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# ON ONE ANALOGUE OF LEBESGUE THEOREM ON THE DIFFERENTIATION OF INDEFINITE INTEGRAL FOR FUNCTIONS OF SEVERAL VARIABLES

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Abstract. It is proved that an indefinite $n$-tuple integral of a summable on the unit $n$-dimentional cube function is differentiable almost everywhere,moreover, it has a strong gradient almost everywhere.

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## 1. Definitions and Notation

Let us assume that $L(0,1)^n = \{f \in L(\mathbb{R}^n) : \operatorname{supp} f \subset (0,1)^n\}$. *The indefinite integral of a function* $f \in L(0,1)^n$ denote by $F_f$, i.e., for every point $x = (x_1, \dots, x_n)$ from $(0,1)^n$

$$F_f(x) = \int\limits_{(0,x_1)\times\dots\times(0,x_n)} f.$$

For $t \in \mathbb{R}^{n-1}$, $\tau \in \mathbb{R}$ and $i \in \overline{1,n}$ denote by $(t,\tau)^i$ the point in $\mathbb{R}^n$ for which $(t,\tau)^i_j = t_j$ if $1 \le j < i$, $(t,\tau)^i_i = \tau$ and $(t,\tau)^i_j = t_{j-1}$ if $i < j \le n$.

Let a function $f$ is defined on $(0,1)^n$, $\tau \in \mathbb{R}$ and $i \in \overline{1,n}$.Denote by $f_{\tau,i}$ the function defined on $(0,1)^{n-1}$ by the equality

$$f_{\tau,i}(t) = f\big((t,\tau)^i\big), \quad t \in (0,1)^{n-1}.$$

Denote for $n \ge 2$ and $x \in (0,1)^n$

$$Q_1(x) = (0,x_2)\times\dots\times(0,x_n), Q_n(x) = (0,x_1)\times\dots\times(0,x_{n-1});$$

and for $n \ge 3$, $x \in (0,1)^n$, $2 \le i \le n-1$

$$Q_i(x) = (0,x_1)\times\dots\times(0,x_{i-1})\times(0,x_{i+1})\times\dots\times(0,x_n).$$

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Let $n \geq 2$ and let $f \in L(0,1)^n$. By virtue of Fubini theorem for a.e., $x \in (0,1)^n$ we have that $f_{x_i,i} \in L(0,1)^{n-1}$ for every $i \in \overline{1,n}$, thus for a.e., $x \in (0,1)^n$ it makes sense the integrals $\int\limits_{Q_i(x)} f_{x_i,i}$, $i \in \overline{1,n}$.

For $n \geq 2, h \in \mathbb{R}^n$ and $i \in \overline{1,n}$ denote by $h(i)$ the point in $\mathbb{R}^n$ such that $h(i)_j = h_j$ for every $j \in \overline{1,n}\backslash\{i\}$ and $h(i)_i = 0$.

Let $n \geq 2$ and $f$ be a function defined in a neighborhood of a point $x \in \mathbb{R}^n$.If for $i \in \overline{1,n}$ there exists the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x+h(i))}{h_i}$$

then let us call its value as *the $i$-th strong partial derivative of $f$ at $x$* and denote it by $D_{[i]}f(x)$.If $f$ has finite $D_{[i]}f(x)$ for every $i \in \overline{1,n}$ then let us say that *there exists a strong gradient of $f$ at $x$ or $f$ has a strong gradient at $x$.*

It is not difficult to verify that if a function $f$ has a strong gradient at a point $x$ then it is differentiable at $x$, and the converse assertion is not true:the function $f(x_1, x_2) = |x_1 x_2|^{\frac{2}{3}}$ is differentiable at the point $(0,0)$, but $\overline{D}_{[1]}f(0,0) = \overline{D}_{[2]}f(0,0) = +\infty$ (see [1] for details). Thus the condition of differentiability at the fixed point is weaker then the condition of the existence of a strong gradient in the same point. Note that the same conclusion remains true even while comparison on the sets of positive measure, namely, according [2] there exists a continuous function such that the set of all points at which $f$ is differentiable but does not have a strong gradient is of full measure.

## 2. Result

According to the well-known Lebesgue theorem *for every $f \in L(0,1)$ its indefinite integral $F_f$, at almost every point $x \in (0,1)$, is differentiable and $F'_f(x) = f(x)$.*

The following statement is a multidimensional analogue of Lebesgue theorem.

**Theorem.** *For every $n \geq 2$ and $f \in L(0,1)^n$ the indefinite integral of $f$, at almost every point $x \in (0,1)^n$, is differentiable, moreover, has a strong gradient and $D_{[i]}F_f(x) = \int\limits_{Q_i(x)} f_{x_i,i}$* for every $i \in \overline{1,n}$.

This theorem in two-dimensional case was proved in [1].

## 3. One Lemma

For $x \in \mathbb{R}^2$ denote by $\mathbb{I}(x)$ the collection of all two-dimensional intervals containing $x$ and for $I \in \mathbb{I}(x)$ by $\delta(I)$ denote the smallest among lengths of its sides.

The proof of Theorem is based on Lebesgue theorem and on the following statement that was established in [1](see also [3] for more general result)

**Lemma.** *Let $f \in L(\mathbb{R}^2)$. Then for almost every $x \in \mathbb{R}^2$*

$$\lim_{I \in \mathbb{I}(x) \, \mathrm{diam}\, I \to 0} \frac{1}{\delta(I)} \int_I |f| = 0.$$

## 4. Proof of Theorem

For any $i \in \overline{1,n}$ let us show that $D_{[i]} F_f(x) = \int\limits_{Q_i(x)} f_{x_i,i}$ for almost every $x \in (0,1)^n$. Consequently, Theorem will be proved. For the simplicity of entries let us consider the case $i = n$.

For the numbers $a$ and $b$ by $J(a,b)$ denote the segment $[\min(a,b), \max(a,b)]$.

Let $x \in (0,1)^n, h \in \mathbb{R}^n, |h_1| < 1, \dots, |h_n| < 1$ and $x + h \in (0,1)^n$. It is easy to check that

$$\frac{F_f(x+h) - F_f(x+h(n))}{h_n} = \frac{\mathrm{sign}\,(h_n)}{h_n} \int\limits_{Q_n(x+h) \times J(x_n, x_n+h_n)} f =$$

$$= \frac{\mathrm{sign}\,(h_n)}{h_n} \int\limits_{Q_n(x) \times J(x_n, x_n+h_n)} f +$$

$$+ \frac{\mathrm{sign}\,(h_n)}{h_n} \left( \int\limits_{Q_n(x+h) \times J(x_n, x_n+h_n)} f \quad \int\limits_{Q_n(x) \times J(x_n, x_n+h_n)} f \right) =$$

$$= \eta_1 + \eta_2; \tag{1}$$

$$\eta_2 \le \frac{1}{|h_n|} \int\limits_{(Q_n(x+h) \times J(x_n, x_n+h_n)) \triangle (Q_n(x) \times J(x_n, x_n+h_n))} |f| =$$

$$= \frac{1}{|h_n|} \int\limits_{(Q_n(x+h) \triangle Q_n(x)) \times J(x_n, x_n+h_n)} |f| = \eta_3; \tag{2}$$

$$(Q_n(x+h_n) \triangle Q_n(x)) \times J(x_n, x_n+h) \subset \bigcup_{j=1}^{n-1} S_j(x,h), \tag{3}$$

where $S_j(x,h) = \{y \in \mathbb{R}^n : y_n \in J(x_n, x_n+h_n),\ y_j \in J(x_j, x_j+h_j);$ $y_k \in (0, x_k+1), k \in \overline{1,n} \setminus \{n,j\}\}$;

$$\eta_3 \le \sum_{j=1}^{n-1} \frac{1}{|h_n|} \int\limits_{S_j(x,h)} |f|. \tag{4}$$

Let us prove that

$$\lim_{h\to 0}\frac{\operatorname{sign}(h_n)}{h_n}\int\limits_{Q_n(x)\times J(x_n,x_n+h_n)} f=\int\limits_{Q_n(x)} f_{x_n,n}$$

for almost every $x\in(0,1)^n$ and for any $j\in\overline{1,n-1}$

$$\lim_{h\to 0}\frac{1}{h_n}\int\limits_{S_j(x,h)}|f|=0 \tag{5}$$

for almost every $x\in(0,1)^n$. Therefore taking into account (1)–(4) we come to the validity of Theorem.

Due to Fubini theorem there is a set $E\subset\mathbb{R}$ with full measure such that for any $t\in E$ the function $\mathbb{R}^{n-1}\ni(r_1,\dots,r_{n-1})\mapsto f(r_1,\dots,r_{n-1},t)$ is summable on $\mathbb{R}^{n-1}$. So for given $y\in(0,1)^{n-1}$ we can consider the function $g_y$ defined as follows: $g_y(t)=0$ for $t\in\mathbb{R}\setminus E$ and for $t\in E$

$$g_y(t)=\int\limits_{(0,y_1)\times\cdots\times(0,y_{n-1})} f(r_1,\dots,r_{n-1},t)dr_1\cdots dr_{n-1}.$$

By virtue of Fubini theorem for any $y\in(0,1)^{n-1}$ we have that $g_y\in L(0,1)$, therefore due to Lebesgue theorem

$$\lim_{\alpha\to 0}\frac{\operatorname{sign}(\alpha)}{\alpha}\int\limits_{J(t,t+\alpha)} g_y(\tau)d\tau=g_y(t)$$

for almost every $t\in(0,1)$. Consequently, taking into account Fubini theorem we conclude that for almost every $x\in(0,1)^n$

$$\lim_{h\to 0}\frac{\operatorname{sign}(h_n)}{h_n}\int\limits_{Q_n(x)\times J(x_n,x_n+h_n)} f=$$

$$=\lim_{h\to 0}\frac{\operatorname{sign}(h_n)}{h_n}\int\limits_{J(x_n,x_n+h_n)} g_{(x_1,\dots,x_{n-1})}(\tau)d\tau=$$

$$=g_{(x_1,\dots,x_{n-1})}(x_n)=\int\limits_{Q_n(x)} f_{x_n,n}.$$

For the simplicity of entries let us prove (5) in the case $j=n-1$. If $n=2$ we have only one possibility: $j=1$, and $S_1(x,h)=J(x_1,x_1+h_1)\times(x_2,x_2+h_2)$. Consequently, by virtue of Lemma for almost every $x\in(0,1)^2$ we have

$$\lim_{h\to 0}\frac{1}{h_2}\int\limits_{S_1(x,h)}|f|=\lim_{h\to 0}\frac{1}{h_2}\int\limits_{J(x_1,x_1+h_1)\times J(x_2,x_2+h_2)}|f|=0.$$

Let now $n \geq 3$. Repeating arguments given for $g_y$, for any fixed $y \in (0,1)^{n-2}$ we can consider the function $l_y$ summable on $(0,1)^2$ which at almost every point $(t_1, t_2) \in (0,1)^2$ is defined as follows

$$l_y(t_1,t_2) = \int\limits_{(0,y_1+1)\times\cdots\times(0,y_{n-2}+1)} |f(r_1,\dots,r_{n-2},t_1,t_2)|dr_1\cdots dr_{n-2}.$$

According to Fubini theorem for any $y \in (0,1)^{n-2}$ we have that $l_y \in L(0,1)^2$. Consequently, by virtue of Lemma

$$\lim_{(\alpha_1,\alpha_2)\to 0} \frac{1}{\alpha_2} \int\limits_{J(t_1,t_1+\alpha_1)\times J(t_2,t_2+\alpha_2)} l_y(\tau_1,\tau_2)d\tau_1 d\tau_2 = 0$$

for almost every $(t_1,t_2) \in (0,1)^2$. Where from taking into account Fubini theorem we conclude that for almost every $x \in (0,1)^n$

$$\lim_{h\to 0} \frac{1}{h_n} \int\limits_{S_{n-1}(x,h)} |f| =$$

$$= \lim_{h\to 0} \frac{1}{h_n} \int\limits_{J(x_{n-1},x_{n-1}+h_{n-1})\times J(x_n,x_n+h_n)} l_{(x_1,\dots,x_{n-2})}(\tau_1,\tau_2)d\tau_1 d\tau_2 = 0.$$

Theorem is proved.

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