Proc. A. Razmadze Math. Inst. 138(2005), 103–105

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NECESSARY AND SUFFICIENT CONDITIONS FOR $\mathbb{C}^n$-DIFFERENTIABILITY AND THE HARTOGS MAIN THEOREM

(Reported on March 20, 2005)

1. Introduction

It is well-known that the notion of holomorphy of functions of one complex variable is based on the notion of derivability, whose n-dimensional generalization is the notion of $\mathbb{C}^n$-differentiability (see Definition 1). It is natural, that the notion of $\mathbb{C}^n$-holomorphy is based on that $\mathbb{C}^n$-differentiability (see Definition 2). But due to the absence of necessary and sufficient conditions for $\mathbb{C}^n$-differentiability, the notion of $\mathbb{C}^n$-holomorphy was introduced by many authors under different assumptions on functions (see, for example, [1]-[10]).

In the present article we give necessary and sufficient conditions for $\mathbb{C}^n$-differentiability (see Theorem 1). These conditions are closely connected with the necessary and sufficient conditions for differentiability of many real variables functions, established by the author recenty [11, 12]. Using Theorem 1, we obtain the necessary and sufficient conditions for $\mathbb{C}^n$-holomorphy both at a given point (see Corollary 1) and in an open set (see Corollary 2). Finally, gives one more proof of the Hartogs Main Theorem.

To this end we will need the following definitions.

Definition 1 [13]. The function $f(z) = u(z) + i\nu(z)$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, \dots, n$, is called $\mathbb{C}^n$-differentiable at the point $z^0 = (z_1^0, \dots, z_n^0) \in \mathbb{C}^n$ if there exist finite numbers $c_k \in \mathbb{C}^1$, such that the equality

$$\lim_{z \to z^0} \frac{f(z) - f(z^0) - \sum\limits_{k=1}^{n} c_k (z_k - z_k^0)}{\sum\limits_{j=1}^{2n} |x_j - x_j^0|} = 0 \tag{1}$$

is fulfilled.

Definition 2 [4]. The function $f(z)$ is called $\mathbb{C}^n$-holomorphic at the point z^0, if f is $\mathbb{C}^n$-differentiable at each point z in a neighbourhood of z^0. The function f is called $\mathbb{C}^n$-holomorphic in the open set $G \subset \mathbb{C}$, if f is $\mathbb{C}^n$-holomorphic at each point of G.

The $\mathbb{C}^n$-differentiable at all points $z \in G$ function is $\mathbb{C}^n$-holomorphic in the open set $G \subset \mathbb{C}^n$.

Definition 3 [8, 15]. The function f is called $\mathbb{C}^n$-analytic in the open set $G \subset \mathbb{C}$, if in some neighbourhood of every point $z^0 \in G$, the value $f(z)$ is a sum of an convergent n-tuple power series.

The function $f(z_1, \dots, z_n)$ is called $\mathbb{C}^1$- analytic ($\mathbb{C}^1$-holomorphic) at the point $z^0 \in \mathbb{C}^n$ with respect to the variable z_k, if the function $f(z^0(z_k))$ of one z_k variable is analytic (holomorphic) at z_k^0, where $z^0(z_k) = (z_1^0, \dots, z_{k-1}^0, z_k, z_{k+1}^0, \dots, z_n^0)$.

2000 *Mathematics Subject Classification:* 32A20.

Key words and phrases. $\mathbb{C}^n$-differentiable, $\mathbb{C}^n$-holomorphic, separately $\mathbb{C}^1$-holomorphic, separately $\mathbb{C}^1$-analytic, $\mathbb{C}^n$-analytic.

The function $f(z_1 \dots, z_n)$ is called separately $\mathbb{C}^1$-analytic (separately $\mathbb{C}^1$-holomorphic) in the open set $G \subset \mathbb{C}^n$, if the functions $f(z^0(z_1)), \dots, f(z^0(z_n))$ are $\mathbb{C}^1$- analytic ($\mathbb{C}^1$-holomorphic) at all points $z^0 \in G$.

Hartogs' Main Theorem [16, 6, 9, 10, 13-15]. The separately $\mathbb{C}^1$ -analytic function in the open set $G \subset \mathbb{C}^n$ is $\mathbb{C}^n$-analytic in G.

2. The Main Results

Now we introduce the notation

$$D_{\widehat{x}_k} f(z^0) = \lim_{\substack{x_k \to x_k^0 \\ |x_j - x_j^0| \le |x_k - x_k^0| \\ j \ne k}} \frac{f(z) - f(z(x_k^0))}{x_k - x_k^0}, \quad k = 1, \dots, 2n, \tag{2}$$

where, along with the points $z = (x_1, x_2, \dots, x_{2n-1}, x_{2n})$ and $z^0 = (x_1^0, x_2^0, \dots, x_{2n-1}^0, x_{2n}^0)$, we introduce the notation $z(x_k^0) = (x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_{2n})$.

The main result is formulated as follows.

Theorem 1. *For the function $f(z)$ to be $\mathbb{C}^n$-differentiable at the point $z^0 = (z_1^0, \dots, z_n^0)$, it is necessary and sufficient that the following equality holds*

$$D_{\widehat{x}_{2k-1}} f(z^0) + D_{\widehat{x}_{2k}} f(z^0) = 0 \tag{3}$$

for all $k = 1, \dots, n$.

Corollary 1. *For the function $f(z)$ to be $\mathbb{C}^n$-holomorphic at the point z^0, it is necessary and sufficient the fulfilment of the condition*

$$D_{\widehat{x}_{2k-1}} f(z) + i D_{\widehat{x}_{2k}} f(z) = 0 \tag{4}$$

for all $k = 1, \dots, n$ at all points z in some neighbourhood of the point z^0.

Corollary 2. *The function $f(z)$ is $\mathbb{C}^n$-holomorphic in the open set $G \subset \mathbb{C}^n$ if and only if the equality* (4) *holds for all $k = 1, \dots, n$ at all points $z \in G$.*

In particular case, where $n = 1$ we have

Corollary 3 [12]. *The function $f(z)$, $z = x + iy = (x, y)$, has the finite derivative $f'(z_0)$ at the point $z_0 = x_0 + iy_0 = (x_0, y_0)$, if and only if the equality*

$$D_{\widehat{x}} f(z_0) + i D_{\widehat{y}}(z_0) = 0 \tag{5}$$

is fulfilled, where

$$D_{\widehat{x}} f(z_0) = \lim_{\substack{x \to x_0 \\ |y - y_0| \le |x - x_0|}} \frac{f(x, y) - f(x_0, y)}{x - x_0}$$

and

$$D_{\widehat{y}} f(z_0) = \lim_{\substack{y \to y_0 \\ |x - x_0| \le |y - y_0|}} \frac{f(x, y) - f(x, y_0)}{y - y_0}.$$

Corollary 4. *For the function $f(z)$ to be holomorphic in the open set $G \subset \mathbb{C}^1$, it is necessary and sufficient that the equality* (5) *be fulfilled at all points $z_0 \in G$.*

3. A New Proof of the Hartogs Main Theorem

Since the function $f(z_1, \dots, z_n)$ is separately $\mathbb{C}^1$-analytic in the open set $G \subset \mathbb{C}^n$, therefore equality (3) is fulfilled for all $k = 1, \dots, n$ at all points $z^0 \in G$, according to Corollary 3. Hence the function f is $\mathbb{C}^n$-differentiable at all points $z^0 \in G$, by virtue of Theorem 1. Therefore f in G is $\mathbb{C}^n$-holomorphic and in particular, continuous. Thus we obtain that the n-repeated Cauchy's integral formula is reduced to the n-tuple Cauchy's integral formula. Consequently, the function f is $\mathbb{C}^n$-analytic in G.

Corollary 5. *The notions of $\mathbb{C}^n$-holomorphy and $\mathbb{C}^n$-analyticity of n complex variables functions are equivalent.*

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