1}{2}\big[\partial^{+}_{[x_1]}\psi(x^0) + \partial^{-}_{[x_1]}\psi(x^0)\big]. \end{aligned} $$

The function $\varphi(x_1,x_2) = |x_1|^{1/2} + |x_2|^{1/2}$ in the neighborhood of the point $x^0 = (0,0)$ fits for the second part of the theorem. We have

$$\partial^{(1)}_{[x_1]}\varphi(x^0) = \lim_{(h_1,h_2)\to(0,0)} \frac{\varphi(h_1,h_2)-\varphi(-h_1,h_2)}{2h_1} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{|h_1|^{1/2}+|h_2|^{1/2}-|-h|^{1/2}-|h_2|^{1/2}}{2h_1} = 0,$$

$$\partial^{+}_{[x_1]}\psi(x^0) = \lim_{\substack{h_1\to 0+\\ h_2\to 0}} \frac{\varphi(h_1,h_2)-\varphi(0,h_2)}{h_1} =$$

$$= \lim_{\substack{h_1\to 0+\\ h_2\to 0}} \frac{|h_1|^{1/2}+|h_2|^{1/2}-|h|^{1/2}}{|h_1|} = +\infty,$$

$$\partial^{-}_{[x_1]}\psi(x^0) = \lim_{\substack{h_1\to 0-\\ h_2\to 0}} \frac{|h_1|^{1/2}}{h_1} = -\lim_{\substack{h_1\to 0-\\ h_2\to 0}} \frac{|h_1|^{1/2}}{|h_1|} = -\infty. \qquad \square$$

§ II.2. The Unilateral in the Angular Partial Derivatives

Below we shall introduce partial in the angular $\pm$ derivatives with respect to the variables x_1 and x_2 at the point $x^0 = (x_1^0, x_2^0)$ for the function $\psi(x)$, $x = (x_1, x_2)$ by means of equalities (1)–(4) under the condition that each of the limits exists and does not depend on the constant therein:

$$\partial^{+}_{\widehat{x_1}}\psi(x^0) = \lim_{\substack{h_1\to 0+\\ |h_2|\le a h_1}} \frac{\psi(x_1^0+h_1, x_2^0+h_2)-\psi(x_1^0, x_2^0+h_2)}{h_1}, \quad a>0, \tag{1}$$

$$\partial^{+}_{\widehat{x_2}}\psi(x^0) = \lim_{\substack{h_2\to 0+\\ h_2\ge b|h_1|}} \frac{\psi(x_1^0+h_1, x_2^0+h_2)-\psi(x_1^0, x_2^0+h_2)}{h_2}, \quad b>0, \tag{2}$$

$$\partial^{-}_{\widehat{x_1}}\psi(x^0) = \lim_{\substack{h_1\to 0-\\ |h_2|\le -c h_1}} \frac{\psi(x_1^0+h_1, x_2^0+h_2)-\psi(x_1^0, x_2^0+h_2)}{h_1}, \quad c>0, \tag{3}$$

$$\partial^{-}_{\widehat{x_2}}\psi(x^0) = \lim_{\substack{h_2\to 0-\\ h_2\le -d|h_1|}} \frac{\psi(x_1^0+h_1, x_2^0+h_2)-\psi(x_1^0+h_1, x_2^0)}{h_2}, \quad d>0. \tag{4}$$

The existence and equality of the values $\partial^{-}_{\widehat{x_i}}\psi(x^0)$ and $\partial^{-}_{\widehat{x_i}}\psi(x^0)$ is the necessary and sufficient condition for the existence of $\partial_{\widehat{x_i}}\psi(x^0)$ ([2], Definition 2.1).

The angular $\pm$ gradients at the point x^0 for the function $\psi(x)$ are introduced by the equalities, if there exist their components:

$$^{+}\operatorname{ang\,grad}\psi(x^0) = \big(\partial^{+}_{\widehat{x_1}}\psi(x^0),\ \partial^{+}_{\widehat{x_2}}\psi(x^0)\big), \tag{5}$$

$$^{-}\operatorname{ang\,grad}\psi(x^0) = \big(\partial^{-}_{\widehat{x_1}}\psi(x^0),\ \partial^{-}_{\widehat{x_2}}\psi(x^0)\big). \tag{6}$$

This implies that for the angular gradient ([2], Definition 2.2)

$$\operatorname{ang\,grad}\psi(x^0)=\big(\partial^{+}_{\widehat{x_1}}\psi(x^0),\,\partial^{+}_{\widehat{x_2}}\psi(x^0)\big),\tag{7}$$

we arrive at the following

Theorem II.2.1. *For the existence of $\psi(x^0)$ it is necessary and sufficient that there exist equal values ${}^{+}\operatorname{ang\,grad}\psi(x^0)$ and ${}^{-}\operatorname{ang\,grad}\psi(x^0)$, and if they are equal we have*

$${}^{-}\operatorname{ang\,grad}\psi(x^0)=\operatorname{ang\,grad}\psi(x^0)={}^{+}\operatorname{ang\,grad}\psi(x^0).\tag{8}$$

From the above-said, by virtue of Theorem 2.2 from [2], we get

Theorem II.2.2. *For the total differential $d\psi(x^0)$ to exist, it is necessary and sufficient that there exist finite and equal between themselves angular $\pm$ gradients at the point x^0 specified by equalities* (5) *and* (6).

§ II.3. The Unilateral Differentials

Since the finiteness of $\operatorname{ang\,grad}\psi(x^0)$ is the necessary and sufficient condition for the existence of a total differential $d\psi(x^0)$ ([2], Theorem 2.2 and [3], Theorem 1), using the angular $\pm$ gradients, we introduce the following

Definition II.3.1. The function $\psi(x)$ is said to be $+$ differentiable at the point x^0, if ${}^{+}\operatorname{ang\,grad}\psi(x^0)$ is finite, while the $+$ differential, in symbols $d^{+}\psi(x^0)$, for the function $\psi(x)$ at the point x^0 is defined by the equality

$$d^{+}\psi(x^0)=\partial^{+}_{\widehat{x_1}}\psi(x^0)dx_1+\partial^{+}_{\widehat{x_2}}\psi(x^0)dx_2.\tag{1}$$

Analogously, the $-$ differential for the finite ${}^{-}\operatorname{ang\,grad}\psi(x^0)$ is defined by the equality

$$d^{-}\psi(x^0)=\partial^{-}_{\widehat{x_1}}\psi(x^0)dx_1+\partial^{-}_{\widehat{x_2}}\psi(x^0)dx_2.\tag{2}$$

Thus we have the following

Theorem II.3.1. *For the existence of the total differential $d\psi(x^0)$ it is necessary and sufficient that there exist the equal between themselves $\pm$ differentials $d^{-}\psi(x^0)$ and $d^{+}\psi(x^0)$, and if they are equal, we have*

$$d^{-}\psi(x^0)=d\psi(x^0)=d^{+}\psi(x^0).\tag{3}$$

Remark. The finiteness of ${}^{+}\operatorname{str\,grad}\psi(x^0)$ implies that of ${}^{+}\operatorname{ang\,grad}\psi(x^0)$ and, consequently, the existence of the $+$ differential $d^{+}\psi(x^0)$. The fact analogous to the above is available for the case of ${}^{-}\operatorname{str\,grad}\psi(x^0)$.

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