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On One Analogue of Lebesgue Theorem on the Differentiation of Indefinite Integral for Functions of Several Variables

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Definitions and notation. For $x \in (0,1)^n$ denote

$$Q(x) = (0,x_1) \times \cdots \times (0,x_n),$$
$$Q_1(x) = (0,x_2) \times \cdots \times (0,x_n),$$
$$Q_i(x) = (0,x_1) \times \cdots \times (0,x_{i-1}) \times (0,x_{i+1}) \times \cdots \times (0,x_n) \quad (2 \le i \le n-1),$$
$$Q_n(x) = (0,x_1) \times \cdots \times (0,x_{n-1}).$$

The indefinite integral of a function $f \in L(0,1)^n$ is denoted by F_f and is defined as follows

$$F_f(x) = \int\limits_{Q(x)} f(t)dt, \quad x \in (0,1)^n.$$

For $t \in \mathbb{R}^{n-1}$, $\tau \in \mathbb{R}$ and $i \in \overline{1,n}$ denote by (t,τ,i) the point in $\mathbb{R}^n$ for which $(t,\tau,i)_j = t_j$ if $1 \le j < i$, $(t,\tau,i)_i = \tau$, and $(t,\tau,i)_j = t_{j-1}$ if $i < j \le n$.

Let a function f is defined on $(0,1)^n$, $\tau \in \mathbb{R}$ and $i \in \overline{1,n}$. Denote by $f_{\tau,i}$ the function defined on $(0,1)^{n-1}$ by the equality

$$f_{\tau,i}(t) = f(t,\tau,i), \quad t \in (0,1)^{n-1}.$$

Note that by virtue of Fubini's theorem for $f \in L(0,1)^n$ for a.e. $x \in (0,1)^n$ we have that $f_{x_i,i} \in L(0,1)^{n-1}$ for every $i \in \overline{1,n}$. Thus for a.e. $x \in (0,1)^n$ it has sense the integrals

$$\int\limits_{Q_i(x)} f_{x_i,i}(t)dt, \quad i \in \overline{1,n}.$$

For $n \ge 2, h \in \mathbb{R}^n$ and $i \in \overline{1,n}$ denote by $h(i)$ the point in $\mathbb{R}^n$ such that $h(i)_j = h_j$ for every $j \in \overline{1,n}\backslash\{i\}$ and $h(i)_i = 0$.

Let $n \ge 2$ and f be a function defined in a neighborhood of a point $x \in \mathbb{R}^n$. If for $i \in \overline{1,n}$ there exists the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x+h(i))}{h_i},$$

then let us call its value as *the i-th strong partial derivative of f at x* and denote it by $D_{[i]}f(x)$.If f has finite $D_{[i]}f(x)$ for every $i \in \overline{1,n}$ then let us say that *there exists a strong gradient of f at x or f has a strong gradient at x.*

In [1] and [2] it is noted that if a function f has a strong gradient at a point x then it is differentiable at x, and the converse assertion is not true: the function $f(x_1,x_2) = |x_1x_2|^{\frac{2}{3}}$ is differentiable at the point $(0,0)$,but $\overline{D}_{[1]}f(0,0) = \overline{D}_{[2]}f(0,0) = +\infty$.Thus the condition of differentiability at the fixed point is weaker then the condition of the existence of a strong gradient in the same point. Note that the same conclusion remains true even while comparison on the sets of positive measure, namely, in [3] it is proved

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that there exists a continuous function such that the set of all points at which f is differentiable but does not have a strong gradient is of full measure.

Result. By virtue of the well-known theorem of Lebesgue for every $f \in L(0,1)$ its indefinite integral F_f, at almost every point x, is differentiable and $F'_f(x) = f(x)$.

The following assertion is a multidimensional analogue of Lebesgue theorem.

Theorem. *For every $n \geq 2$ and $f \in L(0,1)^n$ the indefinite integral of f, at almost every point x, is differentiable, moreover, has a strong gradient and*

$$D_{[i]}F_f(x) = \int\limits_{Q_i(x)} f_{x_i,i}(t)dt \quad \text{for every} \quad i \in \overline{1,n}.$$

This assertion in two-dimensional case was proved in [1].

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