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SOME NEW RESULTS ON THE CONTINUITY AND DIFFERENTIABILITY OF FUNCTIONS OF SEVERAL REAL VARIABLES

Abstract. For the functions of several real variables we establish: the necessary and sufficient conditions for continuity, the necessary and sufficient conditions for differentiability, the differentiability almost everywhere of an indefinite double integral and of absolutely continuous functions of two variables. The notion of an Lebesgue's intense point of summable functions of two variables is introduced. It is stated that almost every point is an Lebesgue's intense point at which an indefinite double integral and absolutely continuous functions of two variables have a finite strong gradient. The notions of the continuity and the limit in the wide are introduced and their connection with the continuity and the existence of the limit is established.

რეზიუმე. მრავალი ნამდვილი ცვლადის ფუნქციებისთვის დადგენილია: უწყვეტობის აუცილებელი და საკმარისი პირობები, დიფერენცირებადობის აუცილებელი და საკმარისი პირობები, განუსაზღვრელი ორმაგი ინტეგრალის და ორი ცვლადის აბსოლუტურად უწყვეტი ფუნქციის თითქმის ყველგან დიფერენცირებადობა. შემოღებულია ლებეგის ინტენსური წერტილის ცნება ორი ცვლადის ჯამებადი ფუნქციისთვის და დადგენილია: თითქმის ყველა წერტილი ლებეგის ინტენსური წერტილია, ლებეგის ინტენსურ წერტილებში სასრული ძლიერი გრადიენტი აქვთ განუსაზღვრელ ორმაგ ინტეგრალს და ორი ცვლადის აბსოლუტურად უწყვეტ ფუნქციას. შემოღებულია ფართო აზრით უწყვეტობის და ზღვრის ცნებანი და დადგენილია მათი კავშირი უწყვეტობასთან და ზღვრის არსებობასთან.

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შინაარსი

Preface

By the end of the XIX-th century we have already been familiar with the examples of functions of two variables which are discontinuous at the given point and continuous at the same point with respect to each of independent variables.

By the same time it was shown that for a total differential of a function of two variables to exist at the point x^0 it is not necessarily that its partial derivatives are continuous at that point, but their finiteness at the same point is insufficient for the function itself to be continuous.

It is significant that each of the above-mentioned "pathological" properties of these functions can be observed only at separate points.

In the middle of the XX-th century G. P. Tolstov has proved that impressive functions of two variables do exist and the sets of "pathological" points for them are very massives.

These facts gave serious impact on the investigation of functions of several variables for their continuity, as well as for the existence of a total differential.

Thus two basic problems on the necessary and sufficient conditions for functions of several variables have been outlines: for the continuity of a function on the one hand, and for the existence of a total differential on the other hand. Resolution of these problems will evidently allow one to solve some other problems which form the third group of problems. This group of problems involve the introduced here notions of the continuity in the wide and the limit in the wide which are tightly connected with the continuity and the existence of a limit.

The same group involves the proof that an indefinite double integral and an absolutely continuous function of two variables possess a total differential. These facts are realized at almost all points, that is at the points, which are called by the author as Lebesgue's intense points.

In the present monograph we investigate the above-mentioned problems and those which are tightly connected with them.

The monograph consists of four chapters.

Each chapter is supplied with an abstract and introduction and divided into sections and subsections. Theorems, propositions, formulas, etc. in subsections have individual numbering. References inside subsections are one-digital, and multi-digital outside subsections.

CHAPTER I

Separately Partial Continuity from Different Points of View and Continuity

The goal of the present chapter is to resolve the problem on the continuity at a point of a function of several variables.

For the function of several variables there arises the question whether there is some notion or property at the point x^0 with respect to a separate independent variable such that the presence of that property at the point x^0 for the function with respect to each of independent variables is the necessary and sufficient condition for the function to be continuous at that point?

Introduction

1. The notion continuity of a function of one variable extends automatically to the functions of several variables in two variants: the continuity (which is sometimes called joint continuity), and continuity with respect to every independent variable for all the remaining fixed independent variables.

Evidently, the continuity implies the continuity in each variable, and the inverse statement, as is known, is incorrect.

In greater detail, there exists a function of two variable which is discontinuous at the given point and continuous at that point with respect to every variable. Moreover, there exists a function of two variables which is discontinuous at a separate point and continuous along (with respect to) every straight line passing through that point. Further, there exists a function of two variables which is discontinuous at the given point, and continuous along every analytic curve passing through that point ([14]).

It is also known that the continuity follows from the continuity along all singly differentiable curves which pass through the given point, and the continuity not follows from the continuity along all twice differentiable curves ([20]).

It should be noted that these functions have neither the property to be continuous, nor even the limit at a “pathological” point.

That “pathology” for such function can be observed only at a single point.

There naturally arises the question as to what extent may be wide the set of such “pathological” points?

G. P. Tolstov answered this question and formulated his result in the form of the following

Theorem A ([27], 432–433). *There exists the function of two variables which is discontinuous at almost every point of the unit square and at every point of that square continuous with respect to every variable**.

Note that this function cannot be discontinuous everywhere because it belongs, by the Lebesgue theorem, to the first Baire class, and hence, it has everywhere, by the Baire theorem, a dense set of points of continuity.

Detailed exposition of those and analogous facts can be found in Z. Piotrowski's review paper ([19]).

2. The present chapter is organized as follows: solution in two variants of the above–formulated problem are given in Sections 2 and 3. Two new notions: (a) separately strong partial continuity and (b) separately angular partial continuity are introduced herein.

It is shown that each of those notions is equivalent to the notion of continuity.

Here we present in short the content of the remaining sections.

§ 4. Connection between the continuity and the existence of a finite limit for the function of one variable becomes exhausted in that the continuity implies the existence of a finite limit and not vice versa.

For functions of several variables it became possible to establish the necessary and sufficient conditions for the continuity, one of the conditions is the existence of a finite limit.

§ 5. The functions of two variables take a special place in the mathematical analysis even because they are tightly connected with analytic functions of a complex variable. In this section, for the functions of two variables we formulate the results which have already been established for functions of n variables.

§ 6. A total increment is, in most cases, connected with the continuity of a corresponding function. An increment of another type can be obtained by composing successively strong partial increments with respect to every independent variable. The obtained in such a way expression is called by the author an increment in the wide. Using this notion, we introduce the notion of the continuity in the wide. The sufficient condition for the continuity in the wide is established. In particular, the continuity implies the continuity in the wide and not vice versa.

For functions of two variables it is stated that the continuity with respect to every variable together with the continuity in the wide is equivalent to the continuity.

§ 7. In the analysis the notion of the limit precedes the notion of the continuity. Herein, for the continuity in the wide we have found the notion

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*This Tolstov's function does not possess almost everywhere even the property of continuity in the wide (see Remark 6.6.1 below).

of the limit in the wide. It is proved that the existence of a finite limit implies the existence of an equal limit in the wide and not vice versa.

For the functions of two variables it is proved that if there exist equal finite separated partial limits and if a finite limit in the wide is equal to them, then the limit is also equal to them.

§ 8. For the continuity of functions of two variables the sufficient conditions is given consistings in the continuity with respect to one of variables, uniformly with respect to the other variable and partial continuity with respect to the same other variable.

§ 9. The necessary and sufficient conditions for the continuity of functions of two variables allow us to introduce the notions of unilateral limit and unilateral continuity of functions of two variables. The obtained results maintain, in principal, all interconnections between the notions and the existence of a limit or continuity for functions of one variable.

§ 1. Preliminaries

1.1. Basic Notions

1. For the point $x = (x_1, \dots, x_n)$ from the n-dimensional real Euclidean space $\mathbb{R}^n$ by $\|x\|$ we denote any of the three equivalent norms

$$\|x\|_1 = \max_{1 \le i \le n} |x_i|, \tag{1}$$

$$\|x\|_2 = \sum_{i=1}^{n} |x_i|, \tag{2}$$

$$\|x\|_3 = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}, \tag{3}$$

which are connected by the well-known relations

$$\|x\|_1 \le \|x\|_2 \le n\|x\|_1, \tag{4}$$

$$\|x\|_1 \le \|x\|_3 \le \sqrt{n}\|x\|_1, \tag{5}$$

$$\|x\|_2 \le n\|x\|_3. \tag{6}$$

By $U(x^0, \delta)$, $\delta > 0$ we denote a δ-neighborhood of the point $x^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$, i.e., $U(x^0, \delta) = \{x \in \mathbb{R}^n : \|x - x^0\| < \delta\}$. $U^0(x^0, \delta)$ denotes a punctured δ-neighborhood of the point x^0 (without center x^0), i.e., $U^0(x^0, \delta) = U(x^0, \delta) \setminus \{x^0\} = \{x \in \mathbb{R}^n : 0 < \|x - x^0\| < \delta\}$.

The symbols $U(x^0)$ and $U^0(x^0)$ denote, in general, a neighborhood and punctured neighborhood of the point x^0.

In what follows, it will be, unless otherwise stated, assumed that values of functions are real and finite.

Let the function $u = \varphi(x)$, $x = (x_1, \dots, x_n)$, be defined in $U^0(x^0)$. A finite number A is said to be a limit at the point x^0 of the function $\varphi(x)$, symbolically

$$\lim_{x\to x^0}\varphi(x)=A \ \text{ or } \ \lim_{\substack{x_1\to x_1^0\\ \cdots\cdots\\ x_n\to x_n^0}}\varphi(x_1,\dots,x_n)=A, \tag{7}$$

if for every number $\varepsilon>0$ there exists the number $\delta=\delta(x^0,\varepsilon,\varphi)>0$ such that for all points $x\in U^0(x^0,\delta)$ the inequality

$$\big|\varphi(x)-A\big|<\varepsilon, \quad x\in U^0(x^0,\delta) \tag{8}$$

holds.

For the given in $U(x^0)$ function $f(x)$ a total increment, or briefly, an increment at the point x^0 is called the difference

$$\Delta_{x^0}f(x)=f(x)-f(x^0), \quad x\in U(x^0). \tag{9}$$

The function $f(x)$ is called continuous at the point x^0 if the value $f(x^0)$ is finite and

$$\lim_{x\to x^0}\Delta_{x^0}f(x)=0. \tag{10}$$

This equality means that for every $\varepsilon>0$ there exists a number $\delta=\delta(x^0,\varepsilon,f)>0$ such that

$$\big|f(x)-f(x^0)\big|<\varepsilon, \quad x\in U(x^0,\delta). \tag{11}$$

In this case the point x^0 is called the point of continuity of the function $f(x)$ and according to (7) we write

$$\lim_{x\to x^0}f(x)=f(x^0). \tag{12}$$

2. Let the function $\psi(x)$ be defined on the set $E\subset\mathbb{R}^n$ and let the set e be the subset of E, $e\subset E$. The cases are available when the function $\psi(x)$ on the subset e or, what comes to the same thing, along the subset e has better property than on E. Therefore it is advisable to consider the function $\psi|e$ defined only on e which is called a restriction of the function ψ on e. Hence $(\psi|e)(x)=\psi(x)$ for all $x\in e$. In that case they say that the function ψ is an extension of the function $\psi|e$ from e to E.

1.2. General Theorem on the Continuity and Existence of a Finite Limit

1. We have the following

Theorem 1.2.1. *The function $f(x)$ is continuous at the point x^0 or has at x^0 a finite limit A, if and only if there is a neighborhood $U(x^0,\delta)$ representable by a union of a finite number of sets $\mathcal{M}_k$ for which the restriction $f|\mathcal{M}_k$ along $\mathcal{M}_k$ is continuous at x^0, or has the limit A, $k=1,\dots,p$, $p=p(x^0,f)<+\infty$.*

Proof. For the continuity or for the existence of a finite limit A the number p is equal to 1. Conversely, let the restrictions $f|\mathcal{M}_k$ along $\mathcal{M}_k$ be continuous at the point x^0, $k = 1, \dots, p$. Therefore for every $\varepsilon > 0$ there exists $\delta_k > 0$ such that

$$\big|(f|\mathcal{M}_k)(x) - f(x^0)\big| < \varepsilon \quad \text{for} \quad x \in U(x^0, \delta_k) \cap \mathcal{M}_k.$$

Obviously, the number $\delta = \min\limits_{1 \le k \le p} \delta_k$ is positive (due to the finiteness of the number p), and hence

$$\big|(f|\mathcal{M}_k)(x) - f(x^0)\big| < \varepsilon$$

for

$$x \in U(x^0, \delta) \bigcap \Big(\bigcup_{k=1}^{p} \mathcal{M}_k\Big) = U(x^0, \delta) \bigcap \big(U(x^0, \delta)\big) = U(x^0, \delta).$$

Thus $|f(x) - f(x^0)| < \varepsilon$ for $x \in U(x^0, \delta)$, and the continuity of the function $f(x)$ at the point x^0 is obvious.

The case for the finite limit A is considered analogously. □

2. In the sequel we shall need the following easily verifiable

Proposition 1.2.1. *Let the function of n variables $f(x_1, \dots, x_n)$ be defined in the neighborhood $U(x^0)$ of the point $x^0 = (x_1, \dots, x_n^0) \in \mathbb{R}^n$. We take any finite numbers $x_{n+1}^0, \dots, x_m^0$ and consider the point $\overline{x}^{\,0} = (x_1^0, \dots, x_n^0, x_{n+1}^0, \dots, x_m^0)$. In the neighborhood $U(\overline{x}^{\,0})$ of the point $\overline{x}^{\,0} \in \mathbb{R}^{n+m}$ we define the function F by the equality*

$$F(x_1, \dots, x_n, x_{n+1}, \dots, x_m) = f(x_1, \dots, x_n).$$

Then:

1) *the continuity of the function $f(x_1, \dots, x_n)$ at the point x^0 implies the same for the function $F(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ at the point $\overline{x}^{\,0}$;*

2) *if $f(x_1, \dots, x_n)$ has a finite limit A at the point x^0, then A is the limit at the point $\overline{x}^{\,0}$ for the function $F(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ as well.*

1.3. Separated Partial Limits and the Notion of Separately Partial Continuity

Along with the points $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $x^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$ we introduce the following notation ([5]):

$$x(x_i^0) = \big(x_1, \dots, x_{i-1}, x_i^0, x_{i+1}, \dots, x_n\big), \tag{1}$$

$$x^0(x_j) = \big(x_1^0, \dots, x_{i-1}^0, x_j, x_{i+1}^0, \dots, x_n^0\big), \tag{2}$$

which will allow us to write long expressions in a short form.

Let the function $f(x)$ be defined in the neighborhood $U(x^0)$ of the point x^0. The function of one x_i variable $f(x^0(x_i))$ is called the i-th partial function at the point x^0 of the function $f(x)$, or the i-th coordinate function of $f(x)$ at x^0, which sometimes is denoted symbolically as ${}^i f(x_i)$. Consequently, we have the function of one x_i variable

$${}^i f(x_i) = f(x^0(x_i)). \tag{3}$$

1. The limit

$$\lim_{x_i\to x_i^0} {}^i f(x_i) = \lim_{x_i\to x_i^0} f(x^0(x_i)), \tag{4}$$

either finite or infinite of fixed sign, is called the i-th partial limit, or the limit with respect to x_i variable at x^0 for the function $f(x)$.

The relations

$${}^i f(x_i^0) = f(x^0) = f\big(x^0(x_i^0)\big) \tag{5}$$

are obvious.

Definition 1.3.1 ([6]). We say that the function $f(x_1,\dots,x_n)$ possesses separated partial limits (perhaps not equal) at the point x^0 if limit (4) exists for all $i=1,\dots,n$.

The absence of the limit at the point x^0 for the function $f(x)$ is obvious if limit (4) does not exist at least for one value i. But the existence of all separated partial limits (4) and even their equality does not imply the existence of limit for the function $f(x)$ at the point x^0 (see, for e.g., the function $\psi(x_1,x_2)$ from 7.3).

2. Besides the notion of the continuity at the point x^0 which is sometimes called joint continuity at that point, there exists the notion of a separately partial continuity at the given point.

If the partial function ${}^i f(x_i) = f(x^0(x_i))$ is continuous at the point x_i^0, then the function $f(x)$ is called partial continuous at the point x^0 with respect to the variable x_i.

In this case the value $f(x^0)$ is finite and according to (5) we have

$$\lim_{x_i\to x_i^0} {}^i f(x_i) = f(x^0) = \lim_{x_i\to x_i^0} f\big(x^0(x_i)\big). \tag{6}$$

Equality (6) means that for every $\varepsilon>0$ there exists the number $\delta_i = \delta_i(x_i^0,\varepsilon,f)>0$ such that the inequality

$$\big|{}^i f(x_i) - f(x^0)\big| < \varepsilon \tag{7}$$

is fulfilled for all $x_i \in (x_i^0-\delta_i,\, x_i^0+\delta_i)$.

If we apply the notion of a partial increment at the point x^0 for the function $f(x)$ with respect to the variable x_k,

$$\Delta_{x_k^0} f(x) = f\big(x^0(x_k)\big) - f(x^0), \tag{8}$$

which, due to the function ${}^k f(x_k)$, takes the form

$$\Delta_{x_k^0} f(x) = {}^k f(x_k) - f(x^0), \tag{9}$$

then we can write (6) and (7) respectively as

$$\lim_{x_i\to x_i^0} \Delta_{x_i^0} f(x) = 0 \tag{10}$$

and

$$\big|\Delta_{x_i^0} f(x)\big| < \varepsilon, \quad x_i \in \big(x_i^0-\delta_i,\, x_i^0+\delta_i\big). \tag{11}$$

If the function $f(x)$ at the point x^0 is partial continuous with respect to every variable, i.e., if equality (6) is fulfilled for all $i = 1, \dots, n$, then the function $f(x)$ is called separately partial continuous at the point x^0.

Obviously, the separately partial continuity of the function $\psi(x)$ at the point x^0 implies the existence of finite separated partial limits of $\psi(x)$ at x^0, equal to $\psi(x^0)$.

It is easily seen that every continuous at x^0 function is separately partial continuous at the point x^0.

Indeed, the continuity of the function $f(x_1, \dots, x_n)$ at the point $x^0 = (x_1^0, \dots, x_n^0)$ is equivalent to equality 1.1.(12). Of all natural numbers $1, \dots, n$ we take any i and suppose in equality 1.1.(12) that x_j is equal to x_j^0 for all $j \neq i$. As a result we get (6). Hence $f(x)$ with respect to x_i is continuous at x^0. Thus we have established the desired result due to the arbitrariness of i.

The converse statement is invalid: the separately partial continuity does not imply the continuity. In detail this problem has been considered by Z. Piotrowski in [19]. Among such facts Tolstov's Theorem A in Introduction is of special attention [27].

§ 2. The Continuity is Equivalent to a Separately Strong Partial Continuity

For the function of several variables we introduce the notion of the continuity with respect to individual independent variable whose fulfilment with respect to all independent variables ensures continuity, and vice versa.

To this end, for the function $f(x)$, $x = (x_1, \dots, x_n)$ defined in the neighborhood $U(x^0)$ of the point $x^0 = (x_1^0, \dots, x_n^0)$ we introduce some definitions using the symbols of 1.3(1) and 1.3(2).

2.1. Separately Strong Partial Continuity

Definition 2.1.1 ([6]). The difference

$$\Delta_{[x_k^0]} f(x) = f(x) - f\big(x(x_k^0)\big) \tag{1}$$

is called a strong partial increment at the point x^0 of the function $f(x)$ with respect to the variable x_k.

Definition 2.1.2 ([4], [6]). The function $f(x)$ is called a strongly partial continuous with respect to the variable x_k at the point x^0, if the equality

$$\lim_{x \to x^0} \Delta_{[x_k^0]} f(x) = 0 \tag{2}$$

is fulfilled.

Under equality (2) we mean that for every $\varepsilon > 0$ there exists the number $\delta_k = \delta_k(x^0, \varepsilon, f) > 0$ such that

$$\big|\Delta_{[x_k^0]} f(x)\big| < \varepsilon, \quad x \in U(x^0, \delta_k). \tag{3}$$

Proposition 2.1.1. *The strong partial continuity with respect to the variable x_k at the point x^0 of the function $f(x)$ implies the partial continuity of $f(x)$ at x^0 with respect to the same variable x_k, and not vice versa.*

Proof. The fulfilment of equality (2) implies that the function $\lambda(x_1,\dots,x_n)= \Delta_{[x_k^0]} f(x_1,\dots,x_n)$ of variables $x_1,\dots,x_n$, defined in $U(x^0)$ has zero limit at the point $x^0=(x_1^0,\dots,x_n^0)$. In particular, the function $\lambda(x_1,\dots,x_n)$ will have zero limit at x^0 along the Ox_k axis, i.e., for partial values $x_j=x_j^0$, $j\neq k$. Hence $f(x^0(x_k))-f(x^0)\to 0$ as $x_k\to x_k^0$. Thus we have obtained equality 1.3.(10) for $i=k$.

The impossibility of the converse statement is illustrated by an example of the function

$$\varphi(x_1,x_2)=\begin{cases}0 & \text{for } x_1\cdot x_2=0\\ 1 & \text{for } x_1\cdot x_2\neq 0\end{cases}. \tag{4}$$

Obviously, this function is separately partial continuous at the origin $O=(0,0)$.

It can be easily verified that the function $\varphi(x_1,x_2)$ does not satisfy condition (2) at the point O for $k=1,2$. This follows from the equality

$$\varphi(x_1,x_2)-\varphi(0,x_2)=\begin{cases}1 & \text{for } x_2\neq 0\\ 0 & \text{for } x_2=0\end{cases}, \tag{5}$$

$$\varphi(x_1,x_2)-\varphi(x_1,0)=\begin{cases}1 & \text{for } x_1\neq 0\\ 0 & \text{for } x_1=0\end{cases}. \tag{6}$$

Thus the function $\varphi(x_1,x_2)$ is separately partial continuous at the point O and does not possess the property of the strong partial continuity with respect to each of variables at O. □

Regarding Definition 2.1.2 we make the following remarks.

Remark 2.1.1. Every function of one variable $a(x_1)$ can be considered as function of variables $x_1,\dots,x_n$, assuming $A(x_1,\dots,x_n)=a(x_1)$, $n>1$. Therefore the strong partial continuity at the point $(x_1^0,\dots,x_n^0)$ of the function $A(x_1,\dots,x_n)$ with respect to the variable x_1 means the continuity of the function $a(x_1)$ at x_1^0.

Thus the notion of the strong partial continuity with respect to individual variable is a convenient generalization of the notion of the continuity of functions of one variable.

Remark 2.1.2. The fulfilment of equality (2) implies the same equality for all functions of type $f+\omega$, where ω is arbitrary finite in $U(x^0)$ function, independent from the variable x_k.

Definition 2.1.3 ([4], [6]). The function $f(x)$ is separately strong partial continuous at the point x^0, if $f(x)$ with respect to every variable is strongly partial continuous at x^0, i.e., equality (2) is fulfilled for all $k = 1, \dots, n$.

Remark 2.1.3. Equality (2) for partial values $x_j = x_j^0$ for $j \neq k$ implies the equality

$$\lim_{x_k \to x_k^0} \big[f(x^0(x_k)) - f(x^0)\big] = 0, \quad k = 1, \dots, n, \tag{7}$$

which shows that the function $f(x)$ possessing the finite limit A at the point x^0 will satisfy equality (7) only in the case $A = f(x^0)$.

2.2. The First Basic Theorem on the Continuity

Theorem 2.2.1 ([4]). *For the continuity of the function $f(x)$ at the point x^0, it is necessary and sufficient that it possess separately strong partial continuity at x^0.*

Proof. To establish the necessity of the condition of our theorem, we take arbitrary number k from the numbers $1, \dots, n$ and write following obvious equality

$$f(x) - f\big(x(x_k^0)\big) = \big[f(x) - f(x^0)\big] + \big[f(x^0) - f(x(x_k^0))\big]. \tag{1}$$

Since the function $f(x)$ is continuous at the point x^0, the expressions in square brackets are arbitrarily small when the point x is close enough to the point x^0. This is equivalent to equality 2.1.(2) for natural number k, which is taken arbitrarily. Hence the function $f(x)$ is separately strong partial continuous at the point x^0.

To establish the sufficiency of the condition of our theorem, we start with the fact that the function $f(x)$ with respect to the variable x_1 is strongly partial continuous at the point x^0. Thus equality 2.1.(2) holds for $k = 1$, or what is the same thing, we have the equality

$$\lim_{x \to x^0} \big[f(x) - f(x_1^0, x_2, \dots, x_n)\big] = 0. \tag{2_1}$$

Similarly, the strong partial continuity of the function $f(x)$ at the point x^0 with respect to the variable x_2 is equivalent to the equality

$$\lim_{x \to x^0} \big[f(x) - f(x_1, x_2^0, x_3, \dots, x_n)\big] = 0,$$

which in a particular case $x_1 = x_1^0$ takes the form

$$\lim_{x \to x^0} \big[f(x_1^0, x_2, \dots, x_n) - f(x_1^0, x_2^0, x_3, \dots, x_n)\big] = 0. \tag{2_2}$$

This process, according to strong partial continuity of the function $f(x)$ with respect to the variable x_n at the point x^0, ends with the equality

$$\lim_{x \to x^0} \big[f(x_1, \dots, x_{n-1}, x_n) - f(x_1, \dots, x_{n-1}, x_n^0)\big] = 0,$$

which for partial values $x_j = x_j^0$ for $j = 1, \dots, n-1$, takes the form

$$\lim_{x \to x^0} \left[f(x_1^0, \dots, x_{n-1}^0, x_n) - f(x^0) \right] = 0. \qquad (2)_n$$

From equalities $(2)_1$–$(2)_n$ we easily find that

$$\lim_{x \to x^0} \left[f(x) - f(x^0) \right] = 0,$$

which by the finiteness of $f(x^0)$ is equivalent to the equality

$$\lim_{x \to x^0} f(x) = f(x^0). \qquad \square$$

Remark 2.2.1. The importance in Theorem 2.2.1 is that the equality 2.1.(2) must be fulfilled for all values $k = 1, \dots, n$. If equality 2.1.(2) is fulfilled only $n-1$ values, then the function may turn out to be discontinuous at the point x^0. This can be illustrated by an example of the following function:

$$\mu(x_1, x_2) = \begin{cases} 1 & \text{for} \quad x_2 \neq 0 \\ 0 & \text{for} \quad x_2 = 0 \end{cases}. \qquad (3)$$

Obviously, the function $\mu(x_1, x_2)$ is discontinuous at all points belonging to the Ox_1-axis.

Besides, the function $\mu(x_1, x_2)$ is strongly partial continuous with respect to the variable x_1 at every point $(x_1^0, 0)$.

Indeed, by equality 2.1.(2) we have

$$\Delta_{[x_1^0]} \mu(x_1, x_2) = \mu(x_1, x_2) - \mu(x_1^0, x_2). \qquad (4)$$

This difference is equal to $1 - 1 = 0$ for $x_2 \neq 0$ and to $0 - 0 = 0$ for $x_2 = 0$. Hence for arbitrary point (x_1, x_2) we have the equality

$$\Delta_{[x_1^0]} \mu(x_1, x_2) = 0. \qquad (5)$$

Consequently, the function $\mu(x_1, x_2)$ is strongly partial continuous at every point $(x_1^0, 0)$ with respect to the variable x_1.

The function $\mu(x_1, x_2)$ does not possess the property of the strong partial continuity with respect to the variable x_2 at the points $(x_1^0, 0)$. This follows from the fact that for all points (x_1, x_2) with $x_2 \neq 0$ the equality $\mu(x_1, x_2) - \mu(x_1, 0) = 1 - 0 = 1$ holds. Hence $\mu(x_1, x_2)$ does not satisfy equality 2.1.(2) at the points $(x_1^0, 0)$ for $k = 2$.

2.3. Statements Following From the First Basic Theorem

From identity 2.2.(1) we get the following

Theorem 2.3.1 ([4], [6]). *For the continuity of the function $f(x)$ at the point $x^0 = (x_1^0, \dots, x_n^0)$, the following two conditions are necessary and sufficient:*

1) *the function $f(x)$ with respect to some one variable, say to x_k, is strongly partial continuous at the point x^0;*

2) *the function* $f(x(x_k^0))$ *depending on the rest* $n-1$ *variables is continuous at the point* x^0.

Along with the symbols 1.3.(1) and 1.3.(2) we introduce one more symbol. If in $x=(x_1,\dots,x_n)$ we make change $x_k=x_k^0$ and $x_j=x_j^0$ for $j\neq k$, then the obtained point will be denoted by the symbol

$$x(x_k^0,x_j^0). \tag{1}$$

If now we apply Theorem 2.3.1 to the function $f(x(x_k^0))$ from the same theorem, we will obtain the following result.

Theorem 2.3.2 ([6]). *For the function* $f(x_1,\dots,x_n)$ *of* n *variables to be continuous at the point* $x^0=(x_1^0,\dots,x_n^0)$, *the following joint conditions are necessary and sufficient:*

(i_k) *the function* $f(x_1,\dots,x_n)$ *is strongly partial continuous at the point* x^0 *with respect to some variable* x_k;

(i_{kl}) *the function* $f(x(x_k^0))$ *of* $n-1$ *variables is strongly partial continuous at* x^0 *with respect to some variable* x_ℓ *with* $\ell\neq k$;

(i_{kls}) *the function* $f(x(x_k^0,x_\ell^0))$ *of* $n-2$ *variables is strongly partial continuous at the point* x^0 *with respect to some variable* x_s *for* $s\neq k$ *and* $s\neq\ell$.

The obtained in such away function of only one variable is continuous at the point x^0.

In the sequel, the use will be made of the following

Proposition 2.3.1 ([6]). *The strong partial continuity of the function* $f(x_1,\dots,x_n)$ *at the point* $x^0=(x_1^0,\dots,x_n^0)$ *with respect to some variable* x_ℓ *implies the strong partial continuity of the function* $f(x(x_k^0))$ *at the point* x^0 *with respect to the same variable* x_ℓ, *no matter whatever* $k\neq\ell$ *is.*

Proof. The existence of a finite limit for the function $\Phi(x)$ at the point x^0 along some set E implies existence of the same limit for $\Phi(x)$ at x^0 along every subset $M\subset E$ with the limiting point x^0. Therefore the strong partial continuity of the function $f(x)$ at the point x^0 with respect to the variable x_ℓ implies the same for the function $f(x(x_k^0))$ at the point x^0 with respect to x_ℓ with $\ell\neq k$.

In greater detail, in equality 2.1.(2) written for x_ℓ^0 we replace the point x by a partial point $x(x_k^0)$. As a result, we obtain the equality

$$\lim_{x\to x^0}\left[f(x(x_k^0))-f(x(x_k^0,x_\ell^0))\right]=0. \tag{2}$$

This means that the function $f(x(x_k^0))$ is strongly partial continuous at the point x^0 with respect to the variable x_ℓ. □

On the base of Proposition 2.3.1, from Theorem 2.3.2 we obtain the following sufficient conditions for the continuity.

Theorem 2.3.3 ([6]). *If the function $f(x_1,\dots,x_n)$ with respect to some one variable is partial continuous at the point $x^0=(x_1^0,\dots,x_n^0)$ and, besides, strongly partial continuous at x^0 with respect to every of the remaining $n-1$ variables, then the function $f(x_1,\dots,x_n)$ is continuous at the point x^0, and hence $f(x_1,\dots,x_n)$ is strongly partial continuous at x^0 with respect to the same variable, mentioned at the begining of the theorem* (*by Theorem* 2.2.1).

§ 3. The Continuity is Equivalent to a Separately Angular Partial Continuity

For the function of several variables we again introduce a new notion of continuity with respect to a individual independent variable, whose fulfilment with respect to all variables is equivalent to the continuity.

In the previous Section 2, the variable point $x=(x_1,\dots,x_n)$ tends to the fixed point $x^0=(x_1^0,\dots,x_n^0)$ without some condition on that tending. This fact was accompanied with the word “strongly”.

Now the tending will be considered under certain conditions. We begin our investigation with the introduction of notions which will be needed in the sequel.

3.1. Separately Angular Partial Continuity

Let a finite function $f(x)$ be defined in the neighborhood $U(x^0)$ of the point $x^0=(x_1^0,\dots,x_n^0)$, $x=(x_1,\dots,x_n)\in U(x^0)$.

Definition 3.1.1. The expression

$$\Delta^c_{\widehat{x}_k^0} f(x)=f(x)-f(x(x_k)) \ \text{ for } \ |x_j-x_j^0|\le c_j|x_k-x_k^0|,\ \ j\neq k \qquad (1)$$

depending on the variables $x_1,\dots,x_n$, is called an angular partial increment of the function $f(x)$ at the point x^0 with respect to the variable x_k, corresponding to the collection $c=(c_1,\dots,c_{k-1},c_{k+1},\dots,c_n)$ of positive constants.

Definition 3.1.2 ([2], [4]). The angular partial continuity of the function $f(x)$ at the point x^0 with respect to the variable x_k means the fulfilment of the equality

$$\lim_{x_k\to x_k^0}\Delta^c_{\widehat{x}_k^0} f(x)=0 \qquad (2)$$

for every collection $c=(c_1,\dots,c_{k-1},c_{k+1},\dots,c_n)$ of positive constants*.

Obviously, in this case we likewise have remarks analogous to Remark 2.1.1 and 2.1.2.

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*From equality (1) we can see that: (1) the angular partial continuity with respect to the given variable is defined by $n-1$ arbitrary positive constants; (2) from $x_k\to x_k^0$ it follows that $x_j\to x_j^0$ for all $j\neq k$.

It is also evident that the strong partial continuity of the function $f(x)$ at the point x^0 with respect to the variable x_k implies its angular partial continuity with respect to the same x_k at x^0.

Definition 3.1.3 ([2], [4]). The function $f(x)$ is separately angular partial continuous at the point x^0, if with respect to every variable the function $f(x)$ possesses the property of angular partial continuity at the point x^0, i.e., if for all $k = 1, \dots, n$ and for every collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants, equality (2) holds.

3.2. The Second Basic Theorem on the Continuity

Theorem 3.2.1 ([2], [4]). *For the continuity of the function $f(x)$ at the point x^0, the necessary and sufficient condition is its separately angular partial continuity at x^0.*

Proof. The necessity of the condition follows from the fact that the continuity of the function $f(x)$ at the point x^0 implies the strong partial continuity of the function $f(x)$ at x^0 with respect to every variable, by Theorem 2.2.1. In its turn, this implies the angular partial continuity of the function $f(x)$ at x^0 with respect to every variable. Hence we have separately angular partial continuity of the function $f(x)$ at the point x^0.

Let us now prove the sufficiency of the condition of the above theorem. Let the function $f(x)$ be separately angular partial continuous at the point x^0. Hence equality 3.1.(2) holds for every variable $x_k (k = 1, \dots, n)$ and for arbitrary collection of positive constants. In particular, equality 3.1.(2) will take place for every $x_k (k = 1, \dots, n)$ and for $c_j = 1$, $j \neq k$.

We represent the space $\mathbb{R}^n$ as a union of pyramids $P_1, \dots, P_n$ with common vertices at the point x^0. Every pyramid P_k is defined by a system of inequalities $|x_j - x_j^0| \leq |x_k - x_k^0|$ for all $j \neq k$. The pyramid here is understood as two-sheeted, i.e., extending infinitely to both sides from the vertex x^0.

To establish the continuity of the function $f(x)$ at the point x^0, it is necessary and sufficient to prove that, by virtue of Theorem 1.2.1, the equality

$$\lim_{\substack{x \to x^0 \\ x \in P_k}} f(x) = f(x^0) \tag{1}$$

is valid for every $k = 1, \dots, n$.

Without loss of generality, it will be sufficient to prove equality (1) for the case $k = 1$, so our subsequent reasoning is connected with that case. Thus it will be assumed that the point $x = (x_1, \dots, x_n)$ tends from the neighborhood $U(x^0)$ to the point $x^0 = (x_1^0, \dots, x_n^0)$ under the condition $x \in P_1$.

Since the function $f(x)$ possesses the property of angular partial continuity at the point x^0 with respect to the variable x_1, we have the equality

$$\lim_{\substack{x\to x^0\\ x\in P_1}} \big[f(x)-f\big(x(x_1^0)\big)\big]=0. \tag{$(2)_1$}$$

The point $x(x_1^0)=(x_1^0,x_2,\dots,x_n)$ belongs to some pyramid* from $P_2,\dots,P_n$. Suppose that the point $x(x_1^0)$ belongs to Δ_{i_1} with $i_1\neq 1$. Along with equality $(2)_1$ and the fact that the function $f(x)$ is angular partial continuous with respect to the variable x_{i_1} at the point x^0, we have the equality

$$\lim_{\substack{x\to x^0\\ x(x_1^0)\in P_{i_1}}} \big[f\big(x(x_1^0)\big)-f\big(x(x_1^0,x_{i_1}^0)\big)\big]=0, \tag{$(2)_{i_1}$}$$

where $x(x_1^0,x_{i_1}^0)$ denotes that point $(x_1^0,\dots)$ from P_{i_1}, whose i_1-th coordinate is $x_{i_1}^0$ (see notation 2.3.(1)).

Now the point $x(x_1^0,x_{i_1}^0)$ belongs to some P_{i_2} with $i_2\neq 1$ and $i_2\neq i_1$. Continuing this process, as a result we obtain a point whose all coordinates are fixed, except one.

This variable coordinate is $x_{i_{n-1}}$. Thus the point belongs to the pyramid $P_{i_{n-1}}$, and it can be written as $x^0(x_{i_{n-1}})$ (see notation 1.3.(2)). As a result, we have the equality

$$\lim_{\substack{x\to x^0\\ x^0(x_{i_{n-1}})\in P_{i_{n-1}}}} \big[f\big(x^0(x_{i_{n-1}}\big)-f(x^0)\big]=0. \tag{$(2)_{i_{n-1}}$}$$

Consequently, starting from the point $x=(x_1,\dots,x_n)\in P_1$ belonging to $U(x^0)$, we took by one point from every pyramid P_m and finally arrived at the point x^0, $m=2,\dots,n$. In such a way we have taken into account the behavior of the function $f(x)$ in $U(x^0)$ with respect to every independent variable.

Comparing equalities $(2)_1$, $(2)_{i_1}$–$(2)_{i_{n-1}}$, we obtain equality (1) for $k=1$. □

3.3. The Third Theorem on the Continuity

In proving the sufficiency of the condition of Theorem 3.2.1, we have revealed the fact that which was formulated in the form of the following

Theorem 3.3.1. *For the function $f(x)$, $x=(x_1,\dots,x_n)$ to be continuous at the point $x^0=(x_1^0,\dots,x_n^0)$, it is necessary and sufficient that*

$$\lim_{\substack{x_k\to x_k^0\\ |x_j-x_j^0|\le|x_k-x_k^0|\\ j\neq k}} \big[f(x)-f\big(x(x_k^0)\big)\big]=0 \tag{1}$$

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*In case $n=2$ we have the points $x=(x_1,x_2)$, $x^0=(x_1^0,x_2)$, and the point $x(x_1^0)=(x_1^0,x_2)$ belongs to the pyramid P_2, which in this case represents the angle $\{(x_1,x_2):|x_2-x_2^0|\ge|x_1-x_1^0|\}$. More precisely, the point $x(x_1^0)$ belongs to the straight line $x_1=x_1^0$. In case $n=3$, the point $x(x_1^0)=(x_1^0,x_2,x_3)$ belongs to P_2, or to P_3.

for every $k = 1, \dots, n$.

3.4. The Summarizing Theorem on the Continuity

On the base of Theorems 2.2.1, 3.2.1 and 3.3.1 we obtain the following summarizing theorem.

Theorem 3.4.1 ([4]). *Separately strong partial continuity of the function* $f(x)$, $x = (x_1, \dots, x_n)$, *at the point* $x^0 = (x_1^0, \dots, x_n^0)$ *and separately angular partial continuity of the same function* $f(x)$ *at the point* x^0 *are equivalent, and each of them is equivalent to the continuity of the function* $f(x)$ *at the point* x^0, *which in its turn is equivalent to the fulfilment of equality* 3.3.(1) *for every* $k = 1, \dots, n$.

Remark 3.4.1. The content of Sections 2 and 3 shows that to the notion of the continuity of the function of one variable there correspond separately strong partial continuity and separately angular partial continuity.

Remark 3.4.2. If the equality

$$\lim_{\substack{x_k \to x_k^0 \\ |x_j - x_j^0| = |x_k - x_k^0| \\ j \neq k}} \big[f(x) - f\big(x(x_k^0)\big)\big] = 0,$$

does not hold at least for one value k from $1, \dots, n$, then the function $f(x)$ is not continuous at the point x^0.

§ 4. The Nonexistence of the Limit and the Continuity

4.1. The Nonexistence of the Limit

Theorem 2.3.1 states, in particular, that the continuity of the function $f(x)$, $x = (x_1, \dots, x_n)$ at the point $x^0 = (x_1^0, \dots, x_n^0)$ follows from the following two facts:

1) the function $f(x)$ with respect to some one variable x_j is strongly partial continuous at the point x^0;

2) the function of the remaining $n-1$ variables $f(x(x_j^0))$ is continuous at the point x^0.

There naturally arises the question: Does the function $f(x)$, $x = (x_1, \dots, x_n)$, $n > 1$, possess the limit at the point $x^0 = (x_1^0, \dots, x_n^0)$, if condition 1) is fulfilled, and instead of condition 2) the condition that

2^1) the function $f(x(x_j^0))$ has the limit at the point x^0, is fulfilled?

The answer is negative. This can be illustrated by an example of the function $\mu(x_1, x_2)$ given by equality 2.2.(3), which is strong partial continuous with respect to the variable x_1 at all points $(x_1^0, 0)$.

Hence condition 1) is fulfilled for $j = 1$ at every point $(x_1^0, 0)$. Moreover, the function $\mu(x_1, x_2)$ has no limit at all points $(x_1^0, 0)$. In fact, the function $\mu(x_1, x_2)$ at the points $(x_1^0, 0)$ has numbers 0 and 1 as partial limits with respect to variables x_1 and x_2, respectively. Hence the function $\mu(x_1, x_2)$

has at all points $(x_1^0, 0)$ the finite separated partial limits, different from each other.

This implies that the function $\mu(x_1, x_2)$ has no limit at all points $(x_1^0, 0)$.

All the above-said can be summarized in the form of the following

Proposition 4.1.1 ([6]). *The strong partial continuity of the function $f(x_1, \dots, x_n)$, $n > 1$, with respect to some one variable x_j at the point $x^0 = (x_1^0, \dots, x_n^0)$ and the existence of the finite limit for the function $f(x(x_j^0))$ with respect to the remaining $n-1$ variables collectively at the point x^0, do not guarantee the existence of the limit for the function $f(x_1, \dots, x_n)$ at the point x^0.*

Remark 4.1.1. In connection with Proposition 4.1.1, it should be noted that the necessary and sufficient conditions for the existence of a finite limit will be given in subsection 7.3.

4.2. The Continuity Under the Finite Limit

There arises the question: if the finite limit does exist, what kind of additional conditions are necessary and sufficient for the function of several variables to be continuous?

Theorem 4.2.1 ([6]). *For the function $f(x)$, $x = (x_1, \dots, x_n)$ to be continuous at the point $x^0 = (x_1^0, \dots, x_n^0)$, the following two conditions are necessary and sufficient:*

1) *the function $f(x)$ at the point x^0 is partial continuous with respect to some one variable (see equality 1.3.(6));*

2) *the function $f(x)$ has finite limit at the point x^0.*

Corollary 4.2.1 ([6]). *For the continuity of the function $f(x)$ at the point x^0, the following two conditions are necessary and sufficient:*

1) *the function $f(x)$ at the point x^0 has either the property of a strong partial continuity, or that of an angular partial continuity with respect to some one variable;*

2) *the function $f(x)$ has a finite limit at the point x^0.*

§ 5. Results on the Continuity of Functions of Two Variables

It seems to us advisable to collect all the results on the continuity of functions of two variables which have been stated for functions of n variables. This is convenient owing to the fact that the two-dimensional case is more transparent from the geometrical viewpoint.

Thus let a finite function of two variables $\varphi(x_1, x_2)$ be defined in the neighborhood $U(x^0)$ of the point $x^0 = (x_1^0, x_2^0)$.

5.1. Separately Strong Partial Continuity, and Continuity

The strong partial continuity of the function $\varphi(x_1, x_2)$ at the point $x^0 = (x_1^0, x_2^0)$ with respect to the variable x_1 means the fulfilment of the equality

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} \big[\varphi(x_1, x_2) - \varphi(x_1^0, x_2)\big] = 0 \tag{1}$$

and the strong partial continuity of the same function with respect to the variable x_2 means the fulfilment of the equality

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} \big[\varphi(x_1, x_2) - \varphi(x_1, x_2^0)\big] = 0. \tag{2}$$

The function $\varphi(x_1, x_2)$ is separately strong partial continuous at the point x^0, if equalities (1) and (2) are fulfilled.

For that case, the first basic Theorem 3.2.1 on the continuity reads as follows:

Theorem 5.1.1 ([2]). *For the function $\varphi(x_1, x_2)$ to be continuous at the point x^0, i.e., for the equality*

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} \varphi(x_1, x_2) = \varphi(x_1^0, x_2^0) \tag{3}$$

to be fulfilled, it is necessary and sufficient that the function $\varphi(x_1, x_2)$ be separately strong partial continuous at the point x^0.

Theorems 2.3.1 and 2.3.2 are especially simple for the case $n = 2$, where they coincide and take the form of

Theorem 5.1.2 ([2], [4]). *For the function $\varphi(x_1, x_2)$ to be continuous at the point x^0, it is necessary and sufficient that $\varphi(x_1, x_2)$ be strongly partial continuous at x^0 with respect to one of the variables, and partial continuous at the point x^0 with respect to the other variable. That is, it is necessary and sufficient that either equality* (1) *and the equality*

$$\lim_{x_2\to x_2^0} \varphi(x_1^0, x_2) = \varphi(x_1^0, x_2^0) \tag{4}$$

or equality (2) *and the equality*

$$\lim_{x_1\to x_1^0} \varphi(x_1, x_2^0) = \varphi(x_1^0, x_2^0) \tag{5}$$

be satisfied pairwise.

This theorem results in the following

Corollary 5.1.1 ([4]). *Let the function $\varphi(x_1, x_2)$ be separately partial continuous at the point x^0, i.e., let equalities* (4) *and* (5) *be fulfilled. Then for the function $\varphi(x_1, x_2)$ to be continuous at the point x^0 it is necessary and sufficient that $\varphi(x_1, x_2)$ to have at the point x^0 the property of the strong partial continuity with respect to one of the variables.*

Corollary 5.1.1 and Theorem 5.1.1 lead to

Corollary 5.1.2 ([4]). *If the function $\varphi(x_1, x_2)$ at the point x^0 possesses the properties of strong partial continuity with respect to one of the variables and partial continuity with respect to to the other variable, then the function $\varphi(x_1, x_2)$ possesses the property of the strong partial continuity at the point x^0 with respect to that latter variable.*

For $n = 2$, Theorem 4.2.1 takes the form of

Theorem 5.1.3 ([6]). *For the continuity of the function $\varphi(x_1, x_2)$ at the point $x^0 = (x_1^0, x_2^0)$, the following two conditions are necessary and sufficient:*

1) *the function $\varphi(x_1, x_2)$ has finite limit at the point x^0;*
2) *either equality* (4), *or equality* (5) *is fulfilled.*

5.2. Separately Angular Partial Continuity, and Continuity

The angular partial continuity of the function $\varphi(x_1, x_2)$ at the point $x^0 = (x_1^0, x_2^0)$ with respect to the variable x_1 means that the equality*

$$\lim_{\substack{x_1 \to x_1^0 \\ |x_2 \to x_2^0| \le c_2 |x_1 - x_1^0|}} \big[\varphi(x_1, x_2) - \varphi(x_1^0, x_2)\big] = 0 \tag{1}$$

is fulfilled for every constant $c_2 > 0$.

The angular partial continuity of the function $\varphi(x_1, x_2)$ at the point x^0 with respect to the variable x_2 means that the equality

$$\lim_{\substack{x_2 \to x_2^0 \\ |x_1 - x_1^0| \le c_1 |x_2 - x_2^0|}} \big[\varphi(x_1, x_2) - \varphi(x_1, x_2^0)\big] = 0 \tag{2}$$

is fulfilled for every constant $c_1 > 0$.

The separately angular partial continuity of the function $\varphi(x_1, x_2)$ at the point x^0 means that the function $\varphi(x_1, x_2)$ has at the point x^0 the properties of angular partial continuity both with respect to x_1 and to x_2.

For $n = 2$, the second basic Theorem 3.2.1 on the continuity takes the form of

Theorem 5.2.1. ([2]). *For the function $\varphi(x_1, x_2)$ to be continuous at the point x^0, it is necessary and sufficient that this function be separately angular partial continuous at the point x^0.*

The following theorem is also valid.

Theorem 5.2.2 ([2], Corollary 1.1). *For the function $\varphi(x_1, x_2)$ to be continuous at the point x^0, it is necessary and sufficient that the following*

*The relation $x_2 \to x_2^0$ in equality (1) follows from the relations $x_1 \to x_1^0$ and $|x_2 - x_2^0| \le c_2 |x_1 - x_1^0|$. Similarly in (2).

two equalities

$$\lim_{\substack{x_1\to x_1^0\\ |x_2-x_2^0|\le c|x_1-x_1^0|}} \big[\varphi(x_1,x_2)-\varphi(x_1^0,x_2)\big]=0 \tag{3}$$

and

$$\lim_{\substack{x_2\to x_2^0\\ |x_2-x_2^0|\ge c|x_1-x_1^0|}} \big[\varphi(x_1,x_2)-\varphi(x_1,x_2^0)\big]=0 \tag{4}$$

be fulfilled at least for one constant $c>0$.

The third Theorem 3.3.1 on the continuity has the form of

Theorem 5.2.3. *For the continuity of the function* $\varphi(x_1,x_2)$ *at the point* x^0, *the following two equalities*

$$\lim_{\substack{x_1\to x_1^0\\ |x_2-x_2^0|\le|x_1-x_1^0|}} \big[\varphi(x_1,x_2)-\varphi(x_1^0,x_2)\big]=0 \tag{5}$$

and

$$\lim_{\substack{x_2\to x_2^0\\ |x_1-x_1^0|\le|x_2-x_2^0|}} \big[\varphi(x_1,x_2)-\varphi(x_1,x_2^0)\big]=0 \tag{6}$$

are necessary and sufficient.

Remark 5.2.1. The fulfilment of equality (3) separately for one of the constant $c>0$, in particular of equality (5), is not equivalent to angular partial continuity of the function $\varphi(x_1,x_2)$ at the point x^0 with respect to the variable x_1.

Similar situation takes place for the variable x_2.

Remark 5.2.2. The sufficient condition for the continuity of the function of two variables, when it is continuous with respect to each of variables, can be found Theorem 2.2.3 of Chapter III.

§ 6. Continuity in the Wide and Its Application

6.1. Increment in the Wide

Let the function $f(x)$, $x=(x_1,\dots,x_n)$ be defined in the neighborhood $U(x^0)$ of the point $x^0=(x_1^0,\dots,x_n^0)$.

Construction of the increment

$$\Delta_{x^0}f(x)=f(x)-f(x^0) \tag{1}$$

for the function $f(x)$ at the point x^0 (see equality 1.1.(9)) shows that the increments $x_j-x_j^0$, $j=1,\dots,n$ are attached simultaneously to all coordinates x_j^0 of the point x^0.

We shall now construct an expression of another structure, which will be called an increment in the wide sense.

It is clear that the expression, defined by equality 2.1.(1), depends on the variables $x_1, \dots, x_n$:

$$\Delta_{[x_k^0]} f(x) = f(x) - f\big(x(x_k^0)\big). \tag{2}$$

We introduce an auxiliary function

$$\lambda(x_1, \dots, x_n) = \Delta_{[x_k^0]} f(x) \tag{3}$$

and write equality (2) for the function $\lambda(x_1, \dots, x_n)$ at the point x^0 with respect to some variable x_j with $j \neq k$. As a result we obtain the new function

$$\mu(x_1, \dots, x_n) = \Delta_{[x_j^0]} \lambda(x), \tag{4}$$

which again depends on $x_1, \dots, x_n$ variables.

For the function $\mu(x_1, \dots, x_n)$ we likewise write equality (2) at the point x^0 for some variable x_ℓ, where $\ell \neq k$ and $\ell \neq j$. This procedure will be continued until all $x_1, \dots, x_n$ variables are exhausted. Final result is called an increment in the wide of the function $f(x)$ at the point x^0 and denoted by $\Delta^n_{[x^0]} f(x)$.

An important property of the above procedure is that the final result does not depend on the order of forming strong partial increments at the point x^0 by formula (2) for the required functions. Therefore the following definition is correct.

Definition 6.1.1 ([4], [7]). An increment in the wide at the point x^0 for the given in $U(x^0)$ finite function $f(x)$, $x = (x_1, \dots, x_n)$, is called the expression

$$\Delta^n_{[x^0]} f(x) = \big(\Delta_{[x_1^0]} \circ \Delta_{[x_2^0]} \circ \dots \circ \Delta_{[x_n^0]}\big)(f)(x), \tag{5}$$

where

$$\Delta_{[x_j^0]} F(x) = F(x) - F\big(x(x_j^0)\big). \tag{6}$$

Hence to get $\Delta^n_{[x^0]} f(x)$ we have to take in the equality (6) instead of F and j the f and $j = n$, $\Delta_{[x_n^0]} f$ and $j = n - 1$, and so on, and finally $\Delta_{[x_2^0]} \circ \dots \circ \Delta_{[x_n^0]} f$ and $j = 1$.

The case $n = 2$ we distinguish separately. If a finite function of two variables $\varphi(x_1, x_2)$ is defined in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$, then the strong partial increment at the point x^0 with respect to the variable x_1 for $\varphi(x_1, x_2)$ is

$$\Delta_{[x_1^0]} \varphi(x_1, x_2) = \varphi(x_1, x_2) - \varphi(x_1^0, x_2) \tag{7}$$

and the strong partial increment with respect to x_2 is equal to

$$\Delta_{[x_2^0]} \varphi(x_1, x_2) = \varphi(x_1, x_2) - \varphi(x_1, x_2^0). \tag{8}$$

Hence the increment in the wide at the point x^0 for $\varphi(x_1, x_2)$ is equal to

$$\Delta^2_{[x^0]} \varphi(x_1, x_2) = \varphi(x_1, x_2) - \varphi(x_1^0, x_2) - \varphi(x_1, x_2^0) + \varphi(x_1^0, x_2^0). \tag{9}$$

It should be noted that expression (9) takes place in the definition of the function of bounded variation (see, e.g., [11], § 254).

6.2. The Continuity in the Wide

Definition 6.2.1 ([4], [7]). The given in $U(x^0)$ finite function $f(x)$ is said to be continuous in the wide at the point x^0, if

$$\lim_{x\to x^0} \Delta^n_{[x^0]} f(x) = 0. \tag{1}$$

In what follows, it is expedient to write the continuity in the wide for the function of two variables, in the form of two equivalent equalities.

The function $\varphi(x_1, x_2)$, finite in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$ is said to be continuous in the wide at the point x^0, if

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} \big[\varphi(x_1,x_2) - \varphi(x_1^0,x_2) - \varphi(x_1,x_2^0) + \varphi(x_1^0,x_2^0)\big] = 0 \tag{2}$$

or what is the same thing, if

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}} \big[\varphi(x_1^0,x_2) + \varphi(x_1,x_2^0) - \varphi(x_1,x_2)\big] = \varphi(x_1^0,x_2^0). \tag{3}$$

6.3. The Increment in the Wide for the Sum

If in the neighborhood $U(x^0)$ of the point $x^0 = (x_1^0, \dots, x_n^0)$ there are finite functions $f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)$, then the equality

$$\Delta^n_{[x^0]} \sum_{j=1}^m f_j(x) = \sum_{j=1}^m \Delta^n_{[x^0]} f_j(x), \quad x = (x_1, \dots, x_n) \in U(x^0). \tag{1}$$

holds.

6.4. The Increment in the Wide for the Special Sum

Consider finite functions of special type:

$$\psi_k : \mathbb{R}^{n-1} \to \mathbb{R}, \quad k = 1, \dots, n, \tag{1}$$

each of which depends on $n-1$ variables. Consider the summary function

$$\begin{aligned}\psi(x) = \psi_1(x_2, x_3, \dots, x_n) + \psi_2(x_1, x_3, \dots, x_n) +\\ + \dots + \psi_n(x_1, x_2, \dots, x_{n-1})\end{aligned} \tag{2}$$

depending on $x = (x_1, \dots, x_n)$.

From equality 6.1.(2) it follows that

$$\Delta_{[x_k^0]} \psi_k(x) = 0, \quad k = 1, \dots, n. \tag{3}$$

Taking into account the fact that succession order in equality 6.1.(5) does not play an important role, from equalities (3) we obtain

$$\Delta^n_{[x^0]} \psi_k(x) = 0, \quad k = 1, \dots, n. \tag{4}$$

Now, equalities 6.3.(1) and (4) yield

$$\Delta^n_{[x^0]}\psi(x) = 0. \tag{5}$$

Consequently, every finite in $U(x^0)$ function $\psi(x)$ of type (2) is continuous in the wide at every point $x^0 = (x^0_1, \dots, x^0_n)$, even for discontinuous in the ordinary sense at x^0 functions $\psi_k(x)$, $k = 1, \dots, n$.

6.5. The Sufficient Condition for the Continuity in the Wide*

Theorem 6.5.1 ([4], [7]). *If the function $f(x)$ is strongly partial continuous with respect to some one variable at the point x^0, then $f(x)$ is continuous in the wide at the point x^0, and not vice versa.*

Proof. Suppose that the function $f(x)$ is strongly partial continuous at the point x^0 with respect to the variable x_j. Then the equality

$$\lim_{x\to x^0} \big[f(x) - f\big(x(x^0_j)\big)\big] = 0 \tag{1}$$

holds.

Since $\Delta^n_{[x^0]}f(x)$ does not depend on the order of making up strong partial increments, to construct the right-hand side of equality 6.1.(5) we shall start with that of $\Delta_{[x^0_j]}f(x)$. Our next step is to construct a strong partial increment for the function $\Delta_{[x^0_j]}f(x)$ at the point x^0 with respect to some variable x_ℓ with $\ell \neq j$. Having realized all this, we have

$$\begin{aligned}\Delta_{[x^0_\ell]}\big(\Delta_{[x^0_j]}f\big)(x) &= \Delta_{[x^0_j]}f(x) - \big(\Delta_{[x^0_j]}f(x)\big)_{x_\ell = x^0_\ell} = \\ &= \big[f(x) - f\big(x(x^0_j)\big)\big] - \big[f(x) - f\big(x(x^0_j)\big)\big]_{x_\ell = x^0_\ell} = \\ &= \big[f(x) - f\big(x(x^0_j)\big)\big] - \big[f\big(x(x^0_\ell)\big) - f\big(x(x^0_j, x^0_\ell)\big)\big].\end{aligned}$$

By equality (1), the both differences in the square brackets tend to zero as $x \to x^0$.

As a result of finite number of steps, we obtain equality 6.2.(1), i.e., the function $f(x)$ will turn to be continuous in the wide at the point x^0.

By means of formula 6.4.(5) we find that the converse statement is invalid. Indeed, for arbitrary finite functions $\alpha(x_1)$ and $\beta(x_2)$, not necessarily continuous, we have

$$\Delta^2_{[x^0]}\omega(x_1, x_2) = 0, \tag{2}$$

where $x^0 = (x^0_1, x^0_2)$ is arbitrary point from $\mathbb{R}^2$, and

$$\omega(x_1, x_2) = \alpha(x_1) + \beta(x_2). \tag{3}$$

Hence the function $\omega(x_1, x_2)$ is continuous in the wide at every point from $\mathbb{R}^2$. □

Theorems 2.2.1 and 6.5.1 lead to

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*Another sufficient condition for the continuity in the wide for functions of two variables will be given in Chapter III (see Proposition 2.2.1 of Chapter III).

Corollary 6.5.1 ([4], [7]). *If the function $f(x)$ is continuous at the point x^0, then it is continuous in the wide at the same point x^0. The converse statement is invalid.*

Remark 6.5.1. The function $\omega(x_1, x_2)$, defined by equality (3), is continuous at the point $x^0 = (x_1^0, x_2^0)$, iff $\alpha(x_1)$ is continuous at x_1^0, and $\beta(x_2)$ is continuous at x_2^0.

This follows from Theorem 5.1.1 and from the fact that

$$\omega(x_1, x_2) - \omega(x_1^0, x_2) = \alpha(x_1) - \alpha(x_1^0), \tag{4}$$

$$\omega(x_1, x_2) - \omega(x_1, x_2^0) = \beta(x_2) - \beta(x_2^0). \tag{5}$$

Remark 6.5.2. The function $\mu(x_1, x_2)$, defined by equality 2.2.(3), is continuous in the wide at every point from $\mathbb{R}^2$.

Indeed, the continuity in the wide of the function $\mu(x_1, x_2)$ at the point $(x_1^0, 0)$ follows from Theorem 6.5.1, with regard for the fact that the function $\mu(x_1, x_2)$ at these points is strongly partial continuous with respect to the variable x_1. The function $\mu(x_1, x_2)$ at the remaining points from $\mathbb{R}^2$ is continuous and therefore is continuous in the wide, by Corollary 6.5.1.

Remark 6.5.3. The statement of Theorem 6.5.1 can be easily realized for functions of two variables.

Indeed, if the function $\varphi(x_1, x_2)$ at the point $x^0 = (x_1^0, x_2^0)$ is strongly partial continuous with respect to the variable x_1, then we write the right-hand side of equality 6.1.(9) in the form

$$\big[\varphi(x_1, x_2) - \varphi(x_1^0, x_2)\big] - \big[\varphi(x_1, x_2^0) - \varphi(x_1^0, x_2^0)\big],$$

which tends to zero as $(x_1, x_2) \to (x_1^0, x_2^0)$.

However, if the function $\varphi(x_1, x_2)$ at the point x^0 is strongly partial continuous with respect to the variable x_2, then we write the right-hand side of the same equality in the form

$$\big[\varphi(x_1, x_2) - \varphi(x_1, x_2^0)\big] - \big[\varphi(x_1^0, x_2) - \varphi(x_1^0, x_2^0)\big],$$

which also tends to zero as $(x_1, x_2) \to x^0$.

6.6. The Continuity of the Function of Two Variables Under Its Continuity in the Wide

From subsection 1.3 it is well-known that separately partial continuity does not imply the continuity*.

Moreover, as we see, the property of continuity in the wide is far from that of the continuity.

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*Below will be given one sufficient condition for the continuity of the function, when this function is separately partial continuous (see Theorem 2.2.3 in Chapter III).

Let us now prove that these two properties simultaneously guarantee the continuity, and vice versa.

Just this is the answer to the question: what useful information does the notion of the continuity in the wide carry?

Theorem 6.6.1 ([7]). *The finite function $\varphi(x_1, x_2)$ defined in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$ is continuous at x^0, iff the function $\varphi(x_1, x_2)$ at the point x^0 is both separately partial continuous and continuous in the wide.*

Proof. If the function $\varphi(x_1, x_2)$ possesses both the above-mentioned properties at the point x^0, then its continuity at x^0 follows from the equality

$$\begin{aligned} &\varphi(x_1^0 + h, x_2^0 + k) - \varphi(x_1^0, x_2^0) = \\ &= \big[\varphi(x_1^0 + h, x_2^0 + k) - \varphi(x_1^0, x_2^0 + k) - \varphi(x_1^0 + h, x_2^0) + \varphi(x_1^0, x_2^0)\big] + \\ &\quad + \big[\varphi(x_1^0 + h, x_2^0) - \varphi(x_1^0, x_2^0)\big] + \big[\varphi(x_1^0, x_2^0 + k) - \varphi(x_1^0, x_2^0)\big]. \qquad (1) \end{aligned}$$

It is then obvious that the continuous at the point x^0 function $\varphi(x_1, x_2)$ possesses both properties, mentioned in Theorem 6.6.1. □

The following corollaries of that theorem are worth mentioning.

Corollary 6.6.1. *Let the function $\varphi(x_1, x_2)$ be separately partial continuous at the point $x^0 = (x_1^0, x_2^0)$. Then for the continuity of the function $\varphi(x_1, x_2)$ at the point x^0, it is necessary and sufficient that this function be continuous in the wide at x^0.*

Corollary 6.6.2. *Let the function $\varphi(x_1, x_2)$ be continuous in the wide at the point $x^0 = (x_1^0, x_2^0)$. Then for the continuity of the function $\varphi(x_1, x_2)$ at the point x^0, it is necessary and sufficient that this function be separately partial continuous at x^0.*

Remark 6.6.1. Discontinuous at the given point functions of two variables, mentioned in subsection 1.3 are not continuous in the wide at that point. G. Tolstov's functions from Theorem A fail to have the property of the continuity in the wide at the points of discontinuity, i.e., at almost all points (see Introduction in Chapter I).

§ 7. The Limit in the Wide and Its Application

7.1. The Notion of the Limit in the Wide

In the analysis, the notion of the continuity is introduced on the base of the notion of the limit. For the notion of the continuity in the wide we can refer preceding notion of the limit in the wide.

It is obvious that the increment in the wide defined by equality 6.1.(5) for $f(x)$ at the point x^0 involves the $f(x^0)$.

Now we replace the $f(x^0)$ in $\Delta^n_{[x^0]} f(x)$ by finite B and the obtained expression write symbolically as

$$\Delta^n_{[x^0]} f(x)\big|_{f(x^0)=B}\,. \tag{1}$$

Introduce the following

Definition 7.1.1 [([7]). The finite B is said to be the limit in the wide for the function $f(x)$ at the point x^0, if the equality

$$\lim_{x\to x_0}\Big(\Delta^n_{[x^0]} f(x)\big|_{f(x^0)=B}\Big)=0 \tag{2}$$

holds.

The following proposition is obvious.

Proposition 7.1.1. *The function $f(x)$ continuous in the wide at the point x^0 has the limit in the wide at x^0, which is equal to $f(x^0)$.*

Definition 6.2.1 can now be rephrases in the form of

Definition 7.1.2 ([7]). The function $f(x)$ is continuous in the wide at the point x^0, if the $f(x^0)$ is finite and $f(x^0)$ is the limit in the wide for the function $f(x)$ at the point x^0.

On the base of equality 6.1.(9), the finite B will be the limit in the wide for the function $\varphi(x_1,x_2)$ at the point $x^0=(x^0_1,x^0_2)$, if

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}\big[\varphi(x_1,x_2)-\varphi(x^0_1,x_2)-\varphi(x_1,x^0_2)+B\big]=0. \tag{3}$$

Generally number L, finite or infinite of fixed sign, will be the limit in the wide for the function $\varphi(x_1,x_2)$ at the point (x^0_1,x^0_2), if

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}\big[\varphi(x^0_1,x_2)+\varphi(x_1,x^0_2)-\varphi(x_1,x_2)\big]=L. \tag{4}$$

Proposition 7.1.2 ([7]). *If the function $f(x)$ has a finite limit in the wide at the point x^0, then this limit is unique.*

In order to show this, it is necessary write equality (2) for finite B and B_1 and then consider their difference.

7.2. The Existence of the Limit in the Wide

Theorem 7.2.1 ([7]). *If the function $f(x)$ has finite limit B at the point x^0, then B is likewise the limit in the wide for the function $f(x)$ at the point x^0.*

Proof. If the function $f(x)$ is continuous at the point x^0, then $f(x)$ is continuous in the wide at x^0, by Corollary 6.5.1. Therefore the finite $f(x^0)$ is the limit in the wide for $f(x)$ at x^0, by Definition 7.1.2.

Suppose now that the function $f(x)$, possessing at the point x^0 the finite limit B, is discontinuous at x^0.

If we introduce a new function $f^*(x) = f(x)$ for $x \neq x^0$ and $f^*(x^0) = B$, then $f^*(x)$ is continuous at x^0, and inequality $f^*(x) \neq f(x)$ is fulfilled only for $x = x^0$. Thus

$$\Delta^n_{[x^0]} f(x) - \Delta^n_{[x^0]} f^*(x) = f(x^0) - f^*(x^0) = f(x^0) - B,$$

whence

$$\Delta^n_{[x^0]} f(x)\Big|_{f(x^0)=B} - \Delta^n_{[x^0]} f^*(x) = B - B = 0. \tag{1}$$

But according to Corollary 7.5.1, we have

$$\lim_{x \to x^0} \Delta^n_{[x^0]} f^*(x) = 0. \tag{2}$$

Equality 7.1.(2) is obtained now from equalities (1) and (2). □

Proposition 7.2.1 ([7]). *The existence of the finite limit in the wide for the function $\mu(x_1, \ldots, x_n)$, $n > 1$, at the point x^0 does not imply the existence the limit for $\mu(x_1, \ldots, x_n)$ at x^0, neither finite, nor infinite.*

Proof. The function $\mu(x_1, x_2)$ given by equality 2.2.(3) is continuous in the wide at every point from $\mathbb{R}^2$, in particular at the points $(x_1^0, 0)$, by Remark 6.5.1. Therefore the function $\mu(x_1, x_2)$ at the points $(x_1^0, 0)$ has the limit in the wide equal to $\mu(x_1^0, 0) = 0$, by Proposition 7.1.1. On the other hand, the function $\mu(x_1, x_2)$ has no limit at the points $(x_1^0, 0)$, as is mentioned in subsection 4.1. □

7.3. The Necessary and Sufficient Conditions for the Existence a Finite Limit for Functions of Two Variables

As is known, the existence of the finite limit does not follows from existence of equal finites separated partial limits. This can be illustrated, for example, by means of the function $\psi(x_1, x_2) = \frac{x_1 \cdot x_2}{x_1^2 + x_2}$ for $(x_1, x_2) \neq (0, 0)$ and $\psi(0, 0) = 0$. This function is separately partial continuous at the origin $O = (0, 0)$. In particular, $\psi(x_1, x_2)$ has zero separated partial limits at the point O. The absence of the limit for $\psi(x_1, x_2)$ at O follows from the equality $\psi(r \cos\theta, r \sin\theta) = \frac{1}{2} \sin 2\theta$.

Besides, the existence of the finite limit does not follows from the existence of the finite limit in the wide (see Proposition 7.2.1).

Remarkable is that the both properties together implies the existence of the finite limit, and vice versa.

Thus we have shown what informational load carries the notion of the limit in the wide.

Theorem 7.3.1 ([7]). *The finite number A will be the limit for the function $\varphi(x_1, x_2)$ at the point $x^0 = (x_1^0, x_2^0)$, iff the function $\varphi(x_1, x_2)$ has simultaneosly at the point x^0 separated partial limits and the limit in the wide which are equal to A.*

Proof. If A is the limit at the point x^0 for the function $\varphi(x_1, x_2)$, then A for $\varphi(x_1, x_2)$ is both the limit in the wide (see Theorem 7.2.1) and, obviously, separated partial limits.

The converse follows from the equality, which will be obtained by substituting $\varphi(x_1^0, x_2^0)$ by A in equality 6.6.(1). □

Remark 7.3.1. The function $\psi(x_1, x_2)$ from subsection 7.3, devoid of the limit at the point $O = (0, 0)$, has no zero limit in the wide at the point O, by Theorem 7.3.1. Majority functions mentioned in subsection 1.3 of Piotrowski's work ([19]) possess analogous property.

§ 8. Partial Continuity with Respect to One of the Variables, Uniformly with Respect to the Other Variable

We have already got acquainted with the necessary and sufficient conditions for the continuity of functions of two variables (see § 5 and subsection 6.6). In this section we give somewhat different sufficient condition for the continuity of functions of two variables. The facts stated here will be applied to the questions of Chapter IV (see § 6, § 7 and § 9).

Let the finite function of two variables $\varphi(x, y)$ be defined on the rectangle $Q = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$, and let the point $(x_0, y_0) \in Q$.

Definition 8.1 ([2]). The function $\varphi(x, y)$ is called partial continuous with respect to x at the point x^0, uniformly with respect to y on the $[c_1, d_1]$, $c \leq c_1 < d_1 \leq d$, if the equality

$$\lim_{x \to x^0} \big[\varphi(x, y) - \varphi(x_0, y)\big] = 0 \tag{1}$$

takes place uniformly with respect to the variable $y \in [c_1, d_1]$.

Similarly, the function $\varphi(x, y)$ is partial continuous with respect to y at the point y_0, uniformly with respect to x on the $[a_1, b_1]$, $a \leq a_1 < b_1 \leq b$, if the equality

$$\lim_{y \to y^0} \big[\varphi(x, y) - \varphi(x, y_0)\big] = 0 \tag{2}$$

is fulfilled uniformly with respect to the variable $x \in [a_1, b_1]$.

Proposition 8.1 ([2]). *If the function $\varphi(x, y)$ is partial continuous with respect to x at the point x_0, uniformly with respect to y on the $[c_1, d_1]$, then $\varphi(x, y)$ is strongly partial continuous with respect to x at every point (x_0, y^*) with $c_1 < y^* < d_1$.*

Proof. For sufficiently small k, the point $y^* + k$ belongs to $[c_1, d_1]$, and by equality (1) we have

$$\lim_{\substack{x \to x^0 \\ k \to 0}} \big[\varphi(x, y^* + k) - \varphi(x_0, y^* + k)\big] = 0.$$

Thus equality 5.1.(1) is fulfilled. □

Proposition 8.2 ([2]). *If the function $\varphi(x,y)$ is partial continuous with respect to x at the point x^0, uniformly with respect to y on the $[c_1,d_1]$ and the function of one variable $\varphi(x_0,y)$ is continuous at some point y^* with $c_1 < y^* < d_1$, then the function $\varphi(x,y)$ is continuous at the point (x_0,y^*).*

Proof needs Proposition 8.1 and Theorem 5.1.2.

Remark 8.1. If the function $\varphi(x_0,y)$ is not continuous at the point y^*, then the conclusion of Proposition 8.2 is invalid.

Indeed, let the function $\alpha(x)$ be continuous at the point $x_0 \in [a,b]$, and let the finite function $\beta(y)$ be discontinuous at the point $y^* \in (c_1,d_1)$. Then the function $g(x,y) = \alpha(x) + \beta(y)$ is discontinuous at the point (x_0,y^*), although

$$g(x,y) - g(x_0,y) = \alpha(x) + \beta(y) - \alpha(x_0) - \beta(y) = \alpha(x) - \alpha(x_0) \to 0$$

uniformly with respect to y from the neighborhood of the point y^*, as $x \to x_0$.

Theorem 8.1 ([2]). *Let the function $\varphi(x,y)$ be continuous on the rectangle Q. Then $\varphi(x,y)$ is:*

1) *partial continuous with respect to x at every point $x_0 \in [a,b]$, uniformly with respect to y on the $[c,d]$;*

2) *partial continuous with respect to y at every point $y_0 \in [c,d]$, uniformly with respect to x on the $[a,b]$.*

Proof. Since the function $\varphi(x,y)$ is continuous on the bounded closed set Q, then $\varphi(x,y)$ is uniformly continuous on the Q, by the classical Cantor's theorem. Therefore for every $\varepsilon > 0$ there exists the number $\delta = \delta(\varepsilon,\varphi) > 0$ such that

$$\big|\varphi(x_1,y_1) - \varphi(x_2,y_2)\big| < \varepsilon \tag{3}$$

for $(x_1,y_1) \in Q$ and $(x_2,y_2) \in Q$ under $|x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta$.

We take arbitrary points $x_0 \in [a,b]$ and $y_0 \in [c,d]$, suppose that $x_2 = x_0$ in inequality (3) and replace x_1 by any point $x \in [a,b]$ with the property $|x - x_0| < \delta$. Moreover, we take arbitrary point $y \in [c,d]$ and suppose that in inequality (3) $y_1 = y = y_2$. So, inequality (3) takes the form

$$\big|\varphi(x,y) - \varphi(x_0,y)\big| < \varepsilon, \quad |x - x_0| < \delta, \quad c \le y \le d. \tag{4}$$

Hence we have established statement 1).

Analogously we obtain the inequality

$$\big|\varphi(x,y) - \varphi(x,y_0)\big| < \varepsilon, \quad a \le x \le b, \quad |y - y_0| < \delta. \tag{5}$$

Consequently, statement 2) holds. □

Corollary 8.1 ([2]). *If the function $\varphi(x,y)$ is partial continuous with respect to x at every point $x_0 \in [a,b]$, uniformly with respect to y on the $[c,d]$, and the function $\varphi(x_0,y)$ is continuous at every point $y_0 \in [c,d]$, then $\varphi(x,y)$ is uniformly continuous on the Q.*

Proof. In the above suppositions the function $\varphi(x,y)$ is continuous at every point $(x_0,y_0)\in Q$ and uniformly continuous on Q, by Cantor's theorem. □

Corollary 8.2 ([2]). *Let the function $\varphi(x,y)$ be partial continuous with respect to x at every point $x_0\in[a,b]$, uniformly with respect to y on the $[c,d]$ and let the function $\varphi(x_0,y)$ be continuous at every point $y_0\in[c,d]$. Then $\varphi(x,y)$ is partial continuous with respect to y at every point $y_0\in[c,d]$, uniformly with respect to x on the $[a,b]$.*

Proof. It is sufficient to take advantage of Corollary 8.1 and Theorem 8.1. □

Remark 8.2. The continuous on the Q function $\varphi(x,y)$ is characterized by inequalities (4) and (5).

Theorem 8.2 ([2]). *Let the function $\varphi(x,y)$ have bounded partial derivative $\varphi'_x(x,y)$ on the rectangle*

$$r(x_0,\delta)=\big\{(x,y)\in\mathbb{R}^2: x_0-\delta<x<x_0+\delta,\ c_1\le y\le d_1\big\}\subset Q. \qquad (6)$$

Then $\varphi(x,y)$ is partial continuous with respect to x at the point x_0, uniformly with respect to y on the $[c_1,d_1]$.

Proof. Arbitrary point $(x,y)\in r(x_0,\delta)$ we connect with the point (x_0,y) through the linear segment and write for it the Lagrange formula

$$\varphi(x,y)-\varphi(x_0,y)=(x-x_0)\varphi'_x(\xi,y).$$

Due to the boundedness of $\varphi'_x(x,y)$ on the $r(x_0,\delta)$, there exists the constant $c>0$ such that $|\varphi'_x(x,y)|<c$ for all points $(x,y)\in r(x_0,\delta)$. Taking arbitrary number $\varepsilon>0$, we chose a number $\eta>0$ with the properties $\eta<\delta$ and $\eta\cdot c<\varepsilon$.

If now x is so close to x_0 that $|x-x_0|<\eta$, then

$$\big|\varphi(x,y)-\varphi(x_0,y)\big|<\varepsilon \quad \text{for} \quad |x-x_0|<\eta \quad \text{and} \quad c_1\le y\le d_1. \qquad \square$$

§ 9. Unilateral Limit and Continuity of Functions of Two Variables

The notions of unilateral limit and unilateral continuity for functions of one variable are well-known.

From the right limit and the from the right continuity at the point $t_0=0$ can be called as the $^+$limit and the $^+$continuity at the point $t_0=0$. In the sequel, this terminology will be retained for every point t_0.

Thus we can easily get the notions for the $^+$limit [$^-$limit], as well as for the $^+$continuity [$^-$continuity] in the given point with respect to each independent variable of functions of two variables, but they are ineffective.

Namely, let the function $\varphi(x)$, $x=(x_1,x_2)$ be defined in the neighborhood $U(x^0)$, or in the punctured neighborhood $U^0(x^0)=U(x^0)\setminus\{x^0\}$ of the point $x^0=(x_1^0,x_2^0)$. If the function ${}^i\varphi(x_i)$, $i=1,2$, defined by equality

2.1.(3) has at the point x_i^0 unilateral $^+$limit

$$\lim_{x_i \to x_i^0+} {}^i\varphi(x_i),$$

then the function $\varphi(x)$ has at the point x^0 partial $^+$limit with respect to the variable x_i; this can be written in the form

$$\lim_{x_1 \to x_1^0+} \varphi(x_1, x_2^0) \quad \text{and} \quad \lim_{x_2 \to x_2^0+} \varphi(x_1^0, x_2).$$

If the limit

$$\lim_{x_i \to x_i^0+} {}^i\varphi(x_i)$$

is equal to ${}^i\varphi(x_i^0) = \varphi(x^0)$, then the function $\varphi(x)$ is partial $^+$continuous at the point x^0 with respect to the variable x_i, $i = 1, 2$.

Analogously are defined the partial $^-$limit and the partial $^-$continuity for the function $\varphi(x)$ at the point x^0 with respect to the variable x_i, $i = 1, 2$.

The existence of partial $^\pm$limits with respect to x_1 and x_2 and also their equality is, in general, insufficient for the function $\varphi(x)$, $x = (x_1, x_2)$, to have limit at the point $x^0 = (x_1^0, x_2^0)$.

Similarly, synchronous $^\pm$continuity does not, generally speaking, imply the continuity.

As is seen, the notions of the $^+$limit [$^-$limit] and of the $^+$continuity [$^-$continuity] for one-dimensional case are introduced in a natural manner. This natural character is due to the natural partitioning of the neighborhood of the point.

By analogy with the above notions we have introduced the insignificants notions of partial $^+$limit [partial $^-$limit] and of partial $^+$continuity [partial $^-$continuity] with respect to the given independent variable, the functions of two variables.

A unique method of partitioning the neighborhood of the point is inavailable for two-dimensional case: two-dimensional interval can be divided into portions by different ways. Which of these partitionings is more suitable for the problem posed?

Using the notions of strong partial continuity and angular partial continuity (see subsections 2.1 and 3.1), we shall introduce below the notions of unilateral limit and unilateral continuity.

9.1. Strong Unilateral Limit and Continuity

Let the function $f(x)$, $x = (x_1, x_2)$, be defined in the neighborhood $U(x^0)$, or in the punctured neighborhood $U^0(x^0) = U(x^0) \setminus \{x^0\}$ of the point $x^0 = (x_1^0, x_2^0)$.

Introduce the following sets:

$$A_1^+ = \big\{(x_1, x_2) \in U(x^0) : x_1 > x_1^0\big\}, \quad A_2^+ = \big\{(x_1^0, x_2) \in U(x^0) : x_2 > x_2^0\big\},$$
$$A_1^- = \big\{(x_1, x_2) \in U(x^0) : x_1 < x_1^0\big\}, \quad A_2^- = \big\{(x_1^0, x_2) \in U(x^0) : x_2 < x_2^0\big\},$$
$$A_{12}^+ = A_1^+ \cup A_2^+, \quad A_{12}^- = A_1^- \cup A_2^-.$$

Obviously, $A_{12}^{+} \cap A_{12}^{-} = \varnothing$ and

$$A_{12}^{+} \cup A_{12}^{-} = U^0(x^0). \tag{1}$$

Hence the punctured neighborhood is represented as a union of two nonintersecting sets, and the limit with respect to each of them will be called respectively the strong $^{+}$limit and the strong $^{-}$ limit, according to the definitions given below.

1. Definition 9.1.1 ([8]). We say that the function $f(x)$ has at the point x^0 the strong $^{+}$limit, symbolically $f(x^0[+])$, if there exists finite or infinite of fixed sign limit

$$f(x^0[+]) = \lim_{\substack{x \to x^0 \\ x \in A_{12}^{+}}} f(x). \tag{2}$$

The strong $^{-}$limit

$$f(x^0[-]) = \lim_{\substack{x \to x^0 \\ x \in A_{12}^{-}}} f(x) \tag{3}$$

is defined analogously.

If there exist $f(x^0[+])$ and $f(x^0[-])$, then we say that the function $f(x)$ has strong $^{\pm}$limits at the point x^0.

Taking into account (1), the above reasoning allows us to arrive at

Proposition 9.1.1 ([8]). *For the function $f(x)$ to have the limit at the point x^0, it is necessary and sufficient that the $^{\pm}$limits for $f(x)$ at x^0 be equal.*

If these assumptions are fulfilled, we have

$$f(x^0[-]) = \lim_{x \to x^0} f(x) = f(x^0[+]). \tag{4}$$

2. Definition 9.1.2 ([8]). The function $f(x)$ is called strongly $^{+}$continuous at the point $f(x^0)$, if $f(x^0)$ is finite and

$$f(x^0[+]) = f(x^0). \tag{5}$$

Analogously, the function $f(x)$ is called strongly $^{-}$continuous at the point x^0, if $f(x^0)$ is finite and

$$f(x^0[-]) = f(x^0). \tag{6}$$

The function $f(x)$ is called strongly $^{\pm}$continuous at the point x^0, if $f(x)$ at x^0 is both strongly $^{+}$continuous and strongly $^{-}$ continuous.

The following proposition is obvious.

Proposition 9.1.2 ([8]). *For the function $f(x)$ to be continuous at the point x^0, it is necessary and sufficient that $f(x)$ be strongly $^{\pm}$continuous at the point x^0.*

9.2. Strong Jump

1. Definition 9.2.1 ([8]). If $f(x^0[-])$ and $f(x^0[+])$ are finite for the function $f(x)$, then we call

$$\Omega(f,x^0)=\bigl|f(x^0[+])-f(x^0[-])\bigr| \tag{1}$$

a strong jump of the function $f(x)$ at the point x^0, and x^0 is called a point of a finite strong jump of $f(x)$.

The following proposition is obvious.

Proposition 9.2.1 ([8]). *For the function $f(x)$ to have finite limit at the point x^0, it is necessary and sufficient that*

$$\Omega(f,x^0)=0 \tag{2}$$

and in case this equality is fulfilled, we shall have

$$f(x^0[-])=\lim_{x\to x^0}f(x)=f(x^0[+]). \tag{3}$$

2. If x^0 is the point of discontinuity of the function $f(x)$, i.e., $f(x)$ is not continuous at x^0 and equality (2) holds, then x^0 is called strongly removable point of discontinuity of the function $f(x)$: if the limit

$$\lim_{x\to x^0}f(x),$$

being finite and equal to the strong $^{\pm}$limits of $f(x)$ at x^0 is taken as the value of the function f at the point x^0, then as a result of such correction the newly obtained function will be continuous at the point x^0.

This procedure is called strong correction for continuity of the function f at the point x^0, and the point x^0 itself is called strongly correctable point of discontinuity of the function $f(x)$.

If there exist finite $f(x^0[-])$ and $f(x^0[+])$, but $f(x^0[-])\neq f(x^0[+])$, or what is the same thing, there is a bilateral inequality

$$0<\Omega(f,x^0)<+\infty, \tag{4}$$

then x^0 is called the point of strongly first kind discontinuity of the function $f(x)$.

If there does not exist at least one of $f(x^0[-])$ and $f(x^0[+])$, or there exist both, but at least one of them is infinite with a fixed sign, then x^0 is called the point of strongly second kind discontinuity of the function $f(x)$.

9.3. Angular Limit and Angular Continuity

In subsection 9.1 we have considered strong unilateral $^{\pm}$limits and $^{\pm}$continuities. Our consideration was based on such partitioning of the neighborhood of the point, which was dictated by the notion of separately strong partial continuity.

Moreover, the continuity is likewise equivalent to separately angular partial continuity (see subsection 5.2). This allows us to introduce angular limit, angular continuity and unilateral angular limit, unilateral angular continuity.

Let the function $\varphi(x)$, $x = (x_1, x_2)$, be defined in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$.

1. We start with introducing the notion of an angular limit with respect to the given variable.

Definition 9.3.1 ([8]). We say that the function $\varphi(x)$ at the point x^0 has angular limit with respect to the variable x_1, symbolically $\varphi(x^0 \wedge (x_1))$, if for every constant $c > 0$ there exists an independent of c finite or infinite limit

$$\varphi(x^0 \wedge (x_1)) = \lim_{\substack{h_1 \to 0 \\ |h_2| \le c|h_1|}} \varphi(x_1^0 + h_1, x_2^0 + h_2). \tag{1}$$

Similarly, the function $\varphi(x)$ at the point x^0 has angular limit with respect to the variable x_2, if for every constant $\ell > 0$ there exists an independent of ℓ finite or infinite limit

$$\varphi(x^0 \wedge (x_2)) = \lim_{\substack{h_2 \to 0 \\ |h_1| \le \ell|h_2|}} \varphi(x_1^0 + h_1, x_2^0 + h_2). \tag{2}$$

If $\varphi(x^0 \wedge (x_1))$ and $\varphi(x^0 \wedge (x_2))$ do exist, we say that the function $\varphi(x)$ has separated angular limits at the point x^0.

Theorem 9.3.1 ([8]). *The function $\varphi(x)$ will possess the limit at the point x^0, iff at the point x^0 there exist equal separated angular limits for $\varphi(x)$. If these conditions are fulfilled, we shall have*

$$\varphi(x^0 \wedge (x_1)) = \lim_{x \to x^0} \varphi(x) = \varphi(x^0 \wedge (x_2)). \tag{3}$$

Proof. The existence for the function $\varphi(x)$ of the limit at the point x^0 implies the existence of the same limit for the function $\varphi(x)$ at the point x^0 with respect to every subset with limiting point at x^0. In particular, as such are the sets under the limit sign, indicated in equalities (1) and (2). Therefore $\lim\limits_{x \to x^0} \varphi(x)$ is equal to each of the limits (1) and (2).

If $\varphi(x^0 \wedge (x_1)) = \varphi(x^0 \wedge (x_2))$, then the function $\varphi(x)$ at the point x^0 has equal limits with respect to those two subsets, which correspond to particular cases $c = 1$ and $\ell = 1$. But union of the two subsets gives the neighborhood of the point x^0. $\square$

The process of proving Theorem 9.3.1 and Proposition 9.1.1 leads us to the following

Theorem 9.3.2. *The existence for the function $\varphi(x)$, $x = (x_1, x_2)$, of the limit at the point $x^0 = (x_1^0, x_2^0)$ is equivalent to each of the following two statements:*

1) *the function $\varphi(x)$ at the point x^0 has equal strong ${}^{\pm}$limits;*

2) *the right-hand sides of equalities* (1) *and* (2) *are equal for particular cases* $c = 1$ *and* $\ell = 1$.

The limit $\lim\limits_{x \to x^0} \varphi(x)$ *is equal to each value from statements* 1) *and* 2).

2. Below we shall introduce the notion of an angular continuity with respect to the given variable. This notion differs from that adopted by us for angular partial continuity with respect to the same variable (see equalities 5.2.(1) and 5.2.(2)).

The matter is that the angular partial continuity was introduced due to the specific difference. The subtrahend of that difference is obtained by substitution of a partial value of the given variable into the function. Moreover, the minuend of the same difference is value of the function at the point inside the angle, while the subtrahend is value of the function at the point not belonging to the given angle.

Here we shall give the notion of angular continuity with respect to the given variable. This notion involves values of the function only at those points, which belong to the angle which corresponds to the given variable.

Definition 9.3.2 ([8]). Angular continuity with respect to the variable x_1 of the function $\varphi(x)$ at the point x^0 means that $\varphi(x^0)$ is finite and

$$\varphi\big(x^0 \wedge (x_1)\big) = \varphi(x^0). \tag{4}$$

Similarly, angular continuity with respect to the variable x_2 of the function $\varphi(x)$ at the point x^0 means that $\varphi(x^0)$ is finite and

$$\varphi\big(x^0 \wedge (x_2)\big) = \varphi(x^0). \tag{5}$$

The function $\varphi(x)$ is separately angular continuous at the point x^0, if $\varphi(x)$ at x^0 is angular continuous with respect to the variables x_1 and x_2 (separately angular partial continuity took place in subsection 5.2).

Theorem 9.3.3 ([8]). *For the function* $\varphi(x)$ *to be continuous at the point* x^0, *it is necessary and sufficient that* $\varphi(x)$ *be separately angular continuous at* x^0.

Proof. If the function $\varphi(x)$ is continuous at the point x^0, then $\varphi(x^0)$ is finite and $\lim\limits_{x \to x^0} \varphi(x) = \varphi(x^0)$. Limits (1) and (2) are the particular cases of the left-hand side of that equality, and therefore

$$\varphi\big(x^0 \wedge (x_1)\big) = \varphi(x^0) = \varphi\big(x^0 \wedge (x_2)\big).$$

Hence the function $\varphi(x)$ is separately angular continuous at the point x^0.

Conversely, if each of the limits (1) and (2) is equal to the finite $\varphi(x^0)$, then to this $\varphi(x^0)$ are equal limits (1) and (2) for the cases $c = 1$ and $\ell = 1$. Union of these sets gives the neighborhood of the point x^0. Hence the limit of the function $\varphi(x)$ at the point x^0 is equal to the finite $\varphi(x^0)$, i.e., $\varphi(x)$ is continuous at the point x^0. $\square$

From Theorems 9.3.3, 5.1.1, 5.2.1 and Proposition 9.1.2 follows

Theorem 9.3.4. *The continuity of the function $\varphi(x)$ at the point x^0 is equivalent to:*

1) *separately angular continuity of the function $\varphi(x)$ at the point x^0;*
2) *separately strong partial continuity of the function $\varphi(x)$ at the point x^0;*
3) *separately angular partial continuity of the function $\varphi(x)$ at the point x^0;*
4) *strong $^{\pm}$continuity of the function $\varphi(x)$ at the point x^0.*

9.4. Angular Unilateral Limit and Continuity

Angular $^{\pm}$limits at the point x^0 of the function $\varphi(x)$ with respect to x_1, symbolically $\varphi(x^0 \overset{\wedge}{+} (x_1))$ and $\varphi(x^0 \overset{\wedge}{-} (x_1))$ respectively, will be defined below by means of equalities (1) and (2), in case these limits exist and do not depend on the constants $a > 0$ and $b > 0$:

$$\varphi(x^0 \overset{\wedge}{+} (x_1)) = \lim_{\substack{h_1 \to 0^+ \\ |h_2| \le a h_1}} \varphi(x_1^0 + h_1, x_2^0 + h_2), \tag{1}$$

$$\varphi(x^0 \overset{\wedge}{-} (x_1)) = \lim_{\substack{h_1 \to 0^- \\ |h_2| \le -b h_1}} \varphi(x_1^0 + h_1, x_2^0 + h_2) \tag{2}$$

Angular $^{\pm}$limits at the point x^0 of the function $\varphi(x)$ with respect to the variable x_2 can be defined by equalities (3) and (4) under similar assumptions on $c > 0$ and $d > 0$:

$$\varphi(x^0 \overset{\wedge}{+} (x_2)) = \lim_{\substack{h_2 \to 0^+ \\ h_2 \ge c|h_1|}} \varphi(x_1^0 + h_1, x_2^0 + h_2), \tag{3}$$

$$\varphi(x^0 \overset{\wedge}{-} (x_2)) = \lim_{\substack{h_2 \to 0^- \\ h_2 \le -d|h_1|}} \varphi(x_1^0 + h_1, x_2^0 + h_2). \tag{4}$$

We have the following

Proposition 9.4.1 ([8]). *The function $\varphi(x)$ has angular limit with respect to the variable x_1 at the point x^0, iff there exist equal quantities $\varphi(x^0 \overset{\wedge}{-} (x_1))$ and $\varphi(x^0 \overset{\wedge}{+} (x_1))$. In this case we have the following relations:*

$$\varphi(x^0 \overset{\wedge}{-} (x_1)) = \varphi(x^0 \wedge (x_1)) = \varphi(x^0 \overset{\wedge}{+} (x_1)). \tag{5}$$

Analogous proposition is, obviously, valid for the variable x_2 as well.

From Theorem 9.3.1 and propositions above we arrive at the following

Corollary 9.4.1 ([8]). *The function $\varphi(x)$ has at the point x^0 the limit, iff the quantities defined by equalities* (1)–(4) *exist and all are equal between each other. In case these conditions are fulfilled, their common value is equal to $\lim\limits_{x \to x^0} \varphi(x)$.*

Definition 9.4.1. The function φ with respect to the variable x_1 is angular $^+$continuous at the point x^0, if $\varphi(x^0)$ is finite and the equality

$$\varphi\big(x^0 \overset{\wedge}{+} (x_1)\big) = \varphi(x^0)$$

is fulfilled.

Similarly, $\varphi(x)$ with respect to x_1 is angular $^-$continuous at the point x^0, if $\varphi(x^0)$ is finite and

$$\varphi\big(x^0 \overset{\wedge}{-} (x_1)\big) = \varphi(x^0).$$

Finally, the function $\varphi(x)$ with respect to the variable x_1 is angular $^\pm$continuous at the point x^0, if $\varphi(x)$ at x^0 is angular $^+$continuous and angular $^-$continuous with respect to the variable x_1.

We have the following

Proposition 9.4.2. *The function $\varphi(x)$ at the point x^0 is angular continuous with respect to the variable x_1, iff $\varphi(x)$ at x^0 is angular $^\pm$continuous with respect to x_1.*

Analogously we define angular $^+$continuity, $^-$continuity and $^\pm$continuity at the point x^0 of the function $\varphi(x)$ with respect to the variable x_2.

From Theorem 9.3.2 we arrive at

Theorem 9.4.1. *For the function $\varphi(x)$ to be continuous at the point x^0, it is necessary and sufficient that $\varphi(x)$ be angular $^\pm$continuous with respect both to x_1 and to x_2.*

9.5. Angular Jump

Definition 9.5.1 ([8]). If the function φ has finite $\varphi(x^0 \wedge (x_1))$ and $\varphi(x^0 \wedge (x_2))$, then the value

$$\omega(\varphi, x^0) = \big|\varphi\big(x^0 \wedge (x_1)\big) - \varphi\big(x^0 \wedge (x_2)\big)\big| \tag{1}$$

is called angular jump of the function $\varphi(x)$ at the point x^0.

The following proposition holds.

Proposition 9.5.1 ([8]). *The equality*

$$\omega(\varphi, x^0) = 0 \tag{2}$$

is the necessary and sufficient condition for the function $\varphi(x)$ to have finite limit at the point x^0. If equality (2) is fulfilled, then the common value $\varphi(x^0 \wedge (x_1) = \varphi(x^0 \wedge (x_2)$ is the limit of the function $\varphi(x)$ at the point x^0.

Here, as above, we can introduce the notions of: angularly removable point of discontinuity of the function $\varphi(x)$, angular correction for continuity of the function $\varphi(x)$ at the point x^0, angular correctable point of discontinuity of the function $\varphi(x)$, angular first kind discontinuity point of the function $\varphi(x)$, angular second kind discontinuity point of the function $\varphi(x)$.

9.6. Equivalence of Strong and Angular Corrections

Proposition 9.6.1 ([8]). *If the function $f(x)$ admits strong correction for the continuity at the point x^0, then $f(x)$ likewise admits angular correction for the continuity at the point x^0. The converse statement is also valid.*

Proof. Since $f(x)$ admits strong correction for continuity at the point x^0, equality 9.2.(2) is fulfilled. This implies the existence of the finite limit of the function $f(x)$ at the point x^0, by Proposition 9.2.1, and hence the fulfilment of equality 9.5.(2), by Proposition 9.5.1. Therefore angular correction of the function $f(x)$ for its continuity at the point x^0 is quite possible.

Converse statement can be established in a similar way. □

The above proposition allows us to come to an agreement that the function $f(x)$ is called correctable for continuity at the point x^0, if $f(x)$ admits strong or angular correction for continuity at the point x^0.

Finally, if the function $f(x)$ at the point x^0 has noncorrectable, or what is the same thing, unremovable discontinuity, then x^0 is called the point of essential discontinuity of the function $f(x)$, and the function itself is called essentially discontinuous at the point x^0.

CHAPTER II

Separately Partial Differentiability in Various Senses and Differentiability

The main goal of the present chapter is to resolve the problem on the existence at a point of a total differential.

Regarding a separate independent variable there arises the question: does there exist a notion, or a property at the point x^0 for the function f of several variables, such that the fulfilment of that property at the point x^0 with respect to all independent variables is the necessary and sufficient condition for the function f to be differentiable at the point x^0?

Introduction

The notion of a derivative of functions of one variable can be extended automatically to functions of several variables, and we obtain the notion of a partial derivative with respect to the given variable.

The existence of all finite at the point x^0 partial derivatives of the function f, or what is the same thing, the finiteness at the point x^0 of a gradient of f does not imply the existence at the point x^0 of a total differential of the function f. Moreover, the function, possessing a finite gradient at the point x^0, may be discontinuous at x^0. Such, for example, are at the point $(0, 0)$ the most of the functions of two variables indicated in Piotrowski's work [19].

It is remarkable that this fact can be realized at all points of a set, whose plane measure is arbitrarily close to a total measure. Due to its significance, this fact, stated by Tolstov, can be formulated in the form of

Theorem B ([27], § 4). *For every positive number $\mu < 1$ there exists the function $F(x, y)$, defined on the square $Q = \{(x, y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$, possessing at all points of the Q finite partial derivatives of all orders, and moreover $F(x, y)$ is discontinuous on some set $E \subset Q$ of plane measure μ^2.*

In particular, the $\operatorname{grad} F(x, y)$ of the Tolstov's function $F(x, y)$ is finite in neighborhoods of many points (from the set E), but $F(x, y)$ fails to have a total differential at these points. The function (see equality 2.4.(12) below)

$$\psi(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2} & \text{for } x^2+y^2>0\\ 0 & \text{for } x=0=y\end{cases},$$

is the realization of that fact at one point.

Further, the function may be differentiable at some point (x_0,y_0), and moreover, in every punctured neighborhood of the point (x_0,y_0) there exist points at which its gradient is devoid of sense. Hence its gradient is not continuous at (x_0,y_0). For the point $(0,0)$ such is the function (see equality 2.4.(10) below)

$$g(x,y)=\begin{cases}xy\sin\dfrac{1}{xy} & \text{for } x\cdot y\neq 0\\ 0 & \text{for } x\cdot y=0\end{cases}.$$

Moreover, differentiable at the point x^0 function may be disconontinuous at all points of the punctured neighborhood of x^0 (see equalities 2.4.(5), 2.4.(7) and 2.4.(9) below).

It is well known for a long time that the continuity of the $\operatorname{grad} f(x)$ at the point $x^0=(x_1^0,\dots,x_n^0)$ is the sufficient condition for the function $f(x_1,\dots,x_n)$ to be differentiable at the point x^0. The above-mentioned functions confirm that the continuity of the gradient is only the sufficient condition for the differentiability.

The content of Chapter II is presented by sections.

§ 1 is devoted to the well-known elementary statements.

In § 2 we introduce the notion of an angular gradient of the function f at the point x^0 and the main results sounds as follows: the finiteness of the angular gradient of the function f at the point x^0 is the necessary and sufficient condition for the function f to have a total differential at the point x^0.

Some examples of functions for their differentiability are considered herein.

In § 3 we likewise introduce a new notion of a strong gradient of the function f at the point x^0, whose finiteness implies the differentiability of the function f at the point x^0. The converse statement may turn out to be invalid almost everywhere.

It is proved that the continuity of the gradient at a point implies the finiteness of a strong gradient at the same point. Falsity of the converse statement is realizable almost everywhere.

§ 4 illustrates that the notions of strong and angular partial derivatives allow one to consider the corresponding unilateral partial derivatives and differentials for functions of two variables.

§ 5 shows that the necessary and sufficient conditions for the differentiability of functions of two real variables (see Theorem 2.5.3 below) together with the Cauchy–Riemann condition allow us to formulate in the form of one equation

$$D_{\widehat{x}}F(z_0) + iD_{\widehat{y}}F(z_0) = 0$$

the necessary and sufficient condition for the complex function $F(z)$ of the complex variable $z = x+iy$ to have at the point $z_0 = x_0+iy_0$ finite derivative $F'(z_0)$.

§ 1. Differentiability and Separately Partial Differentiability

1.1. Partial Derivative and Separately Partial Differentiability

Let the finite function $u = f(x)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, be defined in the neighborhood $U(x^0)$ of the point $x^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$. Here we use the functions of one variable ${}^i f(x_i) = f(x^0(x_i))$, $i = 1, \dots, n$, which are connected with $f(x)$ (see I, 1.3). If ${}^i f(x_i)$, or what is the same, $f(x^0(x_i))$ has at the point $x_i^0 \in \mathbb{R}$ a derivative $({}^i f(x_i))'(x_i^0)$, respectively $(f(x^0(x_i)))'(x_i^0)$, which is finite or infinite of fixed sign (i.e., $+\infty$ or $-\infty$), then this derivative is called a partial derivative at the point x^0 of the function f with respect to the variable x_i. We denote it symbolically $f'_{x_i}(x^0)$, $\partial_{x_i} f(x^0)$, $\frac{\partial f}{\partial_{x_i}}(x^0)$. Hence

$$f'_{x_i}(x^0) = \lim_{x_i \to x_i^0} \frac{f(x^0(x_i)) - f(x^0)}{x_i - x_i^0} = \lim_{x_i \to x_i^0} \frac{{}^i f(x_i) - f(x^0)}{x_i - x_i^0}. \qquad (1)$$

It is easily seen that in order to find a partial derivative with respect to the variable x_i at the point $x^0 = (x_1^0, \dots, x_n^0)$ for the function $f(x_1, \dots, x_n)$ it is necessary to replace x_j in $f(x_1, \dots, x_n)$ by x_j^0 for all $j \neq i$. As a result we obtain the function of one variable x_i, and its derivative at x_i^0 is given by equality (1).

If for all $i = 1, \dots, n$ there exist $f'_{x_i}(x^0)$, finite or infinite, then we consider the gradient at the point x^0 of the function $f(x)$, which is defined by the equality

$$\operatorname{grad} f(x^0) = \big(f'_{x_1}(x^0), \dots, f'_{x_n}(x^0)\big). \qquad (2)$$

If all $f'_{x_k}(x^0)$ are finite, $k = 1, \dots, n$, then the function $f(x)$ is called separately partial differentiable at the point x^0, what is equivalent to the finiteness of the $\operatorname{grad} f(x^0)$. In case $f'_{x_i}(x^0)$ is finite, we denote the quantity $f'_{x_i}(x^0)dx_i$, $dx_i = x_i - x_i^0$, by the symbol $d_{x_i} f(x^0)$ and call it the partial differential with respect to the variable x_i at the point $x^0 = (x_1^0, \dots, x_n^0)$ of the function $f(x_1, \dots, x_n)$.

Thus

$$d_{x_i} f(x^0) = f'_{x_i}(x^0)\, dx_i. \qquad (3)$$

1.2. The Notion of the Differentiability

The notion of the differentiability of functions of two variables has taken its complete shape on the junction of the XIX-XX centuries. The modern definition of a total differential has been introduced by Stolz. The advantage of that definition is illustrated in the works due to Pierpont, Frechet and, especially, to Young.

Definition 1.2.1. Let the function $f(x)$, $x=(x_1,\dots,x_n)$ be defined, and finite in the neighborhood $U(x^0)$ of the point $x^0=(x_1^0,\dots,x_n^0)$. The function f is called differentiable at the point x^0, if there exists a collection of finite numbers $A=(A_1,\dots,A_n)$ such that the ratio of the quantity

$$f(x)-f(x^0)-(A,x-x_0) \tag{1}$$

to the positive quantity

$$\|x-x_0\|=\sum_{i=1}^{n}|x_i-x_i^0| \tag{2}$$

tends to zero, as $x\to x_0$, where it is assumed that

$$(A,x-x^0)=\sum_{i=1}^{n}A_i\cdot(x_i-x_i^0). \tag{3}$$

The above ratio is defined for all $x\neq x^0$ and remains undefined for $x=x^0$.

For the notion of the differentiability it is very important to know the behavior of the above-mentioned ratio in the punctured neighborhood $U^0(x^0)$ of the point x^0. Since this ratio has zero limit at the point x^0, its value at that point is assumed to be zero. This fact can be realized by introducing in the neighborhood $U(x^0)$ the function

$$\omega_{x^0}(x)=\begin{cases}\dfrac{f(x)-f(x^0)-(A,x-x_0}{\|x-x^0\|} & \text{for } x\neq x^0\\ 0 & \text{for } x=x^0\end{cases}. \tag{4}$$

Using the function ω_{x^0}, the notion of the differentiability of the function f at the point x^0 will take the following form.

Definition 1.2.2. The function f is called differentiable at the point x^0, if there exist a collection of finite numbers $A=(a_1,\dots,A_n)$ and a function ω_{x^0} such that for every point $x\in U(x^0)$ the equality

$$f(x)=f(x^0)+(A,x-x^0)+\|x-x^0\|\cdot\omega_{x^0}(x) \tag{5}$$

holds, where the function ω_{x^0} is continuous at the point x^0 and equal to zero at x^0:

$$\lim_{x\to x^0}\omega_{x^0}(x)=0=\omega_{x^0}(x^0). \tag{6}$$

Equality (6) means that for any arbitrarily small positive number ε there exists a positive number $\delta=\delta(x^0,\varepsilon,f)$ with the property

$$\left|\omega_{x^o}(x)\right|<\varepsilon \tag{7}$$

for all x satisfying the condition $\|x - x^0\| < \delta$. Therefore equality (5), or what is the same thing, differentiability of the function f at the point x^0 is equivalent to the fulfilment of the inequality

$$\left|f(x) - f(x^0) - (A, x - x^0)\right| < \varepsilon \cdot \|x - x^0\| \tag{8}$$

for all x with the properties $0 < \|x - x^0\| < \delta$ and $\delta = \delta(x^0, \varepsilon, f) > 0$.

In such a case, a total differential, or briefly, a differential at the point x^0 of the function f is called a linear mapping

$$\sum_{i=1}^{n} A_i \cdot (x_i - x_i^0), \tag{9}$$

which corresponds to the increments $x_1 - x_1^0, \dots, x_n - x_n^0$ of independent variables.

A total differential of the function f at the point x^0 is denoted symbolically $df(x^0, dx)$, or in short, $df(x^0)$, where $dx = (dx_1, \dots, dx_n)$. Consequently,

$$df(x^0) = \sum_{i=1}^{n} A_i \cdot dx_i, \tag{10}$$

where dx_i stand for the increment $x_i - x_i^0$ which need not be infinitely small.

In this case the point x^0 is called the point of differentiability of the function $f(x)$, and $df(x^0)$ is sometimes called the first order differential at the point x^0 of the function f.

If the function $\phi(x)$ is differentiable at every point of some set $E \subset \mathbb{R}^n$, then $\phi(x)$ is called a differentiable function on the set E.

The expression "there exists $df(x^0)$" is equivalent to that of "the function $f(x)$ is differentiable at the point x^0".

Definition 1.2.3. The function $f(x_1, \dots, x_n)$ is called continuously differentiable at the point $x^0 = (x_1^0, \dots, x_n^0)$, if the function

$$\operatorname{grad} f(x) = \big(f'_{x_1}(x), \dots, f'_{x_n}(x)\big), \quad x = (x_1, \dots, x_n), \tag{11}$$

is finite in the neighborhood $U(x^0)$ and continuous at the point x^0, i.e., if the first order partial derivatives $f'_{x_j}(x)$, $j = 1, \dots, n$, are continuous functions at the point x^0.

1.3. Elementary Properties of Differentiable Functions

For our exposition to be more complete, we present here the proof of the well-known statement.

Proposition 1.3.1. *The differentiability of the function f at the point x^0 implies:*

1) *the existence and finiteness of all partial derivatives $f'_{x_i}(x^0)$ and of the equality*

$$A_i = f'_{x_i}(x^0), \quad i = 1, \dots, n; \tag{1}$$

2) *the finiteness at the point x^0 of the* $\operatorname{grad} f(x^0)$*;*

3) *separately partial differentiability of the function* f *at the point* x^0*;*

4) *the equality*

$$df(x^0) = \sum_{i=1}^{n} f'_{x_i}(x^0)\, dx_i. \tag{2}$$

Proof. All x_j but x_i in equality 1.2.(5) are replaced by their partial values x_j^0, $j \neq i$. Thus we obtain the equality

$$f\big(x^0(x_i)\big) = f(x^0) + A_i \cdot (x_i - x_i^0) + |x_i - x_i^0| \cdot \omega_{x^0}\big(x^0(x_i)\big). \tag{3}$$

Using a partial increment, the latter will take the form (see I, equality 1.3.(8))

$$\Delta_{x_i^0} f(x) = A_i \cdot (x_i - x_i^0) + (x_i - x_i^0) \cdot \frac{|x_i - x_i^0|}{x_i - x_i^0} \cdot \omega_{x^0}\big(x^0(x_i)\big). \tag{4}$$

Here we introduce an auxiliary function

$$\omega^i_{x^0}(x_i) = \frac{|x_i - x_i^0|}{x_i - x_i^0} \cdot \omega_{x^0}\big(x^0(x_i)\big), \quad x_i \neq x_i^0. \tag{5}$$

Since the function $|x_i - x_i^0|/(x_i - x_i^0)$ of the variable x_i is bounded in the punctured neighborhood of the point x_i^0, and the left-hand side of relation 1.2.(6) is likewise valid for the partial value $x = x^0(x_i)$, from (5) we obtain the equality

$$\lim_{x_i \to x_i^0} \omega^i_{x^0}(x_i) = 0 \tag{6}$$

and (4) takes the form

$$\Delta_{x_i^0} f(x) = A_i \cdot (x_i - x_i^0) + (x_i - x_i^0) \cdot \omega^i_{x^0}(x_i). \tag{7}$$

The last two equalities imply that the function $f(x^0(x_i))$ has a derivative at the point x_i^0, and the equality $(f(x^0(x_i)))'(x_i^0) = A_i$ holds.

Since the function $f(x)$ is differentiable at the point x^0, all numbers A_i are finite. Hence there exist finite partial derivatives $f'_{x_i}(x^0) = A_i$, $i = 1, \dots, n$. This in its turn means that the function $f(x)$ is, by the definition, separately partial differentiable at the point x^0, and the grad $f(x^0)$ is finite. Thus according to equality 1.2.(10) we obtain equality (2). □

For $df(x^0)$ we have the following well-known

Proposition 1.3.2. 1) *The* $df(x^0)$ *is linear function of differentials of independent variables* dx_i*,* $i = 1, \dots, n$*;*

2) *the* $df(x^0)$ *is the sum of partial* $d_{x_i} f(x^0)$ *differentials*

$$df(x^0) = \sum_{i=1}^{n} d_{x_i} f(x^0); \tag{8}$$

3) *the increment* $\Delta_{x^0} f(x) = f(x) - f(x^0)$ *for the function* f *at the point* x^0 *admits the representation*

$$\Delta_{x^0} f(x) = df(x^0) + \|x - x^0\| \cdot o(1), \tag{9}$$

where $o(1)$ *denotes an infinitesimal at the point* x^0 *function* $\omega_{x^0}(x)$ *satisfying condition* 1.2.(6).

The summand $df(x^0)$ in the right-hand side of equality (9) is called a principal part of the increment $\Delta_{x^0} f(x)$ of the function f at the point x^0, which is equipped with the property

$$\lim_{x \to x^0} df(x^0) = 0 \tag{10}$$

(due to the fact that all $dx_i \to 0$ as $x \to x^0$).

For the second summand from equality (9) the equality

$$\lim_{x \to x^0} \|x - x^0\| \cdot o(1) = 0 \tag{11}$$

is obvious.

From the last two equalities we obtain the equality

$$\lim_{x \to x^0} \Delta_{x^0} f(x) = 0, \tag{12}$$

which means that the function f is continuous at the point x^0.

This fact can be formulated in the form of the following

Proposition 1.3.3. *If* x^0 *is the point of differentiability of the function* f*, then* f *is continuous at* x^0.

This proposition is likewise evident from the relations

$$\begin{aligned} \big|f(x) - f(x^0)\big| &\le \big|f(x) - f(x^0) - (A, x - x^0)\big| + \big|(A, x - x^0)\big| < \\ &< \varepsilon\|x - x^0\| + \|x - x^0\| \cdot \max_{1 \le i \le n} \big|f'_{x_i}(x^0)\big|. \end{aligned}$$

If along with 1.1.(2) we introduce the vector $dx = (dx_1, \dots, dx_n)$, then equality (2) can be written in the form of the scalar product

$$df(x^0) = \big(\operatorname{grad} f(x^0),\, dx\big). \tag{13}$$

Proposition 1.3.4. *Let the function* $f(x)$, $x = (x_1, \dots, x_n)$ *be differentiable at the point* $x^0 = (x_1^0, \dots, x_n^0)$. *We take arbitrary natural number* $m > n$ *and define for* $x \in U(x^0)$ *and* $(x_{n+1}, \dots, x_m) \in \mathbb{R}^{m-n}$ *the function* F *by the equality*

$$F(x_1, \dots, x_n, x_{n+1}, \dots, x_m) = f(x_1, \dots, x_n). \tag{14}$$

Then the function $F(\overline{x})$, *where* $\overline{x} = (x_1, \dots, x_n, x_{n+1}, \dots, x_m)$, *is differentiable at the point* $\overline{x}^0 = (x_1^0, \dots, x_n^0, x_{n+1}^0, \dots, x_m^0)$, *no matter how the point* $(x_{n+1}^0, \dots, x_m^0)$ *is, and the equality*

$$dF(\overline{x}^0)(dx_1, \dots, dx_m) = df(x^0)(dx_1, \dots, dx_n). \tag{15}$$

takes place.

Proof. The property appearing in inequality 1.2.(8) can now be written as

$$\Big|f(x) - f(x^0) - \sum_{i=1}^{n} f'_{x_i}(x^0) \cdot (x_i - x_i^0)\Big| < \varepsilon\|x - x^0\| \tag{16}$$

for all x with the properties $0<|x_k-x_k^0|<\delta/m$, $k=1,\dots,n$.

As far as the function F is constant with respect to the variables $x_{n+1},\dots,x_m$, $F'_{x_j}(\overline{x}^0)=0$ for $j=n+1,\dots,m$. On the other hand, it is obvious that $\|x-x^0\|<\|\overline{x}-\overline{x}^0\|$ for any system $|x_{n+1}-x_{n+1}^0|,\dots,|x_m-x_m^0|$. In particular, take $|x_j-x_j^0|<\delta/m$, $j=n+1,\dots,m$. Thus we have

$$\Big|F(\overline{x})-F(\overline{x}^0)-\sum_{k=1}^m (x_k-x_k^0)\cdot F'_{x_k}(\overline{x}^0)\Big|<\varepsilon\|\overline{x}-\overline{x}^0\| \tag{17}$$

under $0<\|\overline{x}-\overline{x}^0\|=\sum\limits_{k=1}^m |x_k-x_k^0|<m\cdot\delta/m=\delta$.

Hence the function F is differentiable at the point $\overline{x}^0$, and

$$dF(\overline{x}^0)(dx_1,\dots,dx_m)=$$
$$=\sum_{k=1}^m F'_{x_k}(\overline{x}^0)dx_k=\sum_{k=1}^n f'_{x_k}(x^0)dx_k=df(x^0)(dx_1,\dots,dx_n). \qquad \Box$$

Corollary 1.3.1. *Let the functions $a(x_1)$ and $b(x_2)$ have finite derivatives $a'(x_1^0)$ and $b'(x_2^0)$ at the points x_1^0 and x_2^0, respectively. Then the functions $\varphi(x_1,x_2)=a(x_1)+b(x_2)$, $\psi(x_1,x_2)=a(x_1)\cdot b(x_2)$ and $\omega(x_1,x_2)=a(x_1)/b(x_2)$ (if $b(x_2)\neq 0$ in the neighborhood of the point x_2^0) have total differentials at the point (x_1^0,x_2^0), and the equalities*

$$d\varphi(x_1^0,x_2^0)=a'(x_1^0)dx_1+b'(x_2^0)dx_2, \tag{18}$$

$$d\psi(x_1^0,x_2^0)=a'(x_1^0)\cdot b(x_2^0)dx_1+b'(x_2^0)\cdot a(x_1^0)dx_2, \tag{19}$$

$$d\omega(x_1^0,x_2^0)=\frac{1}{b^2(x_2^0)}\big[a'(x_1^0)b(x_2^0)dx_1-b'(x_2^0)a(x_1^0)dx_2\big]. \tag{20}$$

are valid.

Proof. The function $a(x_1)$, being as the function of two variables (x_1,x_2), is differentiable at all points (x_1^0,x_2), by Proposition 1.3.3. Analogously, the function $b(x_2)$ has the total differential at all points (x_1,x_2^0). The both functions $a(x_1)$ and $b(x_2)$, being the functions of two variables, have total differentials at the point (x_1^0,x_2^0). Next, we use the well-known formulas:

$$\begin{aligned} d(u\pm v)&=du\pm dv,\\ d(uv)&=udv+vdu,\\ d\Big(\frac{u}{v}\Big)&=\frac{1}{v^2}\big[vdu-udv\big],\ \ v\neq 0. \qquad \Box\end{aligned} \tag{21}$$

Remark 1.3.1. 1) We know from Proposition 1.3.1 that the differentiability of the function f at the point x^0 implies the finiteness of the expression

$$\sum_{i=1}^n f'_{x_i}(x^0)\,dx_i=\big(\operatorname{grad} f(x^0),\,dx\big). \tag{22}$$

The converse statement is invalid because from the finiteness of (22) we cannot conclude that (22) is the differential of the function f at the point x^0 for a very simple reason that f need not be differentiable and even continuous at the point x^0 (see Introduction in Chapter II).

2) The differentiability is defined by means of the norm 1.2.(2). We have made such a choice for a very simple reason that different estimations can be easily performed by using this norm, and moreover, the notion of the differentiability does not depend on the norms 1.1.(1)–1.1.(3) from Chapter I. This follows from the fact that the ratio of each of the above-mentioned norms to another norm is bounded below by an absolute or dependent only of the dimension n of the space $\mathbb{R}^n$ a positive constant (see estimates 1.1.(4)–1.1.(6) of Chapter I).

3) For the function of one variable $\lambda(t)$, the existence at the point t_0 of a finite derivative $\lambda'(t_0)$ implies that the equality

$$\lim_{t\to t_0} \frac{\lambda(t) - \lambda(t_0) - (t - t_0)\cdot\lambda'(t_0)}{t - t_0} = 0. \tag{23}$$

is fulfilled.

For the same function $\lambda(t)$, the notion of the differentiability 1.2.1 at the point t_0 implies the fulfilment of the equality (in fact, for the function of one variable the differentiability is equivalent to the finiteness of its derivative)

$$\lim_{t\to t_0} \frac{\lambda(t) - \lambda(t_0) - (t - t_0)\cdot\lambda'(t_0)}{|t - t_0|} = 0. \tag{24}$$

The last two equalities are equivalent.

Indeed, if $t > t_0$, then equalities (23) and (24) coincide. If $t < t_0$, then the denominator in equality (24) is $|t - t_0| = -(t - t_0)$, and these equalities are again coincide because $-0 = 0$.

4) It is evident that the results established for real-valued functions $u = f(x)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ are extended to vector-valued functions $u = (u_1, \dots, u_m)$, $u \in \mathbb{R}^m$, if every function $u_j = u_j(x_1, \dots, x_n)$ has the needed properties, $j = 1, \dots, n$.

1.4. The Differentiability with Respect to a Subcollection of Variables

As is known, the existence with respect to x_i of the partial differential $d_{x_i} f(x^0)$ at the point $x^0 = (x_1^0, \dots, x_n^0)$ for the function $f(x_1, \dots, x_n)$ is equivalent to the existence at the point x^0 of a finite partial derivative $f'_{x_i}(x^0)$ with respect to the same variable x_i.

This situation can be widened, if we consider the problem on the existence at the point x^0 of a differential of the function f with respect to arbitrary subcollection of variables from the principal collection $(x_1, \dots, x_n)$.

To avoid formal complications, we will consider the existence at the point x^0 of a differential of the function f with respect to the subcollection $(x_2, \dots, x_n)$.

To this end, we have in $f(x_1,\dots,x_n)$ to replace x_1 by x_1^0 and then to consider the new function $\phi(x_2,\dots,x_n)=f(x_1^0,x_2,\dots,x_n)$.

Definition 1.4.1. If the differential at the point $(x_2^0,\dots,x_n^0)$ of the function $\phi(x_2,\dots,x_n)$ exists, then we say that the function $f(x_1,\dots,x_n)$ with respect to the subcollection $(x_2,\dots,x_n)$ has the differential at the point x^0.

It can be easily seen that statement 1) in Proposition 1.3.1 admits the following generalization.

Proposition 1.4.1. *If the function $f(x_1,\dots,x_n)$ is differentiable at the point $x^0=(x_1^0,\dots,x_n^0)$, then the function f with respect to every subcollection from the collection $(x_1,\dots,x_n)$ is differentiable at x^0.*

The converse statement is invalid. This is understood in a sense that the differentiability at the point x^0 does not follow from the differentiability at the point x^0 with respect to every subcollection consisting of a lesser number of independent variables than the principal collection. This is seen by an example of the function

$$\varphi(x,y)=\begin{cases}1 & \text{for } x\cdot y\neq 0\\ 0 & \text{for } x\cdot y=0\end{cases}. \tag{1}$$

This function is discontinuous and, the more so, non-differentiable at the point $(0,0)$, although the function $\varphi(x,y)$ has zero partial derivatives at the point $(0,0)$.

The sufficient conditions allowing one to conversing Proposition 1.4.1, will be given in Theorem 3.4.1.

§ 2. Differentiability is Equivalent the Finiteness of an Angular Gradient

Before proving the basic theorem on the necessary and sufficient condition for the differentiation, we will cite some definitions.

Let the function $f(x)$, $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, be defined and finite in the neighborhood $U(x^0)$ of the point $x^0=(x_1^0,\dots,x_n^0)\in\mathbb{R}^n$.

2.1. Angular Partial Derivative and Angular Gradient

Definition 2.1.1 ([2], [5]). We say that the function f has at the point x^0 an angular partial derivative with respect to the variable x_k, symbolically $f'_{\widehat{x}_k}(x^0)$, if for every collection $c=(c_1,\dots,c_{k-1},c_{k+1},\dots,c_n)$ of positive $n-1$ constants there exists an independent of the c finite or infinite (of fixed sign) limit

$$f'_{\widehat{x}_k}(x^0)=\lim_{x_k\to x_k^0}\frac{\Delta^c_{\widehat{x}_k^0}f(x)}{x_k-x_k^0}, \tag{1}$$

where (see I, equality 3.1.(1))

$$\Delta^c_{\widehat{x}^0_k} f(x) = f(x) - f\big(x(x^0_k)\big) \quad \text{for} \quad |x_j - x^0_j| \le c_j|x_k - x^0_k|, \quad j \ne k. \tag{2}$$

Relations (1) and (2) can be written in short as follows:

$$f'_{\widehat{x}_k}(x^0) = \lim_{\substack{x_k \to x^0_k \\ |x_j - x^0_j| \le c_j|x_k - x^0_k| \\ j \ne k}} \frac{f(x) - f(x(x^0_k))}{x_k - x^0_k}. \tag{3}$$

If the angular partial derivative $f'_{\widehat{x}_k}(x^0)$ is finite, then equality (3) means that for every arbitrarily small number $\varepsilon > 0$ and for every collection of positive constants $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ there exists a number $\delta = \delta(x^0, \varepsilon, c, f)$ such that

$$\Big|\frac{f(x) - f(x(x^0_k))}{x_k - x^0_k} - f'_{\widehat{x}_k}(x^0)\Big| < \varepsilon \tag{4}$$

for all x's with the properties $0 < \|x - x^0\| < \delta$ and $|x_j - x^0_j| \le c_j|x_k - x^0_k|$ for all $j \ne k$.

The existence of $f'_{\widehat{x}_k}(x^0)$ implies existence of the partial derivative $f'_{x_k}(x^0)$, and the equality $f'_{\widehat{x}_k}(x^0) = f'_{x_k}(x^0)$. To show this, we have to put in (3) $x_j = x^0_j$ for all $j \ne k$.

The existence of the angular partial derivative does not, in general, follow from existence of the partial derivative. Indeed, the function $\varphi(x_1, x_2)$ defined by equality 1.4.(1) has finite partial derivatives at the point $O = (0, 0)$ and has no angular partial derivatives at O. In fact, the absence at O of an angular partial derivative with respect to the variable x_1 follows from that the ratio in equality (3) for $\varphi(x_1, x_2)$ has the form $\varphi(x_1, x_2)/x_1$, which has no limit at O, if the conditions mentioned in (3) are satisfied (this ratio along the Ox_1 axis has the form $0/x_1 = 0$, while along the line $x_2 = x_1$ it has the form $1/x_1$, $0 \ne x_1 \to 0$).

If $f'_{\widehat{x}_k}(x^0)$ is finite, then the function $f(x)$ with respect to the variable x_k has the property of angular partial continuity at the point x^0 (see Chapter I, Section 3.1).

Arbitrary finite function $a(x_1)$ of one variable can be considered as a function of several variables $\psi(x_1, \dots, x_n)$, which is equal to $a(x_1)$ for arbitrary $x_1, \dots, x_n$. Therefore the derivative $a'(x^0_1)$, if it is, coincides with $\psi'_{\widehat{x}_1}(x^0_1, x_2, \dots, x_n)$ for arbitrary $x_2, \dots, x_n$.

Definition 2.1.2 ([2], [5]). If there exist $f'_{\widehat{x}_k}(x^0)$, $k = 1, \dots, n$, finite or infinite (of fixed signs), then we call $f(x)$ the function possessing an angular gradient at the point x^0 and write

$$\operatorname{ang\,grad} f(x^0) = \big(f'_{\widehat{x}_1}(x^0), \dots, f'_{\widehat{x}_n}(x^0)\big). \tag{5}$$

Definition 2.1.3. We say that the function $f(x)$ with respect to the variable x_k has the property of angular partial differentiability at the point

x^0, if $f'_{\widehat{x}_k}(x^0)$ is finite and we write

$$d_{\widehat{x}_k} f(x^0) = f'_{\widehat{x}_k}(x^0)\, dx_k. \tag{6}$$

Moreover, $d_{\widehat{x}_k} f(x^0)$ is called an angular partial differential with respect to the variable x_k of the function $f(x)$ at the point x^0.

Definition 2.1.4 [2], [5]. The function f has at the point x^0 the property of separately angular partial differentiability, if $f'_{\widehat{x}_k}(x^0)$ are finite for all $k = 1, \dots, n$, and this means that $\operatorname{ang\,grad} f(x^0)$ is finite.

2.2. The First Basic Theorem on the Differentiability

Theorem 2.2.1 ([2], [5]). *For the function $f(x)$ to be differentiable at the point x^0, it is necessary and sufficient that* $\operatorname{ang\,grad} f(x^0)$ *is finite, i.e., it is necessary and sufficient that the function $f(x)$ is separately angular partial differentiable at the point x^0.*

Proof. The necessity. Suppose that the function $f(x)$ is differentiable at the point x^0 and establish that $f'_{\widehat{x}_k}(x^0)$ is finite for all $k = 1, \dots, n$.

We write the identity

$$f(x) - f\big(x(x_k^0)\big) - (x_k - x_k^0) f'_{x_k}(x^0) = \big[f(x) - f(x^0)\big] - \\ - \sum_{j=1}^{n} (x_j - x_j^0) f'_{x_j}(x^0) - \Big[f\big(x(x_k^0)\big) - f(x^0) - \sum_{j\neq k} (x_j - x_j^0) f'_{x_j}(x^0)\Big]. \tag{1}$$

Suppose we have an arbitrary positive number ε and a collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants. Since the function $f(x)$ is differentiable at the point x^0, then for $\varepsilon^* = \varepsilon/4(1 + \sum\limits_{j\neq k} c_j)$ there exists a number $\delta = \delta(x^0, c, \varepsilon, f) > 0$ such that an absolute value in the right-hand side of identity (1) will be, by estimate 1.2.(8), smaller than the value

$$\varepsilon^* \sum_{j=1}^{n} |x_j - x_j^0| + \varepsilon^* \sum_{j\neq k} |x_j - x_j^0| \leq 2\varepsilon^* \sum_{j=1}^{n} |x_j - x_j^0| = \\ = 2\varepsilon^* \Big(|x_k - x_k^0| + \sum_{j\neq k} |x_j - x_j^0|\Big) \tag{2}$$

for all x's with the properties $0 < \|x - x^0\| < \delta$.

If along with the condition $0 < \|x - x^0\| < \delta$ the point $x = (x_1, \dots, x_n)$ is subjected to the conditions $|x_j - x_j^0| \leq c_j |x_k - x_k^0|$ for all $j \neq k$, then the absolute value in the left-hand side of identity (1) will be, by virtue of (2), smaller then the value

$$|x_k - x_k^0| \cdot 2\varepsilon^* \Big(1 + \sum_{j\neq k} c_j\Big) = \frac{1}{2} \varepsilon |x_k - x_k^0| < \varepsilon |x_k - x_k^0|.$$

Hence inequality 2.1.(4) is fulfilled, and $f'_{\widehat{x}_k}(x^0) = f'_{x_k}(x^0)$.

Since the function $f(x)$ is differentiable at the point x^0, all partial derivatives $f'_{x_k}(x^0)$ are finite, $k = 1, \dots, n$.

Consequently, the function f possesses at the point x^0 the property of separately angular partial differentiability, or what is the same thing, the $\operatorname{ang}\operatorname{grad} f(x^0)$ is finite.

Moreover, it is stated that the total differential $df(x^0)$ of the differentiable at the point x^0 function f admits the following two representations:

$$df(x^0) = \sum_{k=1}^{n} f'_{\widehat{x}_k}(x^0)\, dx_k, \tag{3}$$

and

$$df(x^0) = \sum_{k=1}^{n} d_{\widehat{x}_k} f(x^0). \tag{4}$$

Sufficiency. Let the function f possesses at the point x^0 the property of separately angular partial differentiability, i.e., all $f'_{\widehat{x}_k}(x^0)$ are finite, $k = 1, \dots, n$. Therefore limit 2.1.(3) is finite for all values of k and for arbitrary collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants, in particular, for $c_j = 1$, $j \neq k$.

By P_k we denote a set of all points $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ each of which satisfies the conditions $|x_j - x_j^0| \leq |x_k - x_k^0|$ for all $j \neq k$. A set of all points $x \in P_k$ likewise satisfying the condition $\|x - x^0\| < \eta$, $\eta > 0$, we denote by P_k^η. Thus the neighborhood $U(x^0, \eta)$ is the union of a finite number of sets P_k^η, $k = 1, \dots, n$.

Take arbitary number $\varepsilon > 0$. Since all $f'_{\widehat{x}_k}(x^0)$ and all values $k = 1, \dots, n$ are finite, there is a number $\delta = \delta(x^0, \varepsilon, f) > 0$, suitable for all values $k = 1, \dots, n$, such that the inequality

$$\left| f(x) - f\big(x(x_k^0)\big) - (x_k - x_k^0) \cdot f'_{\widehat{x}_k}(x^0) \right| < \varepsilon |x_k - x_k^0| \tag{5}$$

will be satisfied by all points $x \in P_k^\delta \setminus \{x^0\}$.

To establish that the function f is differentiable at the point x^0, we have to state that inequality 1.2.(8) in the punctured neighborhood $U(x^0, \delta) \setminus \{x^0\}$ is fulfilled. Towards this end, in its turn it is sufficient that inequality 1.2.(8) be fulfilled for each of $P_k^\delta \setminus \{x^0\}$ separately, $k = 1, \dots, n$.

Without loss of generality, we may be satisfied with the proof of the same inequality 1.2.(8) for one set, say for $P_1^\delta \setminus \{x^0\}$. Therefore in the sequel we will assume that the point x, tending to the point x^0, always belongs to $P_1^\delta \setminus \{x^0\}$.

Since the point x belongs to $P_1^\delta \setminus \{x^0\}$, inequality (5) is fulfilled for $k = 1$. Hence the inequality

$$\left| f(x) - f\big(x(x_1^0)\big) - (x_1 - x_1^0) f'_{\widehat{x}_1}(x^0) \right| < \varepsilon |x_1 - x_1^0| \tag*{$(6)_1$}$$

is fulfilled for all x with the properties $0 < \|x - x^0\| < \delta$ and $|x_j - x_j^0| \leq |x_1 - x_1^0|$ for all $j = 2, \dots, n$.

The point $x(x_1^0)=(x_1^0,x_2,\dots,x_n)$ appearing in the punctured neighborhood $U(x^0,\delta)\setminus\{x^0\}$ does not belong to the set $P_1^\delta\setminus\{x^0\}$; it will belong* to some $P_{\ell_1}^\delta\setminus\{x^0\}$ with $\ell_1\neq 1$. Therefore again, by virtue of inequality (5) for $k=\ell_1$, the inequality

$$\big|f\big(x(x_1^0)\big)-f\big(x(x_1^0,x_{\ell_1}^0)\big)-(x_{\ell_1}-x_{\ell_1}^0)f'_{\widehat{x}_{\ell_1}}(x^0)\big|< \\ <\varepsilon|x_{\ell_1}-x_{\ell_1}^0|, \qquad (6)_{\ell_1}$$

will be fulfilled, where $x(x_1^0,x_{\ell_1}^0)$ denotes that point $(x_1^0,\cdots)$ whose ℓ_1-th coordinate is equal to $x_{\ell_1}^0$ (see I, notation 2.3.(1)).

Now the point $x(x_1^0,x_{\ell_1}^0)$ whose two coordinates are already fixed, will belong to some $P_{\ell_2}^\delta\setminus\{x^0\}$ with $\ell_2\neq 1$ and $\ell_2\neq\ell_1$.

Continuing this process, as a result we obtain a point whose all, but one, coordinates are fixed. This single varying coordinate is $x_{\ell_{n-1}}$. Then using symbol 1.3.(2) from Chapter I, we can write this point as $x^0(x_{\ell_{n-1}})$. This means that the point $x^0(x_{\ell_{n-1}})$ belongs to the set $P_{\ell_{n-1}}^\delta\setminus\{x^0\}$. Applying inequality (5) for $k=\ell_{n-1}$, we have

$$\big|f\big(x^0(x_{\ell_{n-1}})\big)-f(x^0)-(x_{\ell_{n-1}}-x_{\ell_{n-1}}^0)f'_{\widehat{x}_{\ell_{n-1}}}(x^0)\big|< \\ <\varepsilon|x_{\ell_{n-1}}-x_{\ell_{n-1}}^0|. \qquad (6)_{\ell_{n-1}}$$

Write the following identity

$$f(x)-f(x^0)-\sum_{k=1}^{n}(x_k-x_k^0)f'_{\widehat{x}_k}(x^0)= \\ =\big[f(x)-f\big(x(x_1^0)\big)-(x_1-x_1^0)f'_{\widehat{x}_1}(x^0)\big]+ \\ +\big[f\big(x(x_1^0)\big)-f\big(x(x_1^0,x_{\ell_1}^0)\big)-(x_{\ell_1}-x_{\ell_1}^0)f'_{\widehat{x}_{\ell_1}}(x^0)\big]+ \\ +\cdots+\big[f\big(x^0(x_{\ell_{n-1}}^0)\big)-f(x^0)-(x_{\ell_{n-1}}-x_{\ell_{n-1}}^0)f'_{\widehat{x}_{\ell_{n-1}}}(x^0)\big]. \qquad (7)$$

It follows from inequalities $(6)_1$–$(6)_{\ell_{n-1}}$ that the absolute value in the left-hand side of equality (7) is less than the value

$$\varepsilon\big(|x_1-x_1^0|+|x_{\ell_1}-x_{\ell_1}^0|+\cdots+|x_{\ell_{n-1}}-x_{\ell_{n-1}}^0|\big)=\varepsilon\|x-x^0\|.$$

Hence the function $f(x)$ is differentiable at the point x^0, and its differential $df(x^0)$ at x^0 is equal to the sum

$$\sum_{k=1}^{n}f'_{\widehat{x}_k}(x^0)\,dx_k. \qquad (8)$$

Thus the proof of Theorem 2.2.1 is complete. □

Corollary 2.2.1. *The finiteness of the expression* (8) *is the necessary and sufficient condition in order that,* (8) *to be the differential of the function* f *at the point* x^0.

---

*If $n=2$, then the point $x(x_1^0)=(x_1^0,x_2)$ will necessarily belong to the set $P_2^\delta\setminus\{x^0\}$, which in this case has the form $\{(x_1,x_2):|x_2-x_2^0|\geq|x_1-x_1^0|\}$. In case $n=3$, the point $x(x_1^0)=(x_1^0,x_2,x_3)$ will belong to $P_2^\delta\setminus\{x^0\}$, or to $P_3^\delta\setminus\{x^0\}$.

2.3. The Second Theorem on the Differentiability

As we have seen, every angular partial derivative is defined by using a collection of arbitrary positive constants. If the function depends on m independent variables, then for that function we have m angular partial derivatives, and the definition of every angular partial derivative involves a collection of $(m-1)$ arbitrary positive constants. The arbitrariness of these positive constants is needed for the definition, i.e., for the existence of a separate angular partial derivative.

But while proving the second part of Theorem 2.2.1 we have revealed the following fact: if in the definitions of all angular partial derivatives of the function $f(x)$ at the point x^0 one takes all constants c_j equal to 1, then the finiteness of all the obtained in such a way values is sufficient for the function f to be differentiable at the point x^0.

This fact will be used in the sequel in investigating of functions for their differentiability (see Section 2.4 below). We formulate it in the form of the following

Theorem 2.3.1 ([5]). *For the function $f(x_1,\dots,x_n)$ to be differentiable at the point $x^0=(x_1^0,\dots,x_n^0)$, it is necessary and sufficient that*

$$D_{\widehat{x}_k} f(x^0) = \lim_{\substack{x_k\to x_k^0 \\ |x_j-x_j^0|\le|x_k-x_k^0| \\ j\neq k}} \frac{f(x)-f(x(x_k^0))}{x_k-x_k^0} \tag{1}$$

is finite for all $k=1,\dots,n$.

Corollary 2.3.1 ([5]). *The finiteness of all $D_{\widehat{x}_k} f(x^0)$ implies finiteness of all $f'_{\widehat{x}_k} f(x^0)$, and the equality*

$$f'_{\widehat{x}_k}(x^0) = D_{\widehat{x}_k} f(x^0), \quad k=1,\dots,n, \tag{2}$$

$$df(x^0) = \sum_{k=1}^{n} D_{\widehat{x}_k} f(x^0)\,dx_k. \tag{3}$$

Corollary 2.3.2. *From the finiteness of all $D_{\widehat{x}_k} f(x^0)$ it follows that the function $f(x)$ possesses at the point x^0 the property of separately angular partial differentiability, or what is the same, the function $f(x)$ has a finite* ang grad $f(x^0)$.

Introduce the notation

$$\widehat{D} f(x^0) = \big(D_{\widehat{x}_1} f(x^0),\dots,D_{\widehat{x}_n} f(x^0)\big), \tag{4}$$

whose finiteness is understood in a sense that every component $D_{\widehat{x}_k} f(x^0)$, $k=1,\dots,n$, is finite.

Now Theorem 2.3.1 can be rephrased as follows.

Theorem 2.3.2 ([5]). *For the existence of* $df(x^0)$ *it is necessary and sufficient that* $\widehat{D}f(x^0)$ *be finite. If* $\widehat{D}f(x^0)$ *is finite, we have the equality*

$$df(x^0) = \big(\widehat{D}f(x^0),\, dx\big). \tag{5}$$

2.4. Examples on the Differentiability

Using Theorem 2.3.1, we can establish the differentiability as well as non-differentiability of concrete functions.

On the differentiability we investigate some appearing frequently functions.

Proposition 2.4.1 ([5]). *Suppose the numbers* α_j *are positive,* $j = 1,\dots,n$. *Then the condition*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n > 1 \tag{1}$$

is necessary and sufficient for the everywhere continuous function

$$\varphi(x_1,\dots,x_n) = |x_1|^{\alpha_1}\cdot|x_2|^{\alpha_2}\cdots|x_n|^{\alpha_n} \tag{2}$$

to be differentiable at the point $x^0 = (0,\cdots,0)$.

In particular, the function $\nu(x_1,\dots,x_n) = (|x_1|\cdots|x_n|)^{\alpha}$ *is differentiable at the point* x^0 *if and only if* $\alpha > \frac{1}{n}$.

Proof. Sufficiency. By equality 2.3.(3) we have

$$D_{\widehat{x}_k}\varphi(x^0) = \lim_{\substack{x_k\to 0\\ |x_j|\le|x_k|\\ j\ne k}} \frac{|x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n}}{x_k} = \lim_{\substack{x_k\to 0\\ |x_j|\le|x_k|\\ j\ne k}} \frac{|x_k|}{x_k}\cdot\frac{|x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n}}{x_k}.$$

Under the above conditions it follows that

$$\frac{|x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n}}{|x_k|} \le |x_k|^{\alpha_1}\cdots|x_k|^{\alpha_{k-1}}\cdot|x_k|^{\alpha_k-1}\cdot|x_k|^{\alpha_{k+1}}\cdots|x_k|^{\alpha_n} =$$
$$= |x_k|^{(\alpha_1+\cdots+\alpha_{k-1}+\alpha_k+\alpha_{k+1}+\cdots+\alpha_n)-1} \to 0,\ \ x_k\to 0.$$

Hence $D_{\widehat{x}_k}\varphi(x^0) = 0$ for all $k = 1,\dots,n$, and by equality 2.3.(3) we obtain

$$d\varphi(x^0) = 0. \tag{3}$$

Necessity. For $\alpha_j > 0$ and $\alpha_1+\alpha_2+\cdots+\alpha_n \le 1$ there exists none finite $D_{\widehat{x}_k}\varphi(x^0)$, in particular, $\varphi(x)$ is not differentiable at the point x^0. Indeed, should the finite $D_{\widehat{x}_k}\varphi(x^0)$ exist for some k, the expression

$$\frac{|x_k|^{\alpha_1}\cdots|x_k|^{\alpha_n}}{x_k} = \frac{|x_k|^{\alpha_1+\cdots+\alpha_n}}{x_k} =$$
$$= \begin{cases} \dfrac{|x_k|}{x_k} & \text{for } \ \alpha_1+\cdots+\alpha_n = 1\\[2mm] \dfrac{|x_k|}{x_k}\cdot|x_k|^{-\beta} & \text{for } \ \alpha_1+\cdots+\alpha_n < 1,\ \ \beta = 1-(\alpha_1+\cdots+\alpha_n) \end{cases},$$

would have a finite limit, but this is not the case.

In particular, the function

$$\mu(x_1, x_2) = \sqrt{|x_1| \cdot |x_2|} \tag{4}$$

is non-differentiable at the point $(0,0)$. □

Remark 2.4.1. By Proposition 2.4.1, the function

$$\lambda(x_1, x_2) = |x_1 \cdot x_2|^{2/3} \tag{5}$$

is differentiable at the point $(0,0)$, while $\lambda(x_1,x_2)$ is non-differentiable at the points $(a,0)$ and $(0,b)$, where $a \neq 0$ and $b \neq 0$.

Indeed, should the function $\lambda(x_1,x_2)$ be differentiable at the point $(a,0)$, $a \neq 0$, there would exist a finite partial derivative

$$\lambda'_{x_2}(a,0) = \big(\lambda(a,x_2)\big)'(0) = |a|^{2/3} \cdot \big(|x_2|^{2/3}\big)'(0),$$

but this is not the case*.

Hence the function (4) is differentiable only at those points both coordinates of which are different from zero, or equal to zero.

Proposition 2.4.2. *Suppose the numbers $\beta_j > 1$, $j = 1, \ldots, n$. Then the function*

$$\phi(x_1, \ldots, x_n) = \begin{cases} \sum\limits_{j=1}^{n} |x_j|^{\beta_j} & \textit{for all rational } x_j \\ 0 & \textit{at the remaining points} \end{cases} \tag{6}$$

is differentiable at the point $x^0 = (0, \ldots, 0)$,

$$d\phi(x^0) = 0 \tag{7}$$

and discontinuous at all the remaining points $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$.

Proof. We have

$$D_{\widehat{x}_k}\phi(x^0) = \lim_{\substack{x_k \to 0 \\ |x_j| \leq |x_k| \\ j \neq k}} \frac{\phi(x) - \phi(x(x_k^0))}{x_k} =$$

$$= \lim_{\substack{x_k \to 0 \\ |x_j| \leq |x_k| \\ j \neq k}} \frac{\sum\limits_{j=1}^{n} |x_j|^{\beta_j} - \sum\limits_{j \neq k} |x_j|^{\beta_j}}{x_k} = \lim_{x_k \to 0} \frac{|x_k|}{x_k} \cdot \frac{|x_k|^{\beta_k}}{|x_k|} = 0$$

and equality (7) follows from 2.3.(3).

The discontinuity of the function ϕ at every point $x \neq x^0$ follows from that there exist two sequences of points, which tend to x and the values of the function ϕ tend to zero along one of the sequences and do not tend to zero along the other sequence. □

*And what is more, $(|x_2|^{2/3})'(0+) = +\infty$ and $(|x_2|^{2/3})'(0-) = -\infty$.

Proposition 2.4.3. *The corresponding to the number $q>1$ function*

$$\Psi(x_1,\dots,x_n)=\begin{cases}\Big(\sum\limits_{j=1}^{n}|x_j|\Big)^q & \text{for all rational } x_j\\ 0 & \text{at the remaining points}\end{cases}, \tag{8}$$

possesses the same properties as the function (6).

Proof.

$$D_{\widehat{x}_k}\Psi(x^0)=\lim_{\substack{x_k\to 0\\ |x_j|\le |x_k|\\ j\ne k}}\frac{\big(\sum\limits_{j=1}^{n}|x_j|\big)^q-\big(\sum\limits_{j\ne k}|x_j|\big)^q}{x_k}.$$

But

$$\left|\frac{\big(\sum\limits_{j=1}^{n}|x_j|\big)^q-\big(\sum\limits_{j\ne k}|x_j|\big)^q}{x_k}\right|\le$$

$$\le\frac{2\big(\sum\limits_{j=1}^{n}|x_k|\big)^q}{|x_k|}=\frac{2\cdot(n|x_k|)^q}{|x_k|}\to 0,\quad x_k\to 0.$$

Hence

$$d\Psi(x^0)=0. \tag{9}$$

As regards the discontinuity, the function ψ is similar to the function ϕ. □

Proposition 2.4.4. *The corresponding to the number $\alpha>0$ function*

$$\omega(x_1,\dots,x_n)=\begin{cases}\Big(\sum\limits_{j=1}^{n}x_j^2\Big)^{\frac{1+\alpha}{2}} & \text{for all rational } x_j\\ 0 & \text{at the remaining points}\end{cases}, \tag{10}$$

possesses all properties of functions (6) *and* (8).

Proof. Again,

$$D_{\widehat{x}_k}\omega(x^0)=\lim_{\substack{x_k\to 0\\ |x_j|\le |x_k|\\ j\ne k}}\frac{\big(\sum\limits_{j=1}^{n}x_j^2\big)^{\frac{1+\alpha}{2}}-\big(\sum\limits_{j\ne k}x_j^2\big)^{\frac{1+\alpha}{2}}}{x_k}$$

and

$$\left|\frac{\big(\sum\limits_{j=1}^{n}x_j^2\big)^{\frac{1+\alpha}{2}}-\big(\sum\limits_{j\ne k}x_j^2\big)^{\frac{1+\alpha}{2}}}{x_k}\right|\le\frac{2\big(\sum\limits_{j=1}^{n}x_j^2\big)^{\frac{1+\alpha}{2}}}{|x_k|}\le\frac{2\big(\sum\limits_{j=1}^{n}x_k^2\big)^{\frac{1+\alpha}{2}}}{|x_k|}=$$

$$=\frac{2(nx_k^2)^{\frac{1+\alpha}{2}}}{|x_k|}=\frac{2n^{\frac{1+\alpha}{2}}|x_k|^{1+\alpha}}{|x_k|}=2n^{\frac{1+\alpha}{2}}\cdot|x_k|^{\alpha}\to 0,\quad x_k\to 0. \qquad □$$

Proposition 2.4.5. *The function*

$$g(x_1,x_2)=\begin{cases} x_1x_2\sin\dfrac{1}{x_1x_2} & \textit{for}\quad x_1\cdot x_2\neq 0\\ 0 & \textit{for}\quad x_1\cdot x_2=0\end{cases} \tag{11}$$

is differentiable at the point $x^0=(0,0)$*, and its gradient* $\operatorname{grad} g(x_1,x_2)$ *is indeterminate at the punctured neighborhood of the point* x^0*.*

Proof. We have

$$D_{\widehat{x}_1}g(x^0)=\lim_{\substack{x_1\to 0\\ |x_2|\le|x_1|}}\frac{g(x_1,x_2)-g(0,x_2)}{x_1}=\lim_{\substack{x_1\to 0\\ |x_2|\le|x_1|}}\frac{g(x_1,x_2)}{x_1}.$$

But

$$\Big|\frac{g(x_1,x_2)}{x_1}\Big|=\begin{cases}0 & \text{for}\ \ x_2=0\\ \Big|x_2\cdot\sin\dfrac{1}{x_1x_2}\Big|\le|x_2|\le|x_1| & \text{for}\ \ x_2\neq 0\end{cases}.$$

Therefore $D_{\widehat{x}_1}g(0,0)=0$. Similarly we find that $D_{\widehat{x}_2}g(0,0)=0$. Hence

$$dg(x^0)=0. \tag{12}$$

Next, at all points $(0,b)$ with $b\neq 0$ we have

$$g'_{x_1}g(0,b)=\lim_{x_1\to 0}\frac{g(x_1,b)-g(0,b)}{x_1}=b\cdot\lim_{x_1\to 0}\sin\frac{1}{x_1b}.$$

Therefore there non-exist neither $g'_{x_1}(0,b)$, nor $g'_{x_2}(a,0)$ for $a\neq 0$. Hence the $\operatorname{grad} g(x_1,x_2)$ is indeterminate in the neighborhood of the point x^0, and at this stage we cannot speak about the continuity of the function $\operatorname{grad} g(x_1,x_2)$ at the point $(0,0)$. $\square$

Proposition 2.4.5. *The function*

$$\psi(x_1,x_2)=\begin{cases}\dfrac{x_1^2\cdot x_2}{x_1^2+x_2^2} & \textit{for}\ \ x_1^2+x_2^2>0\\ 0 & \textit{for}\ \ x_1=0=x_2\end{cases} \tag{13}$$

possesses the following properties:

1) $\psi(x_1,x_2)$ *is continuous everywhere;*
2) $\operatorname{grad}\psi(x_1,x_2)$ *is finite everywhere;*
3) $\psi(x_1,x_2)$ *is not differentiable at the point* $(0,0)$*;*
4) $\operatorname{grad}\psi(x_1,x_2)$ *is not continuous at the point* $(0,0)$*.*

Proof. The continuity of the function $\psi(x_1,x_2)$ at the points $(x_1,x_2)\neq(0,0)$ is obvious.

For the function $\psi(x_1,x_2)$ to be continuous at the point $(0,0)$, it is necessary and sufficient that equalities 5.1.(1) and 5.1.(2) from Chapter I are fulfilled for $x_1^0=0$ and $x_2^0=0$, respectively. We have

$$\big|\psi(x_1,x_2)-\psi(0,x_2)\big|=\big|\psi(x_1,x_2)-\psi(x_1,0)\big|=\big|\psi(x_1,x_2)\big|=$$
$$=\frac{x_1^2\cdot|x_2|}{x_1^2+x_2^2}<\frac{x_1^2\cdot|x_2|}{x_1^2}=|x_2|\to 0,\ \ (x_1,x_2)\to(0,0).$$

Thus the function $\psi(x_1,x_2)$ is continuous at the point $(0,0)$ as well.

The finiteness of the $\operatorname{grad}\psi(x_1,x_2)$ at all points $(x_1,x_2)\neq(0,0)$ is obvious, while for the point $(0,0)$ we have $\psi'_{x_1}(0,0)=(\psi(x_1,0))'(0)=0=\psi'_{x_2}(0,0)$. Therefore the $\operatorname{grad}\psi(x_1,x_2)$ is finite everywhere.

The non-differentiability of the function $\psi(x_1,x_2)$ at the point $x^0=(0,0)$ follows from the nonexistence, for e.g., of $D_{\widehat{x}_1}\psi(0,0)$. Indeed, the ratio appearing in equality 2.3.(1) for $k=1$ has the form

$$\frac{x_1^2\cdot x_2}{x_1(x_1^2+x_2^2)}=\frac{x_1\cdot x_2}{x_1^2+x_2^2}. \tag{14}$$

Let us take arbitrary positive number $\ell<1$ and put in equality (14) $x_2=\ell x_1$. Then $|x_2|<|x_1|$, and the obtained for that case ratio is equal to $\frac{\ell}{1+\ell^2}$. This means that $D_{\widehat{x}_1}\psi(0,0)$ and hence $d\psi(x^0)$ do not exist.

Finally, were the $\operatorname{grad}\psi(x_1,x_2)$ continuous at the point x^0, $d\psi(x^0)$ would exist, but this is not the case. □

Remark 2.4.2. The fact that the finite limit

$$\lim_{\substack{x_k\to x_k^0\\ |x_j-x_j^0|=|x_k-x_k^0|\\ j\neq k}}\frac{f(x)-f(x(x_k^0))}{x_k-x_k^0} \tag{15}$$

does not exist at least for one value k from $1,\dots,n$, is the sufficient condition for the function $f(x_1,\dots,x_n)$ is non-differentiable at the point $(x_1^0,\dots,x_n^0)$.

2.5. The Necessary and Sufficient Conditions for the Differentiability of Functions of Two Variables

As far as real functions of two real variables are tightly connected with analytic functions of a complex variable, to simplify our investigation it is more convenient to formulate separately the results which correspond to the case $n=2$.

The differentiability of the function $\varphi(x_1,x_2)$ at the point $x^0=(x_1^0,x_2^0)$ implies the existence of $\varphi'_{x_1}(x^0)$ and $\varphi'_{x_2}(x^0)$ and the fulfilment of the equality

$$\lim_{\substack{x_1\to x_1^0\\ x_2\to x_2^0}}\frac{\varphi(x_1,x_2)-\varphi(x^0)-(x_1-x_1^0)\varphi'_{x_1}(x^0)-(x_2-x_2^0)\varphi'_{x_2}(x^0)}{|x_1-x_1^0|+|x_2-x_2^0|}=0. \tag{1}$$

The existence for the function $\varphi(x_1,x_2)$ of the angular partial derivative $\varphi'_{\widehat{x}_1}(x^0)$ at the point x^0, means that for every constant $c_2>0$ there exists a finite, or an infinite (of fixed sign) limit

$$\varphi'_{\widehat{x}_1}(x^0) = \lim_{\substack{x_1 \to x_1^0 \\ |x_2 - x_2^0| \le c_2 |x_1 - x_1^0|}} \frac{\varphi(x_1, x_2) - \varphi(x_1^0, x_2)}{x_1 - x_1^0}, \tag{2}$$

which does not depend on c_2.

Analogously, the existence of $\varphi'_{\widehat{x}_2}(x^0)$ means that of a finite, or an infinite limit

$$\varphi'_{\widehat{x}_2}(x^0) = \lim_{\substack{x_2 \to x_2^0 \\ |x_1 - x_1^0| \le c_1 |x_2 - x_2^0|}} \frac{\varphi(x_1, x_2) - \varphi(x_1, x_2^0)}{x_2 - x_2^0}, \tag{3}$$

for an arbitrary constant $c_1 > 0$ such that this limit would be independent from c_1.

Moreover (see equality 2.1.(5)),

$$\operatorname{ang\,grad} \varphi(x^0) = \big(\varphi'_{\widehat{x}_1}(x^0), \varphi'_{\widehat{x}_2}(x^0)\big). \tag{4}$$

1. The first basic Theorem 2.2.1 results in

Theorem 2.5.1 ([2]). *For the function $\varphi(x_1, x_2)$ to be differentiable at the point $x^0 = (x_1^0, x_2^0)$, it is necessary and sufficient that angular partial derivatives $\varphi'_{\widehat{x}_1}(x^0)$ and $\varphi'_{\widehat{x}_2}(x^0)$, or what is the same,* $\operatorname{ang\,grad} \varphi(x^0)$, *be finite. The finiteness of the* $\operatorname{ang\,grad} \varphi(x^0)$ *is equivalent to the existence of the equality*

$$d\varphi(x^0) = \varphi'_{\widehat{x}_1}(x^0)\, dx_1 + \varphi'_{\widehat{x}_2}(x^0)\, dx_2. \tag{5}$$

Specific character of a plane set allows one to prove the following

Theorem 2.5.2 ([2]). *For the function $\varphi(x_1, x_2)$ to be differentiable at the point $x^0 = (x_1^0, x_2^0)$, it is necessary and sufficient that the limits*

$$\lim_{\substack{x_1 \to x_1^0 \\ |x_2 \to x_2^0| \le c|x_1 - x_2^0|}} \frac{\varphi(x_1, x_2) - \varphi(x_1^0, x_2)}{x_1 - x_1^0} \tag{6}$$

and

$$\lim_{\substack{x_2 \to x_2^0 \\ |x_2 - x_2^0| \ge c|x_1 - x_1^0|}} \frac{\varphi(x_1, x_2) - \varphi(x_1, x_2^0)}{x_2 - x_2^0}. \tag{7}$$

be finite for some one constant $c > 0$. If these limits are finite, they are equal to $\varphi'_{\widehat{x}_1}(x^0)$ and $\varphi'_{\widehat{x}_2}(x^0)$, respectively.

For the particular case $x_1^0 = 0 = x_2^0$, Theorem 2.5.2 can be interpreted from geometrical viewpoint. When calculating limit (6), the point (x_1, x_2) always belongs to the union of mutually vertical angles, containing the Ox_1-axis,

$$\left.\begin{matrix} x_1 \ge 0 \\ -cx_1 \le x_2 \le cx_1 \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} x_1 \le 0 \\ cx_1 \le x_2 \le -cx_1 \end{matrix}\right. .$$

At the same time, the point $(0,x_2)$ is outside of that union, on the Ox_2-axis.

When calculating limit (7), the point $x^0=(x_1^0,x_2^0)$ belongs to the union of angles

$$\left.\begin{array}{r} x_2\ge 0\\ -\dfrac{1}{c}\,x_2\le x_1\le \dfrac{1}{c}\,x_2\end{array}\right\}\quad\text{and}\quad\left\{\begin{array}{l} x_2\le 0\\ \dfrac{1}{c}\,x_2\le x_1\le -\dfrac{1}{c}\,x_2\end{array}\right.,$$

while the point $(x_1,0)$ lies on the Ox_1-axis.

By Theorem 2.3.1, for the function $\varphi(x_1,x_2)$ to be differentiable at the point $x^0=(x_1^0,x_2^0)$, it is necessary and sufficient that limits

1) (2) and (3) for $c_2=1=c_1$,

or

2) (6) and (8) for $c=1$

be finite.

Thus we have the following

Theorem 2.5.3 ([5]). *For the function $\varphi(x_1,x_2)$ to be differentiable at the point $x^0=(x_1^0,x_2^0)$, it is necessary and sufficient that the quantities*

$$D_{\widehat{x}_1}(x^0)=\lim_{\substack{x_1\to x_1^0\\ |x_2-x_2^0|\le|x_1-x_1^0|}}\frac{\varphi(x_1,x_2)-\varphi(x_1^0,x_2)}{x_1-x_1^0}\tag{8}$$

and

$$D_{\widehat{x}_2}(x^0)=\lim_{\substack{x_2\to x_2^0\\ |x_1-x_1^0|\le|x_2-x_2^0|}}\frac{\varphi(x_1,x_2)-\varphi(x_1,x_2^0)}{x_2-x_2^0}\tag{9}$$

be finite.

If these limits are finite, we have the following equalities:

$$\varphi_{\widehat{x}_1}(x^0)=D_{\widehat{x}_1}\varphi(x^0),\quad \varphi_{\widehat{x}_2}(x^0)=D_{\widehat{x}_2}\varphi(x^0),\tag{10}$$

$$d\varphi(x^0)=D_{\widehat{x}_1}\varphi(x^0)dx_1+D_{\widehat{x}_2}\varphi(x^0)\,dx_2.\tag{11}$$

2. Here we present one somewhat different necessary and sufficient condition for the differentiability of a function of two variables, when beforehand are known finiteness its partial derivatives.

Theorem 2.5.4 ([30], p. 139). *If the* $\operatorname{grad}\varphi(x^0)$ *is finite, then for the differentiability of the function $\varphi(x_1,x_2)$ at the point $x^0=(x_1^0,x_2^0)$ it is necessary and sufficient that the equality*

$$\lim_{\substack{x_1\to x_1^0\\ x_2-x_2^0}}\frac{\Delta^2_{[x^0]}\varphi(x_1,x_2)}{|x_1-x_1^0|+|x_2-x_2^0|}=0\tag{12}$$

be fulfilled, where

$$\Delta^2_{[x^0]}\varphi(x_1,x_2)=\varphi(x_1,x_2)-\varphi(x_1^0,x_2)-\varphi(x_1,x_2^0)+\varphi(x_1^0,x_2^0).\tag{13}$$

Proof. We have

$$\big[\varphi(x_1,x_2)-\varphi(x^0)-(x_1-x_1^0)\varphi'_{x_1}(x^0)-(x_2-x_2^0)\varphi'_{x_2}(x^0)\big]-$$
$$-\Delta^2_{[x^0]}\varphi(x_1,x_2)=\big[\varphi(x_1,x_2^0)-\varphi(x^0)-(x_1-x_1^0)\varphi'_{x_1}(x^0)\big]+$$
$$+\big[\varphi(x_1^0,x_2)-\varphi(x^0)-(x_2-x_2^0)\varphi'_{x_2}(x^0)\big]. \tag{14}$$

Since partial derivatives $\varphi'_{x_1}(x^0)$ and $\varphi'_{x_2}(x^0)$ are finite, for arbitrary $\varepsilon>0$ there exists $\delta=\delta(\varepsilon,x^0,\varphi)>0$ such that the absolute value in the right-hand side of equality (14) is less than $\varepsilon(|x_1-x_1^0)|+|x_2-x_2^0|)$ under $|x_1-x_1^0|<\delta$ and $|x_2-x_2^0|<\delta$. Thus we have

$$\Bigg|\frac{\varphi(x_1,x_2)-\varphi(x^0)-(x_1-x_1^0)\varphi'_{x_1}(x^0)-(x_2-x_2^0)\varphi'_{x_2}(x^0)}{|x_1-x_1^0|+|x_2-x_2^0|}-$$
$$-\frac{\Delta^2_{[x^0]}\varphi(x_1,x_2)}{|x_1-x_1^0|+|x_2-x_2^0|}\Bigg|<\varepsilon,\quad |x_1-x_1^0|<\delta,\quad |x_2-x_2^0|<\delta.$$

It is clear that equalities (1) and (12) are, or are not fulfilled together. □

Remark 2.5.1. Below we will give somewhat different sufficient conditions for the existence of a total differential for functions of two variables (see Theorem 2.2.1 and statement (2) of Theorem 2.2.3 in Chapter III).

§ 3. Finiteness of a Strong Gradient Implies Differentiability

Here we introduce the notion of a strong gradient and state that the finiteness of the strong gradient implies differentiability, and not vice versa. It is also established that the continuity of the gradient implies the existence of a finite strong gradient, and not vice versa.

3.1. A Strong Partial Derivative and a Strong Gradient

For the finite function $f(x)$, $x=(x_1,\dots,x_n)$ defined in the neighborhood $U(x^0)$ of the point $x^0=(x_1^0,\dots,x_n^0)\in\mathbb{R}^n$ we introduce the following

Definition 3.1.1 ([2], [5]). We say that the function $f(x)$ possesses at the point x^0 a strong partial derivative with respect to the variable x_k, symbolically $f'_{[x_k]}(x^0)$, if there exists a finite, or an infinite (of fixed sign) limit

$$f'_{[x_k]}(x^0)=\lim_{x\to x^0}=\frac{\Delta_{[x_k^0]}f(x)}{x_k-x_k^0}, \tag{1}$$

where (see I, equality 2.1.(1))

$$\Delta_{[x_k^0]}f(x)=f(x)-f\big(x(x_k^0)\big). \tag{2}$$

If $f'_{[x_k]}(x^0)$ is finite, equality (1) means that for every $\varepsilon>0$ there exists the number $\delta_k=\delta_k(x^0,\varepsilon,f)>0$ with the property

$$\big|\Delta_{[x_k^0]}f(x)-(x_k-x_k^0)f'_{[x_k]}(x^0)\big|<\varepsilon|x_k-x_k^0| \tag{3}$$

for all $x \in U^0(x^0, \delta_k)$.

The existence of the finite $f'_{[x_k]}(x^0)$ implies, obviously, that with respect to the variable x_k the function f is strongly partial continuous at the point x^0 (see I, equality 2.1.(2)).

Proposition 3.1.1 ([2], [5]). *If the continuous in the neighborhood $U(x^0)$ of the point x^0 function f has in the punctured neighborhood $U^0(x^0)$ a finite partial derivative $f'_{x_k}(x)$ for which there exists at x^0 the limit*

$$\lim_{x \to x^0} f'_{x_k}(x) \tag{4}$$

finite, or infinite, then the equality

$$f'_{[x_k]}(x^0) = \lim_{x \to x^0} f'_{x_k}(x) \tag{5}$$

holds. However, if limit (4) *is finite, then the partial derivative $f'_{x_k}(x)$ is continuous at the point x^0, and there takes place the equality*

$$f'_{[x_k]}(x^0) = f'_{x_k}(x^0). \tag{6}$$

The proof of the above proposition is contained in the proof of Theorem 3.2.1, below.

Note that one can consider the derivative of a function of one variable in terms of a strong partial derivative with respect to the same variable of the same function, but interpreting it as dependent formally of several variables.

Definition 3.1.2 ([2], [5]). We say that the function f has at the point x^0 a strong gradient, symbolically $\operatorname{str\,grad} f(x^0)$, if for every $k = 1, \dots, n$ there exist finite or infinite $f'_{[x_k]}(x^0)$, and we write

$$\operatorname{str\,grad} f(x^0) = \big(f'_{[x_1]}(x^0), \dots, f'_{[x_n]}(x^0)\big). \tag{7}$$

The following proposition is obvious.

Proposition 3.1.2 ([2], [5]). *If there exists a* $\operatorname{str\,grad} f(x^0)$*, then there likewise exists* $\operatorname{ang\,grad} f(x^0)$*, and the equalities*

$$\operatorname{str\,grad} f(x^0) = \operatorname{ang\,grad} f(x^0) = \operatorname{grad} f(x^0) \tag{8}$$

hold.

Definition 3.1.3. The function f is called strongly partial differentiable at the point x^0 with respect to the variable x_k, if there exists finite $f'_{[x_k]}(x^0)$, and in this case we write

$$d_{[x_k]} f(x^0) = f'_{[x_k]}(x^0)\, dx_k. \tag{9}$$

Moreover, $d_{[x_k]}(x^0)$ is called a strong partial differential with respect to the variable x_k of the function f at the point x^0.

Definition 3.1.4. The function $f(x)$ possesses at the point x^0 the property of separately strong partial differentiability, if there all $f'_{[x_k]}(x^0)$ exist and are finite, i.e., if there exists a finite $\operatorname{str}\operatorname{grad} f(x^0)$.

3.2. The Continuity of a Gradient Implies the Existence of a Finite Strong Gradient

Theorem 3.2.1 ([2], [5]). *If in the neighborhood $U(x^0)$ of the point x^0 the function $f(x)$ is continuous and in the punctured neighborhood $U^0(x^0)$ it has a finite* $\operatorname{grad} f(x)$*, for which there exists a finite, or an infinite limit*

$$\lim_{x\to x^0} \operatorname{grad} f(x), \tag{1}$$

then the equality

$$\operatorname{str}\operatorname{grad} f(x^0) = \lim_{x\to x^0} \operatorname{grad} f(x) \tag{2}$$

holds.

However, if limit (1) *is finite, then the* $\operatorname{grad} f(x)$ *is continuous at the point x^0, and we have the equality*

$$\operatorname{str}\operatorname{grad} f(x^0) = \operatorname{grad} f(x^0). \tag{3}$$

Moreover, the existence of the finite $\operatorname{str}\operatorname{grad} f(x^0)$ *does not, in general, imply the continuity of* $\operatorname{grad} f(x)$ *at the point x^0.*

proof. Equality (2) yields

$$\Delta_{[x_k^0]} f(x) =$$
$$= (x_k - x_k^0) f'_{x_k}(x_1,\dots,x_{k-1},x_k^0+\theta_k(x_k-x_k^0),x_{k+1},\dots,x_n),\ \ 0<\theta_k<1.$$

The finiteness of limit (1) implies both the continuity of $\operatorname{grad} f(x)$ at the point x^0 and equality (3) due to the fact that the partial derivative, possessing the finite limit at some point, is continuous at the same point.

That the converse to the concluding statement of the above theorem is invalid can be illustrated by an example of the function $g(x_1,x_2)$ defined by equality 2.4.(11). It is already known that the function $g(x_1,x_2)$ is differentiable at the point $(0,0)$ and its gradient is not continuous at $(0,0)$.

Let us now prove that the strong gradient of the function $g(x_1,x_2)$ at the point $(0,0)$ is finite. By the definition, we have

$$g'_{[x_1]}(0,0) = \lim_{(x_1,x_2)\to(0,0)} \frac{g(x_1,x_2)-g(0,x_2)}{x_1} =$$
$$= \lim_{(x_1,x_2)\to(0,0)} x_2 \sin\frac{1}{x_1x_2} = 0.$$

In a similar way we obtain the equality $g'_{[x_2]}(0,0)=0$. Hence

$$\operatorname{str}\operatorname{grad} g(0,0) = (0,0). \qquad \square \tag{4}$$

3.3. Relation Between the Continuity of a Gradient of a Function and the Finiteness of Its Strong Gradient

1. For a function of one variable we can indicate two properties which are equivalent almost everywhere, though one of the properties is stronger at an individual point.

Such properties are, for example:

1) continuity and symmetric continuity ([22], p. 266);

2) derivability and symmetric derivability ([13], p. 381; [22], p. 249);

3) derivability and existence of a finite upper derivate (see [21], pp. 270 and 108).

Tolstov's Theorem shows that the continuity of a function of two variables and its continuity with respect to each of variables are the properties of that function which may be nonequivalent almost everywhere (see Theorem A in introduction of Chapter I).

2. In proving the concluding part of Theorem 3.2.1 we have stated that the function $g(x_1, x_2)$ defined by equality 2.4.(11) has the finite strong gradient at the point $x^0 = (0,0)$, and moreover, its gradient is not continuous at the point x^0.

Here we prove that the just mentioned nonequivalence can be realized almost everywhere.

Theorem* 3.3.1. *There exists an absolutely continuous function of two variables which has almost everywhere both finite strong and discontinuous gradients.*

Proof. For bounded and everywhere on the $[0,1]$ discontinuous functions $\alpha(x)$ and $\beta(y)$ we consider the corresponding indefinite L-integrals

$$A(x) = \int_0^x \alpha(x)\,dt \quad \text{and} \quad B(y) = \int_0^y \beta(\tau)\,d\tau.$$

The function of two variables $\nu(x,y) = A(x) + B(y)$ on the unit square $Q = [0,1] \times [0,1]$ is absolutely continuous (see Definition 2.1.1 in Chapter IV) and possesses a total differential at almost all points $(x,y) \in Q$ (see Corollary 1.3.1),

$$d\nu(x,y) = \alpha(x)\,dx + \beta(y)\,dy.$$

Since the derivative of the function of one variable is its strong partial derivative with respect to the same variable (see 3.1), then at the points $(x,y) \in Q$, at which the total differential $d\nu(x,y)$ exists, the finite is

$$\operatorname{str}\operatorname{grad}\nu(x,y) = \big(\alpha(x), \beta(x)\big).$$

On the other hand, the $\operatorname{grad}\nu(x,y) = (\alpha(x), \beta(y))$ is discontinuous almost everywhere. Hence the $\operatorname{str}\operatorname{grad}\nu(x,y)$ is finite almost everywhere and the $\operatorname{grad}\nu(x,y)$ is discontinuous almost everywhere. □

*The author's this result is published for the first time.

3.4. The Finitenes of a Strong Gradient Implies Differentiability

Theorem 3.4.1 ([2], [5]). *The existence of a finite* str grad $f(x^0)$ *implies existence of a total differential* $df(x^0)$ *and*

$$\operatorname{str\,grad} f(x^0) = \operatorname{ang\,grad} f(x^0) = \operatorname{grad} f(x^0). \tag{1}$$

The First Proof. In the neighborhood of the point x^0 the expression

$$f(x) - f(x^0) - \sum_{k=1}^{n} (x_k - x_k^0) f'_{[x_k]}(x^0) \tag{2}$$

is finite. It can be represented as

$$\begin{aligned} &\big[f(x) - f\big(x(x_1^0)\big) - (x_1 - x_1^0) f'_{[x_1]}(x^0)\big] + \\ &+\big[f\big(x(x_1^0)\big) - f\big(x(x_1^0, x_2^0) - (x_2 - x_2^0)\big) f'_{[x_2]}(x^0)\big] + \cdots + \\ &+\big[f\big(x^0(x_n)\big) - f(x^0) - (x_n - x_n^0) f'_{[x_n]}(x^0)\big]. \end{aligned} \tag{3}$$

In (3), for every square bracket we make use of estimate 3.1.(3). Note that for the values $k = 2, \dots, n$ we put in 3.1.(3) partial values $x_j = x_j^0$ for $j = 1, \dots, k-1$.

It is clear that the absolute value of (2) is less than

$$\varepsilon\big(|x_1 - x_1^0| + |x_2 - x_2^0| + \cdots + |x_n - x_n^0|\big) = \varepsilon\|x - x^0\|.$$

Hence $df(x^0)$ exists.

The Second Proof is obtained by virtue of Theorem 2.2.1 with regard of Proposition 3.1.2. □

Remark 3.4.1. Classical result that the continuity of a gradient implies the existence of a total differential can be obtained from Theorems 3.2.1 and 3.4.1.

Proposition 3.4.1 ([2], [5]). *The finiteness of* ang grad $f(x^0)$*, or what is the same, the existence of* $df(x^0)$ *does not imply the existence of (neither finite, nor infinite)* str grad $f(x^0)$*.*

Proof. As is known, the function

$$\lambda(x_1, x_2) = |x_1 \cdot x_2|^{2/3} \tag{4}$$

defined by equality 2.4.(4), is differentiable at the point $x^0 = (0, 0)$.

Let us now show that the str grad $\lambda(x^0)$ does not exist. Indeed, for a particular case $x_1 > 0$ we have the expression (here $x_1^0 = 0 = x_2^0$)

$$\frac{\Delta_{[x_1^0]}\lambda(x_1, x_2)}{x_1} = \frac{(x_1|x_2|)^{2/3}}{x_1} = \Big(\frac{x_2^2}{x_1}\Big)^{1/3},$$

which has no limit as $(x_1, x_2) \to (0, 0)$. This follows from the fact that the last expression tends to different numbers for $x_2^2 = x_1$ (to 1) and for $x_2 = x_1$ (to 0). Hence $\lambda'_{[x_1]}(x^0)$ does not exist.

Consequently, the str grad $\lambda(x^0)$ does not exist. □

Remark 3.4.2. The gradient of the function $\lambda(x_1, x_2) = |x_1 \cdot x_2|^{2/3}$ is not continuous at the point $x^0 = (0,0)$. This follows from Theorem 3.2.1 with regard of the fact that the $\operatorname{str}\operatorname{grad}\lambda(x^0)$ does not exist.

Remark 3.4.3. In [16] has been announced the following result: for every $n \geq 2$ there exists a continuous function $f : \mathbb{R}^n \to R$ which is almost everywhere differentiable, but has no almost everywhere a finite strong gradient.

3.5. The Sufficient Condition for Differentiability of a Function, when It Is Differentiable with Respect to a Subcollection of Variables

As is already known, the function $\lambda(x_1, x_2)$ defined by equality 3.4.(4) is differentiable at the point $(0,0)$, but it has no strong partial derivatives at that point. This fact indicates that equality 2.2.(1) does not imply the existence of $f'_{[x_k]}(x^0)$, when $df(x^0)$ does exist.

But despite this fact, equality 2.2.(1) allows nevertheless us to find for one and the same function the connection between its differentiability both for n and for $n-1$ variables at a given point.

The theorem below follows directly from equality 2.2.(1) and Definition 1.4.1.

Theorem 3.5.1 ([5]). *For the function $f(x_1, \dots, x_n)$ to be differentiable at the point $x^0 = (x_1^0, \dots, x_n^0)$, it is sufficient that the function $f(x)$ at the point x^0 have a finite strong partial derivative with respect to some one variable x_k and at the point x^0 differentiable is function*

$$f(x(x_k^0)) = f(x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_n), \tag{1}$$

depending on the remaining $n-1$ variables.

Obviously, we can apply Theorem 3.5.1 to the function $f(x(x_k^0))$ from the same theorem and then continue the procedure until we get a function of one variable.

Therefore the following theorem is valid if we take into account that a derivative of function of one variable can be interpreted as its strong partial derivative with respect to the same variable (see 3.1).

Theorem 3.5.2. *For the function $f(x_1, \dots, x_n)$ to be differentiable at the point $x^0 = (x_1^0, \dots, x_n^0)$, it is sufficient that the following conditions be fulfilled:*

(i_k) *$f(x)$ has at the point x^0 a finite strong partial derivative with respect to one, say x_k, variable;*

($\mathrm{i}_{k\ell}$) *$f(x(x_k^0))$ has at the point x^0 a finite strong partial derivative with respect to the other variable, say with respect to x_ℓ, $\ell \neq k$;*

($\mathrm{i}_{k\ell s}$) *$f(x(x_k^0, x_\ell^0))$ has at the point x^0 a finite strong partial derivative with respect to the variable x_s, different from x_k and x_ℓ;*

And so on, the obtained in such a way function of one variable has at the point x^0 a finite derivative.

It should be noted that the existence at the point x^0 of a finite strong partial derivative with respect to variable x_i for the function $f(x(x_k^0))$, $i \neq k$, is more weak property of the function $f(x)$, than the existence of a finite strong partial derivative for the function $f(x)$ at the point x^0 with respect to the same variable x_i.

Thus we have the following

Theorem 3.5.3 ([5]). *Assume the function $f(x_1, \dots, x_n)$ has at the point $x^0 = (x_1^0, \dots, x_n^0)$ a finite partial derivative with respect to some one variable and at the point x^0 it has finite strong partial derivatives with respect to each of the remaining $n-1$ variables. Then the function $f(x)$:*

(a) *is differentiable at the point x^0;*

(b) *has at the point x^0 a finite angular partial derivative with respect to the variable we have just spoken at the beginning of our theorem.*

Proof. Statement (a) follows from Theorem 3.5.2, and statement (b) follows from Theorem 2.2.1 with regard of statement (a). □

Since the continuity of the partial derivative implies the existence of the finite strong partial derivative with respect to the same variable (see Proposition 3.1.1), form Theorem 3.5.3 we obtain

Theorem 3.5.4. *If from the partial derivatives f'_{x_j}, $j = 1, \dots, n$, some one is finite at the point x^0 and the remaining partial derivatives are continuous at x^0 functions, then statements* (a) *and* (b) *of Theorem* 3.5.3 *hold.*

3.6. The Sufficient Conditions for Differentiability of Functions of Two Variables

For the function of two variables $\varphi(x)$, $x = (x_1, x_2)$ defined in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$ we can determine strong partial derivatives with respect to variables x_1 and x_2 by using respectively the following equalities:

$$\varphi'_{[x_1]}(x^0) = \lim_{\substack{x_1 \to x_1^0 \\ x_2 \to x_2^0}} \frac{\varphi(x_1, x_2) - \varphi(x_1^0, x_2)}{x_1 - x_1^0} \tag{1}$$

and

$$\varphi'_{[x_2]}(x^0) = \lim_{\substack{x_1 \to x_1^0 \\ x_2 \to x_2^0}} \frac{\varphi(x_1, x_2) - \varphi(x_1, x_2^0)}{x_2 - x_2^0}. \tag{2}$$

Moreover (see equality 3.1.(7)),

$$\operatorname{str\,grad} \varphi(x^0) = (\varphi'_{[x_1]}(x^0), \varphi'_{[x_2]}(x^0)). \tag{3}$$

For functions of two variables, both Theorems 3.5.2 and 3.5.3 are identical. We formulate them in the form of one

Theorem 3.6.1 ([2], [5]). *Let the function of two variables $\varphi(x)$, $x=(x_1,x_2)$ have at the point $x^0=(x_1^0,x_2^0)$ a finite partial derivative with respect to one of the variables and at the point x^0 a finite strong partial derivative with respect to the other variable. Then the function $\varphi(x)$ is differentiable at the point x^0, and therefore $\varphi(x)$ has at the point x^0 finite an angular partial derivative with respect to that variable we have just spoken at the beginning of this theorem.*

From the above theorem we immediately arrive at

Theorem 3.6.2. *Let the function $\varphi(x_1,x_2)$ be separately partial differentiable at the point $x^0=(x_1^0,x_2^0)$. Then for the existence of a total differential $d\varphi(x^0)$ it is sufficient that $\varphi(x_1,x_2)$ possess at the point x^0 a finite strong partial derivative with respect to one of the variables.*

For functions of two variables Theorem 3.5.4 can be rewritten in the form of

Theorem 3.6.3. *If the function $\varphi(x_1,x_2)$ has partial derivatives, one of which is finite at the point $x^0=(x_1^0,x_2)$ and the other is continuous at the point x^0, then the following statements take place:*
1) *there exists $d\varphi(x^0)$;*
2) *$\varphi(x_1,x_2)$ has at the point x^0 a finite angular partial derivative with respect to the same variable, mentioned at the begining of to theorem.*

It should be noted that statement 1) of Theorem 3.6.3 is due to K. J. Thomae ([25]; [11], § 310). We formulate this theorem as follows.

Theorem 3.6.4 ([25]). *If for the function $\varphi(x_1,x_2)$ one of partial derivatives is finite and the other is continuous at the point $x^0=(x_1^0,x_2^0)$, then $\varphi(x_1,x_2)$ is differentiable at x^0.*

3.7. Classification of Functions by Various Gradients

The obtained in this section results on the differentiability of functions of several variables allow us to formulate the following summarizing theorem.

Theorem 3.7.1 ([5]). *A class with continuous at the point x^0 gradients of functions is contained strictly in a class with finite at the point x^0 strong gradients of functions, and the latter is contained strictly in a class of functions with finite at the point x^0 angular gradients. This class coincides with the class of differentiable at x^0 functions.*

§ 4. Unilateral in Various Senses Partial Derivatives and Differentials of Functions of Two Variables

For the function $\psi(x)$, $x=(x_1,x_2)$ defined in the neighborhood $U(x^0)$ of the point $x^0=(x_1^0,x_2^0)$ we introduce the functions (see equality 1.1.(1)) ${}^1\psi(x_1)=\psi(x_1,x_2^0)$ and ${}^2\psi(x_2)=\psi(x_1^0,x_2)$.

If the function ${}^i\psi(x_i)$ has at the point x_i^0 a derivative $({}^i\psi(x_i))'(x_i^0)$ called a partial derivative at the point x^0 of the function $\psi(x)$ with respect to the variable x_i, then we denote it now by $\partial_{x_i}\psi(x^0)$. If there exist $\partial_{x_1}\psi(x^0)$ and $\partial_{x_2}\psi(x^0)$, then we consider the gradient of the function $\psi(x)$ at the point x^0 (see equality 1.1.(2))

$$\operatorname{grad}\psi(x^0) = \big(\partial_{x_1}\psi(x^0), \partial_{x_2}\psi(x^0)\big).$$

It is quite possible that the function ${}^i\psi(x_i)$ has no derivative at the point x_i^0, i.e., there is no $\partial x_i\psi(x^0)$, but ${}^i\psi(x_i)$ has ${}^+$derivative at x_i^0, symbolically $\partial_{x_i}^+\psi(x^0)$, which is called the right-hand partial derivative of the function $\psi(x)$ at the point x^0 with respect to the variable x_i. Hence

$$\partial_{x_i}^+\psi(x^0) = \lim_{x_i\to x_i^0+}\frac{{}^i\psi(x_i) - {}^i\psi(x_i^0)}{x_i - x_i^0} = \lim_{x_i\to x_i^0}\frac{{}^i\psi(x_i) - \psi(x^0)}{x_i - x_i^0}.$$

The left-hand partial derivative of the function $\psi(x)$ at the point x^0 with respect to the variable x_i,

$$\partial_{x_i}^-\psi(x^0) = \lim_{x_i\to x_i^0-}\frac{{}^i\psi(x_i) - {}^i\psi(x_i^0)}{x_i - x_i^0} = \lim_{x_i\to x_i^0}\frac{{}^i\psi(x_i) - \psi(x^0)}{x_i - x_i^0}$$

is defined analogously.

It is obvious that for the existence of $\partial_{x_i}\psi(x^0)$ the necessary and sufficient condition is the existence of equal quantities $\partial_{x_i}^+\psi(x^0)$ and $\partial_{x_i}^-\psi(x^0)$, $i = 1, 2$.

In case quantities $\partial_{x_1}^+\psi(x^0)$ and $\partial_{x_2}^+\psi(x^0)$ exist we introduce ${}^+$gradient of the function $\psi(x)$ at the point x^0,

$${}^+\operatorname{grad}\psi(x^0) = \big(\partial_{x_1}^+\psi(x^0),\ \partial_{x_2}^+\psi(x^0)\big).$$

Analogously, if $\partial_{x_1}^-\psi(x^0)$ and $\partial_{x_2}^-\psi(x^0)$ exist, then we introduce ${}^-$gradient of the function $\psi(x)$ at the point x^0 by the equality

$${}^-\operatorname{grad}\psi(x^0) = \big(\partial_{x_1}^-\psi(x^0),\ \partial_{x_2}^-\psi(x^0)\big).$$

For the relations

$${}^-\operatorname{grad}\psi(x^0) = \operatorname{grad}\psi(x^0) = {}^+\operatorname{grad}\psi(x^0)$$

to be valid, it is necessary and sufficient that all components be equal,

$$\partial_{x_i}^-\psi(x^0) = \partial_{x_i}\psi(x^0) = \partial_{x_i}^+\psi(x^0), \quad i = 1, 2.$$

The above equalities are not, in general, sufficient for the existence of an angular, or a strong gradient.

Below we will introduce unilateral strong and angular partial ${}^\pm$derivatives and prove the necessary and sufficient conditions for the existence of a strong and an angular gradient. The conditions for the existence of an angular gradient will at the same time be the conditions for the existence of a total differential.

4.1. Unilateral Strong Partial Derivatives

The notion of strong partial derivatives (see 3.1) with respect to the variables x_1 and x_2 at the point $x^0=(x_1^0,x_2^0)$ for the function $\psi(x)$, $x=(x_1,x_2)$ makes it possible to introduce strong partial $^{\pm}$derivatives with respect to x_1 and x_2 at the point x^0 for $\psi(x)$:

$$\partial^{+}_{[x_1]}\psi(x^0)=\lim_{\substack{(h_1,h_2)\to(0,0)\\ h_1>0}}\frac{\psi(x_1^0+h_1,x_2^0+h_2)-\psi(x_1^0,x_2^0+h_2)}{h_1}\,, \tag{1}$$

$$\partial^{-}_{[x_1]}\psi(x^0)=\lim_{\substack{(h_1,h_2)\to(0,0)\\ h_1<0}}\frac{\psi(x_1^0+h_1,x_2^0+h_2)-\psi(x_1^0,x_2^0+h_2)}{h_1}\,, \tag{2}$$

$$\partial^{+}_{[x_2]}\psi(x^0)=\lim_{\substack{(h_1,h_2)\to(0,0)\\ h_2>0}}\frac{\psi(x_1^0+h_1,x_2^0+h_2)-\psi(x_1^0+h_1,x_2^0)}{h_2}\,, \tag{3}$$

$$\partial^{-}_{[x_2]}\psi(x^0)=\lim_{\substack{(h_1,h_2)\to(0,0)\\ h_2<0}}\frac{\psi(x_1^0+h_1,x_2^0+h_2)-\psi(x_1^0+h_1,x_2^0)}{h_2}\,. \tag{4}$$

It is clear that for the existence of $\partial_{[x_i]}\psi(x^0)$ it is necessary and sufficient that there exist equal quantities $\partial^{+}_{[x_i]}\psi(x^0)$ and $\partial^{-}_{[x_i]}\psi(x^0)$, and if they are equal, we have

$$\partial^{-}_{[x_i]}\psi(x^0)=\partial_{[x_i]}\psi(x^0)=\partial^{+}_{[x_i]}\psi(x^0),\quad i=1,2. \tag{5}$$

Introduce now strong $^{\pm}$gradients at the point x^0 for the function $\psi(x)$ by the equalities

$$^{+}\operatorname{str}\operatorname{grad}\psi(x^0)=\big(\partial^{+}_{[x_1]}\psi(x^0),\partial^{+}_{[x_2]}\psi(x^0)\big), \tag{6}$$

$$^{-}\operatorname{str}\operatorname{grad}\psi(x^0)=\big(\partial^{-}_{[x_1]}\psi(x^0),\partial^{-}_{[x_2]}\psi(x^0)\big), \tag{7}$$

which together with the strong gradient

$$\operatorname{str}\operatorname{grad}\psi(x^0)=\big(\partial_{[x_1]}\psi(x^0),\partial_{[x_2]}\psi(x^0)\big) \tag{8}$$

are connected as follows.

Proposition 4.1.1 ([8]). *For the existence of* $\operatorname{str}\operatorname{grad}\psi(x^0)$ *it is necessary and sufficient that equal* $^{-}\operatorname{str}\operatorname{grad}\psi(x^0)$ *and* $^{+}\operatorname{str}\operatorname{grad}\psi(x^0)$ *exist, and if they are equal, we have*

$$^{-}\operatorname{str}\operatorname{grad}\psi(x^0)=\operatorname{str}\operatorname{grad}\psi(x^0)={}^{+}\operatorname{str}\operatorname{grad}\psi(x^0). \tag{9}$$

Theorem 4.1.1 ([8]). *The existence of finites* $\partial^{-}_{[x_i]}\psi(x^0)$ *and* $\partial^{+}_{[x_i]}\psi(x^0)$ *implies the finiteness of a strong symmetrical partial derivative with respect to the variable* x_i *at the point* x^0 *for the function* $\psi(x_1,x_2)$*, denoted by* $\partial^{(1)}_{[x_i]}\psi(x^0)$*, for which we have the equality*

$$\partial^{(1)}_{[x_i]}\psi(x^0)=\frac{1}{2}\left[\partial^{-}_{[x_i]}\psi(x^0)+\partial^{+}_{[x_i]}\psi(x^0)\right],\quad i=1,2. \tag{10}$$

Moreover, there exists the function for which the left-hand side of equality (10) *is finite , and the summands in the right-hand side of the same equality are infinite, of opposite signs.*

Proof. The first part of the theorem we verify for the variable x_1. Since in the equality (see [17], Definition 3)

$$\partial^{(1)}_{[x_1]}\psi(x^0)=\lim_{(h_1,h_2)\to(0,0)}\frac{\psi(x_1^0+h_1,x_2^0+h_2)-\psi(x_1^0-h_1,x_2^0+h_2)}{2h_1}, \qquad (11)$$

the ratio appearing under the limit sign is an even function with respect to h_1, we can assume that $h_1>0$ and have

$$\begin{aligned}\partial^{(1)}_{[x_1]}\psi(x^0) &= \frac{1}{2}\lim_{(h_1,h_2)\to(0,0)}\frac{\psi(x_1^0+h_1,x_2^0+h_2)-\psi(x_1^0,x_2^0+h_2)}{h_1}+\\ &+\frac{1}{2}\lim_{(h_1,h_2)\to(0,0)}\frac{\psi(x_1^0-h_1,x_2^0+h_2)-\psi(x_1^0,x_2^0+h_2)}{-h_1}=\\ &=\frac{1}{2}\big[\partial^{+}_{[x_i]}\psi(x^0)+\partial^{-}_{[x_i]}\psi(x^0)\big].\end{aligned}$$

The function $\varphi(x_1,x_2)=|x_1|^{1/2}+|x_2|^{1/2}$ in the neighborhood of the point $x^0=(0,0)$ is most convenient for the second part of the above theorem. We have

$$\begin{aligned}\partial^{(1)}_{[x_1]}\varphi(x^0) &= \lim_{(h_1,h_2)\to(0,0)}\frac{\varphi(h_1,h_2)-\varphi(-h_1,h_2)}{2h_1}=\\ &=\lim_{(h_1,h_2)\to(0,0)}\frac{|h_1|^{1/2}+|h_2|^{1/2}-|-h_1|^{1/2}-|h_2|^{1/2}}{2h_1}=0,\\ \partial^{+}_{[x_1]}\varphi(x^0) &= \lim_{\substack{h_1\to0+\\ h_2\to0}}\frac{\varphi(h_1,h_2)-\varphi(0,h_2)}{h_1}=\\ &=\lim_{\substack{h_1\to0+\\ h_2\to0}}\frac{|h_1|^{1/2}+|h_2|^{1/2}-|h_2|^{1/2}}{h_1}=+\infty,\\ \partial^{-}_{[x_1]}\varphi(x^0) &= \lim_{\substack{h_1\to0-\\ h_2\to0}}\frac{|h_1|^{1/2}}{h_1}=-\lim_{\substack{h_1\to0-\\ h_2\to0}}\frac{|h_1|^{1/2}}{|h_1|}=-\infty. \qquad \square\end{aligned}$$

4.2. Unlateral Angular Partial Derivatives

Below, using equalities (1)–(4), we will introduce angular partial $^{\pm}$derivatives with respect to the variables x_1 and x_2 at the point $x^0=(x_1^0,x_2^0)$ for the function $\psi(x)$, $x=(x_1,x_2)$, under the condition that each of the following limits exists and does not depend on the constants, indicated there:

$$\partial^{+}_{\widehat{x}_1}\psi(x^0)=\lim_{\substack{h_1\to0+\\ |h_2|\le a\cdot h_1}}\frac{\psi(x_1^0+h_1,x_2^0+h_2)-\psi(x_1^0,x_2^0+h_2)}{h_1}, \quad a>0, \qquad (1)$$

$$\partial^+_{\widehat{x}_2}\psi(x^0)=\lim_{\substack{h_2\to 0+\\ h_2\geq b\cdot h_1}}\frac{\psi(x^0_1+h_1,x^0_2+h_2)-\psi(x^0_1+h_1,x^0_2)}{h_2}\,,\quad b>0, \tag{2}$$

$$\partial^-_{\widehat{x}_1}\psi(x^0)=\lim_{\substack{h_1\to 0-\\ |h_2|\leq -c\cdot h_1}}\frac{\psi(x^0_1+h_1,x^0_2+h_2)-\psi(x^0_1,x^0_2+h_2)}{h_1}\,,\quad c>0, \tag{3}$$

$$\partial^-_{\widehat{x}_2}\psi(x^0)=\lim_{\substack{h_2\to 0-\\ h_2\leq -d\cdot |h_1|}}\frac{\psi(x^0_1+h_1,x^0_2+h_2)-\psi(x^0_1+h_1,x^0_2)}{h_2}\,,\quad d>0. \tag{4}$$

The existence of equal quantities $\partial^-_{\widehat{x}_i}\psi(x^0)$ and $\partial^+_{\widehat{x}_i}\psi(x^0)$ is the necessary and sufficient condition for the existence of $\partial_{\widehat{x}_i}\psi(x^0)$ (see 2.5).

We introduce also angular $^\pm$gradients at the point x^0 for the function $\psi(x)$ by the equalities (if there exist their components)

$$^+\operatorname{ang\,grad}\psi(x^0)=\big(\partial^+_{\widehat{x}_1}\psi(x^0),\partial^+_{\widehat{x}_2}\psi(x^0)\big), \tag{5}$$

$$^-\operatorname{ang\,grad}\psi(x^0)=\big(\partial^-_{\widehat{x}_1}\psi(x^0),\partial^-_{\widehat{x}_2}\psi(x^0)\big). \tag{6}$$

Thus for the angular gradient (see equality 2.5.(4)) we obtain

Proposition 4.2.1 ([8]). *For the existence of* $\operatorname{ang\,grad}\psi(x^0)$*, it is necessary and sufficient that the quantities* $^+\operatorname{ang\,grad}\psi(x^0)$ *and* $^-\operatorname{ang\,grad}\psi(x^0)$ *be equal, and if they are such we have*

$$^-\operatorname{ang\,grad}\psi(x^0)=\operatorname{ang\,grad}\psi(x^0)={}^+\operatorname{ang\,grad}\psi(x^0). \tag{7}$$

By Theorem 2.5.1 we obtain

Proposition 4.2.2 ([8]). *For the existence of the total differential* $d\psi(x^0)$*, it is necessary and sufficient that the given by equalities* (5) *and* (6) *angular* $^\pm$*gradients are finite and equal.*

4.3. Unilateral Differentials

Since the finiteness of $\operatorname{ang\,grad}\psi(x^0)$ is the necessary and sufficient condition for the existence of the total differential $d\psi(x^0)$ (see Theorem 2.5.1), using angular $^\pm$gradients we can introduce the following

Definition 4.3.1 ([8]). The function $\psi(x)$ is called $^+$differentiable at the point x^0 if $^+\operatorname{ang\,grad}\psi(x^0)$ is finite, and the $^+$differential, symbolically $d^+\psi(x^0)$, for $\psi(x)$ at x^0 is defined by the equality

$$d^+\psi(x^0)=\partial^+_{\widehat{x}_1}\psi(x^0)\,dx_1+\partial^+_{\widehat{x}_2}\psi(x^0)\,dx_2. \tag{1}$$

The $^-$differential under the finite $^-\operatorname{ang\,grad}\psi(x^0)$ is defined analogously by the equality

$$d^-\psi(x^0)=\partial^-_{\widehat{x}_1}\psi(x^0)\,dx_1+\partial^-_{\widehat{x}_2}\psi(x^0)\,dx_2. \tag{2}$$

Thus we have

Proposition 4.3.1 ([8]). *For the existence of the total differential* $d\psi(x^0)$*, it is necessary and sufficient that* $^{\pm}$*differentials* $d^-\psi(x^0)$ *and* $d^+\psi(x^0)$ *be equal, and if they are such we have*

$$d^-\psi(x^0) = d\psi(x^0) = d^+\psi(x^0). \tag{3}$$

Remark 4.3.1 ([8]). The finiteness of the $^+\operatorname{str}\operatorname{grad}\psi(x^0)$ implies fimiteness of the $^+\operatorname{ang}\operatorname{grad}\psi(x^0)$ and hence the existence of the $^+$ differential $d^+\psi(x^0)$. Similar fact can be applied to the $^-\operatorname{str}\operatorname{grad}\psi(x^0)$.

§ 5. Conditions for the $\mathbb{C}$-Differentiability

It is well-known (see, for e.g. [1]) that the fundamental theorem of complex analysis concerning $\mathbb{C}$-differentiability of a complex-valued function $w = F(z)$ of a complex variable $z = x + iy$ consists of two parts and their fulfilment at the point $z_0 = x_0 + iy_0$ is the necessary and sufficient condition for the existence of a finite derivative $F'(z_0)$.

In the first part of this theorem the function $F(z)$, being the function of two real variables (x, y), is required to be differentiable at the point $z_0 = (x_0, y_0)$ (see 2.5).

The second part of the same theorem requires the fulfilment of the Cauchy–Riemenn condition

$$F'_x(z_0) + iF'_y(z_0) = 0. \tag{C-R}$$

5.1. The Necessary and Sufficient Condition for the $\mathbb{C}$-Differentiability

We have already obtained the necessary and sufficient conditions of differentiability of real-values functions of two real variables (see Theorem 2.5.3). Here we present the theorem whose statement somewhat differs from that suggested in [2] and [5].

Theorem 5.1.1. *For the complex-valued function* $w = F(z)$ *of the complex variable* $z = x + iy$ *to have a finite derivative* $F'(z_0)$ *at the point* $z_0 = x_0 + iy_0$*, it is necessary and sufficient that the equality*

$$D_{\widehat{x}}\, F(z_0) + iD_{\widehat{y}}\, F(z_0) = 0 \tag{1}$$

or what is the same thing, the equalities

$$D_{\widehat{x}}\, u(z_0) = D_{\widehat{y}}\, v(z_0), \tag{2}$$

$$D_{\widehat{y}}\, u(z_0) = -D_{\widehat{x}}\, v(z_0), \tag{3}$$

where $F(z) = u(z) + iv(z)$*, be fulfilled.*

Corollary 5.1.1. *For the function* $F(z)$ *to be holomorphic in the open set* $G \subset \mathbb{C}$*, it is necessary and sufficient that the equality* (1) *or, which is the same, equalities* (2) *and* (3) *be fulfilled at all points* $z_0 \in G$*.*

5.2. Sufficient Conditions for the $\mathbb{C}$-Differentiability

The sufficient conditions for differentiability of functions of two real variables are also available (see Theorem 3.6.2). Therefore the following theorem is valid.

Theorem 5.2.1. *If the condition* (C-R) *is fulfilled for the function* $F(z)$ *and either* $F'_{[x]}(z_0)$*, or* $F'_{[y]}(z_0)$ *is finite, then there exists at* z_0 *the finite derivative* $F'(z_0)$ *and equality* 5.1.(1) *holds.*

From Theorem 3.6.3 we obtain

Theorem 5.2.2. *If the function* $F(z)$ *satisfies the condition* (C-R) *and any one of its partial derivatives* $F'_x(z)$ *and* $F'_y(z)$ *is continuous at the point* z_0*, then there exists the finite derivative* $F'(z_0)$*, and equality* 5.1.(1) *holds.*

Obviously, the sufficient conditions of the existence of the finite derivative $F'(z_0)$, mentioned in Theorems 5.2.1 and 5.2.2 can be rephrased in the form of sufficient conditions for the function $F(z)$ to be holomorphic both in the open set $G \subset \mathbb{C}$ and at the given point.

Remark 5.2.1. Since the continuity at the point $z_0 = x_0 + iy_0$ of the complex-values function $\Phi(z) = A(z) + iB(z)$ is equivalent to the simultaneous continuity at the point $z_0 = x_0 + iy_0$ of real functions $A(z) = A(x, y)$ and $B(z) = B(x, y)$ at the point (x_0, y_0), the problem on the continuity at z_0 of the function $\Phi(z)$ and, for e.g., of the partial derivative $\Phi'_x(z) = A'_x(z) + iB'_x(z)$, can be solved by means of earlier stated theorems for the continuity of real-valued functions of two real variables.

CHAPTER III

Twice Differentiability, Bettazzi Derivative and Mixed Partial Derivatives

Introduction

The material of the present chapter is organized as follows.

§ 1. The results obtained in the previous chapter allow us to formulate the necessary and sufficient conditions for the existence of a total differential of arbitrary order. First of all, this will be realized with respect to twice differentiability. The sufficient conditions for functions of two variables to have a total differential of second order are established.

§ 2. The notion of a derivative introduced by Bettazzi in 1884 and for certain reasons called afterwards a strong derivative, is tightly connected with functions of two variables. The connection of Bettazzi derivative with the existence of a total differential as well as with a mixed partial derivative of second order is indicated.

§ 3. This section presents a survey of the results on the interconnection between mixed partial derivatives of second order. First, the classical Young's theorem and Tolstov's two theorems are formulated. The sufficient conditions for the equality of mixed partial derivatives of second order due to Chelidze, are given.

§ 1. The Conditions of Twice Differentiability

1.1. The Necessary and Sufficient Condition of Twice Differentiability

Let the function $f(x)$, $x = (x_1, \dots, x_n)$ defined in the neighborhood $U(x^0)$ of the point $x^0 = (x_1^0, \dots, x_n^0)$ have a total differential $df(x)$ at every point $x \in U(x^0)$.

If the function $df(x)$ of the variable $x \in U(x^0)$ is differentiable at the point x^0, then the function f is, as is known, called twice differentiable at the point x^0.

1. The necessary and sufficient condition is available for the existence of the total differential $df(x)$, $x \in U(x^0)$ which consists in the finiteness of $\operatorname{ang}\operatorname{grad} f(x)$, or what is the same, in the finiteness of the sum $\sum_{k=1}^{n} f'_{\widehat{x}_k}(x)dx_k$.

Here we have the following

Theorem 1.1.1 ([2]). *For the function $f(x)$ to be twice differentiable at the point x^0, it is necessary and sufficient that $(f'_{\widehat{x}_i}(x))'_{\widehat{x}_j}(x^0)$ be finite for all $i = 1, \dots, n$ and $j = 1, \dots, n$.*

If a matrix with elements $(f'_{\widehat{x}_i}(x))'_{\widehat{x}_j}(x^0)$ is called of the second order angular gradient of the function f at the point x^0, symbolically $\operatorname{ang\,grad}^{(2)} f(x^0)$, then Theorem 1.1.1 can be reformulated in the form of

Theorem 1.1.2 ([2], [5]). *For the function f to be twice differentiable at the point x^0, it is necessary and sufficient that a finite* $\operatorname{ang\,grad}^{(2)} f(x^0)$ *exist.*

Using Theorem 2.3.1 from Chapter II, we obtain

Theorem 1.1.3. *For the function f to be twice differentiable at the point x^0, it is necessary and sufficient that $D_{\widehat{x}_i} D_{\widehat{x}_j} f(x^0)$ be finite for all $i = 1, \dots, n$ and $j = 1, \dots, n$.*

2. It is known that the finiteness of a strong gradient of the function F at the point x^0 is the sufficient condition for the differentiability of F at x^0. This means that for the differentiability at the point x^0 of the differential $df(x)$ it is sufficient that $(f'_{\widehat{x}_i}(x))'_{[x_j]}(x^0)$ be finite for all $i = 1, \dots, n$ and $j = 1, \dots, n$.

If the matrix with elements $(f'_{\widehat{x}_i}(x))'_{[x_j]}(x^0)$ is denoted by $\operatorname{str\,grad}^{(2)} f(x^0)$, then we will have

Proposition 1.1.1. *For the function f to be twice differentiable at the point x^0, it is sufficient that* $\operatorname{str\,grad}^{(2)} f(x^0)$ *be finite.*

Next, the function F is called twice continuously differentiable at the point x^0 if the differential $df(x)$ has continuous partial derivatives at x^0.

Thus we have the following proposition in which by $\operatorname{grad}^{(2)} f(x)$ is denoted the matrix with elements $(f'_{\widehat{x}_i}(x))'_{x_j}(x)$, $i = 1, \dots, n$ and $j = 1, \dots, n$.

Proposition 1.1.2. *For the function f to be twice continuously differentiable at the point x^0, it is necessary and sufficient that* $\operatorname{grad}^{(2)} f(x)$ *be continuous at x^0.*

1.2. The Sufficient Conditions of Twice Differentiability of Functions of Two Variables

By Theorem 1.1.2, the function of two variable $\varphi(x)$, $x = (x_1, x_2)$ is twice differentiable at the point $x^0 = (x_1^0, x_2^0)$ if and only if $\operatorname{ang\,grad}^{(2)} \varphi(x^0)$ is finite.

For twice differentiability of $\varphi(x)$ at the point x^0 it is sufficient that the $\operatorname{str\,grad}^{(2)} \varphi(x^0)$ be finite or the $\operatorname{grad}^{(2)} \varphi(x)$ be continuous at x^0.

Here we will give another sufficient conditions of twice differentiability. Recall that the function $\varphi(x)$ is called twice differentiable at the point x^0, if angular partial derivatives $\varphi'_{\widehat{x}_1}(x)$ and $\varphi'_{\widehat{x}_1}(x)$ are finite in the neighborhood

$U(x^0)$ and differentiable at the point x^0. This, by virtue of Theorem 2.5.1 from Chapter II, means that for the function $\varphi(x)$ to be twice differentiable at the point x^0, it is necessary and sufficient that the gradients

$$\operatorname{ang\,grad}\varphi'_{\widehat{x}_1}(x^0) = \big((\varphi'_{\widehat{x}_1})'_{\widehat{x}_1}(x^0),\ (\varphi'_{\widehat{x}_1})'_{\widehat{x}_2}(x^0)\big) \tag{1}$$

and

$$\operatorname{ang\,grad}\varphi'_{\widehat{x}_2}(x^0) = \big((\varphi'_{\widehat{x}_2})'_{\widehat{x}_1}(x^0),\ (\varphi'_{\widehat{x}_2})'_{\widehat{x}_2}(x^0)\big) \tag{2}$$

be finite.

Now let us prove the following

Theorem 1.2.1 ([2]). *Let one component in each of the quantities*

$$\big(\partial^2_{\widehat{x}_1}\varphi(x), \partial_{\widehat{x}_2}\partial_{\widehat{x}_1}\varphi(x)\big) \ \ \text{and} \ \ \big(\partial_{\widehat{x}_1}\partial_{\widehat{x}_2}\varphi(x), \partial^2_{\widehat{x}_2}\varphi(x)\big) \tag{3}$$

*be continuous at the point x^0 and the other component be finite at x^0. Then the function $\varphi(x)$ is twice differentiable at the point x^0, and**

$$\partial_{\widehat{x}_1}\partial_{\widehat{x}_2}\varphi(x^0) = \partial_{\widehat{x}_2}\partial_{\widehat{x}_1}\varphi(x^0). \tag{4}$$

Proof. Since one component in the pair

$$\big(\partial_{\widehat{x}_1}\partial_{\widehat{x}_1}\varphi(x), \partial_{\widehat{x}_2}\partial_{\widehat{x}_1}\varphi(x)\big) \tag{5}$$

is continuous and the other is finite at the point x^0, the angular partial derivative $\partial_{\widehat{x}_1}\varphi(x)$ is the function, differentiable at the point x^0 (see Theorem 3.6.4 of Chapter II).

Analogously we can prove that the function $\partial_{\widehat{x}_2}\varphi(x)$ is differentiable at the point x^0.

Thus the function $\varphi(x)$ is twice differentiable at the point x^0.

Further, according to Young's theorem (see Theorem 3.3.2 below), from the twice differentiability of the function $\varphi(x)$ at the point x^0 we obtain the equality $\partial_{x_1}\partial_{x_2}\varphi(x^0) = \partial_{x_2}\partial_{x_1}\varphi(x^0)$, which with regard for Theorem 1.1.1, results in equality (4). □

§ 2. Properties of Bettazzi Derivative of Functions of Two Variables

2.1. The Notion of Bettazzi Derivative

The notion of a derivative at the point $x^0 = (x_1^0, x_2^0)$ for a function of two variables $\varphi(x_1, x_2)$ has been introduced by Bettazzi in terms of the limit

$$\lim_{(h,k)\to(0,0)} \frac{\Delta^2_{[x^0]}\varphi(h,k)}{hk}, \tag{1}$$

where

$$\Delta^2_{[x^0]}\varphi(h,k) = \\ = \varphi(x_1^0 + h_1, x_2^0 + h_2) - \varphi(x_1^0, x_2^0 + h_2) - \varphi(x_1^0 + h_1, x_2^0) + \varphi(x_1^0, x_2^0). \tag{2}$$

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*That is, the mixed angular partial derivatives are equal.

The expression $\Delta_{[x^0]}\varphi(h,k)$ can be interpreted as a function $\Phi(I)$ of a closed segment $I\subset\mathbb{R}^2$ with principal vertices at the points (x_1^0,x_2^0) and $(x_1^0+h_1,x_2^0+h_2)$. Under such interpretation, the limit

$$\lim_{(h,k)\to(0,0)}\frac{\Phi(I)}{|hk|}, \tag{3}$$

was called a strong derivative at the point x^0 of the function of segment $\Phi(I)$, symbolically $\Phi'_s(x^0)$.

The author of the present work considers it rightful to call limit (1) the Bettazzi derivative of the function $\varphi(x_1,x_2)$ at the point x^0, in honour of its author Bettazzi (1884).

In the sequel, the above symbols will be retained, and the Bettazzi derivative will be denoted by $\varphi'_s(x^0)$. Consequently,

$$\varphi'_s(x^0)=\lim_{(h,k)\to(0,0)}\frac{\Delta^2_{[x^0]}\varphi(h,k)}{hk}\,. \tag{4}$$

2.2. Properties of Bettazzi Derivative

1. It should be here noted that the function may have finite Bettazzi derivative and have no partial derivatives at some point. Such is, for e.g., the function $\omega(x_1,x_2)=\alpha(x_1)+\beta(x_2)$, where finite functions α and β are assumed to have no derivatives. We have $\Delta^2_{[x^0]}\omega(h,k)=0$ at every point $x^0=(x_1^0,x_2^0)$. Therefore $\omega'_s(x^0)=0$ for all x^0.

If the functions $\lambda(x_1)$ and $\mu(x_2)$ have finite derivatives $\lambda'(x_1^0)$ and $\mu'(x_2^0)$, then the function $\psi(x_1,x_2)=\lambda(x_1)\cdot\mu(x_2)$ has at the point (x_1^0,x_2^0) a finite Bettazzi derivative, and

$$\psi'_s(x_1^0,x_2^0)=\lambda'(x_1^0)\cdot\mu(x_2^0). \tag{1}$$

In particular, this yields

$$(x_1\cdot x_2)'_s(x_1^0,x_2^0)=1 \tag{2}$$

at every point (x_1^0,x_2^0).

Proposition 2.2.1. *Let the function $\varphi(x_1,x_2)$ at the point $x^0=(x_1^0,x_2^0)$ have a finite Bettazzi derivative $\varphi'_s(x_1^0,x_2^0)$. Then we have the following statements:*

1) *the function $\varphi(x_1,x_2)$ is continuous in the wide at the point (x_1^0,x_2^0);*

2) *$\Delta^2_{[x^0]}\varphi(h,k)$, being the function of two variables (h,k), is continuous at the point $(0,0)$.*

Proof. First, from the finiteness of $\varphi'_s(x^0)$ follows the equality

$$\lim_{(h,k)\to(0,0)}\Delta^2_{[x^0]}\varphi(h,k)=0, \tag{3}$$

which means statement 1).

Second, we have the relations

$$\Delta^2_{[x^0]}\varphi(0,0)=\Delta^2_{[x^0]}\varphi(h,0)=\Delta^2_{[x^0]}\varphi(0,k)=0. \tag{4}$$

Taking into account that $\Delta^2_{[x^0]}\varphi(0,0) = 0$, equality (3) means that the function $\Delta^2_{[x^0]}\varphi(h,k)$ is continuous at the point $(0,0)$. Thus our proposition is proved. □

2. The existence of a total differential and derivability by Bettazzi are connected as follows.

Theorem 2.2.1 ([2]). *If there exist finite* $\varphi'_{x_1}(x_1^0, x_2^0)$, $\varphi'_{x_2}(x_1^0, x_2^0)$ *and* $\varphi'_s(x_1^0, x_2^0)$, *then there exists the total differential* $d\varphi(x_1^0, x_2^0)$.

Moreover, the existence of the finite $\varphi'_s(x_1^0, x_2^0)$ *does not follow from the finiteness of* $\operatorname{str}\operatorname{grad}\varphi(x_1^0, x_2^0)$ *and, all the more, from the existence of* $d\varphi(x_1^0, x_2^0)$.

Proof. The left-hand side of the equality

$$\varphi'_s(x_1^0, x_2^0) = \lim_{(h,k)\to(0,0)} \frac{\Delta^2_{[x^0]}\varphi(h,k)}{|h|+|k|} \cdot \frac{|h|+|k|}{hk}$$

is finite, and

$$\Big|\frac{|h|+|k|}{hk}\Big| = \frac{1}{|k|} + \frac{1}{|h|} \to +\infty, \quad (h,k)\to(0,0).$$

Therefore equality 2.5.(12) from Chapter II is fulfilled. Hence by Theorem 2.5.4 of Chapter II, $d\varphi(x_1^0, x_2^0)$ exists.

Further, the function $g(x_1, x_2)$ defined by equality 2.4.(9) has finite $\operatorname{str}\operatorname{grad}(0,0)$ (see equality 3.2.(4) in Chapter II).

Moreover,

$$\Delta^2_{[(0,0)]}g(h,k) = hk\sin(hk)^{-1}$$

and hence $g'_s(0,0)$ does not exist. □

3. The theorem below deals with the representation of a function, having a finite Bettazzi derivative.

Theorem 2.2.2 ([2]). *Let the function* $\varphi(x_1, x_2)$ *have at the point* $x^0 = (x_1^0, x_2^0)$ *a finite Bettazzi derivative* $\varphi'_s(x^0)$. *Then there exists the continuous at the point* x^0 *function* $g(x_1, x_2)$ *possessing the following two properties:*

$$\varphi(x_1, x_2) = g(x_1, x_2) + \varphi(x_1^0, x_2) + \varphi(x_1, x_2^0) - \varphi(x_1^0, x_2^0), \tag{5}$$

$$g'_s(x^0) = \varphi'_s(x^0). \tag{6}$$

In addition, if the function $\varphi(x_1, x_2)$ *is separately partial continuous at the point* x^0, *then the function* $\varphi(x_1, x_2)$ *is continuous at the point* x^0.

Proof. In equality 2.1.(2) we put $x_1 = x_1^0 + h_1$, $x_2 = x_2^0 + k$ and write it as

$$\varphi(x_1, x_2) = g(x_1, x_2) + \varphi(x_1^0, x_2) + \varphi(x_1, x_2^0) - \varphi(x_1^0, x_2^0), \tag{7}$$

where the function $g(x_1, x_2)$ is defined by the equality

$$g(x_1, x_2) = \Delta^2_{[x^0]}\varphi(x_1 - x_1^0, x_2 - x_2^0). \tag{8}$$

The function $g(x_1, x_2)$ is continuous at the point x^0, by statement (2) of Proposition 2.2.1.

To define $g'_s(x^0)$, we have to find $\Delta^2_{[x^0]} g(h,k)$. We have

$$\Delta^2_{[x^0]} g(h,k) = g(x_1^0 + h, x_2^0 + k) - g(x_1^0, x_2^0 + k) - g(x_1^0 + h, x_2^0) + g(x_1^0, x_2^0).$$

But by equality (4),

$$g(x_1^0, x_2^0 + k) = g(x_1^0 + h, x_2^0) = g(x_1^0, x_2^0) = 0.$$

Therefore

$$\Delta^2_{[x^0]} g(h,k) = g(x_1^0 + h, x_2^0 + k).$$

From Equality (8) we now find that

$$g(x_1^0 + h, x_2^0 + k) = \Delta^2_{[x^0]} \varphi(h,k).$$

The last two equalities yield

$$\Delta^2_{[x^0]} g(h,k) = \Delta^2_{[x^0]} \varphi(h,k).$$

By the assumption, $\varphi'_s(x^0)$ is finite. So, there exists the finite $g'_s(x^0)$, and equality (6) holds.

The second part of the theorem follows from equality (5). □

From Theorems 2.2.1 and 2.2.2 we arrive at

Theorem 2.2.3 ([2]). *If the function $\varphi(x_1, x_2)$ has at the point $x^0 = (x_1^0, x_2^0)$ finite $\varphi'_s(x^0)$, then the following two statements are valid:*

1) *separately partial continuity of the function $\varphi(x_1, x_2)$ at the point x^0 implies continuity of the function $\varphi(x_1, x_2)$ at the point x^0;*

2) *separately partial differentiability of the function $\varphi(x_1, x_2)$ at the point x^0 implies its differentiability at the point x^0.*

4^0. Interconnection between the Bettazzi derivative and mixed partial derivative consists in the following.

Theorem 2.2.4 ([2]). *Let the function $\varphi(x_1, x_2)$ in the neighborhood $U(x^0)$ of the point $x^0 = (x_1^0, x_2^0)$ have at least one finite mixed derivative. Then for $\varphi'_s(x^0)$ to exist, it is necessary and sufficient that this mixed partial derivative have limit at the point x^0. If this limit exists, then $\varphi'_s(x^0)$ is equal to it.*

Proof. For clarity, we suppose that in $U(x^0)$ there exists finite $\partial_{x_2}\partial_{x_1}\varphi(x_1, x_2)$. We introduce the function $\mu(x_2) = \varphi(x_1^0 + h, x_2) - \varphi(x_1^0, x_2)$ which by the Lagrange formula takes the form $\mu(x_2) = h\varphi'_{x_1}(x_1^0 + \theta_1 h, x_2)$, where $0 < \theta_1 < 1$. Moreover, the difference $\mu(x_2^0 + k) - \mu(x_2^0)$ is equal both to $\Delta^2_{[x^0]}\varphi(h,k)$ and to $k \cdot \mu'(x_2^0 + \theta_2 k) = hk\partial_{x_2}\partial_{x_1}\varphi(x_1^0 + \theta_1 h, x_2^0 + \theta_2 k)$, $0 < \theta_2 < 1$. Hence

$$\Delta^2_{[x^0]}\varphi(h,k) = hk\partial_{x_2}\partial_{x_1}\varphi(x_1^0 + \theta_1 h, x_2^0 + \theta_2 k), \tag{9}$$

which implies the statement of the theorem. □

Corollary 2.2.1. *Let the function $\varphi(x_1, x_2)$ have in the neighborhood of the point $x^0 = (x_1^0, x_2^0)$ the both finite mixed partial derivatives. If they have limits at the point x^0, then there exists $\varphi'_s(x^0)$, and the both limits are equal to it.*

§ 3. On Mixed Partial Derivatives of Second Order

3.1. Preliminaries

Let the function $f(x)$ be defined in the δ-neighborhood $U(x^0, \delta)$ of the point $x^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$. Suppose that the function f has at every point $x \in U(x^0, \delta)$ a finite partial derivative with respect to the variable x_i, symbolically f'_{x_i}, or $\partial_{x_i} f(x)$. Obviously, f'_{x_i} is finite function in $U(x^0, \delta)$, and it is quite possible that it has partial derivative with respect to x_j at the point $x \in U(x^0, \delta)$, symbolically $f''_{x_i x_j}(x)$, or $\partial_{x_j} \partial_{x_i} f(x)$. It is called a second order partial derivative at the point x of the function f with respect first to x_i and then to x_j, or briefly, of a second order partial derivative at the point x with respect to variables x_i, x_j.

If $j = i$, then it is called a second order partial derivative of the function f with respect to x_i at the point x, symbolically $f''_{x_i^2}$, or $\partial^2_{x_i} f(x)$ (here $\partial^2_{x_i} f(x) = \partial_{x_i} \partial_{x_i} f(x)$).

If $j \neq i$, then it is called a second order mixed partial derivative of the function f at the point x with respect to variables* x_i, x_j.

They say that the function f at the point x has partial derivative of arbitrary order with respect to x_k, if for arbitrary values $p = 1, 2, \dots$ the p-th order partial derivatives of the function f at the point x with respect to x_k are finite, symbolically $\partial^p_{x_k} f(x)$, or $\frac{\partial^p f}{\partial^p_{x_k}}(x)$.

Possibly, the function f at the point $x \in U(x^0, \delta)$ has the second order mixed partial derivative with respect to variables x_j, x_i (first to x_j and then to x_i), symbolically $f''_{x_j, x_i}(x)$, or $\partial_{x_i} \partial_{x_j} f(x)$.

The basic problem for mixed partial derivatives of second order involves the question: what properties of the function f at the point x guarantee the fulfilment of the equality

$$\partial_{x_i} \partial_{x_j} f(x) = \partial_{x_j} \partial_{x_i} f(x)? \tag{1}$$

The property expressed by equality (1) is sometimes called either symmetry of mixed partial derivatives of second order of the function f at the point x, or permutability of a sequence in which we take partial derivatives with respect to the variables x_i and x_j.

Various examples show that there exist functions, at some points x for which inequality

$$\partial_{x_i} \partial_{x_j} f(x) \neq \partial_{x_j} \partial_{x_i} f(x). \tag{2}$$

hold.

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*As is seen, a sequence of partial derivatives is fixed symbolically by the sign "comma" between variables.

Here we consider two examples: first, when one hand-side of relation (2) exists and the other does not; second, both hand-sides of relation (2) are finite and different (The second example has another properties as well; see Corollaries 3.1.1 and 3.1.2, and also Remark 3.3.1 below).

Example 3.1.1. Consider the function $\varphi(x_1, x_2) = \alpha(x_1) + \beta(x_2)$, where $\alpha(x_1)$ has a finite derivative for all x_1, and the finite function $\beta(x_2)$ has derivative nowhere. Then $\partial_{x_2}\partial_{x_1}\varphi(x_1, x_2) = \partial_{x_2}\alpha'(x_1) = 0$ for all (x_1, x_2), and the set for the existence of a mixed partial derivative $\partial_{x_1}\partial_{x_2}\varphi(x_1, x_2)$ is empty, since a set for the existence of a partial derivative $\partial_{x_2}\varphi(x_1, x_2) = \beta'(x_2)$ is empty.

The following example is well know.

Example 3.1.2. Let us prove that for the function

$$\psi(x_1, x_2) = \begin{cases} x_1 \cdot x_2 \dfrac{x_1^2 - x_2^2}{x_1^2 x_2^2} & \text{for} \quad x_1^2 + x_2^2 > 0 \\ 0 & \text{for} \quad x_1 = 0 = x_2 \end{cases}, \tag{3}$$

the inequality

$$\partial_{x_2}\partial_{x_1}\psi(0,0) \neq \partial_{x_1}\partial_{x_2}\psi(0,0) \tag{4}$$

holds.

Note at once that $\psi(x_1, 0) = \psi(0, x_2) = \psi(0, 0) = 0$.

To find the left-hand side of inequality (4), we have first to find a partial derivative of the function $\psi(x_1, x_2)$ with respect to x_1 at the point $(0, x_2)$, symbolically $\partial_{x_1}\psi(0, x_2)$. The latter is the function of x_2, and we have to find its derivative at the point $x_2 = 0$, i.e., $(\partial_{x_1}\psi(0, x_2))'(0)$.

Analogously, the right-hand side of inequality (4) is $(\partial_{x_2}\psi(x_1, 0))'(0)$.

We start with the finding of the left-hand side of inequality (4) by using the just mentioned sequence. We have

$$\partial_{x_1}\psi(0, x_2) = \lim_{x_1 \to 0} \frac{\psi(x_1, x_2) - \psi(0, x_2)}{x_1 - 0}.$$

Here $x_1 \neq 0$, and hence the quantities $\psi(x_1, x_2)$ are specified by the upper line of equality (3), due to $x_1^2 + x_2^2 > 0$.

Moreover, $\psi(0, x_2) = 0$. Hence for $x_2 \neq 0$ we have

$$\partial_{x_1}\psi(0, x_2) = \lim_{x_1 \to 0} \frac{\psi(x_1, x_2)}{x_1} = \lim_{x_1 \to 0} x_2 \cdot \frac{x_1^2 - x_2^2}{x_1^2 x_2^2} = x_2 \frac{-x_2^2}{x_2^2} = -x_2.$$

However, if $x_2 = 0$, then $\partial_{x_1}\psi(0, 0)$ is the derivative of the function $\psi(x_1, 0) = 0$ at the point $x_1 = 0$. Thus

$$\partial_{x_1}\psi(0, x_2) = \begin{cases} -x_2 & \text{for} \quad x_2 \neq 0 \\ 0 & \text{for} \quad x_2 = 0 \end{cases}. \tag{5}$$

Next, $\partial_{x_2}\partial_{x_1}\psi(0,0)$ means the derivative $\frac{d}{dx_2}\partial_{x_1}\psi(0, x_2)$ at $x_2 = 0$, and when calculating it we assume that $x_2 \neq 0$ under the corresponding limiting sign. Therefore, taking into account (5), we have

$$\partial_{x_2}\partial_{x_1}\psi(0,0) = \lim_{x_1\to 0}\frac{\partial_{x_1}\psi(0,x_2)-\partial_{x_1}\psi(0,0)}{x_2-0} = \\ = \lim_{x_2\to 0}\frac{-x_2-0}{x_2} = \lim_{x_2\to 0}(-1) = -1$$

and hence

$$\partial_{x_2}\partial_{x_1}\psi(0,0) = -1. \tag{6}$$

Analogously we obtain the equalities

$$\partial_{x_2}\psi(x_1,0) = \begin{cases} x_1 & \text{for } \ x_1 \neq 0 \\ 0 & \text{for } \ x_1 = 0 \end{cases} \tag{7}$$

and

$$\partial_{x_1}\partial_{x_2}\psi(0,0) = 1. \tag{8}$$

Consequently, inequality (4) is established.

Corollary 3.1.1. *For the function* $\psi(x_1,x_2)$ *defined by equality* (3) *the* $\operatorname{grad}\psi(x_1,x_2)$ *is continuous everywhere. In particular, there exists everywhere the total differential* $d\psi(x_1,x_2)$*.*

Proof. The continuity of the function $\operatorname{grad}\psi(x_1,x_2)$ at all the points $(x_1,x_2) \neq (0,0)$ follows from the equalities

$$\psi'_{x_1}(x_1,x_2) = \frac{x_2(x_1^4 - x_2^4 + 4x_1^2x_2^2)}{(x_1^2+x_2^2)^2} \ \text{ for } \ x_1^2+x_2^2>0, \tag{9}$$

$$\psi'_{x_2}(x_1,x_2) = \frac{x_1(x_1^4 - x_2^4 - 4x_1^2x_2^2)}{(x_1^2+x_2^2)^2} \ \text{ for } \ x_1^2+x_2^2>0. \tag{10}$$

Taking into account the equality $\psi'_{x_1}(0,0)=0$, we obtain the continuity of the partial derivative $\psi'_{x_1}(x_1,x_2)$ at the point $(0,0)$ from the following estimates:

$$\left|\psi'_{x_1}(x_1,x_2)\right| \leq \frac{|x_2|(x_1^4+x_2^4+2x_1^2x_2^2+2x_1^2x_2^2)}{(x_1^2+x_2^2)^2} \leq \\ \leq \frac{|x_2|\,[(x_1^2+x_2^2)^2+x_1^4+x_2^4]}{(x_1^2+x_2^2)^2} \leq \\ \leq \frac{|x_2|\,[(x_1^2+x_2^2)^2+(x_1^2+x_2^2)^2]}{(x_1^2+x_2^2)^2} = 2|x_2| \to 0 \ \text{ as } \ x_2 \to 0.$$

Thus

$$\lim_{(x_1,x_2)\to(0,0)}\psi'_{x_1}(x_1,x_2) = \psi'_{x_1}(0,0) = 0, \tag{11}$$

which means that the partial derivative $\psi'_{x_1}(x_1,x_2)$ is continuous at the point $(0,0)$.

The continuity of the partial derivative $\psi'_{x_2}(x_1,x_2)$ at the point $(0,0)$ can be proved analogously.

The existence of the total differential $d\psi(x_1, x_2)$ at all points (x_1, x_2) follows from the continuity everywhere of the function $\operatorname{grad}\psi(x_1, x_2)$. $\square$

Corollary 3.1.2. *The function $\psi(x_1, x_2)$ defined by equality* (3) *possesses the following properties:*

1) *it is not twice differentiable at the point* $(0, 0)$*, i.e., it has no total differential of second order at the point* $(0, 0)$*;*

2) *at every point* $(x_1, x_2) \neq (0, 0)$ *it has total differential of arbitrary order;*

3) *at all points* $(x_1, x_2) \neq (0, 0)$ *the equality*

$$\partial_{x_2}\partial_{x_1}\psi(x_1, x_2) = \partial_{x_1}\partial_{x_2}\psi(x_1, x_2), \quad (x_1, x_2) \neq (0, 0) \tag{12}$$

holds.

Proof. 1) Were the function $\psi(x_1, x_2)$ twice differentiable at the point $(0, 0)$, the equality $\partial_{x_2}\partial_{x_1}\psi(0, 0) = \partial_{x_1}\partial_{x_2}\psi(0, 0)$ would hold, by Young's theorem (see Theorem 3.3.2 below). But this contradicts inequality (4).

2) The function $\psi(x_1, x_2)$ has at the points $(x_1, x_2) \neq (0, 0)$ total differential of arbitrary order because its partial derivatives of any orders are continuous at the points $(x_1, x_2) \neq (0, 0)$ (see equalities (9) and (10)).

3) As far as the function $\psi(x_1, x_2)$ has total differential of arbitrary order, in particular of second order, at all points $(x_1, x_2) \neq (0, 0)$, equality (12) holds at the points $(x_1, x_2) \neq (0, 0)$, by the same Young's theorem. $\square$

3.2. Inequality of Mixed Partial Derivatives

As we see, the function $\psi(x_1, x_2)$ defined by equality 3.1.(3) has everywhere the continuous gradient and satisfies inequality 3.1.(2) at a single point $(0, 0)$.

There naturally arises the problem: for the function having everywhere continuous gradient, how rich may be a set of those points at which inequality 3.1.(2) is fulfilled?

In connection with this problem we have Tolstov's two statements in which by K is denoted the unit square $\{(x_1, x_2) \in \mathbb{R}^2 : \ 0 \leq x_1 \leq 1, \ 0 \leq x_2 \leq 1\}$.

Theorem 3.2.1 ([29], Proposition II). *There exists the function $F(x_1, x_2)$ with the continuous in K gradient and possessing everywhere in K the both mixed partial derivatives for which the inequality*

$$\partial_{x_1}\partial_{x_2}F(x_1, x_2) \neq \partial_{x_2}\partial_{x_1}F(x_1, x_2) \tag{1}$$

is fulfilled at almost all points $(x_1, x_2) \in K$.

Theorem 3.2.2 ([29], Proposition I). *For every positive number $\eta < \frac{1}{2}$ there exists the function $\Psi(x_1, x_2)$ with the continuous in K gradient and possessing in K the both mixed partial derivatives. Moreover, there exists a measurable set $Q \subset K$ of plane measure η^2, such that at every point*

$(x_1, x_2) \in Q$ *the inequality*

$$\partial_{x_1}\partial_{x_2}\Psi(x_1, x_2) \neq \partial_{x_2}\partial_{x_1}\Psi(x_1, x_2) \tag{2}$$

holds, while the mixed partial derivatives outside the set Q *coincide**.

3.3. The Sufficient Conditions for Equality of Mixed Partial Derivatives of Functions of Two Variables

1. Theorem 3.3.1 ([27]). *If the function* $F(x, y)$ *has everywhere in the domain* G *finite partial derivatives* $\partial^2_{x_1}F$, $\partial_{x_1}\partial_{x_2}F$, $\partial_{x_2}\partial_{x_1}F$ *and* $\partial^2_{x_2}F$, *then almost everywhere in* G *the equality*

$$\partial_{x_2}\partial_{x_1}F = \partial_{x_1}\partial_{x_2}F \tag{1}$$

holds.

Remark 3.3.1. Under the conditions of Theorem 3.3.1 one cannot, in general, state that equality (1) is fulfilled everywhere in G. This can be illustrated by an example of the function $\psi(x_1, x_2)$ given by equality 3.1.(3).

Indeed, equalities 3.1.(9) and 3.1.(10) show that all partial derivatives of second order are finite at every point $(x_1, x_2) \neq (0, 0)$, and mixed partial derivatives coincide at such points. First, mixed partial derivatives are finite at the point $(0, 0)$, by equalities 3.1.(6) and 3.1.(8). Second, $\partial^2_{x_1}\psi(0, 0)$ means the derivative $(\partial_{x_1}\psi(x_1, 0))'(0)$. Therefore the condition $x_1 \neq 0$ must be fulfilled in $\partial_{x_1}\psi(x_1, 0)$, but this by virtue of equality 3.1.(9) implies that $\partial_{x_1}\psi(x_1, 0) = 0$. Hence $\partial^2_{x_1}\psi(0, 0) = (0)'(0) = 0$.

Just in the similar way we obtain the equality $\partial^2_{x_2}\psi(0, 0) = 0$.

Consequently, the function $\psi(x_1, x_2)$ satisfies the conditions of Theorem 3.3.1 at every point $(x_1, x_2) \in \mathbb{R}^2$. Despite this fact, inequality 3.1.(4), as is seen, holds.

The above-formulated Tolstov's theorem is, in fact, a generalization of the classical "local" Young's theorem on the validity of inverting the order of taking partial derivatives at those points, at which partial derivatives F'_x and F'_y have total differentials.

Here is the Young's theorem.

Theorem 3.3.2 ([30], pp. 141-2; [11], p. 427). *Let the function* $f(x_1, x_2)$ *have finite partial derivatives* $\partial_{x_1}f(x_1, x_2)$ *and* $\partial_{x_2}f(x_1, x_2)$ *in the neighborhood of the point* $x^0 = (x^0_1, x^0_2)$, *and let these partial derivatives be the functions, differentiable at the point* x^0. *Then the equality*

$$\partial_{x_2}\partial_{x_1}f(x^0) = \partial_{x_1}\partial_{x_2}f(x^0), \tag{2}$$

---

*This statement somewhat differs from that proposed by the author. It is said in [29] that the plane measure of the set Q is positive. But upon proving this theorem it is stated that the plane measure of the set Q is equal to $(\frac{\lambda-3}{\lambda-2})^2$ and outside the set Q the mixed partial derivatives coincide, where $3 < \lambda < 4$ (see [29], p. 33).

holds.

This Young's theorem can be formulated in short as follows: the twice differentiable at the point x^0 function f leads to equality (2).

Remark 3.3.2. Tolstov's Theorem 3.3.1 does not follow from Young's Theorem 3.3.2. Indeed, the function $F(x_1, x_2)$ from Tolstov's Theorem B (see Introduction in Chapter II) satisfies equality (1) almost everywhere in the square Q, by Theorem 3.3.1. Moreover, suppositions of Theorem 3.3.2 are not fulfilled at the points of the set $E \subset Q$.

2. Here we indicate two more theorems on the equality of mixed partial derivatives.

Let the function of two variables $\Phi(x, y)$ be defined in the domain $G \subset \mathbb{R}^2$, and suppose that $\Phi(x, y)$ in G has finite partial derivatives $p(x, y) = \Phi'_x(x, y)$ and $q(x, y) = \Phi'_y(x, y)$.

The following two theorems are valid.

Theorem 3.3.3 ([26]). *Let the functions $p(x, y)$ and $q(x, y)$ be continuous in G, and let one of them, say $p(x, y)$, satisfy the conditions:*

A) *for some summable function $\varphi(x)$ the inequality*

$$\left|\frac{p(x, y+k) - p(x, y)}{k}\right| \leq \varphi(x) \tag{3}$$

is fulfilled;

B) *almost everywhere in G there exists the finite partial derivative $p'_y(x, y)$.*

Then almost everywhere in G there exists the finite partial derivative $q'_x(x, y)$, and for almost all point $(x, y) \in G$ the equality

$$q'_x(x, y) = p'_y(x, y) \tag{4}$$

is valid.

Theorem 3.3.4 ([26]). *If in the neighborhood of the point $M(x_0, y_0)$ there exists the finite partial derivative $p'_y(x, y)$ which is continuous with respect to the variable x at the point M, then there exists the finite $q'_x(x_0, y_0)$, and the equality*

$$p'_y(x_0, y_0) = q'_x(x, y) \tag{5}$$

is valid.

CHAPTER IV

On Double Indefinite Integral and Absolutely Continuous Functions of Two Variables

In the present chapter we prove the finiteness of a strong gradient, in particular, the existence of a total differential, almost everywhere both for an indefinite double integral and for an absolutely continuous function of two variables.

For the function of two variables, summable on a rectangle, we introduce the notion of an Lebesgue's intense points and prove that almost each point is an Lebesgue's intense point for every summable function.

The strong gradient at the Lebesgue's intense points is finite, in particular, the total differential exists both for an indefinite double integral and for an absolutely continuous function of two variables.

For an indefinite integral with a parameter we prove the theorem which contains C. de la Vallée Poussin's theorem on differentiation of an integral with respect to the parameter and Lebesgue's theorem on differentiation of an indefinite integral.

The problem on double differentiability of an indefinite double integral is investigated and the sufficient conditions for an Lebesgue's intense point are established.

§ 1. Differentiability of an Indefinite Double Integral

1.1. Partial and Mixed Partial Derivatives of an Indefinite Integral

Let the function of two variables $f(x,y)$ be summable on the rectangle $Q=\{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ c\le y\le d\}$. Consider for the function f the indefinite double integral

$$F(x,y)=\int_a^x\int_c^y f(t,\tau)\,dt\,d\tau. \tag{1}$$

The finite at every point $(x,y)\in Q$ function $F(x,y)$ we can, by Fubini's theorem, write as

$$F(x,y)=\int_a^x\bigg(\int_c^y f(t,\tau)\,d\tau\bigg)\,dt, \tag{2}$$

$$F(x,y) = \int\limits_c^y \left(\int\limits_a^x f(t,\tau)\,dt \right) d\tau. \tag{3}$$

If we apply Lebesgue's theorem to equality (2), then a set of those x's, for which

$$F'_x(x,y) = \int\limits_c^y f(t,\tau)\,d\tau, \tag{4}$$

will depend on the parameter y. G. P. Tolstov studied this situation in detail and proved the following

Theorem 1.1.1 ([29], § 7; [28], p. 90). *For every function $f \in L(Q)$ there exist measurable sets $e_1 \subset [a,b]$ with $|e_1| = b-a$ and $e_2 \subset [c,d]$ with $|e_2| = d-c$ such that defined by equality* (1) *the function $F(x,y)$ has the following properties:*

1) *at every point (x,y) with $x \in e_1$ and $c \le y \le d$ there exists finite $F'_x(x,y)$ and*

$$F'_x(x,y) = \int\limits_c^y f(x,\tau)\,d\tau; \tag{5}$$

2) *at every point (x,y) with $a \le x \le b$ and $y \in e_2$ there exists finite $F'_y(x,y)$ and*

$$F'_y(x,y) = \int\limits_a^x f(t,y)\,dt; \tag{6}$$

3) *there exists a measurable set $E \subset Q$ with $|E| = |Q|$ such that at every point $(x,y) \in E$ hold the equalities*

$$F''_{x,y}(x,y) = f(x,y) = F''_{y,x}(x,y) \tag{7}$$

with finite terms.*

From statements 1) and 2) of Theorem 1.1.1 it follows that the functions of two variables F'_x and F'_y are finite almost everywhere on the Q. Namely, the function F'_x is finite on the set $\mathcal{E}_1 = \{(x,y) \in Q : x \in e_1,\ c \le y \le d\}$ with $|\mathcal{E}_1| = |Q|$, and the function F'_y is finite on the set $\mathcal{E}_2 = \{(x,y) \in Q : a \le x \le b,\ y \in e_2\}$ with $|\mathcal{E}_2| = |Q|$.

Proposition 1.1.1. *The functions F'_x and F'_y are summable on the Q.*

Proof. Since $f \in L(Q)$,

$$\int\limits_a^b \int\limits_c^d |f(t,\tau)|\,dt\,d\tau \equiv I(f)$$

is finite.

*We have introduced the symbols $F''_{x,y} = \partial y\,\partial x\,F$ and $F''_{y,x} = \partial x\,\partial y\,F$.

The estimates

$$|F'_x(x,y)| \le \int\limits_c^y |f(x,\tau)|\,d\tau \le \int\limits_c^d |f(x,\tau)|\,d\tau$$

obtained from equality (5), show that

$$\int\limits_a^b |F'_x(x,y)|dx \le I(f)$$

whence

$$\int\limits_a^b \int\limits_c^d |F'_x(x,y)|dx\,dy \le (d-c)\,I(f) < +\infty.$$

Hence $F'_x \in L(Q)$. Analogously, $F'_y \in L(Q)$. The embeddings $F''_{x,y} \in L(Q)$ and $F''_{y,x} \in L(Q)$ follow from equalities (7). □

1.2. Differentiability of an Indefinite Double Integral

If the function $\varphi(x)$ is summable on the $[a,b]$, and

$$\Phi(x) = \int\limits_a^b \varphi(t)\,dt, \tag{1}$$

then according to Lebesgue's theorem (1903), there exists the measurable set $e \subset [a,b]$ with $|e| = b-a$ such that the equality $\Phi'(x) = \varphi(x)$, or what is the same, the equality

$$\lim_{h\to 0} \frac{1}{h} \int\limits_x^{x+h} \varphi(t)\,dt = \varphi(x) \tag{2}$$

is fulfilled for all $x \in e$.

All Lebesgue's points of the function φ on the $[a,b]$ belong to the set e. In particular, all points of continuity of the function φ on the $[a,b]$ belong to the set e, if $\varphi(x)$ has such a point on the $[a,b]$.

The following problems are quite natural.

I. Does the indefinite double integral

$$F(x,y) = \int\limits_a^x \int\limits_c^y f(t,\tau)\,dt\,d\tau, \tag{3}$$

corresponding to the summable on the rectangle $Q = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}$ function f, have or have no a total differential almost everywhere?

II. If $F(x,y)$ has a total differential, then at what points and how the set of such points is connected with the function f?

The answer to Problem I will be given here, and Problem II will be considered in Section 4.

We proceed to formulating and proving the following

Lemma 1.2.1 ([3]). *For every function $f \in L(Q)$ there exists the measurable set $E \subset Q$ with $|E| = |Q|$ such that at every point $(x,y) \in E$ the equalities*

$$\lim_{\substack{h\to 0\\ |k|\le c|h|}} \frac{1}{h} \int_x^{x+h}\int_y^{y+k} f(t,\tau)\,dt\,d\tau = 0, \tag{4}$$

$$\lim_{\substack{k\to 0\\ |h|\le l|k|}} \frac{1}{k} \int_x^{x+h}\int_y^{y+k} f(t,\tau)\,dt\,d\tau = 0 \tag{5}$$

are fulfilled, no matter how the constants $c > 0$ and $l > 0$ are.

Proof. Without restriction of generality, we can assume that $h > 0$ and $k > 0$. For every constant $c > 0$ under $k \le ch$ we have

$$\left|\frac{1}{h}\int_x^{x+h}\int_y^{y+k} f(t,\tau)\,dt\,d\tau\right| \le \frac{1}{h}\int_x^{x+h}\int_y^{y+ch} |f(t,\tau)|\,dt\,d\tau =$$

$$= ch\left(\frac{1}{h\cdot ch}\int_x^{x+h}\int_y^{y+ch} |f(t,\tau)|\,dt\,d\tau\right). \tag{6}$$

By virtue of the Lebesgue's theorem ([21], p. 118), there exists the measurable set $E_1 \subset Q$ with $|E_1| = |Q|$, such that the expression in the brackets appearing in (6) has a finite limit at every point $(x,y) \in E_1$, equal to $|f(x,y)|$. Therefore equality (4) is fulfilled at the points $(x,y) \in E_1$.

Equality (5) is likewise fulfilled at every point (x,y) of a specific set $E_2 \subset Q$ with $|E_2| = |Q|$.

It is now clear that equalities (4) and (5) together can be fulfilled at every point $(x,y) \in E$, where $E = E_1 \cap E_2$, $|E| = |Q|$. $\square$

Theorem 1.2.1 ([3]). *Indefinite integral* (3) *has a total differential at almost all points $(x,y) \in Q$ for every function $f \in L(Q)$.*

Proof. To prove this theorem, it is necessary and sufficient to prove that the quantities $D_{\widehat{x}}F(x,y)$ and $D_{\widehat{y}}F(x,y)$ are finite at almost all points $(x,y) \in Q$, by Theorem 2.5.3 of Chapter II.

Let us, for example, prove $D_{\widehat{x}}F(x,y)$ is finite at almost all $(x,y) \in Q$. To this end, we consider the expression

$$\frac{F(x+h,y+k) - F(x,y+k)}{h} - \int_c^y f(x,\tau)d\tau =$$

$$= \frac{1}{h}\int\limits_{x}^{x+h}\int\limits_{c}^{y+k} f(t,\tau)\,dt\,d\tau - \int\limits_{c}^{y} f(x,\tau)\,d\tau =$$

$$= \frac{1}{h}\int\limits_{x}^{x+h}\bigg(\int\limits_{c}^{y}\big[f(t,\tau)-f(x,\tau)\big]d\tau\bigg)dt + \frac{1}{h}\int\limits_{x}^{x+h}\int\limits_{y}^{y+k} f(t,\tau)dt\,d\tau \equiv$$

$$\equiv I_h(x,y) + J_{h,k}(x,y). \tag{7}$$

By Lemma 1.2.1, there exists the measurable set $E \subset Q$ with $|E| = |Q|$, such that at every point $(x,y) \in E$ the equality

$$\lim_{\substack{h\to 0\\ |k|\le|h|}} J_{h,k}(x,y) = 0. \tag{8}$$

is fulfilled.

Now we show that the equality

$$\lim_{h\to 0} I_h(x,y) = 0 \tag{9}$$

is fulfilled at almost all points $(x,y) \in Q$. Towards this end, we make use of the sets e_1 and e_2 and also the sets $\mathcal{E}_1$ and $\mathcal{E}_2$ from Theorem 1.1.1.

On the set $\mathcal{E}_1$ we have the equality 1.1.(5). Hence the equality

$$\lim_{h\to 0}\frac{F(x+h,y)-F(x,y)}{h} = \int\limits_{c}^{y} f(x,\tau)d\tau, \quad (x,y) \in \mathcal{E}_1 \tag{10}$$

is fulfilled. It is clear that under $h \to 0$ the difference

$$\frac{F(x+h,y)-F(x,y)}{h} - \int\limits_{c}^{y} f(x,\tau)\,d\tau =$$

$$= \frac{1}{h}\bigg(\int\limits_{a}^{x+h}\int\limits_{c}^{y} - \int\limits_{a}^{x}\int\limits_{c}^{y}\bigg) - \int\limits_{c}^{y} f(x,\tau)\,d\tau =$$

$$= \frac{1}{h}\int\limits_{x}^{x+h}\int\limits_{c}^{y} f(t,\tau)\,dt\,d\tau - \frac{1}{h}\int\limits_{x}^{x+h}\int\limits_{c}^{y} f(x,\tau)\,d\tau =$$

$$= \frac{1}{h}\int\limits_{x}^{x+h}\bigg(\int\limits_{c}^{y} f(t,\tau)\,d\tau - \int\limits_{c}^{y} f(x,\tau)d\tau\bigg)\,dt =$$

$$= \frac{1}{h}\int\limits_{x}^{x+h}\bigg(\int\limits_{c}^{y}\big[f(t,\tau)-f(x,\tau)\big]\,d\tau\bigg)\,dt = I_h(x,y)$$

tends to zero.

It is now clear that equalities (8) and (9) are fulfilled simultaneously at the points $(x,y) \in E^*$, where $E^* = E \cap \mathcal{E}_1$ and $|E^*| = |Q|$. By equality

(7), all this means that at the points $(x,y)\in E^*$ the quantity $D_{\widehat{x}}F(x,y)$ is finite, and at these points

$$D_{\widehat{x}}F(x,y)=\int\limits_c^y f(t,\tau)d\tau,\ \ (x,y)\in E^*,\quad |E^*|=|Q|. \tag{11}$$

The finiteness of the quantity $D_{\widehat{y}}F(x,y)$ at the points $(x,y)\in E^{**}=E\cap\mathcal{E}_2$, $|E^{**}|=|Q|$ is established analogously, and

$$D_{\widehat{y}}F(x,y)=\int\limits_a^x f(t,y)\,dt,\ \ (x,y)\in E^{**},\quad |E^{**}|=|Q|. \tag{12}$$

Obviously, at the points (x,y) of the set $M=E^*\cap E^{**}$, $|M|=|Q|$,

$$\widehat{D}F(x,y)=\big(D_{\widehat{x}}F(x,y),D_{\widehat{y}}F(x,y)\big) \tag{13}$$

is finite (see equality 2.3.(4) in Chapter II), and

$$dF(x,y)=D_{\widehat{x}}F(x,y)dx+D_{\widehat{y}}F(x,y)dy \tag{14}$$

(see equality 2.5.(11) in Chapter II). □

From here, we obtain the following

Theorem 1.2.2. *The indefinite double integral*

$$\Psi(x,y)=\int\limits_a^x\int\limits_c^y \psi(t,\tau)\,dt\,d\tau \tag{15}$$

for every R-integrable on the Q function ψ has the total differential almost everywhere on the Q.

Theorem 1.2.3 ([4], [5]). *At every point $(x_0,y_0)\in Q$ of differentiability of the indefinite integral* (3) *with $f\in L(Q)$ we have*

$$\lim_{\substack{h\to 0\\ k\to 0}}\frac{1}{|h|+|k|}\int\limits_{x_0}^{x_0+h}\int\limits_{y_0}^{y_0+k} f(t,\tau)\,dt\,d\tau=0. \tag{16}$$

In particular, equality (16) *is fulfilled at almost all points $(x_0,y_0)\in Q$.*

Proof. At the point $(x_0,y_0)\in Q$ of differentiability of the function $F(x,y)$, $D_{\widehat{x}}F(x_0,y_0)$ and $D_{\widehat{y}}F(x_0,y_0)$ are finite. In particular, $F'_x(x_0,y_0)$ and $F'_y(x_0,y_0)$ are finite. Hence the equality

$$\lim_{\substack{h\to 0\\ k\to 0}}\frac{F(x_0+h,y_0+k)-F(x_0,y_0+k)-F(x_0+h,y_0)+F(x_0,y_0)}{|h|+|k|}=0. \tag{17}$$

is fulfilled by Theorem 2.5.4 from Chapter II.

If for the defined by (3) function $F(x,y)$ we calculate the numerator of the equality (17), then we will get equality (16).

The fulfilment of equality (16) almost everywhere on the Q for every function $f \in L(Q)$ follows from Theorem 1.2.1. □

Theorem 1.2.4 ([5]). *Let indefinite integral* (3) *for* $f \in L(Q)$ *have in the neighborhood of the point* $(x_0, y_0) \in Q$ *finite* $F'_x(x, y)$, $F'_y(x, y)$ *and* $F''_{x,y}(x, y)$. *Then for the function* $F(x, y)$ *to be differentiable at the point* (x_0, y_0), *it is necessary and sufficient that*

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk}{|h|+|k|} F''_{x,y}(x_0+\theta_1 h, y_0+\theta_2 k) = 0, \quad 0<\theta_1, \ \ \theta_2<1. \tag{18}$$

If in addition, $F''_{x,y}(x, y) = f(x, y)$ *in the neighborhood of the point* (x_0, y_0), *then for the differentiability of the function* $F(x, y)$ *at* (x_0, y_0) *it is necessary and sufficient that*

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{hk}{|h|+|k|} f(x_0+\theta_1 h, y_0+\theta_2 k) = 0, \quad 0<\theta_1, \ \ \theta_2<1. \tag{19}$$

Proof. Substituting the function φ by F, the assumption of Theorem 2.5.4 from Chapter II is fulfilled. Therefore equality (17) is the necessary and sufficient condition for the function $F(x, y)$ to be differentiable at the point (x_0, y_0). But the numerator in (17) can be written as $F''_{x,y}(x_0+\theta h, y_0+\theta k)$, by equality 2.2.(9) from Chapter III. □

§ 2. Differentiability of an Absolutely Continuous Function of Two Variables

We intend to prove the theorem for absolutely continuous functions of two variables, which will be analogous to the classical Lebesgue's theorem (1903).

2.1. The Notion of an Absolutely Continuous Function of a Two-Dimensional Segment

In the sequel, under a segment will be meant a closed non-degenerated two-dimensional segment, or an empty set.

Let on the rectangle $Q = \{(x, y) \in \mathbb{R}^2 : \ a \le x \le b, \ c \le y \le d\}$ there is a real-valued function of segment Φ. This means that to every segment $I = \{(t, \tau) \in Q : x_1 \le t \le x_2, \, y_1 \le \tau \le y_2\} \subset Q$ there corresponds the unique real number $\Phi(I)$, and for an empty set $\varnothing$ we assume that $\Phi(\varnothing) = 0$.

A function of the segment Φ is said to be continuous on the Q, if for an arbitrarily small number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon, \Phi) > 0$, such that for every segment $I \subset Q$ with the property $|I| < \delta$ the inequality $|\Phi(I)| < \varepsilon$ is fulfilled. This fact is written as

$$\lim_{|I|\to 0} \Phi(I) = 0. \tag{1}$$

Two segments $I_1 \subset Q$ and $I_1 \subset Q$ are said to be non-overlapping (without common inner point) if $I_1^\circ \cap I_2^\circ = \varnothing$, where by E^0 is denoted the interior of the set $E \subset \mathbb{R}^2$.

A function of the segment Φ is said to be additive on Q, if for every finite system of pairwise non-overlapping (having pairwise no common interior points) segments $I_1 \subset Q, \dots, I_p \subset Q$ the equality

$$\Phi\Big(\bigcup_{k=1}^{p} I_k\Big) = \sum_{k=1}^{p} \Phi(I_k) \tag{2}$$

is fulfilled.

Definition 2.1.1. A function of the segment Φ is called an absolutely continuous function on the rectangle Q, if to every number $\varepsilon > 0$ there corresponds the number $\eta = \eta(\varepsilon, \Phi) > 0$, such that for every finite system of pairwise non-overlapping segments $I_1 \subset Q, \dots, I_p \subset Q$ with the property $|I_1| + \dots + |I_p| < \eta$ the inequality

$$\sum_{k=1}^{q} |\Phi(I_k)| < \varepsilon \tag{3}$$

is fulfilled.

Obviously, an absolutely continuous function of a segment on the Q is continuous on the Q.

2.2. The Connection Between Functions of a Point and of a Two-Dimensional Segment

With every function $\varphi(x, y)$ of a point defined on the Q, we can connect the function of the segment $\Phi(I)$, $I \subset Q$. This can be done as follows. Taking arbitrary segment $I = \{(x, y) \in Q : x_1 \le x \le x_2,\ y_1 \le y \le y_2\} \subset Q$, we compare the number $\Phi(I)$ by the rule

$$\Phi(I) = \varphi(x_2, y_2) - \varphi(x_1, y_2) - \varphi(x_2, y_1) + \varphi(x_1, y_1). \tag{1}$$

Note that such an comparison is not perfect. The matter is that the right-hand side of equality (1) remains invariable when replacing the function $\varphi(x, y)$ by the function $\psi(x, y) = \varphi(x, y) + \alpha(x) + \beta(y)$, where $\alpha(x)$ and $\beta(y)$ are arbitrary finite functions on the $[a, b]$ and $[c, d]$, respectively.

This fact shows that for the definition an absolutely continuous on the Q function $\varphi(x, y)$ of a point, there is no need to be satisfied with the absolute continuity on the Q of the corresponding function of the segment $\Phi(I)$. This defect can be corrected as follows.

Definition 2.2.1 ([24], p. 246). A function $\varphi(x, y)$ of a point defined on the Q is said to be absolutely continuous on the Q, if the corresponding function of the segment $\Phi(I)$ defined by equality (1) is absolutely continuous on the Q, and the functions $\varphi(x, c)$ and $\varphi(a, y)$ are absolutely continuous on $[a, b]$ and $[c, d]$, respectively.

Hence equality (1) provides us with the function of the segment by means of the function of a point.

The converse is also possible, i.e. to get the function of a point through the function of a segment. Indeed, let there be a function of the segment $\Psi(I)$, $I \subset Q$. We take any point $(x,y) \in Q$, where $a < x \le b$, $c < y \le d$. Suppose

$$\begin{aligned} \psi(x,y) = \Psi(I_{x,y}) \ \text{ for } \ I_{x,y} = \big\{(t,\tau) \in Q :\ a < t \le x,\ c < \tau \le y\big\}, \\ \psi(x,c) = 0 \ \text{ for } \ a \le x \le b, \\ \psi(a,y) = 0 \ \text{ for } \ c \le y \le d, \end{aligned} \tag{2}$$

The defined in such a way function $\psi(x,y)$ satisfies equality (1) with the left-hand side $\Psi(I)$, $I \subset Q$.

Obviously, the absolutely continuous on the Q function $\varphi(x,y)$ is uniformly continuous, in particular, continuous on the Q.

2.3. Representation of an Absolutely Continuous Function of a Point and Summability of Its Partial Derivatives

To every absolutely continuous on the Q function $\Phi(x,y)$, there corresponds a triple of functions $\varphi \in L(Q)$, $g \in L([a,b])$, $h \in L([c,d])$ such that the equality ([24], p. 246)

$$\Phi(x,y) = \int_a^x \int_c^y \varphi(t,\tau)\,dt\,d\tau + \int_a^x g(t)\,dt + \int_c^y h(\tau)\,d\tau + \Phi(a,c) \tag{1}$$

holds, and vice versa.

Applying statement (1) from Theorem 1.1.1 to the double integral from equality (1) and Lebesgue's theorem (1903) to the first ordinary integral, we establish the existence of a measurable set $e_1^* \subset [a,b]$ with $|e_1^*| = b - a$, such that the function $\Phi'_x(x,y)$ is finite at every point (x,y) with $x \in e_1^*$ and $c \le y \le d$, and

$$\Phi'_x(x,y) = \int_c^y \varphi(x,\tau)\,d\tau + g(x), \quad x \in e_1^*, \quad c \le y \le d. \tag{2}$$

Analogously, applying statement (2) from Theorem 1.1.1 to the double integral and Lebesgue's theorem to the second ordinary integral, we establish the existence of a measurable set $e_2^* \subset [c,d]$ with $|e_2^*| = d - c$, such that the function $\Phi'_y(x,y)$ is finite at every point (x,y) with $a \le x \le b$ and $y \in e_2^*$, and

$$\Phi'_y(x,y) = \int_a^x \varphi(t,y)\,dt + h(y), \quad a \le x \le b, \quad y \in e_2^*. \tag{3}$$

Finally, there exists the measurable set $E^* \subset Q$ with $|E^*| = |Q|$, such that at every point $(x, y) \in E^*$ we have

$$\Phi''_{x,y}(x, y) = \varphi(x, y) = \Phi''_{y,x}(x, y), \tag{4}$$

with finite terms.

The following proposition is obvious.

Proposition 2.3.1. *For the absolutely continuous on the Q function $\Phi(x, y)$, defined by equality* (1)*, the functions Φ'_x, Φ'_y and $\Phi''_{x,y} = \Phi''_{y,x}$, defined by equalities* (2)–(4)*, belong to the space $L(Q)$.*

2.4. Differentiability of an Absolutely Continuous Function of Two Variables

Every absolutely continuous on the Q function $\Phi(x, y)$ admits a representation of type 2.3.(1). For an indefinite double integral in the right-hand side of equality 2.3.(1) we prove, by using Theorem 1.2.1 that a total differential exists almost everywhere on the Q. The remaining two functions of one variable are absolutely continuous and therefore have almost everywhere a total differential*. All this can be summarized in the form of the following

Theorem 2.4.1 ([2]). *Every absolutely continuous on the Q function $\Phi(x, y)$ has a total differential almost everywhere on the Q. Its partial and mixed partial derivatives, given on the corresponding sets by equalities* 2.3.(3)–2.3.(4)*, are summable on the Q functions.*

Remark 2.4.1. Theorems 1.2.1 and 2.4.1 are the analogues of the classical Lebesgue's theorem for functions of two variables. One more analogue of that Lebesgue's theorem will be given in Section 5, in the form of equality (4).

Remark 2.4.2. The function $\psi(x, y)$ defined on the rectangle $Q = [a, b] \times [c, d]$ is said to be separately absolutely continuous on the Q, if ψ is absolutely continuous on $[a, b]$ for every fixed $y \in [c, d]$ and absolutely continuous on $[c, d]$ for every fixed $x \in [a, b]$.

Tolstov ([28], p. 50) constructed an example of a separately absolutely continuous function in the form of a repeated integral, which is discontinuous almost everywhere on the Q.

---

*Assume $F(x, y) = \alpha(x)$ for $(x, y) \in Q$. Then we have the relations

$$\frac{F(x_0 + h, y_0 + k) - F(x_0, y_0) - \alpha'(x_0)h}{|h| + |k|} = \frac{\alpha(x_0 + h) - \alpha(x_0) - \alpha'(x_0)h}{|h| + |k|} =$$
$$= \frac{o(h)}{|h| + |k|} \to 0, \text{ as } (h, k) \to (0, 0).$$

§ 3. The Finiteness of a Strong Gradient of an Indefinite Integral and of an Absolutely Continuous Function

3.1. Separately Strong Differentiability of an Indefinite Integral

As is already known (see Ch.II, 3.3), the property of the function to have a total differential at the given point is weaker than the property to have a finite strong gradient at the same point.

It is also stated that an indefinite double integral has a total differential almost everywhere (see Theorem 1.2.1).

Proceeding from these two statements, there naturally arises a question: whether an indefinite double integral has a finite almost everywhere strong gradient?

The answer is positive: the indefinite double integral is almost everywhere separately strong partial differentiable one (see Ch. II, Definition 3.1.4).

Theorem 3.1.1 ([2]). *Let the function $f(x,y)$ be summable on the rectangle $Q = \{(x,y) \in \mathbb{R}^2 : \ a \le x \le b,\, c \le y \le d\}$. Then the corresponding indefinite integral*

$$F(x,y) = \int\limits_a^x \int\limits_c^y f(t,\tau)\,dtd\tau \tag{1}$$

possesses the following properties:

1) *for almost every $x_0 \in [a,b]$ and for every $y_0 \in [c,d]$ the $F'_{[x]}(x_0,y_0)$ is finite, and*

$$F'_{[x]}(x_0,y_0) = \int\limits_c^{y_0} f(x_0,\tau)\,d\tau; \tag{2}$$

2) *for every $x_0 \in [a,b]$ and for almost every $y_0 \in [c,d]$ the $f'_{[y]}(x_0,y_0)$ is finite, and*

$$F'_{[y]}(x_0,y_0) = \int\limits_a^{x_0} f(t,y_0)\,dt; \tag{3}$$

3) *at almost every point $(x_0,y_0) \in Q$ the $\operatorname{str}\operatorname{grad} F(x_0,y_0)$ is finite.*

In proving this theorem we use the following Tolstov's lemma.

Lemma 3.1.1 ([28] Lemma 15 and Remark on page 89). *For $(x,y) \in Q$, let*

$$\Phi(x,y) = \int\limits_a^x \varphi(t,y)\,dt, \tag{4}$$

where it is assumed that:

A) *for almost every fixed $x \in [a,b]$ the function $\varphi(x,y)$ with respect to y is continuous on the $[c,d]$;*

B) *there exists the summable on the* $[a,b]$ *function* $M(x)$*, such that the inequality*

$$\big|\varphi(x,y)\big| \le \mathcal{M}(x) \tag{5}$$

is fulfilled for almost every $x \in [a,b]$ *and for all* $y \in [c,d]$.

Then for every arbitrarily small number $\eta > 0$ *there exists a perfect set* $E \subset [a,b]$ *with* $|E| > b-a-\eta$*, such that the equality*

$$\lim_{h\to 0} \frac{1}{h}\big[\Phi(x+h,y)-\Phi(x,y)\big] = \varphi(x,y) \tag{6}$$

is fulfilled uniformly with respect to (x,y)*, where* $x \in E$ *and* $c \le y \le d$.

On the base of that lemma we prove

Lemma 3.1.2 ([2]). *Under the conditions of Lemma* 3.1.1*, there exists a measurable set* $e \subset [a,b]$ *with* $|e| = b-a$*, such that at every point* (x_0,y_0) *with* $x_0 \in e$ *and* $y_0 \in [c,d]$ *the* $\Phi'_{[x]}(x_0,y_0)$ *is finite, and*

$$\Phi_{[x]}(x_0,y_0) = \varphi(x_0,y_0). \tag{7}$$

Proof. It follows from Lemma 3.1.1 that for almost every fixed $x \in [a,b]$ the equality

$$\lim_{h\to 0} \frac{1}{h}\big[\Phi(x+h,y)-\Phi(x,y)\big] = \varphi(x,y) \tag{8}$$

is fulfilled uniformly with respect to $y \in [c,d]$. This means that for almost every fixed such x and for any arbitrarily small number $\varepsilon > 0$ there exists $h_0 = h_0(\varepsilon,x) > 0$, such that

$$\Big|\frac{\Phi(x+h,y)-\Phi(x,y)}{h} - \varphi(x,y)\Big| < \varepsilon \tag{9}$$

for $c \le y \le d$ and $0 < h < h_0$.

Let e denote a set of all x's from $[a,b]$, for each of which equality (8) and condition A) from Lemma 3.1.1 are fulfilled simultaneously. The equality $|e| = b-a$ is obvious.

Let us take an arbitrary point (x_0,y_0) with $x_0 \in e$ and $c \le y_0 \le d$. Since the function of one variable $\varphi(x_0,y)$ is continuous at the point y_0, for the same ε there exists the number $\delta = \delta(\varepsilon,x_0,y_0) > 0$, such that the condition $|y-y_0| < \delta$ implies

$$\big|\varphi(x_0,y)-\varphi(x_0,y_0)\big| < \varepsilon. \tag{10}$$

From estimates (9) and (10) follows the equality

$$\lim_{\substack{h\to 0\\ y\in y_0}} \frac{1}{h}\big[\Phi(x_0+h,y)-\Phi(x_0,y)\big] = \varphi(x_0,y_0), \tag{11}$$

which is equivalent to (7). $\square$

Proof of Theorem 3.1.1. We write the function $F(x,y)$ defined by equality (1) as

$$F(x,y)=\int\limits_a^x \varphi(t,y)\,dt, \tag{12}$$

where

$$\varphi(t,y)=\int\limits_c^y f(t,\tau)\,d\tau. \tag{13}$$

By virtue of Fubini's theorem, from the assumption $f\in L(Q)$ follows that for almost all t from the $[a,b]$ the integral $\int\limits_c^d f(t,\tau)\,d\tau$ is finite. Thus the function $\varphi(t,y)$ for such t is continuous with respect to y on the $[c,d]$. Hence for almost all t from the $[a,b]$ the function $\varphi(t,y)$ is continuous with respect to y on the $[c,d]$.

Consequently, condition A) of Lemma 3.1.1 is fulfilled.

The fulfilment of condition B) follows, by Fubini's theorem, from the relations

$$\big|\varphi(t,y)\big|\le\int\limits_c^y \big|f(t,\tau)\big|\,d\tau\le\int\limits_c^d \big|f(t,\tau)\big|\,dt\equiv\mathcal{M}(t)\in L\big([c,d]\big).$$

Thus the conditions of Lemma 3.1.1 are fulfilled. Therefore we have equality (7) in which the function Φ is replaced by F and the value $\varphi(x_0,y_0)$ by $\int\limits_c^{y_0} f(x_0,\tau)d\tau$.

So, statement 1) of Theorem 3.1.1 takes place at every point (x_0,y_0) with $x_0\in e$ and $c\le y_0\le d$.

Statement 2) of Theorem 3.1.1 is established analogously, and statement 3) of the same theorem is the consequence of statements 1) and 2). $\square$

Remark 3.1.1. The extension of Theorem 3.1.1 to the n-dimensional case has been obtained in [9].

3.2. Corollaries from the Finiteness of a Strong Gradient of an Indefinite Integral

On the base of Theorem 3.1.1 we obtain the following

Theorem 3.2.1 ([2]). *For every summable on the rectangle* $Q=\{(x,y)\in\mathbb{R}^2:\ a\le x\le b, c\le y\le d\}$ *function* $f(x,y)$ *the following statements are valid:*

1) *there exists a measurable set* $e_1\subset[a,b]$ *with* $|e_1|=b-a$*, such that at every point* (x_0,y_0) *with* $x_0\in e_1$ *and* $y_0\in[c,d]$ *the integral* $\int\limits_c^{y_0} f(x_0,\tau)d\tau$

is finite, and

$$\lim_{\substack{h\to 0\\ y\to y_0}} \frac{1}{h}\int_{x_0}^{x_0+h}\int_{c}^{y} f(t,\tau)\,dt\,d\tau = \int_{c}^{y_0} f(x_0,\tau)\,d\tau; \tag{1}$$

2) *there exists a measurable set* $e_2\subset[c,d]$ *with* $|e_2|=d-c$*, such that at every point* (x_0,y_0) *with* $x_0\in[a,b]$ *and* $y_0\in e_2$ *the integral* $\int\limits_a^{x_0} f(t,y_0)\,dt$ *is finite, and*

$$\lim_{\substack{k\to 0\\ x\to x_0}} \frac{1}{k}\int_{y_0}^{y_0+k}\int_{a}^{x} f(t,\tau)dtd\tau = \int_{a}^{x_0} f(t,y_0)\,dt; \tag{2}$$

3) *equalities* (1) *and* (2) *are fulfilled simultaneously at the points* $(x_0,y_0)\in E$*, where* $E=e_1\times e_2$*,* $|E|=|Q|$*.*

To formulate this and the subsequent theorems in short, we introduce the following measurable sets:

A)
$$E_1=\bigcup_{x_0\in e_1} m(x_0),\ |E_1|=|Q|, \tag{3}$$

where the measurable set $e_1\subset[a,b]$ with $|e_1|=b-a$ is adopted from statement 1) of Theorem 3.2.1, and the vertical closed interval $m(x_0)$ is defined by the equality

$$m(x_0)=\big\{(x_0,y):\ c\le y\le d\big\}; \tag{4}$$

B)
$$E_2=\bigcup_{y_0\in e_2} n(y_0),\ |E_2|=|Q|, \tag{5}$$

where the measurable set $e_2\subset[c,d]$ with $|e_2|=d-c$ is adopted from statement 2) of Theorem 3.2.1, and the horizontal closed interval $n(y_0)$ is defined by the equality

$$n(y_0)=\big\{(x,y_0):\ a\le x\le b\big\}. \tag{6}$$

Now Theorem 3.2.1 can be rephrased as follows.

Theorem 3.2.2 ([2], Remark 6.2). *For every function* $f\in L(Q)$*, equalities* (1) *and* (2) *take place at the points* $(x_0,y_0)\in E_1$ *and* $(x_0,y_0)\in E_2$*, respectively. Equalities* (1) *and* (2) *are fulfilled simultaneously at the points* $(x_0,y_0)\in E_3$*, where* $E_3=E_1\cap E_2$*,* $|E_3|=|Q|$*.*

Theorem 3.2.3 ([2]). *For every function* $f\in L(Q)$ *the following statements take place:*

1) *at the points* $(x_0,y_0)\in E_1$ *the equality*

$$\lim_{(h,k)\to(0,0)} \frac{1}{h}\int_{x_0}^{x_0+h}\int_{y_0}^{y_0+k} f(t,\tau)\ dt\,d\tau = 0 \tag{7}$$

holds;

2) *at the points* $(x_0, y_0) \in E_2$ *the equality*

$$\lim_{(h,k)\to(0,0)} \frac{1}{k} \int\limits_{x_0}^{x_0+h} \int\limits_{y_0}^{y_0+k} f(t,\tau)\, dt\, d\tau = 0 \tag{8}$$

is valid;

3) *equalities* (7) *and* (8) *are fulfilled simultaneously at the points* $(x_0, y_0) \in E_3$, *where* $E_3 = E_1 \cap E_2$, $|E_3| = |Q|$*;*

4) *at the points* $(x_0, y_0) \in E_3$ *the equality*

$$\lim_{(h,k)\to(0,0)} \frac{h+k}{hk} \int\limits_{x_0}^{x_0+h} \int\limits_{y_0}^{y_0+k} f(t,\tau)\, dt\, d\tau = 0 \tag{9}$$

holds.

Proof. In the left-hand side of equality (1), the integral on the segment $[c, y]$ represent as a sum of integrals on the segments $[c, y_0]$ and $[y_0, y_0 + k]$.

To the first double integral with the coefficient h^{-1} we apply equality 1.2.(2) and the limit in this case will be equal to the right-hand side of equality (1). This means that equality (7) is fulfilled.

Equality (8) can be proved in a similar way.

Statement 3) follows from statements 1) and 2), and equality (9) is obtained from equalities (7) and (8). □

Remark 3.2.1 ([2]). If $S(x, y) \in L(Q)$ is the Saks' function ([10], p. 96; [21], p. 133), then the expression

$$\frac{1}{hk} \int\limits_{x}^{x+h} \int\limits_{y}^{y+k} S(t,\tau) dt d\tau \tag{10}$$

has the strong upper limit $+\infty$ at every point $(x, y) \in Q$. At the same time, Theorem 3.2.3 shows that tending off expression (10) to $+\infty$ is subordinate to equalities (7) and (8) at the points $(x_0, y_0) \in E_1 \cap E_2$, i.e.,

$$\frac{1}{hk} \int\limits_{x_0}^{x_0+h} \int\limits_{y_0}^{y_0+k} S(t,\tau) dt d\tau = o\Big(\frac{1}{\max(h,k)}\Big). \tag{11}$$

3.3. The Finiteness of a Strong Gradient of an Absolutely Continuous Function

Since a derivative of an arbitrary function of one variable can be interpreted as a strong partial derivative (see 3.1 of Chapter II), equalities 2.3.(1)-2.3.(4) and Theorem 3.1.1 allow us to formulate the following result.

Theorem 3.3.1 ([2]). *To every absolutely continuous on the rectangle Q function $\Phi(x,y)$ there corresponds a triple of functions $\varphi \in L(Q)$, $g \in L([a,b])$ and $h \in L([c,d])$, such that the following statements take place:*

1) *for almost every $x_0 \in [a,b]$ and for every $y_0 \in [c,d]$ there exists the finite $\Phi'_{[x]}(x_0,y_0)$, and*

$$\Phi'_{[x]}(x_0,y_0) = \int\limits_c^{y_0} \varphi(x_0,y)\,dy + g(x_0); \tag{1}$$

2) *for every $x_0 \in [a,b]$ and for almost every $y_0 \in [c,d]$ there exists the finite $\Phi'_{[y]}(x_0,y_0)$, and*

$$\Phi'_{[y]}(x_0,y_0) = \int\limits_a^{x_0} \varphi(x,y_0)\,dx + h(y_0); \tag{2}$$

3) *at almost every point $(x_0,y_0) \in Q$ the* $\operatorname{str\,grad}\Phi(x_0,y_0)$, $\Phi''_{x,y}(x_0,y_0)$ *and $\Phi''_{y,x}(x_0,y_0)$ are finite, and*

$$\Phi''_{x,y}(x_0,y_0) = \varphi(x_0,y_0) = \Phi''_{y,x}(x_0,y_0). \tag{3}$$

§ 4. Lebesgue's Intense Points and Finiteness at These Points of a Strong Gradient of an Indefinite Integral

To the Lebesgue's theorem on a regular derivative for an indefinite double integral ([21], p. 118) there corresponds the notion of the Lebesgue's point in the weak sense, or more exactly, we call the point $(x_0,y_0) \in Q$ the Lebesgue's point in the weak of the function $f \in L(Q)$, $Q = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, c \le y \le d\}$, if

$$\frac{1}{|I|} \iint\limits_I \big|f(x,y) - f(x_0,y_0)\big| dxdy \to 0, \quad I \in J(x_0,y_0) \tag{$*$}$$

no matter how regularity contracting to the point $(x_0,y_0) \in Q$ the standard* system $J(x_0,y_0)$ of rectangles or disks, containing the point (x_0,y_0).

A set of such points is sometimes denoted by $L(f)$. It is well known that $|L(f)| = |Q|$ (see, for e.g., [21], p. 118; [10], p. 39; [15], p. 343; [23], p. 12; [18]).

On the base of Jessen–Marcinkiewicz–Zygmund theorem ([12], [21], p. 148), relation $(*)$ remains valid for the standard non-regular system $\{I(x_0,y_0)\}$ under the condition that the function f belongs to a narrow (compared with $L(Q)$) class $L\ln^+ L$.

Consequently, in the first case the function $f \in L(Q)$ is arbitrary, and contracting to the point $(x_0,y_0) \in Q$ the standard system $J(x_0,y_0)$ is regular. In the second case, the standard system $J(x_0,y_0)$ is arbitrary, and on the function f we impose the condition $f \in L\ln^+ L$.

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*A system of rectangles is standard one if the sides of each of rectangle are parallel to the coordinate axes.

Just this the author meant by saying "the Lebesgue's point in the weak".

4.1. Lebesgue's Intense Points of Summable Functions of Two Variables*

Definition 4.1.1 ([2]). Let the function $f(x,y)$ belong to the space $L^p(Q)$ for some $p \geq 1$.

The point $(x_0, y_0) \in Q$ is called jointly Lebesgue's intense point (of p-th degree) of the function f, symbolically $(x_0, y_0) \in \operatorname{int} L^p_{x,y}(f)$, if the following two conditions are fulfilled:

$$\lim_{(h,k)\to(0,0)} \frac{1}{h} \int\limits_{x_0}^{x_0+h} \left| \int\limits_{c}^{y_0+k} f(x,y)\,dy - \int\limits_{c}^{y_0} f(x_0,y)\,dy \right|^p dx = 0, \tag{1}$$

$$\lim_{(h,k)\to(0,0)} \frac{1}{k} \int\limits_{y_0}^{y_0+k} \left| \int\limits_{a}^{x_0+h} f(x,y)\,dx - \int\limits_{a}^{x_0} f(x,y_0)\,dx \right|^p dy = 0. \tag{2}$$

When equality (1) is fulfilled, then the point (x_0, y_0) is called Lebesgue's intense point with respect to the variable x (of p-th degree) of the function f, symbolically $(x_0, y_0) \in \operatorname{int} L^p_x(f)$.

When equality (2) is fulfilled, then the point (x_0, y_0) is called Lebesgue's intense point with respect to the variable y (of p-th degree) of the function f, symbolically $(x_0, y_0) \in \operatorname{int} L^p_y(f)$.

Theorem 4.1.1 ([2]). *Let the function $f(x,y)$ belong to the space $L^p(Q)$ for some $p \geq 1$. Then the following statements take place:*

1) *there exists a measurable set $e_1^* \subset [a,b]$ with $|e_1^*| = b-a$, such that the set of all points (x_0, y_0) with $x_0 \in e_1^*$ and $y_0 \in [c,d]$ forms the set $\operatorname{int} L^p_x(f)$, $|\operatorname{int} L^p_x(f)| = |Q|$;*

2) *there exists a measurable set $e_2^* \subset [c,d]$ with $|e_2^*| = d-c$, such that the set of all points (x_0, y_0) with $x_0 \in [a,b]$ and $y_0 \in e_2^*$ forms the set $\operatorname{int} L^p_y(f)$, $|\operatorname{int} L^p_y(f)| = |Q|$.*

3) *the set of all points (x_0, y_0) with $x_0 \in e_1^*$ and $y_0 \in e_2^*$ forms the set $\operatorname{int} L^p_{x,y}(f)$, $|\operatorname{int} L^p_{x,y}(f)| = |Q|$.*

Proof. 1) By Fubini's theorem, there exists the measurable set $E_0 \subset [a,b]$ with $|E_0| = b - a$, such that for every $x_0 \in E_0$ the integral

$$\int\limits_{c}^{d} \big|f(x_0,y)\big|\,dy \tag{3}$$

is finite.

*In [2], there occurs "in the strong sense".

Let $(Q_n(y))_{n=1}^{\infty}$ be a set of all polynomials with rational coefficients. To the function $|f(x,y)-Q_n(y)|^p \in L(Q)$ we apply statement 1) of Theorem 3.2.1. According to that statement, there exists the measurable set $E_n \subset [a,b]$ with $|E_n| = b-a$, such that at every point (x_0,y_0) with $x_0 \in E_n$ and $y_0 \in [c,d]$ the equality

$$\lim_{(h,k)\to(0,0)} \frac{1}{h} \int\limits_{x_0}^{x_0+h} \int\limits_{c}^{y_0+k} \bigl|f(x,y)-Q_n(y)\bigr|^p \, dxdy = \int\limits_{c}^{y_0} \bigl|f(x_0,y)-Q_n(y)\bigr|^p \, dy$$

holds.

Introduce the set $E^* = \bigcap\limits_{n=1}^{\infty} E_n$. We have $|E^*| = b-a$ and the equality

$$\lim_{(h,k)\to(0,0)} \frac{1}{h} \int\limits_{x_0}^{x_0+h} \int\limits_{c}^{y_0+k} \bigl|f(x,y)-Q_n(y)\bigr|^p \, dxdy = \int\limits_{c}^{y_0} \bigl|f(x_0,y)-Q_n(y)\bigr|^p \, dy \tag{4}$$

is fulfilled for (x_0,y_0) with $x_o \in E^*$, $y_0 \in [c,d]$ and $n = 1,2,\dots$

For every point $x_0 \in E_0$ with finite integral (3) and for every number $\varepsilon > 0$ there exists the polynomial $Q_m(y)$, $m = m(x_0,\varepsilon)$ with rational coefficients, such that

$$\int\limits_{c}^{d} \bigl|f(x_0,y)-Q_m(y)\bigr|^p \, dy < \varepsilon.$$

Hence

$$\int\limits_{c}^{y_0} \bigl|f(x_0,y)-Q_m(y)\bigr|^p \, dy < \varepsilon \tag{5}$$

for all $y_0 \in [c,d]$.

Consider now the set $e_1^* = E^* \cap E_0$, $|e_1^*| = b-a$. Let $x_0 \in e_1^*$ and $y_0 \in [c,d]$. We have

$$\Bigl| \int\limits_{c}^{y_0+k} f(x,y)\, dy - \int\limits_{c}^{y_0} f(x_0,y)\, dy \Bigr| \le \int\limits_{c}^{y_0+k} \bigl|f(x,y)-Q_m(y)\bigr| \, dy +$$

$$+ \int\limits_{c}^{y_0} \bigl|f(x_0,y)-Q_m(y)\bigr| \, dy + \int\limits_{y_0}^{y_0+k} \bigl|Q_m(x,y)\bigr| \, dy \equiv f_1 + f_2 + f_3.$$

Next,

$$\left(\int_{x_0}^{x_0+h}\left|\int_c^{y_0+k} f(x,y)\,dy-\int_c^{y_0} f(x_0,y)\,dy\right|^p dx\right)^{1/p}\le$$

$$\le\left(\int_{x_0}^{x_0+h}\big(f_1+f_2+f_3\big)^p\,dx\right)^{1/p}\le\left(\int_{x_0}^{x_0+h} f_1^p dx\right)^{1/p}+h^{1/p}\cdot(f_2+f_3).$$

Thus

$$\left(\frac{1}{h}\int_{x_0}^{x_0+h}\left|\int_c^{y_0+k} f(x,y)\,dy-\int_c^{y_0} f(x_0,y)\,dy\right|^p dx\right)^{1/p}\le$$

$$\le\left(\frac{1}{h}\int_{x_0}^{x_0+h} f_1^p\,dx\right)^{1/p}+f_2+f_3.$$

But

$$f_1^p=\left(\int_c^{y_0+k}\big|f(x,y)-Q_m(y)\big|\,dy\right)^p\le$$

$$\le\big(y_0+k-c\big)^{p-1}\int_c^{y_0+k}\big|f(x,y)-Q_m(y)\big|^p\,dy,$$

$$f_2=\int_c^{y_0}\big|f(x_0,y)-Q_m(y)\big|\,dy\le\big(y_0-c\big)^{\frac{p-1}{p}}\left(\int_c^{y_0}\big|f(x_0,y)-Q_m(y)\big|^p\,dy\right)^{1/p},$$

$$f_3=\int_{y_0}^{y_0+k}\big|Q_m(y)\big|\,dy\le k^{\frac{p-1}{p}}\left(\int_{y_0}^{y_0+k}\big|Q_m(y)\big|^p\,dy\right)^{1/p}.$$

For all $y_0\in[c,d]$, the last integral is less than the number ε, if values of k are sufficiently small.

Therefore taking into account (4) and (5), we have

$$\frac{1}{h}\int_{x_0}^{x_0+h}\left|\int_c^{y_0+k} f(x,y)\,dy-\int_c^{y_0} f(x_0,y)\,dy\right|^p dx\le$$

$$\le 2\varepsilon\big[(y_0+k-c)^{p-1}+(y_0-c)^{p-1}+k^{p-1}\big]\le 6\varepsilon(d-c)^{p-1}.$$

Thus we have established statement 1). Statement 2) is established analogously. Statement 3) is the consequence of the previous two statements. □

4.2. The Finiteness of a Strong Gradient of an Indefinite Integral at Lebesgue's Intense Points

Taking into account Theorems 3.1.1 and 3.2.1, from Theorem 4.1.1 for the case $p = 1$ we obtain the following

Theorem 4.2.1 ([2]). *Let the function $f(x, y) \in L(Q)$. Then the corresponding indefinite integral*

$$F(x,y) = \int\limits_a^x \int\limits_c^y f(t,\tau)\,dt d\tau \tag{1}$$

possesses the following properties:

1) *at every point $(x_0, y_0) \in \operatorname{int} L_x(f)$ the $F'_{[x]}(x_0, y_0)$ is finite and*

$$F'_{[x]}(x_0,y_0) = \int\limits_c^{y_0} f(x_0,\tau)\,d\tau, \tag{2}$$

or what is the same,

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{h} \int\limits_{x_0}^{x_0+h} \int\limits_c^{y_0+k} f(t,\tau)\,dt d\tau = \int\limits_c^{y_0} f(x_0,\tau)\,d\tau; \tag{3}$$

2) *at every point $(x_0, y_0) \in \operatorname{int} L_y(f)$ the $F'_{[y]}(x_0, y_0)$ is finite and*

$$F'_{[y]}(x_0,y_0) = \int\limits_a^{x_0} f(t,y_0)\,dt, \tag{4}$$

or what is the same,

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{k} \int\limits_{y_0}^{y_0+k} \int\limits_a^{x_0+h} f(t,\tau)\,dt d\tau = \int\limits_a^{x_0} f(t,y_0)\,dt; \tag{5}$$

3) *at every point $(x_0, y_0) \in \operatorname{int} L_{x,y}(f)$ the* $\operatorname{str\,grad} F(x_0, y_0)$ *is finite, in particular, there exists $dF(x_0, y_0)$.*

Definition 4.2.1. One-dimensional segment

$$m^*(x_0) = \big\{(x_0,y):\ x_0 \in e_1^*,\ c \le y \le d\big\}, \tag{6}$$

where e_1^* is adopted from statement 1) of Theorem 4.1.1, is called a vertical Lebesgue's segment of the function $f(x, y) \in L(Q)$, corresponding to x_0.

In just the same way, one-dimensional segment

$$n^*(y_0) = \big\{(x,y_0):\ a \le x \le b,\ y_0 \in e_2^*\big\}, \tag{7}$$

is called a horizontal Lebesgue's segment of the function $f(x, y) \in L(Q)$, corresponding to y_0.

If we introduce measurable sets

$$E_1^* = \bigcup_{x_0 \in e_1^*} m^*(x_0), \quad E_2^* = \bigcup_{y_0 \in e_2^*} n^*(y_0), \quad E_3^* = E_1^* \cap E_2^*, \tag{8}$$

then Theorem 4.1.1 can be formulated as follows.

Theorem 4.2.2. *For every function $f(x,y) \in L^p(Q)$ with $p \geq 1$ the following statements are valid:*

1) $\operatorname{int} L_x^p(f) = E_1^*$, $|E_1^*| = |Q|$;

2) $\operatorname{int} L_y^p(f) = E_2^*$, $|E_2^*| = |Q|$;

3) $\operatorname{int} L_{x,y}^p(f) = E_3^*$, $|E_3^*| = |Q|$.

It is clear that $\operatorname{int} L_x^p(f)$ consists of vertical, while $\operatorname{int} L_y^p(f)$ of horizontal Lebesgue's segments.

Further, Theorem 4.1.1 shows that for every function $f(x,y) \in L(Q)$ almost all vertical segments from Q are the vertical Lebesgue's segments. Just the same can be said about the horizontal Lebesgue's segments.

Now Theorem 3.2.3 can be strengthened as follows.

Theorem 4.2.3 ([2]). *For every function $f(x,y) \in L^p(Q)$ with $p \geq 1$ the following statements are valid:*

1) *at every point $(x_0,y_0) \in \operatorname{int} L_x^p(f)$ the equality*

$$\lim_{(h,k)\to(0,0)} \frac{1}{h} \int_{x_0}^{x_0+h} \left| \int_{y_0}^{y_0+k} f(x,y)dy \right|^p dx = 0 \tag{9}$$

holds;

2) *at every point $(x_0,y_0) \in \operatorname{int} L_y^p(f)$ we have*

$$\lim_{(h,k)\to(0,0)} \frac{1}{k} \int_{y_0}^{y_0+k} \left| \int_{x_0}^{x_0+h} f(x,y)\,dx \right|^p dy = 0; \tag{10}$$

3) *at every point $(x_0,y_0) \in \operatorname{int} L_{x,y}^p(f)$ equalities* (9) *and* (10) *are fulfilled simultaneously.*

Proof. The way of proving equality 4.1.(1) indicates that equality 4.1.(1) remains valid, if we substitute c by an arbitrary value c_1 from $[c,d]$, in particular, by y_0. The same can be said about equality 4.1.(2). As a result we obtain equalities (9) and (10). □

§ 5. The Differentiability of An Indefinite Integral with a Parameter

1. The rule of derivation under the integral sign has been introduced for the first time by Leibniz, when the integrand and its partial derivative with respect to parameter are simultaneously continuous on a rectangle. This rule is known as Leibniz's rule for an integral.

It is of interest that there exists the function $\psi(x,y)$ with the property

$$\frac{d}{dy}\int\limits_a^b \psi(x,y)\,dx \neq \int\limits_a^b \frac{\partial}{\partial y}\psi(x,y)\,dx,$$

although both integrals exist in the sense of Riemann.

C. de la Vallée Poussin (1916) generalized Leibniz's rule to the L-integral (Lebesgue integral) as follows.

Theorem 5.1 ([31], p. 110). *Let the function $f(x,y)$ be summable with respect to x on the segment $[a,b]$ for every fixed $y\in[c,d]$. Consider a finite on the segment $[c,d]$ function, the definite integral with a parameter y,*

$$\varphi(y)=\int\limits_a^b f(x,y)\,dx, \quad y\in[c,d]. \tag{1}$$

Assume that the following conditions are fulfilled:

1) *the function $f(x,y)$ is absolutely continuous with respect to the variable y on the $[c,d]$ for every $x\in[a,b]$;*

2) *the partial derivative $f'_y(x,y)$ with respect to the variable y is the summable function on the rectangle $Q=\{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ c\le y\le d\}$.*

Then for almost all $y\in[c,d]$ the equality

$$\varphi'(y)=\int\limits_a^b f'_y(x,y)\,dx \tag{2}$$

holds.

2. Here we extend the statement of Theorem 5.1 under the same assumptions 1) and 2). This extension in the form of equality (4) involves generalization of the classical Lebesgue's theorem.

Theorem* 5.2. *Let assumptions* 1) *and* 2) *of Theorem* 5.1 *be fulfilled. Then a finite on $[a,b]\times[c,d]$ function, an indefinite integral with a parameter y,*

$$P(x,y)=\int\limits_a^x f(t,y)\,dt \tag{3}$$

possesses the following properties:

1) *for almost all $x_0\in[a,b]$ and for all $y_0\in[c,d]$ the equality*

$$P'_{[x]}(x_0,y_0)=f(x_0,y_0) \tag{4}$$

with finite terms holds;

* This result is published by the author for the first time.

2) *for all* $x_0 \in [a,b]$ *and for almost all* $y_0 \in [a,b]$ *the equality*

$$P'_{[y]}(x_0,y_0) = \int\limits_a^{x_0} \frac{\partial}{\partial y} f(t,y_0)\,dt \tag{5}$$

with finite terms is fulfilled;

3) *at almost all points* $(x_0,y_0) \in Q$ *the* $\operatorname{str\,grad} P(x_0,y_0)$ *is finite, in particular, there exists the total differential* $dP(x_0,y_0)$.

Proof. 1) Taking into account condition 1) of Theorem 5.1, we have the following relations:

$$\frac{P(x_0+h,y_0+k)-P(x_0,y_0+k)}{h} = \frac{1}{h}\int\limits_{x_0}^{x_0+h} f(t,y_0+k)\,dt =$$

$$= \frac{1}{h}\int\limits_{x_0}^{x_0+h} \big[f(t,y_0+k)-f(t,y_0)\big]\,dt + \frac{1}{h}\int\limits_{x_0}^{x_0+h} f(t,y_0)\,dt =$$

$$= \frac{1}{h}\int\limits_{x_0}^{x_0+h}\bigg(\int\limits_{y_0}^{y_0+k} f'_\tau(t,\tau)d\tau\bigg)dt + \frac{1}{h}\int\limits_{x_0}^{x_0+h} f(t,y_0)\,dt \equiv$$

$$\equiv A_{h,k}(x_0,y_0) + B_h(x_0,y_0).$$

By assumption 2) of Theorem 5.1, we have $f'_\tau \in L(Q)$. Therefore

$$A_{h,k}(x_0,y_0) = \frac{1}{h}\int\limits_{x_0}^{x_0+h}\int\limits_{y_0}^{y_0+k} f'_\tau(t,\tau)\,dt\,d\tau. \tag{6}$$

The right-hand side of equality (6) tends to zero for almost all $x_0 \in [a,b]$ and for all $y_0 \in [c,d]$, as $(h,k) \to (0,0)$ (see Theorem 3.2.3).

Moreover, the finiteness of integral (1) for all $y \in [c,d]$ implies the finiteness of the indefinite integral with a parameter (see equality (3))

$$\int\limits_a^x f(t,y)\,dt = P(x,y) \tag{7}$$

for all $(x,y) \in Q$. Assumptions 1) and 2) of Theorem 5.1 result in the equality

$$\int\limits_a^x f(t,y)\,dt = \int\limits_a^x\int\limits_c^y f'_\tau(t,\tau)\,dtd\tau + \int\limits_a^x f(t,c)\,dt \tag{8}$$

for all $(x,y) \in Q$. By statement 1) from Tolstov's Theorem 1.1.1 we have

$$\frac{\partial}{\partial x}\int\limits_a^x\int\limits_c^y f'_\tau(t,\tau)\,dtd\tau = \int\limits_c^y f'_\tau(x,\tau)\,d\tau = f(x,y)-f(x,c) \tag{9}$$

for all (x,y) with $x \in e_1$ and $y \in [c,d]$, where $|e_1| = b-a$.

Next, By Lebesgue's theorem,

$$\Big(\int_a^x f(t,c)\,dt\Big)' = f(x,c) \tag{10}$$

for all $x \in e \subset [a,b]$, where $|e| = b-a$, and the set e depends on the constant c.

Now from equalities (8) and (10) it follows that

$$\Big(\int_a^x f(t,y)\,dt\Big)'_x = f(x,y) \tag{11}$$

for all (x,y) with $x \in e \cap e_1$ and $y \in [c,d]$.

This means that expression $B_h(x_0,y_0)$ for almost all $x_0 \in [a,b]$ and for all $y_0 \in [c,d]$ tends to $f(x_0,y_0)$, as $h \to 0$. Thus statement 1) is established.

2) Now

$$\frac{P(x_0+h,y_0+k)-P(x_0+h,y_0)}{k} = \frac{1}{k}\int_a^{x_0+h} \big[f(t,y_0+k)-f(t,y_0)\big]\,dt =$$
$$= \frac{1}{k}\int_a^{x_0+h}\Big(\int_{y_0}^{y_0+k} f'_\tau(t,\tau)\,d\tau\Big)\,dt = \frac{1}{k}\int_{y_0}^{y_0+k}\int_a^{x_0} f'_\tau(t,\tau)\,dt\,d\tau +$$
$$+\frac{1}{k}\int_{y_0}^{y_0+k}\int_{x_0}^{x_0+h} f'_\tau(t,\tau)\,dtd\tau \equiv C_k(x_0,y_0) + D_{h,k}(x_0,y_0).$$

By statement 2) of Theorem 1.1.1, the equality

$$\lim_{k\to 0} C_k(x_0,y_0) = \int_a^{x_0} f'_\tau(t,y_0)\,dt$$

holds for all $x_0 \in [a,b]$ and for almost all $y_0 \in [c,d]$ (analogous arguments have been presented for $B_h(x_0,y_0)$).

Further, the equality

$$\lim_{(h,k)\to(0,0)} D_{h,k}(x_0,y_0) = 0$$

for all $x_0 \in [a,b]$ and for almost all $y_0 \in [c,d]$ is analogous to that established above for $A_{h,k}(x_0,y_0)$.

Consequently, statement 2) holds.

3) The finiteness of the $\operatorname{str\,grad} P(x_0,y_0)$ for almost all $(x_0,y_0) \in Q$ follows from statements 1) and 2). Further, the existence of the total differential $dP(x_0,y_0)$ at the same points $(x_0,y_0) \in Q$ follows from the finiteness of the $\operatorname{str\,grad} P(x_0,y_0)$.

Finally, if in equality (3) we put $x = b$, then equality (5) for $x_0 = b$ will take the form of equality (2) because the derivative of the function of one variable is its strong partial derivative with respect to the same variable, since this function can be considered as a function of two variables, constant with respect to the introduced second variable. Thus equality (5) is a generalization of equality (2).

Lebesgue's theorem can be obtained analogously from equality (4), if the function f in equality (3) will be assumed to be independent of the variable y. □

§ 6. The Finiteness and the Continuity of Strong Partial Derivatives of an Indefinite Integral

6.1. Points of Finiteness of Strong Partial Derivatives of an Indefinite Integral

Using Definition 8.1 from Chapter I, we proceed to formulating and proving the following

Lemma 6.1.1 ([2]). *Let the function $f(x,y) \in L(Q)$ be measurable with respect to y on $[c,d]$ for every $x \in [a,b]$ and partial continuous at x_0 with respect to x, uniformly with respect to y on $[c_1,d_1]$, where $c \le c_1 < d_1 \le d$. Then the following statements take place:*

1) *there exists a number $\delta = \delta(x_0) > 0$ such that for all x's with $|x - x_0| < \delta$ the integral*

$$\int_{c_1}^{d_1} f(x,\tau)\,d\tau \tag{1}$$

is finite. Hence the integral

$$\int_{c_1}^{y} f(x,\tau)\,d\tau \tag{2}$$

is likewise finite for $|x - x_0| < \delta$ and $c_1 \le y \le d_1$;

2) *the finite function of two variables*

$$\psi(x,y) = \int_{c_1}^{y} f(x,\tau)\,d\tau, \quad |x - x_0| < \delta, \quad c_1 \le y \le d_1 \tag{3}$$

is continuous at all points (x_0,y_0) with $c_1 < y_0 < d_1$.

Proof. 1) By equality 8.(1) of Chapter I, there exists a number $\delta = \delta(x_0) > 0$ such that under $|x - x_0| < \delta$ and $\tau \in [c_1,d_1]$ the relations $-1 < f(x,\tau) - f(x_0,\tau) < 1$ are fulfilled, and hence

$$f(x_0,\tau) - 1 < f(x,\tau) < f(x_0,\tau) + 1. \tag{4}$$

Since $f \in L(Q)$, the integral $\int\limits_{c_1}^{d_1} f(x,\tau)\,d\tau$ is finite for almost all $x \in [a,b]$, in particular, for almost all x's with $|x - x_0| < \delta$. If such x suppose in (4), then the integral $\int\limits_{c_1}^{d_1} f(x_0,\tau) d\tau$ will be finite. This in its turn, with regard tight-hand of estimates (4), guarantees that integral (1) is finite for all x's with $|x - x_0| < \delta$. This, in particular, imply that integral (2) is finite for $|x - x_0| < \delta$ and $c_1 \le y \le d_1$.

2) If we take any number $\varepsilon > 0$, then for $\varepsilon^* = \varepsilon/(d_1 - c_1)$ there exists $\delta^* = \delta^*(\varepsilon, x_0) > 0$ such that

$$f(x_0,\tau) - \varepsilon^* < f(x,\tau) < f(x_0,\tau) + \varepsilon^* \ \text{ under } \ |x - x_0| < \delta^*, \quad \tau \in [c_1, d_1], \tag{5}$$

by equality 8.(1) from Chapter I.

To show that the function $\psi(x,y)$ is continuous at all points (x_0,y_0) with $c_1 < y_0 < d_1$, we use Theorem 5.1.2 from Chapter I. We have

$$\psi(x_0+h, y_0+k) - \psi(x_0, y_0+k) = \int\limits_{c_1}^{y_0+k} \big[f(x_0+h,\tau) - f(x_0,\tau)\big]\,d\tau, \tag{6}$$

$$\psi(x_0, y_0+k) - \psi(x_0, y_0) = \int\limits_{y_0}^{y_0+k} f(x_0,\tau)\,d\tau. \tag{7}$$

Since for some $\eta_1 = \eta_1(y_0) > 0$ the points $y_0 + k$ belong to the segment $[c_1, d_1]$ under $|k| < \eta_1$ and $c_1 < y_0 < d_1$, an absolute value of the integral in equality (6) is less, by virtue of (5), than $\varepsilon^*(d_1 - c_1) = \varepsilon$ under $|h| < \delta^*$ and $|k| < \eta_1$.

An absolute value of the integral in equality (7) is less, by an absolutely continuity of the integral $\int\limits_{c_1}^{t} f(x_0,\tau) d\tau$, than ε under $|k| < \eta_2$, where $\eta_2 = \eta_2(x_0, \varepsilon) > 0$. $\square$

Remark 6.1.1. In what follows, under the symbol F will be meant the function defined by equality 4.2.(1), where $Q = \{(x,y) \in \mathbb{R}^2 : \ a \le x \le b, \ c \le y \le d\}$.

Theorem 6.1.1 ([2]). *Let the function $f(x,y) \in L(Q)$ be measurable with respect to y on $[c,d]$ for every $x \in [a,b]$ and partial continuous with respect to x at the point x_0, uniformly with respect to y on $[c_1,d_1]$, where $c \le c_1 < d_1 \le d$. Then the function $F'_{[x]}(x_0,y)$ is finite under $c_1 < y < d_1$, and*

$$F'_{[x]}(x_0,y) = \int\limits_{c_1}^{y} f(x_0,\tau)\,d\tau, \quad c_1 < y < d_1. \tag{8}$$

Proof. Because of the fact that integral (2) is finite, the integral in equality (8) is likewise finite, and we obtain (8) from the following equality:

$$\frac{F(x_0+h,y+k)-F(x_0,y+k)}{h}-\int\limits_{c_1}^{y} f(x_0,\tau)\,d\tau=$$

$$=\frac{1}{h}\int\limits_{x_0}^{x_0+h}\int\limits_{c_1}^{y+k}\big[f(t,\tau)-f(x_0,\tau)\big]\,dtd\tau+\int\limits_{y}^{y+k} f(x_0,\tau)\,d\tau. \qquad \square$$

The theorem below is proved analogously.

Theorem 6.1.2 ([2]). *Let the function $f(x,y)\in L(Q)$ be measurable with respect to x on $[a,b]$ for every $y\in[c,d]$ and partial continuous with respect to y at the point y_0, uniformly with respect to x on $[a_1,b_1]$, where $a\le a_1<b_1\le b$. Then the function $F'_{[y]}(x,y_0)$ is finite under $a_1<x<b_1$, and*

$$F'_{[y]}(x,y_0)=\int\limits_{a_1}^{x} f(t,y_0)\,dt, \quad a_1<x<b_1. \tag{9}$$

From the last two theorems we arrive at

Theorem 6.1.3 ([2]). *Let the function $f(x,y)\in L(Q)$ be partial continuous with respect to x at the point x_0, uniformly with respect to $y\in[c,d]$ and partial continuous with respect to y at the point y_0, uniformly with respect to $x\in[a,b]$. Besides, we assume that the function f is measurable with respect to each variable, when the other variable is given arbitrary fixed value.*

Then finite are:

1) $F'_{[x]}(x_0,y)=\int\limits_{c}^{y} f(x_0,\tau)\,d\tau$ *for* $c<y<d$; (10)

2) $F'_{[y]}(x,y_0)=\int\limits_{a}^{x} f(t,y_0)dt$ *for* $a<x<b$; (11)

3) $\operatorname{str}\operatorname{grad}F(x_0,y_0)$ *at* $(x_0,y_0)\in Q^0$, *in particular**, *there exists* $dF(x_0,y_0)$.

6.2. Points of Continuity of Strong Partial Derivatives of an Indefinite Integral

Theorem 6.2.1 ([2]). *Let the function $f(x,y)\in L(Q)$ be continuous on the rectangle $r_1=[a_1,b_1]\times[c,d]$, where $a\le a_1<b_1\le b$. Then $F'_{[x]}(x,y)$*

* By the symbol E^0 is denoted the interior of the set E.

is continuous on the r_1^0, *and*

$$F'_{[x]}(x,y) = \int\limits_c^y f(x,\tau)\,d\tau, \quad (x,y) \in r_1^0. \tag{1}$$

Proof. In Theorem 8.1 of Chapter I, in the capacity of Q we take r_1. Now in Theorem 6.1.1 in the capacity of x_0 we can take arbitrary point of an open interval (a_1, b_1). Therefore equality (1) follows from equality 6.1.(8). The continuity of $F'_{[x]}(x,y)$ on the r_1^0 follows from statement 2) of Lemma 6.1.1, upon substituting x_0 and y_0 by $x \in (a_1, b_1)$ and $y \in (c,d)$, respectively. □

The theorem below is proved analogously.

Theorem 6.2.2 ([2]). *Let the function* $f(x,y) \in L(Q)$ *be continuous on the rectangle* $r_2 = [a,b] \times [c_1, d_1]$, *where* $c \le c_1 < d_1 \le d$. *Then* $F'_{[y]}(x,y)$ *is continuous on the* r_2^0, *and*

$$F'_{[y]}(x,y) = \int\limits_a^x f(t,y)\,dt, \quad (x,y) \in r_2^0. \tag{2}$$

The following theorem is obtained from Theorems 6.2.1 and 6.2.2.

Theorem 6.2.3 ([2]). *Let the function* $f(x,y) \in L(Q)$ *be continuous on the union* $[a_1,b_1] \times [c,d] \cup [a,b] \times [c_1,d_1]$, *where* $a \le a_1 < b_1 \le b$ *and* $c \le c_1 < d_1 \le d$. *Then the functions* $F'_{[x]}(x,y)$ *and* $F'_{[y]}(x,y)$ *at interior points* $(x,y) \in r^0$ *of the rectangle* $r = [a_1,b_1] \times [c_1,d_1]$ *are continuous and we have the equalities*

$$F'_{[x]}(x,y) = \int\limits_{c_1}^y f(x,\tau)\,d\tau, \quad F'_{[y]}(x,y) = \int\limits_{a_1}^x f(t,y)\,dt, \quad (x,y) \in r^0. \tag{3}$$

In particular, an indefinite integral $F(x,y)$ *is continuously differentiable on the* r^0.

§ 7. Repeated and Mixed Partial Derivatives of an Indefinite Integral

Theorem 7.1 ([2]). *Let the function* $f \in L(Q)$ *be continuous on the rectangle* $r(x_0,\delta) = [x_0-\delta, x_0+\delta] \times [c,d] \subset Q$, *where* $\delta > 0$. *Assume that the partial derivative* $f'_x(x,y)$ *is summable on the* $r(x_0,\delta)$, *which with respect to* x *is assumed to be partial continuous at* x_0, *uniformly with respect to* $y \in [c,d]$, *and measurable with respect to* y *on* $[c,d]$ *for every* $x \in [x_0-\delta, x_0+\delta]$. *Then* $(F'_{[x]})'(x_0,y)$ *is finite at all points* (x_0,y) *with* $c < y < d$ *and we have*

$$\big(F'_{[x]}\big)'_x(x_0,y) = \int\limits_c^y f'_x(x_0,\tau)\,d\tau. \tag{1}$$

Proof. To the function $f'_x(x,y)$, $(x,y) \in r(x_0,\delta)$ we apply statement (1) of Lemma 6.1.1. Thus $f'_x(x_0,\tau) \in L([c,d])$.

By Theorem 6.2.1 the function $F'_{[x]}(x,y)$ is continuous on the $r^0(x_0,\delta)$ and

$$\int_c^y f(x,\tau)\,d\tau = F'_{[x]}(x,y), \quad (x,y) \in r^0(x_0,\delta). \tag{2}$$

Denote the right-hand side of equality (2) by $\mu(x,y)$ and we have to prove the equality

$$\mu'_x(x,y) = \int_c^y f'_x(x_0,\tau)\,d\tau, \quad c < y < d. \tag{3}$$

We have

$$\frac{\mu(x_0+h)-\mu(x_0,y)}{h} - \int_c^y f'_x(x_0,\tau)\,d\tau =$$

$$= \int_c^y \Big[\frac{f(x_0+h,\tau)-f(x_0,\tau)}{h} - f'_x(x_0,\tau)\Big]\,d\tau =$$

$$= \int_c^y \big[f'_x(x_0+\theta h,\tau) - f'_x(x_0,\tau)\big]\,d\tau,$$

where $0 < \theta = \theta(x_0,h,\tau) < 1$. Taking now into account that the function $f'_x(x,\tau)$ is partial continuous with respect to x at the point x_0, uniformly with respect to $\tau \in [c,d]$, we can conclude that the theorem is complete. □

Corollary 7.1 ([2]). *Let the function $f \in L(Q)$ be continuous on the $r(x_0,\delta) = [x_0-\delta, x_0+\delta] \times [c,d] \subset Q$, $\delta > 0$, and suppose that on the $r(x_0,\delta)$ there exists a bounded partial derivative $f'_x(x,y)$ which is assumed to be partial continuous with respect to x at x_0, uniformly with respect to $y \in [c,d]$ and measurable with respect to y on $[c,d]$ for every $x \in [x_0-\delta, x_0+\delta]$.*

Then equality (1) *holds at all points (x_0,y) with $c < y < d$.*

Taking into account Theorem 8.1 of Chapter I and statement (2) of Lemma 6.1.1, from Corollary 7.1 we obtain the following

Theorem 7.2 ([2]). *Let the function $f \in L(Q)$ and its partial derivative $f'_x(x,y)$ be continuous on the Q^0. Then the function $(F'_{[x]})'_x(x,y)$ is continuous on the Q^0, and*

$$\big(F'_{[x]}\big)'_x(x,y) = \int_c^y f'_x(x,\tau)\,d\tau, \quad (x,y) \in Q^0. \tag{4}$$

For the $(F_{[x]})'_y$ is valid the following

Theorem 7.3 ([2]). *Let the function $f \in L(Q)$ be continuous on the $r_1 = [a_1, b_1] \times [c, d]$, where $a \le a_1 < b_1 \le b$. Then $(F'_{[x]})'_y$ is continuous on the r_1^0 and*

$$\big(F'_{[x]}\big)'_y(x,y) = f(x,y), \quad (x,y) \in r_1^0. \tag{5}$$

Proof. On the r_1^0, equality 6.2.(1) is valid, and we have to prove that the function $\mu(x,y)$ given by equality (2) satisfies the equality $\mu'_y(x,y) = f(x,y)$ at the points $(x,y) \in r_1^0$. This follows from the relations

$$\mu'_y(x,y) = \lim_{k \to 0} \frac{1}{k} \int\limits_y^{y+k} f(x,\tau)\,\tau = f(x,y),$$

owing to the fact that the function $f(x,y)$ is continuous on $[c,d]$ for every fixed $x \in [a_1, b_1]$. □

Now from Theorems 7.2 and 7.3 we obtain

Theorem 7.4 ([2]). *Let the functions $f \in L(Q)$ and f'_x be continuous on the Q^0. Then $(F'_{[x]})'_x$ and $(F'_{[x]})'_y$ are continuous on the Q^0, and hence the function $F'_{[x]}$ is continuously differentiable on the Q^0.*

Obviously, there take place analogues of Theorems 7.1–7.3 about the functions $(F'_{[y]})'_y$ and $(F'_{[y]})'_x$. The analogue of Theorem 7.4 is the form of

Theorem 7.5 ([2]). *Let the functions $f \in L(Q)$ and f'_y be continuous on the Q^0. Then $(F'_{[y]})'_x$ and $(F'_{[y]})'_y$ are continuous on the Q^0, and hence $F'_{[y]}$ is the function which is continuously differentiable on the Q^0.*

From Theorems 7.4 and 7.5 we have

Theorem 7.6 ([2]). *Let the function $f \in L(Q)$ have continuous on the Q^0 partial derivatives f'_x and f'_y. Then the functions $F'_{[x]}$ and $F'_{[y]}$ are continuously differentiable on the Q^0 and hence an indefinite integral $F(x,y)$ is twice continuously differentiable on the Q^0.*

§ 8. Twice Differentiability of an Indefinite Integral

8.1. Twice Differentiability on the Q^0 of an Indefinite Integral

Theorem 8.1.1 ([2]). *Let the function $f \in L(Q)$ be differentiable on the Q^0, and let integrals*

$$\int\limits_c^d f'_x(x,\tau)\,d\tau, \qquad \int\limits_a^b f'_y(t,y)\,dt \tag{1}$$

be finite for all $x \in (a,b)$ and $y \in (c,d)$, respectively. Then the functions $F'_{[x]}$ and $F'_{[y]}$ are differentiable on the Q^0. Hence the indefinite integral F is twice differentiable on the Q^0.

Proof. By Theorem 6.2.3, the functions

$$F'_{[x]}(x,y) = \int_c^y f(x,\tau)\,d\tau \equiv \Phi(x,y)$$

and

$$F'_{[y]}(x,y) = \int_a^x f(t,y)\,dt \equiv \Psi(x,y)$$

are continuous on the Q^0.

Obviously,

$$\frac{\Phi(x+h,y+k)-\Phi(x,y+k)}{h} = \int_c^{y+k} \frac{f(x+h,\tau)-f(x,\tau)}{h}\,d\tau.$$

Since the function f is differentiable at every point $(x,y)\in Q^0$,

$$f(x+p,y+q)-f(x,y) = pf'_x(x,y)+qf'_y(x,y)+(|p|+|q|)\cdot\alpha,$$

where the function α, defined in the neighborhood of the point (x,y) has zero limit at (x,y) and it is assumed to be equal to zero at (x,y). Therefore

$$\frac{\Phi(x+h,y+k)-\Phi(x,y+k)}{h} = \int_c^{y+k} f'_x(x,\tau)\,d\tau + \frac{|h|}{h}\int_c^{y+k}\alpha\,d\tau.$$

For every number $\varepsilon>0$ there exists a rectangle, containing the point (x,y), on which $|\alpha|<\varepsilon$. Hence there exists finite

$$\Phi'_{[x]}(x,y) = \int_c^y f'_x(x,\tau)\,d\tau \quad \big(=\Phi'_{\widehat{x}}(x,y)\big). \tag{2}$$

Next,

$$\frac{\Phi(x+h,y+k)-\Phi(x+h,y)}{k} = \frac{1}{k}\int_y^{y+k} f(x+h,\tau)\,d\tau =$$

$$= \frac{h}{k}\int_y^{y+k}\frac{f(x+h,\tau)-f(x,\tau)}{h}\,d\tau + \frac{1}{k}\int_y^{y+k} f(x,\tau)\,d\tau =$$

$$= \frac{h}{k}\int_y^{y+k} f'_x(x,\tau)\,d\tau + \frac{|h|}{k}\int_y^{y+k}\alpha\,d\tau + \frac{1}{k}\int_y^{y+k} f(x,\tau)\,d\tau,$$

where we get

$$\Phi'_{\widehat{y}}(x,y) = f(x,y), \quad (x,y)\in Q^0. \tag{3}$$

Thus the function $\Phi'_{\widehat{y}}$ is continuous on the Q^0.

Consequently, the function $\Phi = F'_{[x]}$ is differentiable on the Q^0, by Theorem 2.5.1, or Theorem 3.5.3 of Chapter II.

The differentiability on the Q^0 of the function $\Psi = F'_{[y]}$ for which the function $\Psi'_{\hat{x}} = f$ is continuous on the Q^0 and the function

$$\Psi'_{[y]}(x,y) = \int\limits_a^x f'_y(t,y)\,dt \tag{4}$$

is finite on the Q^0, is proved analogously.

Hence the function $F'_{[y]}$ is differentiable on the Q^0.

Consequently, an indefinite double integral F is twice differentiable on the Q^0. □

8.2. Almost Everywhere Twice Differentiability of an Indefinite Integral

Theorem 8.2.1 ([2]). *Let the function $f(x,y) \in L(Q)$ with respect to x be absolutely continuous on the $[a,b]$ for every $y \in [c,d]$, and let $f'_x \in L(Q)$. Then $F'_{[x]}$ is continuous on the Q^0, the equality*

$$F'_{[x]}(x,y) = \int\limits_c^y f(x,\tau)\,d\tau, \quad (x,y) \in Q^0 \tag{1}$$

holds, and the following statements take place:

1) *for almost all $x_0 \in (a,b)$ and for all $y_0 \in (c,d)$,*

$$\big(F'_{[x]}\big)'_{[x]}(x_0,y_0) = \int\limits_c^{y_0} f'_x(x_0,\tau)\,d\tau \tag{2}$$

is finite;

2) *for almost all $(x_0,y_0) \in Q^0$,*

$$\big(F'_{[x]}\big)'_{[y]}(x_0,y_0) = f(x_0,y_0) \tag{3}$$

is finite;

3) $\operatorname{str\,grad} F'_{[x]}$ *is finite almost everywhere on the Q^0;*

4) $F'_{[x]}$*, in particular, F'_x is differentiable almost everywhere on the Q^0.*

Proof. First we will prove equality (1) and then establish that $F'_{[x]}$ is continuous on the Q^0. Towards this end, we write the function F in terms of

$$F(x,y) = \int\limits_a^x \Phi(t,y)\,dt,$$

where

$$\Phi(t,y)=\int\limits_c^y f(t,\tau)\,d\tau=\int\limits_c^y\int\limits_{t_0}^t f_t'(t,\tau)\,dt\,d\tau+\int\limits_c^y f(t_0,\tau)\,d\tau$$

and t_0 is chosen in (a,b) in such a way that $f(t_0,\tau)\in L([c,d])$. It becomes clear that the function Φ is continuous and the equality $F_x'(x,y)=\Phi(x,y)$ is valid on the Q^0. The continuity of F_x' on the Q^0 implies continuity of $F_{[x]}'$ on the Q^0 (see equality 3.1.(6) in Chapter II), and the equalities

$$F_{[x]}'(x,y)=F_x'(x,y)=\Phi(x,y)=\int\limits_c^y f(x,\tau)\,d\tau.$$

Hence equality (1) is proved.

Now common value of equality (1) we denote by $\psi(x,y)$, for which we have:

1) $$\psi_{[x]}'(x_0,y_0)=\lim_{\substack{h\to 0\\ y\to y_0}}\frac{\psi(x_0+h,y)-\psi(x_0,y)}{h}=$$

$$=\lim_{\substack{h\to 0\\ y\to y_0}}\frac{1}{h}\int\limits_{x_0}^{x_0+h}\int\limits_c^y f_x'(x,\tau)\,d\tau\,dx.$$

But this limit is equal to the integral from equality (2) for almost all $x_0\in(a,b)$ and for all $y_0\in(c,d)$, by equality 3.2.(1).

Thus statement 1) is established.

2) $$\psi_{[y]}'(x_0,y_0)=\lim_{\substack{k\to 0\\ x\to x_0}}\frac{\psi(x,y_o+k)-\psi(x,y_0)}{k}=$$

$$=\lim_{\substack{k\to 0\\ x\to x_0}}\frac{1}{k}\int\limits_{y_0}^{y_0+k} f(x,\tau)\,d\tau.$$

Let the point $x_0\in(a,b)$ be such that $f(x_0,\tau)\in L([c,d])$ (almost every point is such), and we write

$$\frac{1}{k}\int\limits_{y_0}^{y_0+k} f(x,\tau)\,d\tau=\frac{1}{k}\int\limits_{y_0}^{y_0+k} f(x_0,\tau)\,d\tau+\frac{1}{k}\int\limits_{y_0}^{y_0+k}\int\limits_{x_0}^{x} f_t'(t,\tau)\,dt\,d\tau.$$

Applying to the last equality the Lebesgue's theorem and equality 3.2.(8), we obtain statement 2).

3) The finiteness of the $\operatorname{str\,grad} F_{[x]}'(x,y)$ for almost all $(x,y)\in Q^0$ follows from statements 1) and 2).

4) The differentiability of the function $F_{[x]}'$ almost everywhere on the Q^0 follows from statement 3) with regard of Theorem 3.4.1 of Chapter II. □

Just in the same way we can prove

Theorem 8.2.2 ([2]). *Let the function $f(x,y)\in L(Q)$ with respect to y be absolutely continuous on the $[c,d]$ for every $x\in[a,b]$, and let $f'_y\in L(Q)$. Then $F'_{[y]}$ is continuous on the Q^0, equality*

$$F'_{[y]}(x,y)=\int\limits_a^x f(x,y)\,dt,\quad (x,y)\in Q^0 \tag{4}$$

is valid, and the following statements take place:

1) *for all $x_0\in(a,b)$ and for almost all $y_0\in(c,d)$,*

$$\big(F'_{[y]}\big)_{[y]}(x_0,y_0)=\int\limits_a^{x_0} f'_y(t,y_0)\,dt \tag{5}$$

is finite;

2) *for almost all $(x_0,y_0)\in Q^0$,*

$$\big(F'_{[y]}\big)_{[x]}(x_0,y_0)=f(x_0,y_0) \tag{6}$$

is finite;

3) $\operatorname{str}\operatorname{grad}F'_{y]}$ *is finite almost everywhere on the Q^0;*

4) $F'_{[y]}$*, in particular, F'_y is differentiable almost everywhere on the Q^0.*

Remark 8.2.1. Under the above-indicated assumptions, equalities (3) and (6) are the strengthenings of relations 1.1.(7).

Theorems 8.2.1 and 8.2.2 yield (see Remark 2.4.2)

Theorem 8.2.3 ([2]). *Let the function $f\in L(Q)$ be separately absolutely continuous on Q, and let $f'_x\in L(Q)$, $f'_y\in L(Q)$. Then for the indefinite integral F of f equalities (1) and (4) with continuous on the Q^0 terms are valid, and the following statements take place:*

1) $\operatorname{str}\operatorname{grad}F'_{[x]}$ *and* $\operatorname{str}\operatorname{grad}F'_{[y]}$ *are finite almost everywhere on the Q^0;*

2) $F'_{[x]}$ *and* $F'_{[y]}$ *are differentiable almost everywhere on the Q^0;*

3) *F is twice differentiable almost everywhere on the Q^0;*

4) *for almost all $(x_0,y_0)\in Q^0$ we have*

$$\big(F'_{[x]}\big)'_{[y]}(x_0,y_0)=f(x_0,y_0)=\big(F'_{[y]}\big)'_{[x]}(x_0,y_0). \tag{7}$$

All the statements of Theorem 8.2.2 can be obtained under somewhat different assumption (the same can be said regarding Theorem 8.2.1).

Theorem 8.2.4 ([2]). *If the function $f\in L(Q)$ with respect to y is absolutely continuous on the $[c,d]$ for every $x\in[a,b]$, and for some constant $c=c(f)>0$ the relation*

$$\iint\limits_Q \big|f(t,\tau+k)-f(t,\tau)\big|\,dt\,d\tau\le c\cdot|k| \tag{8}$$

is fulfilled, then all the statements of Theorem 8.2.2 are also fulfilled.

Proof. It is sufficient to prove that the partial derivative f'_y, existing almost everywhere on the Q, is summable on the Q. To this end, we put in (8) $k = 1/n$ and have

$$\iint\limits_Q n\Big|f\Big(t,\tau+\frac{1}{n}\Big)-f(t,\tau)\Big|\,dt\,d\tau \le c, \tag{9}$$

whence, by virtue of Fatou's lemma (see, for e.g., [21], p. 29), we obtain

$$\iint\limits_Q \big|f'_\tau(t,\tau)\big|\,dt\,d\tau = \iint\limits_Q \lim_{n\to\infty} n\Big|f\Big(t,\tau+\frac{1}{n}\Big)-f(t,\tau)\Big|\,dt\,d\tau \le$$
$$\le \varliminf_{n\to\infty} \iint\limits_Q n\Big|f\Big(t,\tau+\frac{1}{n}\Big)-f(t,\tau)\Big|\,dt\,d\tau \le c. \quad \Box$$

Thus $f'_y \in L(Q)$ and the theorem is complete. □

Remark 8.2.2. To illustrate Theorem 8.2.2, let us consider a function of one variable $\varphi(x) \in L([a,b])$. Then $\varphi(x)$ is absolutely continuous with respect to y on an arbitrary $[c,d]$ for every $x \in [a,b]$, $\varphi'_y = 0$ on the Q, and the corresponding indefinite double integral

$$\Phi(x,y) = \int\limits_a^x \int\limits_c^y \varphi(t)\,dt = (y-c)\int\limits_a^x \varphi(t)\,dt.$$

From this we obtain that

$$\Phi'_{[y]}(x,y) = \int\limits_a^x \varphi(t)\,dt,$$

$$(\Phi'_{[y]})'_{[x]}(x_0,y_0) = \Big(\int\limits_a^x \varphi(t)dt\Big)'(x_0) = \varphi(x_0) \text{ for almost all } x_0 \text{ and for all } y_0,$$

$$(\Phi'_{[y]})'_{[y]}(x_0,y_0) = 0 = \int\limits_a^{x_0} \varphi'_y(t)\,dt \ \text{ for all } \ (x_0,y_0),$$

$$\operatorname{str\,grad}\Phi'_{[y]}(x_0,y_0) = (\varphi(x_0),0) \ \text{ for almost all } \ x_0 \ \text{ and for all } \ y_0.$$

§ 9. On the Sufficient Conditions of the Lebesgue's Intense Point

Theorem 9.1. *Under the assumptions of Theorem* 6.2.1, *every point* $(x,y) \in r_1^\circ$ *belongs to the set* $\operatorname{int} L^p_x(f)$.

Proof. For every point $(x,y) \in r_1^\circ$ we have

$$F'_x(x+h,y+k) - F'_x(x,y) = \int\limits_c^{y+k} f(x,\tau)\,d\tau - \int\limits_c^y f(x,\tau)\,d\tau, \tag{1}$$

whence

$$\frac{1}{h}\int\limits_{x}^{x+h}\left|\int\limits_{c}^{y+k}f(x,\tau)\,d\tau-\int\limits_{c}^{y}f(x,\tau)\,d\tau\right|^{p}dt=$$

$$=\frac{1}{h}\int\limits_{x}^{x+h}\left|F'_x(x+h,y+k)-F'_x(x,y)\right|^{p}dt. \tag{2}$$

Because of the continuity of F'_x on the r_1°, we can make the right-hand side of equality (2) arbitrarily small for sufficiently small h and k. Hence equality 4.1.(1) is fulfilled for $x_0 = x$ and $y_0 = y$. Thus the point (x, y) belongs to the set $\operatorname{int}L^p_x(f)$. $\square$

The following theorem is proved analogously.

Theorem 9.2. *Under the assumptions of Theorem* 6.2.2, *every point* $(x, y) \in r_2^\circ$ *belongs to the set* $\operatorname{int} L^p_y(f)$.

Next, we have the following

Theorem 9.3. *Under the assumptions of Theorem* 6.2.3, *every point* $(x, y) \in r^\circ$ *belongs to the set* $\operatorname{int} L^p_{x,y}(f)$.

Proof. By virtue of Theorems 9.1 and 9.2, every point $(x, y) \in r^\circ$ belongs to the sets $\operatorname{int}L^p_x(f)$ and $\operatorname{int}L^p_y(f)$. Hence the point $(x, y) \in r^\circ$ belongs to the set $\operatorname{int}L^p_{x,y}(f)$. $\square$

Theorem 9.3 yields

Theorem 9.4. *Let the function f be continuous on the Q. Then each point from the every neighborhood $U(M) \subset Q^0$ of the arbitrary point $M \in Q^0$, belongs to the set* $\operatorname{int} L^p_{x,y}(f)$ *for all* $p \geq 1$.

From this theorem we can easily get

Theorem 9.5. *Let the function f belong to the space $L^p(Q)$ for some $p \geq 1$ and is continuous on the some rectangle $R = [\alpha, \beta] \times [\gamma, \delta] \subset Q$. Then for the values $a = \alpha$ and $c = \gamma$ the equalities* 4.1.(1) *and* 4.1.(2) *are fulfilled at every point $(x_0, y_0) \in U(N)$ from the each neighborhood $U(N) \subset R^0$ of the arbitrary point $N \in R^0$. Thus every point $(x_0, y_0) \in U(N)$ belongs to the set* $\operatorname{int} L_{x,y}(f|R)$, *where by $f|R$ is denoted restriction of the function f on R, i.e., we consider f only on R (see* 1.1 *of Chapter* I).

Corollary 9.1. *Let an indefinite integral F, corresponding to the function $f \in L^p(Q)$ for some $p \geq 1$, possess a continuous at the point $(x_0, y_0) \in Q^0$ partial derivative F'_x for which the equality $F'_x(x, y) = \int_c^y f(x, \tau)\, d\tau$ is assumed to be fulfilled at every point (x, y) from some neighborhood of the point (x_0, y_0). Then the point (x_0, y_0) belongs to the set* $\operatorname{int} L^p_x(f)$.

Proof. Under this assumptions, equalities (1), (2) and 4.1.(1) are fulfilled. □

Now from Corollary 9.1 we obtain

Corollary 9.2. *Let the function $f \in L^p(Q)$ with $p \geq 1$ be measurable with respect to y on the $[c,d]$ for every $x \in [a,b]$ and continuous with respect to x at x_0, uniformly with respect to $y \in [c,d]$. If the continuous at the point (x_0, y_0) integral*

$$\int_c^y f(x,\tau)\,d\tau, \quad |x - x_0| < \delta, \quad c < y_0 < d$$

from equality 6.1.(3) *coincides with $F'_x(x,y)$ in some neighborhood of the point (x_0, y_0), then $(x_0, y_0) \in \operatorname{int} L^p_x(f)$.*

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List of Special Symbols

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