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One-dimensional Fourier series of a function of many variables ☆

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Abstract

It is well known that to each summable in the n-dimensional cube $[-\pi, \pi]^n$ function f of variables $x_1, \ldots, x_n$ there corresponds one n-multiple trigonometric Fourier series $S[f]$ with constant coefficients.

In the present paper, with the function f we associate n one-dimensional Fourier series $S[f]_1, \ldots, S[f]_n$, with respect to variables $x_1, \ldots, x_n$, respectively, with nonconstant coefficients and announce the preliminary results. In particular, if a continuous function f is differentiable at some point $x = (x_1, \ldots, x_n)$, then all one-dimensional Fourier series $S[f]_1, \ldots, S[f]_n$ converge at x to the value $f(x)$.

For illustration we consider the well known example of Ch. Fefferman's function $F(x, y)$ whose double trigonometric Fourier series $S[F]$ diverges everywhere in the sense of Prinsheim. Namely, we establish the simultaneous convergence of the one-dimensional Fourier series $S[F]_1$ and $S[F]_2$ at almost all points $(x, y) \in [-\pi, \pi]^2$ to the values $F(x, y)$.

Keywords: One-dimensional Fourier series; Nonconstant Fourier coefficients

1. Notions of one-dimensional fourier series of a function of many variables

Let some function f of variables $x_1, \ldots, x_n$ be defined and summable in the n-dimensional cube $[-\pi, \pi]^n$ and, in addition, be 2π-periodic with respect to each variable.

By Fubini's theorem we know that f is summable on $[-\pi, \pi]$ as a function of one variable x_1 for almost all $(x_2, x_3, \ldots, x_n) \in [-\pi, \pi]^{n-1}$. We denote by E^1 the set of such $(x_2, x_3, \ldots, x_n)$ and by X^1 the point $(x_2, x_3, \ldots, x_n)$, i.e. $X^1 = (x_2, x_3, \ldots, x_n)$, $X^1 \in E^1$.

Thus we have the function $f(x_1, X^1)$ which is summable with respect to the variable x_1 on $[-\pi, \pi]$ for each $X^1 \in E^1$.

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Let us consider a Fourier series corresponds to the function $f(x_1, X^1)$ with respect to the variable x_1 on $[-\pi, \pi]$ and we denote it by $S[f]_1$, i.e.

$$S[f]_1 = \frac{1}{2}a_0(X^1) + \sum_{k=1}^{\infty} a_k(X^1)\cos kx_1 + b_k(X^1)\sin kx_1,$$

where the coefficients $a_0(X^1)$, $a_k(X^1)$ and $b_k(X^1)$ are defined by the Fourier formulas

$$\begin{aligned} a_0(X^1) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t, X^1)dt, \quad a_k(X^1) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t, X^1)\cos kt\,dt, \\ b_k(X^1) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t, X^1)\sin kt\,dt, \quad k = 1, 2, \ldots. \end{aligned} \tag{1}$$

In these relations, anyone of the variables $x_2, x_3, \ldots, x_n$ may play the role of x_1.

Therefore to each summable function f in the n-dimensional cube $[-\pi, \pi]^n$ there correspond one-dimensional Fourier series $S[f]_1, \ldots, S[f]_n$ with nonconstant coefficients.

In what follows we will discuss only the series $S[f]_1$.

2. Necessary and sufficient condition for the convergence of a one-dimensional fourier series of a function of many variables

Let us consider the partial sum of the one-dimensional Fourier series $S[f]_1$

$$S_m(f; (x_1, X^1)) = \frac{1}{2}a_0(X^1) + \sum_{k=1}^{m} a_k(X^1)\cos kx_1 + b_k(X^1)\sin kx_1,$$

which, after substituting in it the coefficients (1), takes the form

$$S_m(f; (x_1, X^1)) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t, X^1)D_m(t - x_1)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_1 + y_1, X^1)D_m(y_1)dy_1,$$

where D_m is the Dirichlet kernel, i.e.

$$D_m(t) = \frac{\sin(m + \frac{1}{2})t}{2\sin\frac{t}{2}} \quad \text{for} \quad t \neq 2k\pi$$

and

$$D_m(2k\pi) = m + \frac{1}{2} \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots.$$

Since the function f is summable with respect to the variable x_1 on $[-\pi, \pi]$ for any $X^1 \in E^1$, the well known necessary and sufficient condition for the Fourier series $S[\varphi]$ of a function $\varphi \in L[-\pi, \pi]$ to be convergent at some point $t \in [-\pi, \pi]$ to the value $\varphi(t)$ (see [1], Ch. I, §37, equality (37.5); [2], p.55)

$$\lim_{m\to\infty}\int_0^{\delta}[\varphi(t + u) + \varphi(t - u) - 2\varphi(t)]\frac{\sin mu}{u}\,du = 0 \tag{2}$$

takes in our case the form

$$\lim_{m\to\infty}\int_0^{\delta}[f(x_1 + y_1, X^1) + f(x_1 - y_1, X^1) - 2f(x_1, X^1)]\,\frac{\sin my_1}{y_1}\,dy_1 = 0, \quad X^1 \in E^1.$$

Hence we can formulate

Proposition 2.1. *For a one-dimensional Fourier series $S[f]_1$ to converge at a point (x_1, X^1) to the value $f(x_1, X^1)$ for some $x_1 \in [-\pi, \pi]$ and $X^1 \in E^1$ it is necessary and sufficient that the equality*

$$\lim_{m\to\infty}\int_0^{\delta}\frac{f(x_1 + y_1, X^1) + f(x_1 - y_1, X^1) - 2f(x_1, X^1)}{y_1}\,\sin my_1 dy_1 = 0 \tag{3}$$

be fulfilled.

3. Sufficient conditions for the convergence of a one-dimensional Fourier series of a function of many variables

As far back as 1853 B. Riemann considered the problem of representation of functions by trigonometric series. In connection with this problem Riemann introduced into consideration a function, say, φ with the property ([3], p. 245; [1], Ch. I, §66)

$$\lim_{h\to 0} \frac{\varphi(x_0+h)+\varphi(x_0-h)-2\varphi(x_0)}{h} = 0 \tag{4}$$

at a point x_0.

Later, A. Zygmund called the function φ having the property (4) a smooth function at the point x_0 ([4]; [2], p. 43).

It is obvious that a smooth function φ at a point x_0 has the property $\varphi(x_0+h)+\varphi(x_0-h)-2\varphi(x_0) \to 0$ as $h \to 0$ which is called the symmetry of the function φ at x_0.

It is the well-established fact that almost all points of symmetry of any function is the point of its continuity ([5], p. 266) and the converse statement is obvious.

Therefore almost all points of smoothness of any function is the point of its continuity. In addition, a smooth function at separate points may be discontinuous, for example, a discontinuous odd function.

It should be said that if the function φ has the finite derivative $\varphi'(x_0)$ at some point x_0, then φ is smooth at x_0 ([3], p. 43; [1], Ch.I, §66), but the converse statement is not true ([2], p. 48).

Note that if a 2π-periodic and summable function on $[-\pi, \pi]$ is smooth at some point x_0, in particular if φ has the finite derivative $\varphi'(x_0)$, then the Fourier series $S[\varphi]$ of the function φ converges at the point x_0 to the value $\varphi(x_0)$ (see the equality (2)).

Following Riemann, we introduce the following notion of smoothness of a function of many variables (the case $n = 2$ is considered in [6]).

Definition 3.1. A function f of n variables $x_1, \dots, x_n$ is called smooth at a point $x = (x_1, \dots, x_n)$ if the equality

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{|h|} = 0 \tag{5}$$

is fulfilled, where $h = (h_1, \dots, h_n)$ and $|h| = |h_1| + \cdots + |h_n|$.

Proposition 3.2. *If a function f is differentiable at some point x, then f is smooth at x.*

Indeed, that this is so follows from the equality

$$\begin{aligned} &\frac{f(x_1+h_1,\dots,x_n+h_n)+f(x_1-h_1,\dots,x_n-h_n)-2f(x_1,\dots,x_n)}{|h_1|+\cdots+|h_n|} \\ &\quad= \frac{f(x_1+h_1,\dots,x_n+h_n)-f(x_1,\dots,x_n)-A_1(h_1)-\cdots-A_n(h_n)}{|h_1|+\cdots+|h_n|} \\ &\qquad+ \frac{f(x_1-h_1,\dots,x_n-h_n)-f(x_1,\dots,x_n)-A_1(-h_1)-\cdots-A_n(-h_n)}{|-h_1|+\cdots+|-h_n|}. \end{aligned}$$

The converse to Proposition 3.2 is not true (for the case $n = 2$ see [6]).

Proposition 3.3. *If a function f is smooth at a point x, then it is smooth at x with respect to each variable x_j, $1 \le j \le n$.*

To verify that this is so it suffices to put (5) $h_i = 0$ for all $i \ne j$.

Proposition 3.4. *If a function f has at a point x the finite partial derivative $\frac{\partial f}{\partial x_j}$ with respect to the variable x_j, then f is smooth at x with respect to the same variable x_j.*

That this is so follows from the corresponding statement for functions of one variable.

Proposition 3.5. *If a function f is smooth with respect to the variable x_1 at the point (x_1, X^1) for some $x_1 \in [-\pi, \pi]$ and $X^1 \in E^1$, then the Fourier series $S[f]_1$ converges at (x_1, X^1) to the value $f(x_1, X^1)$.*

This assertion follows from the equality (3).

Propositions 3.3 and 3.5 give rise to

Theorem 3.6. *If a continuous on $[-\pi,\pi]^n$ function f is smooth at a point x, in particular if f is differentiable at x, then all one-dimensional Fourier series $S[f]_1,\dots,S[f]_n$ converge at the point x to one and the same value $f(x)$.*

Indeed, the function f as a function of the variable x_j is summable on $[-\pi,\pi]$ for any point $X^j = (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$ from $[-\pi,\pi]^{n-1}$. By virtue of Propositions 3.3 and 3.5, the one-dimensional Fourier series $S[f]_j$ converges at the point $(x_j,X^j)=(x_1,\dots,x_n)$ to the value $f(x_j,X^j)=f(x_1,\dots,x_n)$.

4. Almost everywhere convergence of one-dimensional Fourier series $S[F]_1$ and $S[F]_2$ for Ch. Fefferman's function F

It is well known that there exists an everywhere continuous function $F(x,y)$ of two variables and a 2π-periodic with respect to x and y double trigonometric Fourier series $S[F]$ which diverges everywhere in the Prinsheim sense [7].

The function $F(x,y)$ as function of the variable $x_1\in[-\pi,\pi]$ belongs to the class $L^2[-\pi,\pi]$ for each $y\in[-\pi,\pi]$. Therefore by L. Carleson's theorem [8] we have

Proposition 4.1. *A one-dimensional Fourier series $S[F]_1$ converges to values $F(x,y)$ for almost all $x\in[-\pi,\pi]$ and all $y\in[-\pi,\pi]$.*

Analogously, the following assertion is true.

Proposition 4.2. *The one-dimensional Fourier series $S[F]_2$ converges to the values $F(x,y)$ for all $x\in[-\pi,\pi]$ and almost all $y\in[-\pi,\pi]$.*

Propositions 4.1 and 4.2 give rise to

Theorem 4.3. *The one-dimensional Fourier series $S[F]_1$ and $S[F]_2$ simultaneously converges to the values $F(x,y)$ for almost all $(x,y)\in[-\pi,\pi]^2$.*

Finally, Propositions 4.1, 4.2 and Theorem 4.3 can be made stronger as follows.

Theorem 4.4. *For any function $f\in L^2[-\pi,\pi]^2$ there exist measurable sets E_1, E_2 and E_3 from the square $[-\pi,\pi]^2$ with the properties $|E_1|=|E_2|=|E_3|=4\pi^2$, at whose points the following equalities are fulfilled:*
$S[f]_1(x,y)=f(x,y)$ for $(x,y)\in E_1$,
$S[f]_2(x,y)=f(x,y)$ for $(x,y)\in E_2$,
$S[f]_1(x,y)=f(x,y)=S[f]_2(x,y)$ for $(x,y)\in E_3$.

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