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CATEGORICAL, HOMOLOGICAL, AND HOMOTOPICAL PROPERTIES OF ALGEBRAIC OBJECTS

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Abstract. This monograph is based on the doctoral dissertation of the author defended in the Iv. Javakhishvili Tbilisi State University in 2006. It begins by developing internal category and internal category cohomology theories (equivalently, for crossed modules) in categories of groups with operations. Further, the author presents properties of actions in categories of interest, in particular, the existence of an actor in specific algebraic categories. Moreover, the reader will be introduced to a new type of algebras called noncommutative Leibniz–Poisson algebras, with their properties and cohomology theory and the relationship of new cohomologies with well-known cohomologies of underlying associative and Leibniz algebras. The author defines and studies the category of groups with an action on itself and solves two problems of J.-L. Loday. Homotopical and categorical properties of chain functors category are also examined.

Dedicated to my Grandparents

CONTENTS

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INTRODUCTION

The monograph is dedicated to the study of properties of different algebraic objects: internal categories, equivalently crossed modules, their cohomologies, actor objects, noncommutative Leibniz–Poisson algebras, their cohomologies and their dual algebras, groups with action and Leibniz algebras, the solution of two problems of J.-L. Loday, and the category of chain functors.

In Chaps. 1–4, we examine problems of the internal category theory in the category of groups with operations and develop the cohomology theory of such categories and the cohomology theory for crossed modules of the appropriate type. This field of research was suggested by G. Janelidze, and problems presented in these chapters were proposed by him. This kind of investigation became more attractive after the publication of the paper [78] of T. Porters, where the equivalence of internal categories within categories of groups with operations with the category of crossed modules in this category was established. The idea of the definition of categories of groups with operations comes from J. Higgins [48] and G. Orzech [76]. The result of [78] was known for some specific types of categories (e.g., groups, associative rings and algebras, Lie and Jordan algebras) in works of J. L. Verdier, R. Brown, and C. B. Spencer [19], R. Lavendhomme and J. Roisin [58], and J.-L. Loday [60], but the above result led to a study of internal categories within different algebraic categories simultaneously. We hope that the structure of a category and the internalization of the well-known categorical notions and constructions can lead to obtaining interesting properties of new introduced objects and notions. The statements of Chaps. 2–4 give examples of such results: we define internal category cohomologies in categories of groups with operations, calculate the corresponding complex, and describe completely the cohomologies; we characterize cohomologically trivial internal categories and examine relations between internal category equivalence and homological and cohomological equivalences of internal categories; under certain assumptions, we obtain necessary and sufficient conditions for the existence of the internal Kan extensions, which was not known for ordinary categories.

The notion of a crossed module was introduced by J. H. C. Whitehead in 1949 during the study of homotopy systems of connected CW-complexes (see [89]). The notion of an internal category appeared later. It was a pleasant surprise to find the equivalence between internal categories in groups and crossed modules defined by Whitehead. In Chap. 1, we study internal analogs of well-known categorical notions and the relations between them. These notions for groups, in particular cases, give the notions, or special cases of those, defined by Whitehead for homotopy systems; for example, in the case of groups, the study of morphisms between internal functors gives a map called a crossed homomorphism associated with a certain homomorphism. In the case where internal categories correspond to free crossed modules with free groups of operators, the existence of a morphism between internal functors is a special case of the existence of a deformation operator associated with a homomorphism, between the homotopy systems of dimension 2, and it is a special case of the equivalence of these internal functors considered as homomorphisms between homotopy systems in the sense of Whitehead. In the same case, the equivalence of internal categories is a special case of the equivalence of the corresponding homotopy systems (see [89]).

The crossed-module approach to the study of internal categories enables us at the same time to develop the theory of crossed modules from the categorical point of view, for example, to define internal equivalence of crossed modules, Kan extensions, crossed module cohomologies as the cohomologies of the corresponding internal categories, etc. Thus, results obtained for internal categories give the corresponding results for crossed modules. In the special cases where a category of groups with operations is the category of groups, modules over a ring, associative, associative commutative, Lie, Leibniz, or alternative algebras, we obtain the results for internal categories (equivalently, for crossed modules) in these categories.

Another type of categories with structures called G-categories, where G is a group, was studied by R. Gordon (see [47]). This type of categories occurs in the representation theory of finite-dimensional algebras and algebraic topology and is also of independent interest in investigations. Note that crossed modules in groups can be considered as G-categories (see [47]).

Chapter 5 is devoted to actor objects in categories of interest. This type of categories was defined by G. Orzech (see [76]). This problem was posed by J. M. Casas after he was introduced to the work [32] (see Chap. 3), where cohomologically trivial internal categories were characterized and the actions in categories of interest were studied. Actions in algebraic categories were studied by G. Hochschild [50], S. Mac Lane [70], A. S.-T. Lue [67], K. Norrie [75], J.-L. Loday [62], R. Lavendhomme and T. Lucas [57], and others. The authors were looking for the analogs of automorphisms of groups in associative algebras, rings, Lie algebras, crossed modules, and Leibniz algebras. We see different approaches to this problem. Lue and Norrie (based on the results of Lue [68] and Whitehead [88]) associate to any object a certain type of object—the construction in the corresponding category, called an actor of this object [75], that has special properties analogous to group automorphisms, under which is meant that the actor fits into a certain commutative diagram (see Chap. 5, diagram (5.1.7)). Lavendhomme and Lucas introduced the notion of a Γ-algebra of derivations for an algebra A, which is the terminal object in the category of crossed modules under A. Recently F. Borceux, G. Janelidze, and G. M. Kelly [11, 12] proposed a categorical approach to this problem. They study internal object actions defined in [18] and introduce the notion of a representable action, which in the case of a category of interest is equivalent to the definition of an actor given in [21] (see Chap. 5).

We define an actor and a general actor object in categories of interest; we give a construction of a general actor object and study the problem of the existence of actors. The examples of groups, modules over a ring, Lie, Leibniz, associative, associative commutative algebras, crossed modules, and precrossed modules in groups are considered. The construction and the results obtained in this direction enable us to show the existence of an actor in the category of precrossed modules and to define special objects in the categories of Leibniz and associative algebras respectively, for which actors always exist.

In Chap. 6, we study noncommutative Leibniz–Poisson algebras (NLP-algebras), which are generalizations of classical Poisson algebras. The study of this type of algebras was proposed by T. Pirashvili to me and J. M. Casas. Another type of algebras with bracket operation was studied in [22]. We give a construction of free NLP-algebras, define the cohomology of NLP-algebras, study their properties, and relate them to the known cohomologies. The dual algebras of NLP-algebras are also considered. The cohomology defined by us gives in a special case the cohomology of Poisson algebras. Cohomologies of Poisson algebras were defined and studied by J. Hubschmann [52] in a different way.

Graded generalized Poisson algebras (satisfying the graded Leibniz identity) were studied by I. Kanatchikov [53]. Note that a different type of algebras with brackets, namely, algebras with two bracket operations (Lie–Leibniz algebras), appears in the study of the Witt construction for categories of groups with action on itself [36] (see Sec. VII).

Chapters 7 and 8 are devoted to the solution of problems of J.-L. Loday. In 1999, Loday proposed to me three questions, and later he informed me that he had stated these problems in [62, 64]. These problems concern Leibniz algebras. This notion was introduced by Loday himself in 1989 and was considered in a certain sense as a noncommutative analog of Lie algebras. There is a well-known construction of E. Witt [83, 90], due to which we can associate to lower central series of a group the graded object, which has a Lie-algebra structure; this actually defines the functor $\mathbb{G}\mathrm{r} \longrightarrow \mathbb{L}\mathrm{ie}$. The first problem of Loday: to define algebraic objects called "coquecigrues" that would have an analogous role for Leibniz algebras as groups have for Lie algebras. The second problem: Witt's theorem states that if a group is free, then the corresponding associated Lie algebra is also free [83, 90]. "A free coquecigrue should give rise to a free Leibniz algebra." The third problem can be formulated as follows. Coquecigrues should have groups as examples. Thus, it is reasonable to define the homology of the general linear group $GL(A)$ of a ring A as the homology of a coquecigrue in the corresponding category. These homology objects should have certain interesting properties (see [64]). This problem also involves the study of universal central extensions of concrete type of objects; the kernels of such extensions must have the special description in terms of the objects defined by Loday. The solution of this problem leads us to a new notion of the Leibniz K-theory of a ring (for details, see [64]).

Thus, we search for the category and the functor

$$\begin{array}{c} ? \\ \big\downarrow \\ \mathbb{L}\mathrm{eibniz} \end{array}$$

with the properties stated above.

Note that, according to Encyclopedia Britannica, a coquecigrue is an imaginary creature regarded as an embodiment of absolute absurdity.

We introduce the category of groups (abelian groups) with action on itself $\mathbb{G}\mathrm{r}^\bullet$ ($\mathbb{A}\mathrm{b}^\bullet$), the notions of an ideal, commutator, and central series in this category, and Lie–Leibniz algebras ($\mathbb{LL}$). Then we introduce Condition 1 on the action and according to this condition define the full subcategories

$$\mathbb{G}\mathrm{r}^{\mathrm{c}} \hookrightarrow \mathbb{G}\mathrm{r}^\bullet\,, \qquad \mathbb{A}\mathrm{b}^{\mathrm{c}} \hookrightarrow \mathbb{A}\mathrm{b}^\bullet\,.$$

We construct Witt's analogous functor

$$LL : \mathbb{G}\mathrm{r}^{\mathrm{c}} \longrightarrow \mathbb{LL}.$$

This functor leads us to Leibniz algebras over the ring of integers $\mathbb{Z}$ by taking either the composite

$$\mathbb{G}\mathrm{r}^c \xrightarrow{A} \mathbb{A}\mathrm{b}^c \xrightarrow{L} \mathbb{L}\mathrm{eibniz}$$

or the composite

$$\mathbb{G}\mathrm{r}^c \xrightarrow{LL} \mathbb{LL} \xrightarrow{S_2} \mathbb{L}\mathrm{eibniz},$$

where A is the abelianization functor, $L = LL\big|_{\mathbb{A}\mathrm{b}^c}$ and S_2 is the functor that makes the Lie bracket operation trivial. We introduce two more conditions (Conditions 2 and 3) between round and square brackets for the objects of $\mathbb{G}\mathrm{r}^c$ and according to these conditions define subcategories $\overline{\mathbb{G}\mathrm{r}}$ and $\overline{\mathbb{LL}}$ of $\mathbb{G}\mathrm{r}^c$ and $\mathbb{LL}$, respectively. We prove that the functor LL takes free objects from $\overline{\mathbb{G}\mathrm{r}}$ to free objects in $\overline{\mathbb{LL}}$. The composite

$$S_2\, LL\big|_{\overline{\mathbb{G}\mathrm{r}}} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \mathbb{L}\mathrm{eibniz}$$

gives free Leibniz algebras for free objects from $\overline{\mathbb{G}\mathrm{r}}$. It is proved that on free objects in $\overline{\mathbb{G}\mathrm{r}}$ this functor is isomorphic to the composite $L \circ \bar{A}$, where $\bar{A}$ is the abelianization functor; thus it is another isomorphic way which leads us from free objects in $\overline{\mathbb{G}\mathrm{r}}$ to free Leibniz algebras. Here we apply our result that, in particular, the functor $L : \mathbb{A}\mathrm{b}^c \longrightarrow \mathbb{L}\mathrm{eibniz}$ takes free objects to free Leibniz algebras. Note that our proof of the freeness theorem is different; it is not a generalization of E. Witt's proof for the case of groups. We propose constructions of free objects in the categories of groups with action on itself defined by us and free Leibniz algebras. The properties of commutators and related questions are also studied. The results obtained in Chaps. 7 and 8 give solutions to the two above stated problems of J.-L. Loday [62, 64]. The third problem suggests developing the (co)homology theory and to study universal central extensions in $\overline{\mathbb{G}\mathrm{r}}$. We hope that the constructions and the results obtained in these chapters will lead us to interesting investigations in this direction.

In Chap. 9, we study homotopical and categorical properties of chain functors category. This kind of work was proposed by F. W. Bauer, and the material presented here is a part of our joint work on chain functors [6, 7]. Chain functors were introduced by Bauer himself [4] (see Sec. 9.7 for the definition) for calculating *generalized* homology theories by means of chains and cycles like what one does for *ordinary*, simplicially defined homology theories by chain complexes. Like chain complexes, these chain functors form a category displaying interesting properties by themselves. In [6], we introduce a closed model structure in the category $\mathfrak{Ch}$ of chain functors. More precisely, we define fibrations, cofibrations, and weak equivalences satisfying D. Quillen's axioms CM2–CM5 for a closed model category [79]. It turns out that the first Quillen axiom (the existence of finite limits and colimits) fails for $\mathfrak{Ch}$. In particular, not every map has a kernel or a cokernel, and we do not detect arbitrary pullbacks and pushouts in $\mathfrak{Ch}$. The main issue of [7] is to exhibit that (1) all cofibrations have a cokernel, (2) all regular fibrations have a kernel, and (3) every pushout of a cofibration along a cofibration exists in $\mathfrak{Ch}$ (respectively, for pullbacks and fibrations). All this deserves independent interest, constituting a surprising justification for the concepts of fibrations and cofibrations in $\mathfrak{Ch}$. By using these results, we investigate interesting properties of exact sequences for fibrations and cofibrations. We will apply the results presented in this chapter in a forthcoming paper [8] for revealing the given closed model structure as a certain approximation to a *simplicial* one (satisfying, in addition to CM1–CM5, the axioms SM6 and SM7). Simplicial model structures are discussed, for example, in [46, 49, 79].

Homological properties of nontrivial extensions of abelian categories by a functor are defined by the author, and the coherence of such categories are studied in [26, 29]. In the special case, the results obtained in this direction give new results for the category of modules over nontrivial extensions of rings by bimodules.

This monograph consists of the Introduction and thirty two sections that constitute nine chapters.

Chapter 1. In Sec. 1.1, we recall well-known definitions of an internal category, an internal functor, a category of groups with operations (denoted by $\mathbf{C}$), and a crossed module in $\mathbf{C}$. We describe the correspondence between internal functors and crossed module homomorphisms in $\mathbf{C}$. In Sec. 1.2, we describe morphisms between internal functors and relate these mappings to Whitehead's notion of a crossed homomorphism and an equivalence between the homomorphisms of homotopy systems. We show that a morphism between internal functors implies the isomorphism of these functors (Proposition 1.2.1). In Sec. 1.3, we define an adjunction of internal functors, an internal category equivalence, and an adjoint equivalence. We prove necessary and sufficient conditions for the internal adjunction of functors in $\mathrm{Cat}(\mathbf{C})$ (internal categories in $\mathbf{C}$) (Proposition 1.3.4), and show that an internal adjunction implies an internal equivalence (Proposition 1.3.5). We obtain necessary and sufficient conditions for an internal equivalence and and adjoint equivalence in $\mathrm{Cat}(\mathbf{C})$ (Propositions 1.3.6. and 1.3.7), which imply that the existence of an adjoint pair of internal functors is equivalent to the adjoint equivalence of the corresponding internal categories in $\mathbf{C}$ (Proposition 1.3.8). By Proposition 1.3.9, if the corresponding crossed modules of internal categories are homotopy systems in the sense of Whitehead, then the internal equivalence of these categories is a special case of the equivalence of the corresponding homotopy systems in the sense of Whitehead [89]. We define full and faithful internal functors and give necessary and sufficient conditions for these properties (Lemmas 1.3.11 and 1.3.12). We prove necessary and sufficient conditions for an internal functor to be an internal equivalence (Theorem 1.3.13), which is an analog of the well-known theorem for ordinary categories and functors (Theorem 1 in [72, § 4, IV]). At the end of the section, we give necessary and sufficient conditions for an internal category to be equivalent to a discrete internal category (Proposition 1.3.14), which we apply in Chap. 3, in the characterization of cohomologically trivial internal categories.

Chapter 2. In Sec. 2.1, we recall the definition of an internal diagram on C, $\mathrm{C} \in \mathrm{Cat}(\mathbf{C})$ (see [43]), and denote the corresponding category by $\mathbf{C}^{\mathrm{C}}$. Then we consider abelian groups in $\mathbf{C}^{\mathrm{C}}$, study the action properties of C on $A \in Ab(\mathbf{C}^{\mathrm{C}})$, and conclude that A can be considered as a Coker d-module in the sense of [76], where d is the operator homomorphism of the crossed module corresponding to C. In Sec. 2.2, we construct the complex for the definition of a cohomology of $C \in \mathrm{Cat}(\mathbf{C})$ with coefficients in $A \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}})$ in analogy to the definition of the cohomology of ordinary categories. Applying the equivalence of internal categories and crossed modules, we compute completely this complex, which enables us to compute cohomologies (Theorem 2.2.1).

Chapter 3. In Sec. 3.1, we give the definition of a category of interest, which was introduced by G. Orzech (see [76]). For each category of interest $\mathbf{C}$, we define the corresponding general category of interest $\mathbf{C}_G$ and state the necessary and sufficient conditions for a set of actions to be the set of split derived actions in $\mathbf{C}_G$ in the sense of [76] (Proposition3.1.1). We recall the definitions of a (split) B-structure and of a B-module for $B \in \mathbf{C}$ [76]. We present some preliminary results on the extensions in categories of interest. We determine the necessary and sufficient conditions for the splitness of a singular extension (Proposition 3.1.6), which is similar to the case of groups. In Sec. 3.2, we recall the definition of a derivation in $\mathbf{C}$ (see [76]). We give the construction of the object $\mathbf{I}(\mathrm{C})$, $\mathrm{C} \in \mathbf{C}$. We show that this is a universal object that turns derivations into homomorphisms between the structured objects (Proposition 3.2.2). In Sec. 3.3, we investigate under which conditions a short exact sequence of modules over the internal category C (equivalently, Coker d-module) induces the long exact sequence of cohomology groups (Proposition 3.3.1). We describe $\mathrm{H}^0(\mathrm{C}, A)$ and $\mathrm{H}^1(\mathrm{C}, A)$, which we apply in Sec. 3.4. We define homologically and cohomologically equivalent internal categories and prove that internal category equivalence implies homological and cohomological equivalences (Theorem 3.3.4). We prove that in the case where $\mathbf{C}$ is a category of vector spaces over a field k, these three conditions are equivalent (Proposition 3.3.6). At the end of the section, we state the result, which relates G. J. Ellis's cohomology of crossed modules in the category of groups [42] to internal category cohomology (Proposition 3.3.7). In Sec. 3.4, we give the characterization of cohomologically trivial internal categories. The examples show (see Example 3, Sec. 3.4) that $\mathrm{H}^0(\mathrm{C},-)=0$ does not imply $\mathrm{H}^1(\mathrm{C},-)=0$. Thus, the notion of a cohomological dimension has no sense in our case. We describe separately internal categories C for which $\mathrm{H}^0(\mathrm{C},-)=0$, and separately, those C for which $\mathrm{H}^1(\mathrm{C},-)=0$. We obtain

$\mathrm{H}^0(\mathrm{C},-)=0 \Longleftrightarrow \mathbf{I}(\operatorname{Coker} d)=0$;

$\mathrm{H}^1(\mathrm{C},-)=0 \Longleftrightarrow S(\mathrm{C})$ is equivalent to the discrete category $(\operatorname{Coker} d, \operatorname{Coker} d, 1, 1, 1, 1)$, where S is a singularization functor defined in this section (Theorems 3.4.1 and 3.4.3). In the case where $\mathbf{C}$ is the category of groups, the first condition becomes simpler: $\mathrm{H}^0(\mathrm{C},-)=0 \Longleftrightarrow \mathrm{C}$ is a connected internal category (Corollary 3.4.2). In the case where C is the category of abelian groups, the second condition is equivalent to the condition: the homomorphism $d : \operatorname{Ker} d_0 \to \mathrm{C}_0$ (the crossed module corresponding to C) is a split monomorphism (Corollary 3.4.4). At the end of the section we give examples of computations of cohomologies for discrete, antidiscrete, and one-object internal categories.

Chapter 4. In this chapter, we consider the case $\mathbf{C}=\mathbb{G}\mathrm{r}$. In Sec. 4.1, we define Ext^1 in $\mathrm{Cat}(\mathbb{A}\mathrm{b})$ as a pullback of naturally defined diagram. We give the description of these groups in terms of equivalence classes of extensions. In Sec. 4.2, we define an internal Kan extension. We show that in the case where the domain internal categories in the definition of a Kan extension are connected, then the Kan extension is a unique up to an isomorphism extension of a given internal functor (Proposition 4.2.4); and under "extension" we mean here up to an isomorphism extension. In Proposition 4.2.5, we give the necessary and sufficient conditions for the functor to be a Kan extension. We describe $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M,A)$, $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$, as a pullback of certain naturally defined diagram, and $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M,A)$, the abelian group of isomorphic classes of internal functors, as a cokernel of a certain defined morphism (Proposition 4.2.6 and Lemma 4.2.7). We prove that the short exact sequence in $\mathrm{Cat}(\mathbb{A}\mathrm{b})$ induces the Hom-Ext^1 complex of abelian groups. For any short exact sequence in $\mathrm{Cat}(\mathbb{A}\mathrm{b})$ we deduce the commutative diagram (4.2.7) and prove Lemma 4.2.9 on the properties of $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(\ ,\ )$ and $\widetilde{\mathrm{Ext}^1}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(\ ,\ )$. In Sec. 4.3, applying the results obtained in the previous sections, under certain assumptions we prove the theorems on the necessary and sufficient conditions for the existence of internal Kan extensions in the case where the domain internal categories are connected. We prove the statements for two cases, where the Kan extension is taken along the surjective and along the injective internal functors (Theorems 4.3.2, 4.3.4, and 4.3.6). The case where the domain internal categories are nonconnected is considered in Sec. 4.4 (Theorem 4.4.1). In the special case where the internal categories in the diagram of the Kan extension are one-object categories, we show that the Kan extension reduces to the unique extension of a homomorphism in $\mathbb{A}\mathrm{b}$. From our conditions in this special case we obtain the same conditions which we have for abelian groups.

Chapter 5. Let $\mathbf{C}$ be a category of interest with a set of operations $\Omega=\Omega_0\cup\Omega_1\cup\Omega_2$ and a set of identities $\mathbf{E}$. In Sec. 5.1, we present the main definitions and results, which are used in what follows. We introduce the notions of an actor and of a general actor object for the objects of $\mathbf{C}$. In Sec. 5.2, for any object $A\in\mathbf{C}$ we give a construction of the universal algebra $\mathfrak{B}(A)$ with the operations from Ω. We show that, in general, $\mathfrak{B}(A)$ is an object of $\mathbf{C}_G$ (the general category of interest corresponding to $\mathbf{C}$ defined in Chap. 3). For any $A\in\mathbf{C}$, we define an action of $\mathfrak{B}(A)$ on A, which is a $\mathfrak{B}(A)$-structure on A in $\mathbf{C}_G$ (i.e., the split derived action appropriate to $\mathbf{C}_G$). In a well-known way, we define the universal algebra $\mathfrak{B}(A)\ltimes A$ which is an object of $\mathbf{C}_G$. We define the homomorphism $A\longrightarrow\mathfrak{B}(A)$ in $\mathbf{C}_G$, which turned out to be a crossed module in $\mathbf{C}_G$. We prove, that if an object A has an actor in $\mathbf{C}$, then $\mathfrak{B}(A)$=Actor(A) (Proposition 5.2.5). We show that the general actor object always exists and $\mathfrak{B}(A)$=GActor(A) (Theorem 5.2.7). The main theorem states that an object A from $\mathbf{C}$

has an actor in **C** if and only if $\mathfrak{B}(A) \ltimes A$ is an object in **C** and in this case $A \longrightarrow \mathfrak{B}(A)$ is an actor of A in **C** (Theorem 5.2.6). The cases of crossed modules and precrossed modules in groups are considered. From the results of [11] (Theorem 6.3) and Theorem 5.2.6, applying Proposition 5.2.8, we conclude that a category of interest **C** has representable object actions in the sense of [11] if and only if $\mathfrak{B}(A) \ltimes A \in \mathbf{C}$ for any $A \in \mathbf{C}$, and if it is the case, the corresponding representing objects are $\mathfrak{B}(A)$, $A \in \mathbf{C}$. In Sec. 5.3 we consider separately the case $\Omega_2 = \{+, *, *^\circ\}$. In the case of the groups ($\Omega_2 = \{+\}$), we obtain that $\mathfrak{B}(A) \approx \mathrm{Aut}(A)$, $A \in \mathbb{G}\mathrm{r}$. In the case of Lie algebras ($\Omega_2 = \{+, [\ ,\ ]\}$) for $A \in \mathbb{L}\mathrm{ie}$, we obtain $\mathfrak{B}(A) \approx \mathrm{Der}(A)$. In the case of Leibniz algebras, we have $\mathfrak{B}(A) \in \mathbb{L}\mathrm{eibniz}$ for any $A \in \mathbb{L}\mathrm{eibniz}$; $\mathfrak{B}(A)$ has a split derived set of actions on A if and only if for any $B, C \in \mathbb{L}\mathrm{eibniz}$ that has a derived action on A we have $[c,[a,b]] = -[c,[b,a]]$, for any $a \in A$, $b \in B$, $c \in C$, (which we call Condition 1, and it is equivalent to the existence of an $\mathrm{Actor}(A)$). In this case, $\mathfrak{B}(A) = \mathrm{Actor}(A)$ (Proposition 5.3.5). We give examples of such Leibniz algebras. In particular, Leibniz algebras A with $\mathrm{Ann}(A) = (0)$, where $\mathrm{Ann}(A)$ denotes the annulator of A, and perfect Leibniz algebras (i.e., $A = [A, A]$) satisfy Condition 1. We have an analogous picture for associative algebras. In this case, $\mathfrak{B}(A)$ is always an associative algebra, but the action of $\mathfrak{B}(A)$ on A defined by us is not a derived action on A. Here we introduce Condition 2: for any B and $C \in \mathbb{A}\mathrm{ss}$, which has a derived action on A, we have $c * (a * b) = (c * a) * b$ for any $a \in A$, $b \in B$, $c \in C$, where $*$ denotes the action. The action of $\mathfrak{B}(A)$ on A is a derived action if and only if A satisfies Condition 2 and it is equivalent to the existence of an $\mathrm{Actor}(A)$. In this case, $\mathfrak{B}(A) = \mathrm{Actor}(A)$ (Proposition 5.3.6). Associative algebras with conditions $\mathrm{Ann}(A) = (0)$ or with $A^2 = A$ satisfy Condition 2. These kinds of algebras were considered in [57, 70]. For the special types of objects in $\mathbb{A}\mathrm{ss}$ and $\mathbb{L}\mathrm{eibniz}$ noted above, we prove that $\mathfrak{B}(A) \approx \mathrm{Bim}(A)$ and $\mathfrak{B}(A) \approx \mathrm{Bider}(A)$ respectively (see Propositions 5.3.7 and 5.3.8), where $\mathrm{Bim}(A)$ denotes the associative algebra of bimultipliers defined by G. Hochschild and by S. MacLane for rings (called bimultiplications in [70] and multiplications in [50], from where the notion comes) and $\mathrm{Bider}(A)$ denotes the Leibniz algebra of biderivations of A defined in Sec. 5.2, which is isomorphic for these special types of Leibniz algebras to the biderivation algebra defined by J.-L. Loday in [62]. The cases of groups, modules over a ring, and commutative associative algebras are considered.

Chapter 6. Recall that a Poisson algebra is an associative commutative algebra A equipped with a binary bracket operation $[-,-] : A \otimes A \longrightarrow A$ such that $(A, [-,-])$ is a Lie algebra and the following condition holds:

$$[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b$$

for all $a, b, c \in A$.

Here we consider the case where algebras are not commutative and the bracket operation defines the Leibniz algebra structure [65] (see Chap. 5 for the definition of Leibniz algebra). This kind of algebras we call noncommutative Leibniz-Poisson algebras and denote the corresponding category by **NLP**. In Sec. 6.1, we describe preliminary basic material and construct a free NLP-algebra over a set X. In the case where X is a singleton, we give a description of the basis of the underlying abelian group of the free NLP-algebra in terms of planar binary rooted trees. We define actions, representations, and crossed modules in **NLP**, which are special cases of the corresponding notions for categories of groups with operations (see [76]). In Sec. 6.2, we define a cohomology $H^*_{\mathrm{NLP}}(P, \mathrm{M})$ of an NLP-algebra P (over a field k) with coefficients in a representation M over P as the cohomology of the cochain complex obtained by taking

$$C^0_{\mathrm{NLP}}(P, \mathrm{M}) = 0, \quad C^1_{\mathrm{NLP}}(P, \mathrm{M}) = \mathrm{Hom}(P, \mathrm{M}),$$

and in dimensions $n \geq 2$ by means of taking the pushout of cochain injections

$$\mathrm{cone}\,(\alpha^*) \longleftarrow \overline{C}_H^{*-1}(P, \mathrm{M}^e) \longrightarrow \mathrm{cone}\,(-\beta^*),$$

where

$$\alpha^* : C_H^*(P, \mathrm{M}) \longrightarrow C_H^*(P, \mathrm{M}^e)$$

(defined in [22]) and

$$\beta^* : C_L^*(P, \mathrm{M}) \longrightarrow C_H^*(P, \mathrm{M}^e)$$

(defined by means of differentials of Hochschild and Leibniz complexes) are cochain homomorphisms, cone(α^*) and cone$(-\beta^*)$ are the corresponding mapping cones, $\mathrm{M}^e = \mathrm{Hom}(P, \mathrm{M})$ has a natural P-representation structure [22], and C_H and C_L mean Hochschild and Leibniz complexes, respectively. The differentials are defined as follows:

$$\partial^0_{\mathrm{NLP}} = 0, \quad \partial^1_{\mathrm{NLP}} = (\partial^1_H, 0, \partial^1_L), \quad \partial^n_{\mathrm{NLP}} = \partial^n, \quad n \geq 2.$$

We define the restricted 2-dimensional cohomology $\mathbb{H}^2_{\mathrm{NLP}}(P, \mathrm{M})$ and prove that there are one-to-one correspondences between $H^1_{\mathrm{NLP}}(P, \mathrm{M})$ and the $\mathbb{K}$-vector space $\mathrm{Der}_{\mathrm{NLP}}(P, \mathrm{M})$ of derivations from P to M (Lemma 6.2.2), and between $\mathbb{H}^2_{\mathrm{NLP}}(P, \mathrm{M})$ and the set $\mathrm{Ext}_{\mathrm{NLP}}(P, \mathrm{M})$ of the equivalence classes of extensions of P by M (Theorem 6.2.3). We derive the long exact sequence relating NLP-algebra cohomology with Hochschild and AWB (algebras with bracket, see [22]) cohomologies and in a certain sense with Leibniz cohomologies (Proposition 6.2.1). We prove that if P is a free NLP-algebra, then $\mathbb{H}^2_{\mathrm{NLP}}(P, -) = 0$ and $H^n_{\mathrm{NLP}}(P, -) = 0$ for $n > 2$ (Corollary 6.2.4). Following [54], we define a relative cohomology of NLP-algebras over a field and prove that there is a bijection between the set $\mathrm{CExt}_{\mathrm{NLP}}(P, N; L)$ of equivalence classes of 3-fold crossed extensions (with fixed N) and the second restricted relative cohomology $\mathbb{H}^2_{\mathrm{NLP}}(P, N; L)$ (see Theorem 6.2.6). In Sec. 6.3, we consider algebras over the dual operad of NLP-algebras. The corresponding category of this kind of algebras is denoted by $\mathbf{NLP}^!$. We give the construction of free objects in $\mathbf{NLP}^!$. In the case where F is a free $\mathbf{NLP}^!$-algebra over the one element set, we show that there is a one-to-one correspondence between the set of certain type planar binary rooted trees and the basis of the underlying vector space of F.

Chapter 7. In Sec. 7.1, we define the category of groups with action on itself $\mathbb{G}\mathrm{r}^\bullet$, the category of abelian groups with action on itself $\mathbb{A}\mathrm{b}^\bullet$ and the category of groups with the bracket operation $\mathbb{G}\mathrm{r}^{[\,]}$. This kind of groups are Ω-groups in the sense of [55]. We construct adjoint pairs of functors relating the categories $\mathbb{G}\mathrm{r}^\bullet$, $\mathbb{A}\mathrm{b}^\bullet$, $\mathbb{G}\mathrm{r}^{[\,]}$, and $\mathbb{G}\mathrm{r}$. In Sec. 7.2, we define ideals and commutators for the objects of $\mathbb{G}\mathrm{r}^\bullet$ (similarly for $\mathbb{G}\mathrm{r}^{[\,]}$) and show that these notions are equivalent to the special case of the known notions for Ω-groups (see [55]). In Sec. 7.3, we define central series of groups with action on itself and a category of Lie–Leibniz algebras $\mathbb{LL}$. We consider the category of groups with action on itself $\mathbb{G}\mathrm{r}^{\mathrm{c}}$ satisfying the certain condition (see Condition 1). We give an analogue of the Witt construction (see [90]) and prove that it defines the functor $LL : \mathbb{G}\mathrm{r}^{\mathrm{c}} \longrightarrow \mathbb{LL}$ (Theorem 7.3.4); in particular, this gives the functor $\mathbb{G}\mathrm{r}^{\mathrm{c}} \longrightarrow \mathbb{L}\mathrm{eibniz}$. In a similar way, one can construct the functor $\mathbb{A}\mathrm{b}^{\mathrm{c}} \longrightarrow \mathbb{L}\mathrm{eibniz}$, which is actually the restriction of LL on $\mathbb{A}\mathrm{b}^{\mathrm{c}}$. The functorial relations with the classical situation ($\mathbb{G}\mathrm{r} \longrightarrow \mathbb{L}\mathrm{ie}$) is considered, namely by the restriction of LL on $\mathbb{G}\mathrm{r}$ we obtain the result of E. Witt [83, 90] (see diagram (7.3.3)).

Chapter 8. In Sec. 8.1, we introduce Conditions 2 and 3 for groups with action on itself and denote the corresponding full subcategory of $\mathbb{G}\mathrm{r}^{\mathrm{c}}$ by $\overline{\mathbb{G}\mathrm{r}}$. We prove that if A and B are ideals of G in $\overline{\mathbb{G}\mathrm{r}}$, then the commutator $[A, B]$ is also an ideal of G (Proposition 8.1.5). For ideals A, B, and C of G in $\overline{\mathbb{G}\mathrm{r}}$, we prove that

$$\big[A, [B, C]\big] \subset \big[[A, B], C\big] + \big[[A, C], B\big]$$

(Proposition 8.1.6). These two statements are well-known for the case of groups that we do not have generally in $\mathbb{G}\mathrm{r}^{\mathrm{c}}$. Applying these results we prove that for the objects G_n, $n > 1$, in the definition of central series of groups with action from $\overline{\mathbb{G}\mathrm{r}}$ we have $G_n = [G_{n-1}, G]$ for $n > 1$ (Lemma 8.1.11). From this fact we deduce that for the objects $\overline{G}_n = G_n/G_{n+1}$, where G is a free object in $\mathbb{A}\mathrm{b}^{\mathrm{c}}$, we have only those identities that are inherited from the identities of G by identifying the elements $\overline{x^y} = \overline{x}$, where $x \in G_n$, $y \in G$, x^y denotes an action, and $\overline{x}$ denotes the corresponding class in $\overline{G}_n$, which by definition of the category $\mathbb{A}\mathrm{b}^{\mathrm{c}}$ gives the Leibniz identity or its consequences (Lemma 8.1.12 and Proposition 8.1.13). In Sec. 8.2, we construct free objects in the categories $\mathbb{G}\mathrm{r}^{\bullet}$ (respectively, in $\mathbb{G}\mathrm{r}^{\mathrm{c}}$, $\mathbb{A}\mathrm{b}^{\mathrm{c}}$, $\overline{\mathbb{G}\mathrm{r}}$) and $\mathbb{L}$eibniz. In Sec. 8.3, we discuss the questions concerning identities between round and square brackets in $\mathbb{G}\mathrm{r}^{\bullet}$. We consider the certain set of possible identities in $\mathbb{G}\mathrm{r}^{\bullet}$; easy computations show that none of them is valid in $\mathbb{G}\mathrm{r}^{\bullet}$ (even in $\mathbb{G}\mathrm{r}^{c}$). Nevertheless, we cannot claim that there are no more identities between round and square brackets in $\mathbb{G}\mathrm{r}^{\bullet}$ or in $\overline{\mathbb{G}\mathrm{r}}$. We denote the possible set of identities in $\overline{\mathbb{G}\mathrm{r}}$ by E and the corresponding set of identities in $\mathbb{LL}$, inherited from E due to the functor $\overline{LL} = LL\big|_{\overline{\mathbb{G}\mathrm{r}}} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \mathbb{LL}$, by $\overline{\mathrm{E}}$; thus, $\overline{\mathrm{E}}$ is the set of identities that satisfy the objects $\overline{LL}(G)$, where G is a free object in $\overline{\mathbb{G}\mathrm{r}}$. We define the full subcategory $\overline{\mathbb{LL}} \subset \mathbb{LL}$ of all those Lie–Leibniz algebras over $\mathbb{Z}$ that satisfy identities from $\overline{\mathrm{E}}$. We prove that if G is a free object in $\overline{\mathbb{G}\mathrm{r}}$, then $\overline{LL}(G)$ is a free object in $\overline{\mathbb{LL}}$ (Theorem 8.3.2). Applying Proposition 8.1.13, we prove that the functor $L : \mathbb{A}\mathrm{b}^{\mathrm{c}} \longrightarrow \mathbb{L}\mathrm{eibniz}$ preserves the freeness of objects (Theorem 8.3.3). As a consequence, we also obtain that the composites $\overline{S}_2\,\overline{LL}, L\,\overline{A} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \mathbb{L}\mathrm{eibniz}$ correspond to free objects in $\overline{\mathbb{G}\mathrm{r}}$ free Leibniz algebras over $\mathbb{Z}$ (Corollary 8.3.11). Of course, it would be simpler to prove the commutator properties and Lemma 8.1.11 for $\mathbb{A}\mathrm{b}^{\mathrm{c}}$, then to show that the functor L preserves freeness, and since the abelianization functor $A : \mathbb{G}\mathrm{r}^{\mathrm{c}} \longrightarrow \mathbb{A}\mathrm{b}^{\mathrm{c}}$ has the same property, the composite $LA : \mathbb{G}\mathrm{r}^{c} \longrightarrow \mathbb{L}\mathrm{eibniz}$ would also preserve freeness. Nevertheless, we think that the general Lie–Leibniz case is interesting and that under Conditions 2 and 3 we can show that the properties of commutators in $\overline{\mathbb{G}\mathrm{r}}$ prove Lemma 8.1.11 and that the functor $\overline{LL} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \overline{\mathbb{LL}}$ takes free objects to free objects, from which we easily deduce the corresponding result for Leibniz algebras.

Chapter 9. In Sec. 9.1, we recollect the results of [6] and draw some more or less immediate conclusions. Thus, the four equivalent characterizations of a cofibration are the subject of Lemma 9.1.2, while for the dual properties for a fibration (Theorem 9.5.5) we need some information that is not available before Sec. 9.5. In Sec. 9.2, we prove that for an inclusion (i.e., a regular injection; see Sec. 9.1 for the definition),

$$\boldsymbol{A}_* \hookrightarrow \boldsymbol{B}_*$$

the pushout $\boldsymbol{B}_* \cup \mathit{cone}\ \boldsymbol{A}_*$ carries the structure of a chain functor (see Theorem 9.2.1). Section 9.3 is devoted to the verification of the fact that every cofibration admits a cokernel in $\mathfrak{C}h$ (Theorem 9.3.1), where we apply Theorem 9.2.1, and that a pushout diagram with cofibrations admits a pushout in $\mathfrak{C}h$ (Theorem 9.3.2). In addition, Sec. 9.5 contains the results on fibrations, which are dual to assertions that were presented in Secs. 9.2 and 9.3 for cofibrations. In Sec. 9.4, we present the dual to Theorem 9.2.1 (see Theorem 9.4.2), which ensures the existence of a special pullback in $\mathfrak{C}h$ and which is needed for the verification that every regular fibration has a kernel. In Sec. 9.6, we investigate properties of exact sequences for cofibrations and fibrations. Since some proofs in Secs. 9.2, 9.3, and 9.4 consist of the verification of the defining properties of a chain functor CH1–CH7, we include in Sec. 9.7 some basic material about chain functors, together with a lemma on chain functors taken from [5], which is needed to prove CH3) in Secs. 9.2 and 9.4.

Chapter 1

INTERNAL CATEGORIES, ADJUNCTION, AND EQUIVALENCE IN CATEGORIES OF GROUPS WITH OPERATIONS

In the beginning, we recall the well-known definitions of the category of groups with operations denoted by $\mathbf{C}$ and of an internal category. Using the equivalence of categories $\mathrm{Cat}(\mathbf{C}) \simeq X\,\mathrm{Mod}(\mathbf{C})$ obtained by T. Porter [78], where $\mathrm{Cat}(\mathbf{C})$ is the category of internal categories in $\mathbf{C}$ and $X\,\mathrm{Mod}(\mathbf{C})$ is the category of crossed modules of the appropriate type, we describe morphisms (= internal functors) and morphisms between the morphisms in $\mathrm{Cat}(\mathbf{C})$, adjunction of internal functors, and internal equivalence. We show that in this category any internal adjunction implies the internal equivalence and this is the special case of the Whitehead homotopy equivalence of the corresponding crossed modules, which was defined for certain types of complexes in the category of groups called homotopy systems (see [89]). For the internal category equivalence we prove an analog of Theorem 1 from [72, Sec. 4, IV]. The necessary and sufficient conditions for the equivalence of an internal category to the discrete internal category is obtained.

However some of these results are true for more general internal categories, e.g., for internal groupoids in more general categories, than for categories of groups with operations (see the remark at the end of the proof of Proposition 1.2.1); we restrict ourself to this special case to point out our interest in crossed modules and to show how this structure works in proofs. The results obtained in this chapter give similar results for crossed modules and are applied for the characterization of cohomologically trivial internal categories (Chap. 3) and in the study of the existence of internal Kan extensions (Chap. 4).

In Chap. 3, we continue the study of categorical notions for internal situation; we give the definitions of homological and cohomological equivalences and investigate their relations with internal category equivalence (see Sec. 3.3).

1.1. Preliminary Definitions and Results

Let $\mathbf{C}$ be a category with finite limits. We recall the definition of an internal category [43].

An internal category C in $\mathbf{C}$ consists of:

(a) a pair of objects C_0 and C_1,

(b) four morphisms

$$\mathrm{C}_1 \xrightarrow{d_0} \mathrm{C}_0\,, \quad \mathrm{C}_1 \xrightarrow{d_1} \mathrm{C}_0\,, \quad \mathrm{C}_0 \xrightarrow{i} \mathrm{C}_1\,, \quad \text{and} \quad \mathrm{C}_1 \times_{\mathrm{C}_0} \mathrm{C}_1 \xrightarrow{m} \mathrm{C}_1$$

such that

$$d_0 i = d_1 i = 1_{\mathrm{C}_0}, \quad d_0 m = d_0 \pi_2, \quad d_1 m = d_1 \pi_1,$$
$$m(1 \times m) = m(m \times 1) : \mathrm{C}_1 \times_{\mathrm{C}_0} \mathrm{C}_1 \times_{\mathrm{C}_0} \mathrm{C}_1 \longrightarrow \mathrm{C}_1,$$
$$m(1 \times i) = m(i \times 1) = 1_{\mathrm{C}_1}.$$

Here and below $\mathrm{C}_1 \times_{\mathrm{C}_0} \mathrm{C}_1$ denotes the pullback

$$\begin{array}{ccc} \mathrm{C}_1 \times_{\mathrm{C}_0} \mathrm{C}_1 & \xrightarrow{\pi_2} & \mathrm{C}_1 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle d_1} \\ \mathrm{C}_1 & \xrightarrow{d_0} & \mathrm{C}_0\,. \end{array}$$

Let $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m,)$ and $\mathrm{C}' = (\mathrm{C}'_0, \mathrm{C}'_1, d'_0, d'_1, i', m')$ be internal categories and $F = (F_0, F_1) : \mathrm{C} \longrightarrow \mathrm{C}'$ an internal functor; which means that F_0 and F_1 are morphisms of $\mathbf{C}$ and the diagrams

$$
\begin{array}{ccc}
d_0, d_1 : \mathrm{C}_1 & \overset{i}{\underset{\longrightarrow}{\overleftarrow{\longrightarrow}}} & \mathrm{C}_0 \\
{\scriptstyle F_1}\downarrow & {\scriptstyle i'} & \downarrow{\scriptstyle F_0} \\
d'_0, d'_1 : \mathrm{C}'_1 & \overleftarrow{\rightrightarrows} & \mathrm{C}'_0
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{C}_1 \times_{\mathrm{C}_0} \mathrm{C}_1 & \xrightarrow{m} & \mathrm{C}_1 \\
{\scriptstyle (F_1, F_1)}\downarrow & & \downarrow{\scriptstyle F_1} \\
\mathrm{C}'_1 \times_{\mathrm{C}'_0} \mathrm{C}'_1 & \xrightarrow[m']{} & \mathrm{C}'_1
\end{array}
$$

are commutative.

The idea of the definition of categories of groups with operations comes from J. Higgins [48] and G. Orzech [76], and the axioms below are from [76, 78].

From now on, $\mathbf{C}$ will denote a category of groups with a set of operations Ω and with a set $\mathbf{E}$ of identities such that $\mathbf{E}$ includes the group laws, and the following conditions hold: If Ω_i is the set of i-ary operations in Ω, then

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) the group operations written additively: $(0, -, +)$ are elements of Ω_0, Ω_1, and Ω_2, respectively. Let $\Omega'_2 = \Omega_2 \backslash \{+\}$, $\Omega'_1 = \Omega_1 \backslash \{-\}$ and assume that if $* \in \Omega'_2$, then Ω'_2 contains $*^0$ defined by $x *^0 y = y * x$. Assume further that $\Omega_0 = \{0\}$;

(c) for each $* \in \Omega'_2$, $\mathbf{E}$ includes the identify $x * (y + z) = x * y + x * z$;

(d) for each $\omega \in \Omega'_1$ and $* \in \Omega'_2$, $\mathbf{E}$ includes the identities $\omega(x+y) = \omega(x)+\omega(y)$ and $\omega(x)*y = \omega(x*y)$.

A category satisfying conditions (a)–(d) is called a category of groups with operations [76, 78]. The categories of groups, rings, associative, associative commutative, Lie, Leibniz, and alternative algebras are examples of categories of groups with operations.

Let $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ be an internal category in $\mathbf{C}$. Consider the split exact sequence

$$0 \longrightarrow \operatorname{Ker} d_0 \longrightarrow \mathrm{C}_1 \underset{i}{\overset{d_0}{\rightleftarrows}} \mathrm{C}_0 \longrightarrow 0,$$

where $d_0 i = 1$. As usual, we have the maps

$$\mathrm{C}_1 \underset{\theta^{-1}}{\overset{\theta}{\rightleftarrows}} \mathrm{C}_0 \times \operatorname{Ker} d_0$$

defined by

$$\theta(x) = (d_0(x), x - id_0(x)), \quad \theta^{-1}(r, c) = c + i(r),$$

and we have the induced operations in $\mathrm{C}_0 \times \operatorname{Ker} d_0$:

$$
\begin{aligned}
(r', c') + (r, c) &= (r' + r, c' + i(r') + c - i(r')), \\
(r', c') * (r, c) &= (r' * r, c' * c + c' * (i(r)) + (i(r')) * c),
\end{aligned}
$$

where $* \in \Omega'_2$. We shall use the following notation from [78]:

$$
\begin{aligned}
r \cdot c &= i(r) + c - i(r), \\
r * c &= (i(r)) * c, \\
c * r &= c * (i(r))
\end{aligned}
$$

for each $r \in \mathrm{C}_0$, $c \in \operatorname{Ker} d_0$, and $* \in \Omega'_2$. The set $\mathrm{C}_0 \times \operatorname{Ker} d_0$ with the above structure is an object of C; denote it by $\mathrm{C}_0 \ltimes \operatorname{Ker} d_0$. Moreover, we have the internal category $(\mathrm{C}_0, \mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \overline{d}_0, \overline{d}_1, \overline{i}, \overline{m})$, which is isomorphic to C and is obtained from C by θ. Direct computation gives

$$\begin{aligned} &\overline{d}_0(r,c) = r, \quad \overline{d}_1(r,c) = d(c) + r, \quad d = d_1\big|_{\operatorname{Ker} d_0}, \\ &\overline{i}(r) = (r,0), \quad \overline{m}\big((d(c)+r,c'),(r,c)\big) = (r, c'+c). \end{aligned}$$

We recall Lemma A and Proposition 4 of [78].

Lemma 1.1.1. *For* $c \in \operatorname{Ker} d_0$, $c' \in \operatorname{Ker} d_1$, *and* $* \in \Omega_2'$ *we have*

$$\begin{aligned} c + c' &= c' + c, \\ c * c' &= c' * c = 0. \end{aligned}$$

Lemma 1.1.2. *Let* $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ *be an internal category in* $\mathbf{C}$; *then* $d = d_1\big|_{\operatorname{Ker} d_0} : \operatorname{Ker} d_0 \longrightarrow \mathrm{C}_0$ *satisfies the following conditions*:

(i) $d(r \cdot c) = r + d(c) - r$;
(ii) $d(c) \cdot c' = c + c' - c$;
(iii) $d(c) * c' = c * c'$;
(iv) $d(c * r) = d(c) * r$

for each $r \in \mathrm{C}_0$, $c, c' \in \operatorname{Ker} d_0$ *and* $* \in \Omega_2'$.

Let $A, B \in \mathbf{C}$. As is well-known, a split derived action of B on A means that we have a set of actions on A derived from the split exact sequence $0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$ (see [76]). For more details concerning actions in $\mathbf{C}$, see Sec. 3.1.

Recall that a crossed module in $\mathbf{C}$ is a pair of objects $A, B \in \mathbf{C}$, where B has a split derived action on A, together with a homomorphism $d : A \longrightarrow B$ satisfying conditions (i)–(iv) of Lemma 1.1.2.

Denote by $\operatorname{Cat}(\mathbf{C})$ the category of internal categories and functors in $\mathbf{C}$. By [78], we have an equivalence of categories $\operatorname{Cat}(\mathbf{C}) \simeq X\operatorname{Mod}(\mathbf{C})$, where $X\operatorname{Mod}(\mathbf{C})$ denotes the category of crossed modules in $\mathbf{C}$. According to this equivalence, to each internal category $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ corresponds the crossed module

$$\operatorname{Ker} d_0 \xrightarrow{\ d\ } \mathrm{C}_0 \ .$$

Here $d = d_1\big|_{\operatorname{Ker} d_0}$ is a homomorphism in $\mathbf{C}$ that satisfies conditions of Lemma 1.1.2.

Let $F = (F_0, F_1) : \mathrm{C} \longrightarrow \mathrm{C}'$ be an internal functor. From the isomorphisms

$$\mathrm{C} \approx (\mathrm{C}_0, \mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \overline{d}_0, \overline{d}_1, \overline{i}, \overline{m}), \quad \mathrm{C}' \approx (\mathrm{C}_0', \mathrm{C}_0' \ltimes \operatorname{Ker} \overline{d}{}_0', \overline{d}{}_0', \overline{d}{}_0', \overline{i}\,', \overline{m}\,'),$$

we obtain the commutative diagram

$$\begin{array}{ccc} \overline{d}_0, \overline{d}_1 : \mathrm{C}_0 \ltimes \operatorname{Ker} d_0 & \underset{\xleftarrow{\ \overline{i}\ }}{\rightrightarrows} & \mathrm{C}_0 \\ {\scriptstyle \overline{F}_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ \overline{d}_0', \overline{d}_1' : \mathrm{C}_0' \ltimes \operatorname{Ker} d_0' & \underset{\xleftarrow{\ \overline{i}\,'\ }}{\rightrightarrows} & \mathrm{C}_0', \end{array} \qquad (1.1.1)$$

where $\overline{F}_1$ is defined by F_1. Denote by pr_1 and pr_2 the obvious projection maps

$$\mathrm{C}_0 \xleftarrow{\ pr_1\ } \mathrm{C}_0 \ltimes \operatorname{Ker} d_0 \xrightarrow{\ pr_2\ } \operatorname{Ker} d_0 \ .$$

From (1.1.1) we have

$$
\begin{array}{ccc}
(0,c) & \xrightarrow{\overline{d}_0} & 0 \\
\downarrow & & \downarrow \\
\overline{F}(0,c) & \xrightarrow[\overline{d}_0{}']{} & pr_1\overline{F}_1(0,c)
\end{array}
\Longrightarrow pr_1\overline{F}_1(0,c) = 0, \tag{1.1.2}
$$

$$
\begin{array}{ccc}
(r,0) & \xleftarrow{\bar{i}} & r \\
\downarrow & & \downarrow \\
\begin{array}{c}\overline{F}_1(r,0)\\ (F_0(r),0)\end{array} & \xleftarrow[\bar{i}\,']{} & F_0(r)
\end{array}
\Longrightarrow
\begin{array}{c} pr_1\overline{F}_1(r,0) = F_0(r) \\ pr_2\overline{F}_1(r,0) = 0. \end{array} \tag{1.1.3}
$$

Introduce the notation

$$pr_2\overline{F}_1(0,c) = \widetilde{F}_1(c).$$

From (1.1.2) and (1.1.3) and the fact that $\overline{F}_1$ is a morphism of $\mathbf{C}$, we have

$$\overline{F}_1(r,c) = \overline{F}_1\big((0,c)+(r,0)\big) = \overline{F}_1(0,c)+\overline{F}_1(r,0) = (0,\widetilde{F}_1(c)) + (F_0(r),0) = \big(F_0(r),\widetilde{F}_1(c)\big).$$

From the commutativity of diagram 1.1.1 we have

$$F_0\overline{d}_1 = \overline{d}_1'\overline{F}_1.$$

This equality for each $(r,c) \in \mathrm{C}_0 \ltimes \operatorname{Ker} d_0$ gives

$$F_0d(c) + F_0(r) = d'\widetilde{F}_1(c) + F_0(r);$$

so $F_0d(c) = d'\widetilde{F}_1(c)$, which means that the diagram

$$
\begin{array}{ccc}
\operatorname{Ker} d_0 & \xrightarrow{d} & \mathrm{C}_0 \\
{\scriptstyle \widetilde{F}_1}\downarrow & & \downarrow{\scriptstyle F_0} \\
\operatorname{Ker} d_0' & \xrightarrow[d']{} & \mathrm{C}_0'
\end{array} \tag{1.1.4}
$$

is commutative. Again, from the fact that $\overline{F}_1 \in \operatorname{Mor}\mathbf{C}$, we have

$$
\begin{aligned}
\overline{F}_1\big((r,c)+(r_1,c_1)\big) &= \overline{F}_1(r,c)+\overline{F}_1(r_1,c_1) \\
&= \big(F_0(r),\widetilde{F}_1(c)\big) + \big(F_0(r_1),\widetilde{F}_1(c_1)\big) = \big(F_0(r)+F_0(r_1),\widetilde{F}_1(c)+F_0(r)\cdot\widetilde{F}_1(c_1)\big).
\end{aligned}
$$

On the other hand,

$$
\begin{gathered}
\overline{F}_1\big((r,c)+(r_1,c_1)\big) = \overline{F}_1(r+r_1, c+r\cdot c_1) = \\
= \big(F_0(r+r_1),\widetilde{F}_1(c+r\cdot c_1)\big) = \big(F_0(r)+F_0(r_1),\widetilde{F}_1(c+r\cdot c_1)\big).
\end{gathered}
$$

From the above equalities we obtain

$$\widetilde{F}_1(c) + F_0(r) \cdot \widetilde{F}_1(c_1) = \widetilde{F}_1(c + r \cdot c_1).$$

For $c = 0$ and $r = 0$ we have respectively

$$\begin{aligned} \widetilde{F}_1(r \cdot c) &= F_0(r) \cdot \widetilde{F}_1(c_1), \\ \widetilde{F}_1(c + c_1) &= \widetilde{F}_1(c) + \widetilde{F}_1(c_1). \end{aligned}$$

Similarly, for any binary operation $*$, except the addition in **C**, we have

$$\begin{aligned} \overline{F}_1\big((r,c) * (r_1, c_1)\big) &= \overline{F}_1(r * r_1, c * c_1 + r * c_1 + c * r_1) \\ &= \big(F_0(r * r_1), \widetilde{F}_1(c * c_1 + r * c_1 + c * r_1)\big) \\ &= \big(F_0(r) * F_0(r_1), \widetilde{F}_1(c * c_1) + \widetilde{F}_1(r * c_1) + \widetilde{F}_1(c * r_1)\big); \\ \overline{F}_1\big((r,c) * (r_1, c_1)\big) &= \overline{F}_1(r,c) * \overline{F}_1(r_1, c_1) = \big(F_0(r), \widetilde{F}_1(c)\big) * \big(F_0(r_1), \widetilde{F}_1(c_1)\big) \\ &= \Big(F_0(r) * F_0(r_1), \widetilde{F}_1(c) * \widetilde{F}_1(c_1) + \widetilde{F}_1(c) * F_0(r_1) + F_0(r) * \widetilde{F}_1(c_1)\Big). \end{aligned}$$

From the above two equalities we obtain

$$\widetilde{F}_1(c * c_1) + \widetilde{F}_1(r * c_1) + \widetilde{F}_1(c * r_1) = \widetilde{F}_1(c) * \widetilde{F}_1(c_1) + \widetilde{F}_1(c) * F_0(r_1) + F_0(r) * \widetilde{F}_1(c_1),$$

which gives

$$\begin{aligned} \widetilde{F}_1(c * c_1) &= \widetilde{F}_1(c) * \widetilde{F}_1(c_1), \\ \widetilde{F}_1(r * c_1) &= F_0(r) * \widetilde{F}_1(c_1), \\ \widetilde{F}_1(c * r_1) &= \widetilde{F}_1(c) * F_0(r_1), \end{aligned}$$

for each $r, r_1 \in \mathrm{C}_0, c, c_1 \in \operatorname{Ker} d_0$.

Thus, we can consider the internal functor $(F_0, F_1) : \mathrm{C} \longrightarrow \mathrm{C}'$ as a pair $(F_0, \widetilde{F}_1)$, such that $F_0 : \mathrm{C}_0 \longrightarrow \mathrm{C}'_0$ and $\widetilde{F}_1 : \operatorname{Ker} d_0 \longrightarrow \operatorname{Ker} d'_0$ are morphisms of **C** satisfying the conditions

$$\begin{aligned} \widetilde{F}_1(r \cdot c) &= F_0(r) \cdot \widetilde{F}_1(c), \\ \widetilde{F}_1(r * c) &= F_0(r) * \widetilde{F}_1(c), \\ \widetilde{F}_1(c * r) &= \widetilde{F}_1(c) * F_0(r), \\ d' \widetilde{F}_1(c) &= F_0 d(c), \end{aligned} \tag{1.1.5}$$

for each $r \in \mathrm{C}_0, c \in \operatorname{Ker} d_0$. In the case $\mathbf{C} = \mathbf{Grp}$ for such a pair of morphisms $(F_0, \widetilde{F}_1), \widetilde{F}_1$ is called an operator homomorphism associated with F_0 by J. H. C. Whitehead [89].

It is easy to show that every pair of morphisms $(F_0, \widetilde{F}_1)$ satisfying conditions (1.1.5) determines an internal functor $F : \mathrm{C} \longrightarrow \mathrm{C}'$ and this correspondence is one-to-one. This correspondence is involved in the proof given in [78], but we will need this detailed account in what follows.

1.2. Morphisms between Morphisms in Cat(C)

First, we discuss this question in a more general situation and then pass to the case of the category of groups with operations and crossed modules.

1.2.1. General Case. Let $\underline{C}$ and $\underline{C}'$ be (ordinary) categories. Recall (see [72]) that given two functors $S, T : \underline{C} \longrightarrow \underline{C}'$, a natural transformation $\vartheta : S \longrightarrow T$ is a function that assigns to each object $c \in \underline{C}$ a morphism $\vartheta(c) : S(c) \longrightarrow T(c)$ of $\underline{C}'$ in such a way that for each morphism $\gamma : c \longrightarrow c_1$ the diagram

$$\begin{array}{ccc} S(c) & \xrightarrow{\vartheta(c)} & T(c) \\ {\scriptstyle S(\gamma)}\downarrow & & \downarrow{\scriptstyle T(\gamma)} \\ S(c_1) & \xrightarrow[\vartheta(c_1)]{} & T(c_1) \end{array}$$

is commutative. Hence ϑ is a function $|\underline{C}| \longrightarrow \operatorname{Mor} \underline{C}'$, such that

$$\operatorname{dom} \vartheta(c) = S(c), \quad \operatorname{codom} \vartheta(c) = T(c), \quad T(\gamma)\vartheta(c) = \vartheta(c_1)S(\gamma).$$

The composition of two natural transformations of functors

$$S \xrightarrow{\vartheta} T \xrightarrow{\xi} K$$

for the diagram

$$\underline{C} \; \begin{array}{c} \xrightarrow{S} \\ \xrightarrow{T} \\ \xrightarrow{K} \end{array} \; \underline{C}'$$

is defined by $\xi\vartheta(c) = \xi(c)\vartheta(c)$ for each $c \in |\underline{C}|$.

Let $\underline{C}$ be a category with finite limits and $\mathrm{Cat}(\underline{C})$ be the category of internal categories and functors in $\underline{C}$. Let $\mathrm{C}, \mathrm{C}' \in |\mathrm{Cat}(\underline{C})|$, $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i\ m)$, $\mathrm{C}' = (\mathrm{C}'_0, \mathrm{C}'_1\ d'_0, d'_1\ i'\ m')$, and $S, T : \mathrm{C} \longrightarrow \mathrm{C}'$ be internal functors. This means that

$$S_0, T_0 : \mathrm{C}_0 \longrightarrow \mathrm{C}'_0 \in \operatorname{Mor} \underline{C}\,, \quad S_1, T_1 : \mathrm{C}_1 \longrightarrow \mathrm{C}'_1 \in \operatorname{Mor} \underline{C}\,,$$

and the following diagrams are commutative:

$$\begin{array}{ccc} & \xleftarrow{\;i\;} & \\ \mathrm{C}_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & \mathrm{C}_0 \\ {\scriptstyle S_1}\downarrow & & \downarrow{\scriptstyle S_0} \\ \mathrm{C}'_1 & \overset{d'_0}{\underset{d'_1}{\rightrightarrows}} & \mathrm{C}'_0, \\ & \xleftarrow[\;i'\;]{} & \end{array} \qquad \begin{array}{ccc} & \xleftarrow{\;i\;} & \\ \mathrm{C}_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & \mathrm{C}_0 \\ {\scriptstyle T_1}\downarrow & & \downarrow{\scriptstyle T_0} \\ \mathrm{C}'_1 & \overset{d'_0}{\underset{d'_1}{\rightrightarrows}} & \mathrm{C}'_0\,. \\ & \xleftarrow[\;i'\;]{} & \end{array}$$

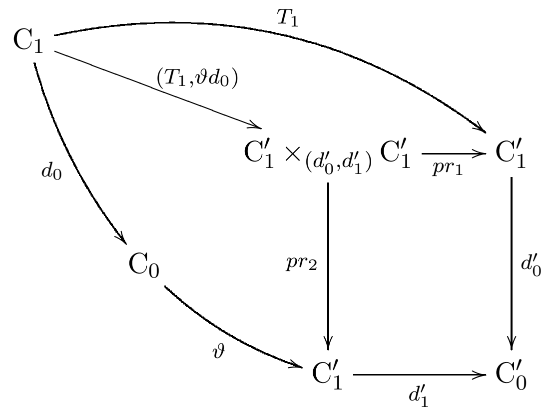
Thus we have

$$\begin{aligned} S_1 i &= i' S_0, & T_1 i &= i' T_0, \\ d'_0 S_1 &= S_0 d_0, & d'_0 T_1 &= T_0 d_0, \\ d'_1 S_1 &= S_0 d_1, & d'_1 T_1 &= T_0 d_1. \end{aligned} \tag{1.2.1}$$

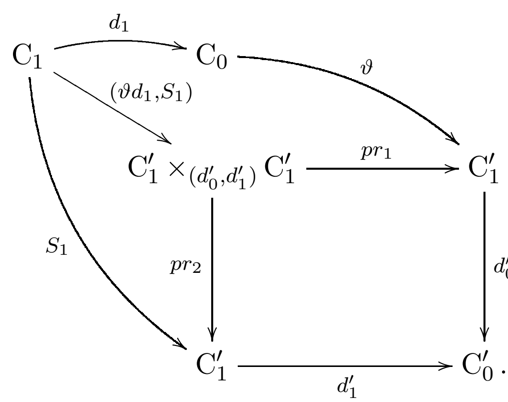
What does the natural transformation $\vartheta : S \longrightarrow T$ mean? In this case, ϑ is a morphism $\mathrm{C}_0 \longrightarrow \mathrm{C}'_1$ of $\underline{C}$, such that the following diagrams are commutative:

$$\begin{array}{ccc} & & \mathrm{C}'_0 \\ & {\scriptstyle S_0}\nearrow & \uparrow{\scriptstyle d'_0} \\ \mathrm{C}_0 & \xrightarrow[\vartheta]{} & \mathrm{C}'_1, \end{array} \qquad \begin{array}{ccc} & & \mathrm{C}'_0 \\ & {\scriptstyle T_0}\nearrow & \uparrow{\scriptstyle d'_1} \\ \mathrm{C}_0 & \xrightarrow[\vartheta]{} & \mathrm{C}'_1, \end{array} \qquad \begin{array}{ccc} \mathrm{C}_1 & \xrightarrow{(T_1, \vartheta d_0)} & \mathrm{C}'_1 \times_{(d'_0, d'_1)} \mathrm{C}'_1 \\ {\scriptstyle (\vartheta d_1, S_1)}\downarrow & & \downarrow{\scriptstyle m'} \\ \mathrm{C}'_1 \times_{(d'_0, d'_1)} \mathrm{C}'_1 & \xrightarrow[m']{} & \mathrm{C}'_1\,. \end{array} \tag{1.2.2}$$

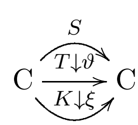
Here $(T_1, \vartheta d_0)$ denotes a morphism induced by T_1 and ϑd_0, since from (1.2.1) and (1.2.2) $d'_0 T_1 = T_0 d_0$, $d'_1 \vartheta d_0 = T_0 d_0$ and thus $d'_0 T_1 = d'_1 \vartheta d_0$. The picture is as follows:



Similarly, $(\vartheta d_1, S_1)$ in (1.2.2) is induced by ϑd_1 and S_1, since from (1.2.1) and (1.2.2) $d'_1 S_1 = S_0 d_1$, $d'_0 \vartheta d_1 = S_0 d_1$ and thus $d'_0 \vartheta d_1 = d'_1 S_1$. The picture is as follows:



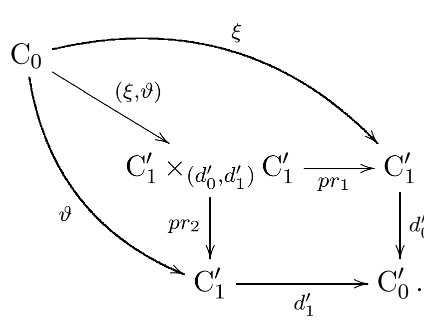
The identity natural transformation $T \xrightarrow{1} T$ is the composite $C_0 \xrightarrow{T_0} C'_0 \xrightarrow{i'} C'_1$ (it satisfies condition (1.2.2)). The composite of two natural transformations of internal functors



is defined as the composite

$$C_0 \xrightarrow{(\xi,\vartheta)} C'_1 \times_{(d'_0,d'_1)} C'_1 \xrightarrow{m'} C'_1,$$

where (ξ, ϑ) is a morphism induced by ϑ and ξ:



(ξ, ϑ) factors through $\mathrm{C}'_1 \times_{(\ )} \mathrm{C}'_1$. It follows from the equalities $d'_1\vartheta = T_0 = d'_0\xi$, which we obtain from (1.2.2) for ϑ and ξ. We must check that the composite $m'(\xi, \vartheta)$ satisfies conditions (1.2.2). The first diagram of (1.2.2) is commutative because it commutes for ϑ and the second is commutative because it commutes for ξ. For the third diagram we must prove that the diagram

$$\begin{array}{ccc} \mathrm{C}_1 & \xrightarrow{(K_1, m'(\xi,\vartheta)d_0)} & \mathrm{C}'_1 \times_{(\ )} \mathrm{C}'_1 \\ {\scriptstyle (m'(\xi,\vartheta)d_1, S_1)}\downarrow & & \downarrow{\scriptstyle m'} \\ \mathrm{C}'_1 \times_{(\ )} \mathrm{C}'_1 & \xrightarrow[m']{} & \mathrm{C}'_1 \end{array}$$

is commutative. For the elements we obtain

$$\begin{array}{ccc} c & \longmapsto & \big(K_1(c), m'(\xi d_0(c), \vartheta d_0(c))\big) \\ \downarrow & & \downarrow \\ \big(m'(\xi d_1(c), \vartheta d_1(c)), S_1(c)\big) & \longmapsto & \begin{array}{l} m'\big(K_1(c), m'\big(\xi d_0(c), \vartheta d_0(c)\big)\big), \\ m'\big(m'\big(\xi d_1(c), \vartheta d_1(c), S_1(c)\big)\big). \end{array} \end{array}$$

Thus, it suffices to show that

$$m'\big(K_1(c), m'\big(\xi d_0(c), \vartheta d_0(c)\big)\big) = m'\big(m'\big(\xi d_1(c), \vartheta d_1(c)\big), S_1(c)\big)$$

for each $c \in \mathrm{C}_1$. From the associativity of composition in C' and commutativity of the third diagram of (1.2.2) for ϑ and ξ, we obtain

$$\begin{aligned} m'\big(m'(\xi d_1(c), \vartheta d_1(c)), S_1(c)\big) &= m'\big(\xi d_1(c), m'(\vartheta d_1(c), S_1(c))\big) \\ &= m'\big(\xi d_1(c), m'(T_1(c), \vartheta d_0(c))\big) = m'\big(m'(\xi d_1(c), T_1(c)), \vartheta d_0(c)\big) \\ &= m'\big(m'(K_1(c), \xi d_0(c)), \vartheta d_0(c)\big) = m'(K_1(c), m'(\xi d_0(c), \vartheta d_0(c))), \end{aligned}$$

which proves the commutativity of the above diagram.

1.2.2. The case of a category of groups with operations. Let $\mathbf{C}$ be the category of groups with operations, $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ and $\mathrm{C}' = (\mathrm{C}'_0, \mathrm{C}'_1, d'_0, d'_1, i', m')$ be internal categories in $\mathbf{C}$, and $S, T : \mathrm{C} \rightrightarrows \mathrm{C}'$ be internal functors. Then as was shown in Sec. 1.1, this yields the commutative diagram

$$\begin{array}{ccc} \mathrm{Ker}\, d_0 & \xrightarrow{d} & \mathrm{C}_0 \\ {\scriptstyle S_1}\downdownarrows{\scriptstyle T_1} & & {\scriptstyle S_0}\downdownarrows{\scriptstyle T_0} \\ \mathrm{Ker}\, d'_0 & \xrightarrow[d']{} & \mathrm{C}'_0, \end{array} \tag{1.2.3}$$

where $S = (S_0, S_1)$ and $T = (T_0, T_1)$ satisfy conditions (1.1.5) ($F_0, \widetilde{F_1}$ are replaced by S_0, S_1 and T_0, T_1, respectively). *A morphism* or *a natural transformation* $\vartheta : (S_0, S_1) \longrightarrow (T_0, T_1)$ is a morphism $\vartheta : \mathrm{C}_0 \longrightarrow \mathrm{C}'_0 \ltimes \mathrm{Ker}\, d'_0$ of $\mathbf{C}$ such that the following diagrams are commutative:

$$\begin{array}{ccc} & & \mathrm{C}'_0 \\ & \nearrow^{S_0} & \uparrow{\scriptstyle \overline{d}_0'} \\ \mathrm{C}_0 & \xrightarrow[\vartheta]{} & \mathrm{C}'_0 \ltimes \mathrm{Ker}\, d'_0, \end{array} \qquad \begin{array}{ccc} & & \mathrm{C}'_0 \\ & \nearrow^{T_0} & \uparrow{\scriptstyle \overline{d}_1'} \\ \mathrm{C}_0 & \xrightarrow[\vartheta]{} & \mathrm{C}'_0 \ltimes \mathrm{Ker}\, d'_0, \end{array}$$

$$
\begin{array}{ccc}
\mathrm{C}_0 \ltimes \operatorname{Ker} d_0 & \xrightarrow{(T_1,\vartheta\overline{d_0})} & (\mathrm{C}'_0 \ltimes \operatorname{Ker} d'_0) \times_{(\overline{d}'_0,\overline{d}'_1)} (\mathrm{C}'_0 \ltimes \operatorname{Ker} d'_0) \\
{\scriptstyle (\vartheta\overline{d}_1{}',S_1)}\big\downarrow & & \big\downarrow{\scriptstyle m'} \\
(\mathrm{C}'_0 \ltimes \operatorname{Ker} d'_0) \times_{(\overline{d}'_0,\overline{d}'_1)} (\mathrm{C}'_0 \ltimes \operatorname{Ker} d'_0) & \xrightarrow[m']{} & \mathrm{C}'_0 \ltimes \operatorname{Ker} d'_0
\end{array}
$$

From the above we have

$$\vartheta(r) = (S_0(r), \alpha(r)), \quad \text{where} \quad \alpha(r) = pr_2\vartheta(r) \in \operatorname{Ker} d'_0;$$

$$d'\alpha(r) + S_0(r) = T_0(r),$$

$$
\begin{array}{ccc}
(r,c) & \longmapsto & ((T_0(r), T_1(c)), S_0(r), \alpha(r)) \\
\big\downarrow & & \big\downarrow \\
 & & (S_0(r), T_1(c) + \alpha(r)) \\
 & & \big\| \\
((S_0(dc+r), \alpha(dc+r)), (S_0(r), S_1(c))) & \longmapsto & (S_0(r), \alpha(dc+r) + S_1(\mathrm{C}))
\end{array}
\tag{1.2.4}
$$

for each $r \in \mathrm{C}_0$ and $c \in \operatorname{Ker} d_0$.

From the fact that ϑ is a morphism of **C** for each $r, r_1 \in \mathrm{C}_0$, we obtain:

1. For each unary operation ω in **C** except the negation,

$$
\begin{aligned}
\vartheta(\omega(r)) &= \omega\vartheta(r), \\
\vartheta(\omega(r)) &= \big(S_0\omega(r), \alpha\omega(r)\big) = \big(\omega S_0(r), \alpha\omega(r)\big), \\
\omega\vartheta(r) &= \omega(S_0(r), \alpha(r)) = \big(\omega S_0(r), \omega\alpha(r)\big), \\
\alpha\omega(r) &= \omega\alpha(r).
\end{aligned}
$$

2. $\vartheta(r + r_1) = \vartheta(r) + \vartheta(r_1)$,

$$\big(S_0(r + r_1), \alpha(r + r_1)\big) = \big(S_0(r) + S_0(r), \alpha(r + r_1)\big), \quad \vartheta(r) + \vartheta(r_1) \quad = (S_0(r), \alpha(r)) + (S_0(r_1), r_1)$$

$$= \big(S_0(r) + S_0(r_1), \alpha(r) + S_0(r) \cdot \alpha(r_1)\big),$$

$$\alpha(r + r_1) = \alpha(r) + S_0(r) \cdot \alpha(r_1). \tag{1.2.5}$$

3. For each binary operation $*$ in **C** except the addition,

$$
\begin{aligned}
\vartheta(r * r_1) &= \vartheta(r) * \vartheta(r_1), \\
\vartheta(r * r_1) &= (S_0(r * r_1), \alpha(r * r_1)) = (S_0(r) * S_0(r_1), \alpha(r * r_1)), \\
\vartheta(r) * \vartheta(r_1) &= (S_0(r), \alpha(r)) * (S_1(r_1), \alpha(r_1)) \\
&= \big(S_0(r) * \big(S_0(r_1), \alpha(r) * \alpha(r_1) + \alpha(r) * S_0(r_1) + S_0(r) * \alpha(r_1)\big)\big), \\
\alpha(r * r_1) &= \alpha(r) * \alpha(r_1) + \alpha(r) * S_0(r_1) + S_0(r) * \alpha(r_1).
\end{aligned}
$$

From (1.2.4) and (1.2.5) we obtain

$$
\begin{aligned}
T_1(c) + \alpha(r) &= \alpha(dc + r) + S_1(c), \\
T_1(c) + \alpha(r) &= \alpha(dc) + S_0(dc) \cdot \alpha(r) + S_1(c).
\end{aligned}
$$

From (1.2.3) we have

$$T_1(c) + \alpha(r) = \alpha(dc) + d'S_1(c) \cdot \alpha(r) + S_1(c).$$

Applying Lemma 1.1.2 we obtain

$$T_1(c) + \alpha(r) = \alpha d(c) + S_1(c) + \alpha(r) - S_1(c) + S_1(c),$$

which gives $\alpha d(c) = T_1(c) - S_1(c)$. Thus, a natural transformation $\vartheta : (S_0, S_1) \longrightarrow (T_0, T_1)$ determines the map $\alpha : \mathrm{C}_0 \longrightarrow \operatorname{Ker} d_0'$ satisfying the following conditions:

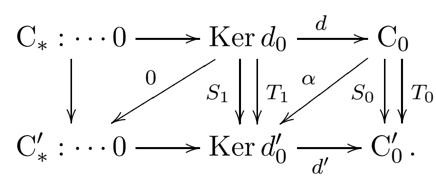
$$
\begin{aligned}
\alpha\omega(r) &= \omega\alpha(r), \\
\alpha(r + r_1) &= \alpha(r) + S_0(r) \cdot \alpha(r_1), && (1.2.6_1) \\
\alpha(r * r_1) &= \alpha(r) * \alpha(r_1) + \alpha(r) * S_0(r_1) + S_0(r) * \alpha(r_1); && \qquad (1.2.6) \\
d'\alpha(r) &= T_0(r) - S_0(r), && \\
\alpha d(c) &= T_1(c) - S_1(c). && (1.2.6_2)
\end{aligned}
$$

It is easy to show that such a map $\alpha : \mathrm{C}_0 \longrightarrow \operatorname{Ker} d_0'$ determines the natural transformation $\vartheta : \mathrm{C}_0 \longrightarrow \mathrm{C}_0' \ltimes \operatorname{Ker} d_0'$ between functors $(S_0, S_1) \longrightarrow (T_0, T_1)$ defined by the correspondence $r \longmapsto (S_0(r), \alpha(r))$, and this correspondence $\alpha \longmapsto \vartheta$ is one-to-one. In what follows under the natural transformation of internal functors we shall mean a map satisfying conditions (1.2.6).

In the case $\mathbf{C} = \mathbb{G}\mathrm{r}$, a map α satisfying conditions $(1.2.6_1)$ is called by Whitehead a crossed homomorphism associated with S_0 (see [89]). Consider the case where the complexes

$$\mathrm{C}_* : 0 \longrightarrow \operatorname{Ker} d_0 \xrightarrow{d} \mathrm{C}_0 \quad \text{and} \quad \mathrm{C}_*' : 0 \longrightarrow \operatorname{Ker} d_0' \xrightarrow{d'} \mathrm{C}_0'$$

are homotopy systems (for definition, see [89]). Then such type of a map $\alpha : \mathrm{C}_0 \longrightarrow \operatorname{Ker} d_0'$ is a deformation operator associated with a homomorphism S_0 in the sense of Whitehead [89]. The picture is as follows:



In this case, the existence of a map α satisfying conditions (1.2.6) in Whitehead's terminology means that homomorphism $S = (S_0, S_1, 0, \ldots)$ is equivalent to a homomorphism $T = (T_0, T_1, 0, \ldots)$. (In formula (4.3) of [89],

$$\omega' g_n - f_n = d_{n+1}' \xi_{n+1} + \xi_n d_n, \quad n \geq 1,$$

we must take $\omega' = 0$, ω' is an element of C_0' in our case, $g_1 = T_0$, $g_2 = T_1$, and $g_i = 0$ for $i > 2$, $f_1 = S_0$, $f_2 = S_1$, and $f_i = 0$ for $i > 2$, $\xi_1 = 0$, $\xi_2 = \alpha$, and $\xi_i = 0$ for $i > 2$, $d_2 = d$, $d_i = 0$, $d_2' = d'$, and $d_i' = 0$ for $i \neq 2$, which gives $(1.2.6_2)$ for $n = 1$ and $n = 2$, respectively).

Thus, in this case the existence of a morphism between internal functors is a special case of the equivalence of these internal functors, considered as homomorphisms of homotopy systems in the sense of Whitehead [89].

The identity natural transformation of internal functors $(T_0, T_1) \xrightarrow{1} (T_0, T_1)$ is the composite

$$\mathrm{C}_0 \xrightarrow{T_0} \mathrm{C}_0' \xrightarrow{\overline{i'}} \mathrm{C}_0' \ltimes \operatorname{Ker} d_0',$$

and the corresponding map $\alpha : \mathrm{C}_0 \longrightarrow \operatorname{Ker} d_0'$ is zero. The composite of two natural transformations of internal functors

$$(S_0, S_1) \xrightarrow{\vartheta} (T_0, T_1) \xrightarrow{\xi} (K_0, K_1)$$

is the composite

$$\mathrm{C}_0 \xrightarrow{(\xi, \vartheta)} (\mathrm{C}_0' \ltimes \operatorname{Ker} d_0') \times_{(d_0', d_1')} (\mathrm{C}_0' \ltimes \operatorname{Ker} d_0') \xrightarrow{m'} (\mathrm{C}_0' \ltimes \operatorname{Ker} d_0')\ .$$

For the elements we have

$$r \longmapsto \big((T_0(r), \beta(r)), (S_0(r), \alpha(r))\big) \longmapsto \big(S_0(r), \beta(r) + \alpha(r)\big),$$

where $\alpha, \beta, \beta + \alpha : \mathrm{C}_0 \longrightarrow Kerd'_0$ correspond to ϑ, ξ and $\xi \cdot \vartheta$, respectively. It is easy to show that $\beta + \alpha$ satisfies conditions (1.2.6).

Proposition 1.2.1. *Let* $\mathrm{C}, \mathrm{C}' \in \mathrm{Cat}\,(\mathbf{C})$ *and* $\alpha : (S_0, S_1) \longrightarrow (T_0, T_1)$ *be a natural transformation between internal functors* $S, T : \mathrm{C} \longrightarrow \mathrm{C}'$. *Then* S *and* T *are naturally isomorphic.*

Proof. We show that there is a map $\alpha' : \mathrm{C}_0 \longrightarrow \mathrm{Ker}\, d'_0$ such that for each $r, r_1 \in \mathrm{C}_0$, $c \in \mathrm{Ker}\, d_0$ the following conditions are satisfied:

1. $\alpha + \alpha' = 0$;
2. $\alpha'\omega(r) = \omega\alpha'(r)$, for each unary operation ω in $\mathbf{C}$ except the negation;
3. $\alpha'(r + r_1) = \alpha'(r) + T_0(r) \cdot \alpha'(r_1)$;
4. $\alpha'(r * r_1) = \alpha'(r) * \alpha'(r_1) + \alpha'(r) * T_0(r_1) + T_0(r) * \alpha'(r_1)$ for each binary operation $*$ in $\mathbf{C}$ except the addition;
5. $d'\alpha'(r) = S_0(r) - T_0(r)$;
6. $\alpha' d(c) = S_1(c) - T_1(c)$.

Define $\alpha'(r) = -\alpha(r)$. The first condition is satisfied by the definition of α'; ω is the homomorphism for the addition, hence, $\omega(0) = 0$ and $\omega(-r) = -\omega(r)$. Thus, for the next conditions we have

2. $\alpha'(\omega(r)) = -\alpha(\omega(r)) = -\omega(\alpha(r)) = \omega(-\alpha(r)) = \omega(\alpha'(r))$.
3. Computing the left and right sides of Condition 3 we can see that they are equal:

$$\begin{aligned}
\alpha'(r + r_1) &= -\alpha(r + r_1) = -(\alpha(r) + S_0(r) \cdot \alpha(r_1)) \\
&= -(S_0(r) \cdot \alpha(r_1)) - \alpha(r) = -((-d'\alpha(r) + T_0(r)) \cdot \alpha(r_1)) - \alpha(r) \\
&= -(d'(-\alpha(r)) \cdot (T_0(r) \cdot \alpha(r_1))) - \alpha(r) = -(-\alpha(r) + T_0(r) \cdot \alpha(r_1) + \alpha(r)) - \alpha(r) \\
&= -\alpha(r) - T_0(r) \cdot \alpha(r_1) + \alpha(r) - \alpha(r) = -\alpha(r) - T_0(r) \cdot \alpha(r_0), \\
\alpha'(r) + T_0(r) \cdot \alpha'(r_1) &= -\alpha(r) + T_0(r) \cdot (-\alpha(r_1)) = -\alpha(r) - T_0(r) \cdot \alpha(r_1).
\end{aligned}$$

4. $$\begin{aligned}
\alpha'(r * r_1) &= -\alpha(r * r_1) = -(\alpha(r) * \alpha(r_1) + \alpha(r) * S_0(r_1) + S_0(r) * \alpha(r_1)) \\
&= -S_0(r) * \alpha(r_1) - \alpha(r) * S_0(r_1) - \alpha(r) * \alpha(r_1) \\
&= -\big((-d'\alpha(r) + T_0(r)) * \alpha(r_1)\big) - \big(\alpha(r) * (-d'\alpha(r) + T_0(r_1))\big) - \alpha(r) * \alpha(r_1) \\
&= -\big((-d'\alpha(r)) * \alpha(r_1) + T_0(r) * \alpha(r_1)\big) - \big(\alpha(r) * (-d'\alpha(r_1)) + \alpha(r) * T_0(r_1)\big) - \alpha(r) * \alpha(r_1) \\
&= -\big((-\alpha(r)) * \alpha(r_1) + T_0(r) * \alpha(r_1)\big) - \big(\alpha(r) * (-\alpha(r_1) + \alpha(r) * T_0(r_1)) - \alpha(r)\alpha(r_1)\big) \\
&= -T_0(r) * \alpha(r_1) + \alpha(r) * \alpha(r_1) - \alpha(r) * T_0(r_1) + \alpha(r) * \alpha(r_1) - \alpha(r) * \alpha(r_1), \\
\alpha'(r) * \alpha'(r_1) + \alpha'(r) * T_0(r_1) + T_0(r) * \alpha'(r_1) &= \alpha(r) * \alpha(r_1) - \alpha(r) * T_0(r_1) - T_0(r) * \alpha(r_1).
\end{aligned}$$

Now it suffices to mention that for each object C of $\mathbf{C}$ and for elements $a, b, c, d \in \mathrm{C}$ we have $a*d+b*c = b*c+a*d$, which implies the analogous identity for actions (see Chap. 3, Proposition 3.1.1, condition 12). Conditions 5 and 6 are obviously satisfied by the definition of α', which completes the proof of Proposition 1.2.1. □

Remark. A similar statement is valid for more general internal categories, if $\mathrm{Im}\,\widetilde{\alpha}$ ($\widetilde{\alpha} : \mathrm{C}_0 \longrightarrow \mathrm{C}'_1$ is defined by α) is a subobject of "internally" invertible morphisms, and in particular, for internal groupoids in more general categories than $\mathbf{C}$. Thus, Proposition 1.2.1 can be obtained from this and the fact that internal categories in $\mathbf{C}$ are internal groupoids. Proceeding from our interest in crossed modules expressed in the title, we gave a detailed proof of this proposition, which somehow contains the proofs of both mentioned statements ($\mathrm{Im}\,\widetilde{\alpha}$ is a subobject of "internally" invertible morphisms and α is an isomorphism) in terms of crossed modules.

Let $\alpha : (F_0, F_1) \longrightarrow (G_0, G_1)$ be a natural transformation of internal functors $F, G : \mathrm{C} \longrightarrow \mathrm{C}'$, and $S : \overline{\mathrm{C}} \longrightarrow \mathrm{C}$ be an internal functor. As in the case of ordinary categories we have the natural transformation $\alpha S : (F_0 S_0, F_1 S_1) \longrightarrow (G_0 S_0, G_1 S_1)$, defined by αS_0. We must show that αS_0 satisfies conditions (1.2.6); α is a natural transformation, so it satisfies conditions $(1.2.6_1)$ (S_0 replaced by F_0); S_0 is a morphism of $\mathbf{C}$. From the above, we conclude that αS_0 satisfies conditions $(1.2.6_1)$. Similarly,

$$d'\alpha(r) = G_0(r) - F_0(r) \quad \text{for each} \quad r \in \mathrm{C}_0\,.$$

Take $r = S_0(\overline{r})$; then we obtain

$$d'\alpha S_0(\overline{r}) = G_0 S_0(\overline{r}) - F_0 S_0(\overline{r}) \quad \text{for each} \quad \overline{r} \in \overline{\mathrm{C}}_0.$$

For the second condition of (2.6_2) we have

$$\alpha d(c) = G_1(c) - F_1(c) \quad \text{for each} \quad c \in \mathrm{Ker}\, d_0.$$

Take $c = S_1(\overline{c})$; then we have $\alpha d S_1(c) = G_1 S_1(c) - F_1 S_1(c)$; but $dS_1 = S_0 d$, which gives the desired equality.

Let $(T_0, T_1) : \mathrm{C}' \longrightarrow \widetilde{\mathrm{C}}$ be an internal functor; then we can also define

$$T\alpha : (T_0 F_0, T_1 F_1) \longrightarrow (T_0 G_0, T_1 G_1)$$

as $T_1\alpha$. We must show that $T_1\alpha$ satisfies conditions (1.2.6); α satisfies these conditions and T_1 is a morphism of $\mathbf{C}$; this proves the first equality of $(1.2.6_1)$. For the second and third conditions, we apply (1.1.5). For $(1.2.6_2)$, we again apply commutativity $T_0 d' = \widetilde{d} T_1$ and the fact that T_1 is a morphism of $\mathbf{C}$.

1.3. Adjunction of Internal Functors, Internal Category Equivalence, and Whitehead Equivalence of Homotopy Systems

Let $\underline{\mathrm{C}}$ be a category with finite limits, C and C′ be internal categories in $\underline{\mathrm{C}}$, and S and T be internal functors

$$\mathrm{C} \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{C}'\,.$$

We have the following well-known category theory notions for the case of internal categories.

Definition 1.3.1. We say that T is left adjoint to S if there are natural transformations of internal functors $\Phi : TS \longrightarrow 1_{\mathrm{C}'}$, $\Psi : 1_{\mathrm{C}} \longrightarrow ST$ such that the composites

$$S \xrightarrow{\Psi S} STS \xrightarrow{S\Phi} S\,, \qquad T \xrightarrow{T\Psi} TST \xrightarrow{\Phi T} T$$

are the identity natural transformations.

Definition 1.3.2. We say that an internal category C is equivalent to C′ if there is a pair of internal functors $\mathrm{C} \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{C}'$ together with natural isomorphisms of internal functors

$$\Phi : TS \xrightarrow{\approx} 1_{\mathrm{C}}\,, \qquad \Psi : 1_{\mathrm{C}} \xrightarrow{\approx} ST\,.$$

In this case, we say that T and S are equivalences of internal categories.

Definition 1.3.3. We say that we have an adjoint equivalence of internal categories C and C′ if there is a pair of adjoint internal functors $\mathrm{C} \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{C}'$ such that the corresponding natural transformations $\Phi : TS \longrightarrow 1_{\mathrm{C}'}$ and $\Psi : 1_{\mathrm{C}} \longrightarrow ST$ are isomorphisms.

From the results of Sec. 2 we have the following assertion.

Proposition 1.3.4. *Let* $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ *and* $\mathrm{C}' = (\mathrm{C}'_0, \mathrm{C}'_1, d'_0, d'_1, i', m')$ *be internal categories in* $\mathbf{C}$ *and* $T = (T_0, T_1)$ *and* $S = (S_0, S_1)$ *be internal functors*

$$\begin{array}{ccc} \mathrm{Ker}\, d_0 & \xrightarrow{d} & \mathrm{C}_0 \\ S_1 \uparrow\downarrow T_1 & & S_0 \uparrow\downarrow T_0 \\ \mathrm{Ker}\, d'_0 & \xrightarrow[d']{} & \mathrm{C}'_0 . \end{array}$$

T is left adjoint to S if and only if there are maps $\varphi : \mathrm{C}'_0 \longrightarrow \ker d'_0$ *and* $\psi : \mathrm{C}_0 \longrightarrow \mathrm{Ker}\, d_0$ *satisfying the following conditions:*

$$\begin{cases} \varphi\omega(r') = \omega\varphi(r'), \\ \varphi(r' + r'_1) = \varphi(r') + T_0S_0(r') \cdot \varphi(r'_1), \qquad (1.3.1_1) \\ \varphi(r' * r'_1) = \varphi(r') * \varphi(r'_1) + \varphi(r') * T_0S_0(r'_1) + T_0S_0(r') * \varphi(r'_1), \end{cases} \tag{1.3.1}$$

$$\begin{cases} d'\varphi(r') = r' - T_0S_0(r'), \\ \varphi d'(c') = c' - T_1S_1(c') \end{cases} \qquad (1.3.1_2)$$

for each $r', r'_1 \in \mathrm{C}'_0$ *and* $c' \in \mathrm{Ker}\, d'_0$*;*

$$\begin{cases} \psi\omega(r) = \omega\psi(r) \\ \psi(r + r_1) = \psi(r) + r \cdot \psi(r_1), \\ \psi(r * r_1) = \psi(r) * \psi(r_1) + \psi(r) * r_1 + r * \psi(r_1), \\ d\psi(r) = S_0T_0(r) - r, \\ \psi d(c) = S_1T_1(c) - c \end{cases} \tag{1.3.2}$$

for each $r, r_1 \in \mathrm{C}_0$ *and* $c \in \mathrm{Ker}\, d_0$*;*

$$S_1\varphi + \psi S_0 = 0, \quad \varphi T_0 + T_1\psi = 0. \tag{1.3.3}$$

From Proposition 1.2.1 for the case of categories of groups with operations $\mathbf{C}$ we obtain

Proposition 1.3.5. *In the category* $\mathbf{C}$ *internal adjunction implies equivalence of internal categories.*

Proposition 1.3.6. *Internal categories*

$$\mathrm{C} : \mathrm{Ker}\, d_0 \xrightarrow{d} \mathrm{C}_0 \quad \text{and} \quad \mathrm{C}' : \mathrm{Ker}\, d'_0 \xrightarrow{d'} \mathrm{C}'_0$$

in $\mathbf{C}$ *are equivalent if and only if there are internal functors*

$$\mathrm{C} \underset{(S_0, S_1)}{\overset{(T_0, T_1)}{\rightleftarrows}} \mathrm{C}'$$

and maps

$$\varphi : \mathrm{C}'_0 \longrightarrow \mathrm{Ker}\, d'_0 \quad \text{and} \quad \psi : \mathrm{C}_0 \longrightarrow \mathrm{Ker}\, d_0$$

satisfying conditions (1.3.1) *and* (1.3.2).

Proposition 1.3.7. *Internal categories* C *and* C′ *in* **C** *are adjointly equivalent if and only if there are internal functors*

$$\mathrm{C} \underset{(S_0,S_1)}{\overset{(T_0,T_1)}{\rightleftarrows}} \mathrm{C}'$$

and maps

$$\varphi : \mathrm{C}'_0 \longrightarrow \operatorname{Ker} d'_0, \quad \psi : \mathrm{C}_0 \longrightarrow \operatorname{Ker} d_0$$

satisfying conditions (1.3.1)*,* (1.3.2)*, and* (1.3.3)*.*

Thus by Proposition 1.2.1 Φ and Ψ are always isomorphisms and we do not require the existence of the maps $\varphi' : \mathrm{C}'_0 \longrightarrow \operatorname{Ker} d'_0$ and $\psi' : \mathrm{C}_0 \longrightarrow \operatorname{Ker} d_0$ satisfying conditions $\varphi + \varphi' = 0$, $\psi + \psi' = 0$, and the following ones:

$$\begin{aligned}
\varphi'\omega(r') &= \omega\varphi'(r'),\\
\varphi'(r' + r'_1) &= \varphi'(r') + r' \cdot \varphi'(r'_1),\\
\varphi'(r' * r'_1) &= \varphi'(r') * \varphi'(r'_1) + \varphi'(r') * r'_1 + \varphi'(r'_1),\\
d'\varphi'(r') &= T_0S_0(r') - r',\\
\varphi'd'(c') &= T_1S_1(c') - c';
\end{aligned} \tag{1.3.1′}$$

$$\begin{aligned}
\psi'\omega(r) &= \omega\psi'(r),\\
\psi'(r + r_1) &= \psi(r) + S_0T_0(r) \cdot \psi'(r_1),\\
\psi'(r * r_1) &= \psi'(r) * \psi'(r_1) + \psi'(r) * S_0T_0(r_1) + S_0T_0(r) * \psi'(r_1),\\
d\psi'(r) &= r - S_0T_0(r),\\
\psi'd(c) &= c - S_1T_1(c).
\end{aligned} \tag{1.3.2′}$$

Proposition 1.3.8. *The following conditions are equivalent in* **C**:

(i) *We have an adjoint pair of internal functors* $\mathrm{C} \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{C}'$;

(ii) *We have an adjoint equivalence of internal categories* C *and* C′*.*

The following proposition follows from the definition of equivalence of homotopy systems given by Whitehead [89], Proposition 1.3.6, and the note concerning natural transformations of internal functors and deformation operators given in Sec. 1.2.

Proposition 1.3.9. *Let* C *and* C′ *be internal categories in the category of groups. If the corresponding crossed modules*

$$\mathrm{C}_* : \operatorname{Ker} d_0 \xrightarrow{\ d\ } \mathrm{C}_0 \qquad \text{and} \qquad \mathrm{C}'_* : \operatorname{Ker} d_0 \xrightarrow{\ d'\ } \mathrm{C}'_0$$

are homotopy systems (see [89])*, then the equivalence of internal categories* C *and* C′ *is a special case of the equivalence of* C_* *and* C'_* *in the sense of Whitehead* (see [89])*.*

Recall that a functor $S : \underline{A} \longrightarrow \underline{A}'$ between (ordinary) categories is called faithful if for each pair of objects $(A_1, A_2) \in |\underline{A}| \times |\underline{A}|$; the map

$$S : \operatorname{Hom}_{\underline{A}}(A_1, A_2) \longrightarrow \operatorname{Hom}_{\underline{A}'}(S_0(A_1), S_0(A_2))$$

defined by $f \longmapsto S(f)$ is injective; S is called full if the above map is surjective, and S is full and faithful if this map is bijective.

For internal categories in categories of groups with operations we obtain the analogous definition.

Definition 1.3.10. An internal functor $S = (S_0, S_1) : \mathrm{C} \longrightarrow \mathrm{C}'$ is called faithful if for each pair $(r, r_1) \in \mathrm{C}_0 \times \mathrm{C}_0$ the map $\mathrm{C}(r, r_1) \longrightarrow \mathrm{C}'(S_0(r), S_0(r_1))$ defined by $(r, c) \longmapsto (S_0(r), S_1(c))$ for each $(r, c) \in \mathrm{C}_0 \ltimes \operatorname{Ker} d_0$ with $d(c) + r = r_1$, is injective. (S_0, S_1) is called full if this map is surjective and S is called full and faithful if this map is a bijection.

Lemma 1.3.11. *Internal functor $S = (S_0, S_1) : \mathrm{C} \longrightarrow \mathrm{C}'$ is faithful if and only if*

$$S_1|_{\operatorname{Ker} d} : \operatorname{Ker} d \longrightarrow \operatorname{Ker} d'$$

is a monomorphism (here as above $d = d_1|_{\operatorname{Ker} d_0}$).

Proof. By the definition of faithful functor the equality $(S_0(r), S_1(c)) = (S_0(r), S_1(c_1))$ implies $(r, c) = (r, c_1)$ for each $c, c_1 \in \operatorname{Ker} d_0$, and $r \in \mathrm{C}_0$ with $d(c) = d(c_1)$. This condition is equivalent to the following one: $S_1(c) = S_1(c_1)$ implies $c = c_1$ for each $c, c_1 \in \operatorname{Ker} d$, which means that $S_1|_{\operatorname{Ker} d}$ is a monomorphism. □

Lemma 1.3.12. *Internal functor $S = (S_0, S_1) : \mathrm{C} \longrightarrow \mathrm{C}'$ is full if and only if for each $c' \in \operatorname{Ker} d'_0$ with $d'c' = S_0(r)$ for some $r \in \mathrm{C}_0$, there is an element $c \in \operatorname{Ker} d_0$ such that $d(c) = r$ and $S_1(c) = c'$.*

Proof. By the definition of the full functor for an arbitrary element $(S_0(r), c')$ with $d'(c') + S_0(r) = S_0(r_1)$ there is an element $c \in \operatorname{Ker} d_0$ such that $d(c) + r = r_1$ and $S_1(c) = c'$, where $r, r_1 \in \mathrm{C}_1$ and $c' \in \operatorname{Ker} d_0$. Taking $r = 0$, we obtain the desired condition.

Let $(S_0(r), c') \in \mathrm{C}'(S_0(r), S_0(r_1))$. We have

$$d'(c') = S_0(r_1) - S_0(r) = S(r_1 - r).$$

By the conditions of the lemma there is an element $c \in \operatorname{Ker} d_0$ with $S_1(c) = c'$ and $d(c) = r_1 - r$. So $d(c) + r = r_1$, $(r, c) \in \mathrm{C}(r, r_1)$ and $(S_0(r), S_1(c)) = (S_0(r), c')$, which proves the lemma. □

For internal categories we have the following analogue of Theorem 1 from [72], §4, IV.

Theorem 1.3.13. *The following properties of an internal functor $T : \mathrm{C} \longrightarrow \mathrm{C}'$ in $\mathbf{C}$ are equivalent:*

(i) *T is an equivalence of internal categories.*

(ii) *T is a part of an adjoint equivalence $\mathrm{C} \rightleftarrows \mathrm{C}'$.*

(iii) *T is full and faithful; for each $r' \in \mathrm{C}'_0$ there is an element $r \in \mathrm{C}_0$ such that we have an isomorphism*

$$(T_0(r), c') : T_0(r) \xrightarrow{\approx} r' .$$

The correspondence $r' \longmapsto r$ defines a homomorphism $S_0 : \mathrm{C}'_0 \longrightarrow \mathrm{C}_0$, and the isomorphisms $(T_0(r), c')$ can be chosen in such a way that the map $\varphi : r' \longmapsto c'$ satisfies conditions $(1.3.1_1)$.

Proof. By the definitions, (ii) implies (i). To prove that (i) implies (iii), let $c \in \operatorname{Ker} d$. From (1.3.2), we have $\psi d(c) = S_1T_1(c) - c$; $dc = 0$, so if $T_1(c) = 0$, then $c = 0$ which by Lemma 1.3.11 means that T is faithful. Similarly, from (1.3.1) we prove that S is faithful. To prove that T is full, let $c' \in \operatorname{Ker} d'_0$ and $d'c' = T_0(r)$ for some $r \in \mathrm{C}_0$. Take $c = -\psi(r) + S_1(c')$, $c \in \operatorname{Ker} d_0$. By Lemma 1.3.12 we must show that $dc = r$ and $T_1(c) = c'$. We have

$$\begin{aligned} dc &= d(-\psi(r + S_1(c'))) = -d\psi(r) + dS_1(c') \\ &= -(S_0T_0(r) - r) + dS_1(c') = r - S_0T_0(r) + S_0d'c' + r - S_0T_0(r) + S_0T_0(r) = r. \end{aligned}$$

Thus, by the definition $c = -\psi(r) + S_1(c')$; on the other hand, by (1.3.2) for each $c \in \operatorname{Ker} d_0$ we have $c = -\psi(r) + S_1T_1(c)$. From the above two equalities we obtain that $S_1T_1(c) = S_1(c')$. But S is a faithful internal functor. This proves that $T_1(c) = c'$. The other conditions of (iii) are trivially satisfied.

Now we shall prove that (iii) implies (ii). By (iii) we have a homomorphism $S_0 : \mathrm{C}_0 \longrightarrow \mathrm{C}_0'$ and a map $\varphi : \mathrm{C}_0' \longrightarrow \operatorname{Ker} d_0'$ satisfying conditions (1.3.1$_1$). For an arbitrary morphism $(0, c') : 0 \longrightarrow r'$ we have an isomorphism $0 \longrightarrow T_0S_0(r')$ defined as the composite $(T_0S_0(r'), \varphi(r'))^{-1} \cdot (0, c')$. The latter has the form $(0, T_1(c))$ for some $c : 0 \longrightarrow S_0(r')$, because T is full and this c is unique because T is faithful. We can define $S_1(c') = c$. Thus we have $S_1(c') = T^{-1}(-\varphi(d'c') + c')$. We must show that S_1 is a morphism in $\mathbf{C}$ and $S = (S_0, S_1)$ satisfies conditions (1.1.5) ($\widetilde{F}_1$ and F_0 are replaced by S_1 and S_0, respectively).

1. From the definition follows that $S_1(\omega(c)) = \omega(S_1(c))$.
2. $S_1(c_1' + c_2') = T_1^{-1}(-\varphi(d(c_1' + c_2')) + c_1' + c_2')$;
   $S_1(c_1') + S_1(c_2') = T_1^{-1}(-\varphi dc_1' + c_1') + T_1^{-1}(-\varphi dc_2' + c_2')$.

Thus it is sufficient to show that $-\varphi(d(c_1' + c_2')) + c_1' + c_2' = -\varphi dc_1' + c_1' - \varphi dc_2' + c_2'$. We have

$$\begin{aligned}
-\varphi(d(c_1' + c_2')) + c_1' + c_2' &= -\varphi((dc_1') + (dc_2')) + c_1' + c_2') \\
&= -(\varphi dc_1') + T_0S_0(d'c_1') \cdot \varphi(dc_2') + c_1' + c_2' = -(T_0S_0(dc_1') \cdot \varphi(dc_2')) - \varphi dc_1' + c_1' + c_2'.
\end{aligned}$$

From (1.3.1) we have $T_0S_0(d'c_1') = -d'\varphi(d'c_1') + d'c_1' = d'(-\varphi d'c_1' + c_1')$; thus we obtain

$$\begin{aligned}
-\varphi(d(c_1' + c_2')) + c_1' + c_2' &= -((-d'\varphi(d'c_1') + d'c_1') \cdot \varphi(d'c_2')) - \varphi dc_1' + c_1' + c_2' \\
&= -((d'(-\varphi(d'c_1') + c_1')) \cdot \varphi(d'c_2')) - \varphi d'c_1' + c_1' + c_2' \\
&= -\varphi d'c_1' + c_1' - \varphi d'c_2' - c_1' + \varphi d'c_1' - \varphi d'c_1' + c_1' + c_2' = -\varphi d'c_1' + c_1' - \varphi d'c_2' + c_2'.
\end{aligned}$$

3. $S_1(c_1' * c_2') = T_1^{-1}(-\varphi(d'(c_1' * c_2')) + c_1' * c_2'$;

$$\begin{aligned}
S_1(c_1') * S_1(c_2') &= T_1^{-1}(-\varphi d'(c_1') + c_1') * T_1^{-1}(-\varphi d'(c_2') + c_2') \\
&= T_1^{-1}(\varphi d'(c_1') * \varphi d'(c_2') - \varphi d'(c_1') * c_2' - c_1' * \varphi d'(c_2') + c_1' * c_2');
\end{aligned}$$

$$\begin{aligned}
-\varphi(d'(c_1' * c_2')) + c_1' * c_1' &= -\varphi((d'c_1') * d'(c_2')) + c_1' * c_2' \\
&= -(\varphi d'(c_1') * \varphi d'(c_2') + \varphi d'(c_1') * T_0S_0d'(c_2') + T_0S_0d'(c_1') * \varphi d'(c_2')) + c_1' * c_2' \\
&= -T_0S_0d'(c_1') * \varphi d'(c_2') - \varphi d'(c_1') * T_0S_0d'(c_2') - \varphi d'(c_1') * \varphi d'(c_2') + c_1' * c_2' \\
&= (-d'c_1' + d'\varphi d'(c_1')) * \varphi d'(c_2') + \varphi d'(c_1') * (-d'c_2' + d'\varphi d'(c_2')) - \varphi d'(c_1') * \varphi d'(c_2') + c_1' * c_2' \\
&= -d'(c_1') * \varphi d'(c_2') + \varphi d'(c_1') * \varphi d'(c_2') - \varphi d'(c_1') * d'(c_2') \\
&\quad + \varphi d'(c_1') * \varphi d'(c_2') - \varphi d'(c_1') * \varphi d'(c_2') + c_1' + c_2' \\
&= -c_1' * \varphi d'(c_2') + \varphi d'(c_1') * \varphi d'(c_2') - \varphi d'(c_1') * c_2' + c_1' * c_2',
\end{aligned}$$

   which gives the desired equality (here we again apply the equality $a * c + b * d = b * d + a * c$ for the elements of C and its consequence for the actions).

4. We must show $S_1(r' * c') = S_0(r') * S_1(c')$. We have

$$S_1(r' * c') = T_1^{-1}(-\varphi(d'(r' * c')) + r' * c');$$

$$S_0(r') * S_1(c') = S_0(r') * T_1^{-1}(-\varphi d'(c') + c') = T_1^{-1}(T_0S_0(r') * (-\varphi d'(c') + c'),$$

$$\begin{aligned}
-\varphi(d'(r' * c')) + r' * c' &= -\varphi(r' * d'c') + r' * c' \\
&= -\varphi(r') * \varphi d'(c') + \varphi(r') * T_0S_0d'(c') + T_0S_0(r') * \varphi(d'c') + r' * c' \\
&= -T_0S_0(r') * \varphi d'(c') - \varphi(r') * T_0S_0d'(c') - \varphi(r') * \varphi d'(c') + r' * c' \\
&= -T_0S_0(r') * \varphi d'(c') - \varphi(r') * (d'\varphi d'(c') + TSd'(c')) + r' * c' \\
&= -T_0S_0(r') * \varphi d'(c') - \varphi(r') * d'(c') + r' * c'
\end{aligned}$$

$$= -T_0S_0(r') * \varphi d'(c') + (-d'\varphi(r') + r') * c'$$
$$= -T_0S_0(r') * \varphi d'(c') + T_0S_0(r') * c' = T_0S_0(r') * (-\varphi d'(c') + c').$$

5. We must show $S_1(r' \cdot c') = S_0(r') \cdot S_1(c')$. We have

$$S_1(r' \cdot c') = T_1^{-1}(-\varphi d'(r' \cdot c') + r' \cdot c');$$
$$-\varphi d'(r' \cdot c') + r' \cdot c' = -\varphi(r' + d'c' - r') + r' \cdot c' = -(\varphi(r') + T_0S_0(r') \cdot \varphi(d'c' - r'))$$
$$= -\varphi(r') + (-d'\varphi(r') + r') \cdot (\varphi(d'c') + T_0S_0(d'c') \cdot \varphi(-r')) + r' \cdot c'$$
$$= -(\varphi(r') + (-d'\varphi(r') + r') \cdot (\varphi(d'c') + (-d'\varphi(d'c') + d'c') \cdot \varphi(-r'))) + r' \cdot c'$$
$$= -(\varphi(r') + (-d'\varphi(r') + r') \cdot (\varphi d'(c') + d'(-\varphi(d'c')) \cdot (c' + \varphi(-r) - c') + r'c'$$
$$= -(\varphi(r') + (-d\varphi(r') + r') \cdot (\varphi d'(c') - \varphi d'(c') + c' + \varphi(-r') - c' + \varphi d'(c'))) + r' \cdot c'$$
$$= -(\varphi(r') + (-d'\varphi(r')) \cdot (r' \cdot c' + r' \cdot \varphi(-r') + r'(-c') + r' \cdot \varphi d'(c')) + r' \cdot c'$$
$$= -(\varphi(r') - \varphi(r') + r' \cdot c' + r' \cdot \varphi(-r') - r' \cdot c' + r' \cdot \varphi d'(c') + \varphi(r')) + r' \cdot c'$$
$$= -\varphi(r') - r' \cdot \varphi d'(c') - r' \cdot (-c') - r' \cdot \varphi(-r') - r' \cdot c + r' \cdot c'$$
$$= -\varphi(r') - r' \cdot \varphi d'(c') + r' \cdot c' - r' \cdot \varphi(-r').$$

From (1.3.1) we obtain

$$\varphi(0) = \varphi(-r') + T_0S_0(-r') \cdot \varphi(r'), \quad 0 = \varphi(-r') + (-T_0S_0(r')) \cdot \varphi(r'),$$
$$\varphi(-r') = -(-T_0S_0(r')) \cdot \varphi(r') = -(-r' + d'\varphi(r')) \cdot \varphi(r')$$
$$= -((-r') \cdot (\varphi(r') + \varphi(r') - \varphi(r'))) = -(-r') \cdot \varphi(r').$$

Thus,

$$r' \cdot \varphi(-r') = r' \cdot \big(- (-r') \cdot \varphi(r')\big) = r' \cdot \big((-r') \cdot (-\varphi(r'))\big) = (r' + (-r')) \cdot (-\varphi(r')) = -\varphi(r').$$

Applying this to the above equality, we conclude

$$S_1(r' \cdot c') = T_1^{-1}(-\varphi(r') - r' \cdot \varphi d'(c') + r' \cdot c' + \varphi(r')).$$

On the other hand, we have

$$S_0(r') \cdot S_1(c') = S_0(r') \cdot T_1^{-1}(-\varphi d'c' + c')$$
$$= T_1^{-1}(T_0S_0(r') \cdot (-\varphi d'(c') + c')) = T_1^{-1}((d'\varphi(r') + r') \cdot (-\varphi d'(c') + c'))$$
$$= T_1^{-1}(-\varphi(r') + r'(-\varphi d'(c') + c') + \varphi(r')) = T_1^{-1}(-\varphi(r') - r' \cdot \varphi d'(c') + r' \cdot c' + \varphi(r')),$$

which proves the equality.

We have also to show that $S_0d' = d'S_1$. For each $c' \in \operatorname{Ker} d_0'$,

$$d'S_1(c') = d'T_1^{-1}(-\varphi d'(c') + c');$$

c' is a morphism

$$0 \longrightarrow d'(c'),$$

$-\varphi d'(c') + c'$ is a morphism

$$0 \longrightarrow T_0S_0(d'c'),$$

so $T_1^{-1}(-\varphi d'(c') + c')$ is a morphism

$$0 \longrightarrow S_0d'(c')$$

and hence

$$d'T_1^{-1}(-\varphi d'(c') + c') = S_0d'(c')$$

which gives the desired equality. Note that by the definition of S_1 the last condition of (1.3.1) is also satisfied.

To define $\psi : \mathrm{C}_0 \longrightarrow \operatorname{Ker} d_0$ note that

$$d'\varphi T_0(r) = T_0(r - S_0T_0(r)).$$

Since T is full and faithful, there is an unique element $c \in \operatorname{Ker} d_0$ such that

$$d(c) = r - S_0T_0(r), \quad T_1(c) = \varphi T_0(r).$$

Define ψ by $\psi(r) = T_1^{-1}(-\varphi T_0(r))$. It is not difficult to show that ψ satisfies conditions (1.3.2) and (1.3.3). We shall demonstrate the proof of the first equality of (1.3.3). For this we need to prove that $\varphi T_0 S_0 = T_1 S_1 \varphi$. We have

$$\begin{aligned}
\varphi T_0S_0(r') &= \varphi(-d'\varphi(r') + r') = \varphi(-d'\varphi(r')) + T_0S_0(-d'\varphi(r')) \cdot \varphi(r') \\
&= \varphi(-d'\varphi(r')) - d'\varphi(-d'\varphi(r') + (-d'\varphi(r')) \cdot \varphi(r') \\
&= \varphi(-d'\varphi(r')) - \varphi(-d'\varphi(r')) - \varphi(r') + \varphi(r') + \varphi(r') + \varphi(-d'\varphi(r')) \\
&= \varphi(r') - ((T_0S_0d'\varphi(r')) \cdot \varphi(d'\varphi(r'))) = \varphi(r') - \big((-d'\varphi(r') + d'\varphi d'\varphi(r')) \cdot \varphi(d'\varphi(r'))\big) \\
&= \varphi(r') - \big(- \varphi(r') + \varphi d'\varphi(r') + \varphi d'\varphi(r') - \varphi d'\varphi(r') + \varphi(r')\big).
\end{aligned}$$

We apply here the equality

$$\varphi(-r') = -((-T_0S_0(r')) \cdot \varphi(r')),$$

which can be obtained from the second condition of $(1.3.1_1)$. From $(1.3.1_2)$ we have

$$T_1S_1\varphi(r') = -\varphi d'\varphi(r') + \varphi(r'),$$

which gives the desired equality. Applying this, we shall prove the first equality of (1.3.3). We have

$$\begin{aligned}
S_1\varphi(r') + \psi S_0(r') &= T_1^{-1}(-\varphi d'\varphi(r') + \varphi(r')) + T_1^{-1}(-\varphi T_0S_0(r')) \\
&= T_1^{-1}(-(\varphi(r') - T_1S_1\varphi(r')) + \varphi(r')) + T_1^{-1}(-T_1S_1\varphi(r')) \\
&= T_1^{-1}(T_1S_1\varphi(r') - \varphi(r') + \varphi(r')) - S_1\varphi(r') = 0.
\end{aligned}$$

This completes the proof of Theorem 1.3.13. □

Note that (ii) $\Longrightarrow$ (i) can be proved directly. For the given T, S, φ, ψ we must define $\overline{\varphi}$ by $\overline{\varphi} = S_1^{-1}(-\psi S_0(r'))$; it is proved that $\overline{\varphi}$ satisfies conditions (1.3.1) and (1.3.3) and $(T, S, \overline{\varphi}, \psi)$ is an adjoint equivalence of internal categories C and C′. Also note that by Lemmas 1.3.11 and 1.3.12, the condition *T is full and faithful, and for each $r' \in \mathrm{C}'_0$ there is an element $r \in \mathrm{C}_0$ such that* $T_0(r) \approx r'$ is equivalent to the following: *T induces the isomorphisms* $\operatorname{Ker} d \approx \operatorname{Ker} d'$, $\operatorname{Coker} d \approx \operatorname{Coker} d'$; this kind of a morphism between crossed modules is usually called a weak equivalence (see, e.g., [42]).

Proposition 1.3.14. *Let* $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i\ m)$ *be an internal category in the category of groups with operations* $\mathbf{C}$. C *is equivalent to the discrete internal category, if and only if* $d = d_1\big|_{\operatorname{Ker} d_0}$ *is a monomorphism and the natural epimorphism* $\pi : \mathrm{C}_0 \longrightarrow \operatorname{Coker} d$ *has a section.*

Proof. Let the conditions of the Proposition hold. We shall prove that C is equivalent to the discrete internal category $\mathrm{C}' = (\operatorname{Coker} d, \operatorname{Coker} d, 1, 1, 1, 1)$. Let $u : \operatorname{Coker} d \longrightarrow \mathrm{C}_0$ be a section of $\pi, \pi u = 1_{\operatorname{Coker} d}$. For the category C′ we have: $\mathrm{C}'_0 = \operatorname{Coker} d$, $\operatorname{Ker} d'_0 = 0$, $d' = 0$. Define internal functors $(T_0, T_1) : \mathrm{C} \longrightarrow \mathrm{C}'$, $(S_0, S_1) : \mathrm{C}' \longrightarrow \mathrm{C}$ by $T_0 = \pi$, $T_1 = 0$, $S_0 = u$, $S_1 = 0$. It is easy to see that these maps satisfy conditions (1.1.5). The picture is the following

$$\begin{array}{ccc}
\operatorname{Ker} d_0 & \xrightarrow{\ d\ } & \mathrm{C}_0 \\
\Big\downarrow{\scriptstyle T_1=0} & & \Big\downarrow{\scriptstyle T_0=\pi} \\
0=\operatorname{Ker} d_0' & \xrightarrow{\ d'=0\ } & \mathrm{C}_0'=\operatorname{Coker} d \\
\Big\downarrow{\scriptstyle S_1=0} & & \Big\downarrow{\scriptstyle S_0=u} \\
\operatorname{Ker} d_0 & \xrightarrow[\ d\ ]{} & \mathrm{C}_0
\end{array} .$$

For the split extension

$$0 \longrightarrow \operatorname{Im} d \longrightarrow \mathrm{C}_0 \underset{\pi}{\overset{u}{\leftrightarrows}} \operatorname{Coker} d \longrightarrow 0 \tag{1.3.4}$$

we have $\mathrm{C}_0 \approx \operatorname{Coker} d \ltimes \operatorname{Im} d$; so each $r \in \mathrm{C}_0$ has a form $r = (\pi(r), dc)$. Define $\psi' : \mathrm{C}_0 \longrightarrow \operatorname{Ker} d_0$ by $\psi'(\pi(r), dc) = c$; d is a monomorphism, from which it follows that ψ' is defined correctly. We must show that the maps ψ' and $\varphi = 0$ satisfy conditions (1.3.2′) and (1.3.1) respectively. It is easy to see that φ satisfies (1.3.1). For (1.3.2′) we have:

1. For each unary operation ω, except the negation

$$\psi'(\omega(r)) = \psi'(\omega(\pi(r), dc)) = \psi'(\omega\pi(r), \omega d(c)) = \psi'(\pi\omega(r), d(\omega(c))) = \omega(c);$$
$$\omega\psi'(r) = \omega\psi'(\pi(r), dc) = \omega(c).$$

2. $\psi'(r_1 + r_2) = \psi'((\pi(r_1), dc_1) + (\pi(r_2), dc_2)) = \psi'(\pi(r_1) + \pi(r_2), dc_1 + \pi(r_1) \cdot dc_2)$; here the action $\pi(r_1) \cdot dc_2$ is induced from the extension (1.3.4):

$$\pi(r_1) \cdot dc_2 = (\pi(r_1), 0) + (0, dc_2) - (\pi(r_1), 0).$$

Consider the action $(\pi(r_1), 0) \cdot c_2$ induced from the extension

$$0 \longrightarrow \operatorname{Ker} d_0 \longrightarrow \mathrm{C}_1 \xrightarrow{d_0} \mathrm{C}_0 \longrightarrow 0; \tag{1.3.5}$$

we have $(\pi(r_1), 0) \in \mathrm{C}_0$, $c_2 \in \operatorname{Ker} d_0$. From Lemma 1.1.2 we obtain

$$d((\pi(r_1), 0) \cdot c_2) = (\pi(r_1), 0) + (0, dc_2) - (\pi(r_1), 0).$$

From the above we conclude

$$\psi'(r_1 + r_2) = \psi'(\pi(r_1) + \pi(r_2), dc_1 + d((\pi(r_1, 0) \cdot c_2))$$
$$= (\pi(r_1 + r_2), d\big(c_1 + (\pi(r_1), 0) \cdot c_2\big) = c_1 + (\pi(r_1), 0) \cdot c_2 = \psi'(r_1) + S_0T_0(r_1) \cdot \psi'(r_2).$$

3. $\psi'(r_1 * r_2) = \psi'((\pi(r_1), dc_1) * (\pi(r_2), dc_2)) = \psi'(\pi(r_1) * \pi(r_2), dc_1 * dc_2 + (dc_1) * \pi(r_2) + \pi(r_1) * dc_2)$.

The action $(dc_1) * \pi(r_2)$ is induced from the extension (1.3.4). Thus,

$$(dc_1) * \pi(r_2) = (0, dc_1) * (\pi(r_2), 0).$$

Consider the action $c_1 * (\pi(r_2), 0)$ induced from (1.3.5). Here $(\pi(r_2), 0) \in \mathrm{C}_0$, $c_1 \in \operatorname{Ker} d_0$. Thus from Lemma 1.1.2 we obtain

$$d(c_1 * (\pi(r_2), 0)) = d((0, c_1) * (\pi(r_2), 0) = (0, dc_1) * (\pi(r_2), 0).$$

So

$$(dc_1) * \pi(r_2) = d(c_1 * (\pi(r_2), 0)).$$

Similarly,

$$\pi(r_1) * dc_2 = d((\pi(r_1), 0) * c_2).$$

From the above arguments we have

$$
\begin{aligned}
\psi'(r_1 * r_2) &= \psi'(\pi(r_1 * r_2), d(c_1 * c_2) + d(c_1 * (\pi(r_2), 0)) + d((\pi(r_1), 0) * c_2) \\
&= \psi'(\pi(r_1 * r_2), d(c_1 * c_2 + c_1 * (\pi(r_2), 0) + (\pi(r_1), 0) * c_2)) \\
&= c_1 * c_2 + c_1 * (\pi(r_2), 0) + (\pi(r_1), 0) * c_2 \\
&= \psi'(\pi(r_1), dc_1) * \psi'(\pi(r_1), dc_1) + \psi'((\pi(r_1), dc_1) * (\pi(r_2), 0) + (\pi(r_1), 0) * \psi'(\pi(r_2), dc_2) \\
&= \psi'(r_1) * \psi'(r_2) + \psi'(r_1) * S_0T_0(r_2) + S_0T_0(r_1) * \psi'(r_2).
\end{aligned}
$$

4. $d\psi'(r) = d\psi'(\pi(r), dc) = (0, dc)$, $r - S_0T_0(r) = (\pi(r), dc) - (\pi(r), 0) = (0, dc)$.
5. $\psi' dc = \psi'(0, dc) = c$, $c - S_1T_1(c) = c - 0 = c$.

Thus ψ' satisfies conditions (1.3.2′). Now we shall prove the converse statement. Let C be equivalent to the discrete category $C' = (C_0', C_0', 1, 1, 1, 1)$. Denote by T and S internal functors $C \underset{S}{\overset{T}{\rightleftarrows}} C'$, which induce the equivalence of the given categories. By Proposition 3.3.4 of Chap. 3 (where we do not use in the proof the statement of Proposition 1.3.14) C and C′ are homologically equivalent. Hence we have the isomorphisms $\operatorname{Ker} d \approx \operatorname{Ker} d' = 0$ and $\operatorname{Coker} d \approx \operatorname{Coker} d' = C_0'$ induced by S and T and the following commutative diagram:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{Ker} d & \longrightarrow & \operatorname{Ker} d_0 & \xrightarrow{d} & C_0 & \xrightarrow{\pi} & \operatorname{Coker} d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle T_1=0} & & \downarrow{\scriptstyle T_0} & & {\scriptstyle \alpha}\downarrow{\scriptstyle \approx} & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \xrightarrow{d'=0} & C_0' & \xrightarrow{=} & C_0' & \longrightarrow & 0\ . \\
 & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle S_1=0} & & \downarrow{\scriptstyle S_0} & & {\scriptstyle \beta}\downarrow{\scriptstyle \approx} & & \\
0 & \longrightarrow & \operatorname{Ker} d & \longrightarrow & \operatorname{Ker} d_0 & \xrightarrow{d} & C_0 & \xrightarrow{\pi} & \operatorname{Coker} d & \longrightarrow & 0
\end{array}
$$

Thus d is a monomorphism. From the commutativity of the diagram we have $\pi S_0 \alpha = \beta\alpha = 1_{\operatorname{Coker} d}$, which proves that $S_0\alpha$ is a section of π. □

Chapter 2

COHOMOLOGY OF INTERNAL CATEGORIES IN CATEGORIES OF GROUPS WITH OPERATIONS

In this chapter we define and study the cohomology $H^n(C, -)$ of an internal category C in the category $\mathbf{C}$ of groups with operations [78], [76] (see Sec. 1.1 for the definition). As in Chap. 1, using the equivalence of categories $\operatorname{Cat}(\mathbf{C}) \xrightarrow{\sim} X\operatorname{Mod}(\mathbf{C})$ [78], we describe completely the cohomology $H^n(C, -)$ and the corresponding complex $\{K^n(C, -), \partial^n, n \geq 0\}$. In particular cases this gives the description of the cohomology of internal categories in the category of groups, associative algebras, Lie algebras, etc. Regarding the internal category cohomology as the cohomology of the corresponding crossed module, we obtain the equivalent results for the crossed module cohomology.

2.1. Abelian Groups in the Category $\mathbf{C}^{C}$ of Internal Diagrams

Let $\mathbf{C}$ be a category of groups with operations and $C = (C_0, C_1, d_0, d_1, i, m)$ an internal category in $\mathbf{C}$. Recall that [43] an internal diagram F on C consists of an object $F_0 \xrightarrow{\gamma_0} C_0$ of $\mathbf{C}/C_0$ and a morphism $e : C_1 \times_{C_0} F_0 \longrightarrow F_0$ such that $\gamma_0 e = d_1\pi_1$, $e(i \times 1) = 1_{F_0}$, and $e(1 \times e) = e(m \times 1)$:

$C_1 \times_{C_0} C_1 \times_{C_0} F_0 \longrightarrow F_0$. By an abelian group in the category $\mathbf{C}^{C}$ of internal diagrams on C is meant a quintuple $A = (A_0, \pi, e, \eta, \mu)$, where $A_0 \xrightarrow{\pi} C_0$ is an object of $\mathbf{C}/C$, $e : C_1 \times_{C_0} A_0 \longrightarrow A_0$ an action of C on A, $\eta : C_0 \longrightarrow A_0$ a group identity and $\mu : A_0 \times_{C_0} A_0 \longrightarrow A_0$ an addition, which are morphisms of $\mathbf{C}$ satisfying standard conditions. We can consider A as an internal category of the form $A = (C_0, A_0, \pi, \pi, \eta, \mu)$ with the additional structure e, which is isomorphic to the internal category $(C_0, C_0 \ltimes \operatorname{Ker}\pi, \pi', \pi', \eta', \mu')$ with the additional structure $e' : (C_0 \ltimes \operatorname{Ker} d_0) \times_{C_0} (C_0 \ltimes \operatorname{Ker}\pi) \longrightarrow C_0 \ltimes \operatorname{Ker}\pi$; here $\pi'(r,a) = r$, $\eta'(r) = (r,0)$, $\mu'((r,a'),(r,a)) = (r, a'+a)$. Note that by Lemma 1.1.1 applied to A, $\operatorname{Ker}\pi$ is an abelian group and $a_1 * a_2 = 0$ for all $a_1, a_2 \in \operatorname{Ker}\pi$. By the definition of an internal diagram, e' is a morphism in C satisfying the conditions

$$\pi' e'((r,c),(r,a)) = d(c) + r,$$
$$e'((r,0),(r,a)) = (r,a),$$
$$e'\big((d(c)+r, c'), e'((r,c),(r,a))\big) = e'((r, c'+c),(r,a));$$

moreover, e' satisfies the distributivity condition (for the abelian group structure on A), from which follows

$$e'((r,c),(r,0)) = (d(c)+r, 0).$$

From the above and from the fact that e' is a morphism in $\mathbf{C}$ we obtain

$$\begin{aligned} e'((r,c),(r,a)) &= e'\big(((0,0),(0,a)) + ((r,c),(r,0))\big) = e'((0,0),(0,a)) + e'((r,c),(r,0)) = \\ &= (0,a) + (d(c)+r, 0) = (d(c)+r, a); \end{aligned}$$

on the other hand,

$$\begin{aligned} e'((r,c),(r,a)) &= e'\big(((0,c),(0,0)) + ((r,0),(r,a))\big) = e'((0,c),(0,0)) + e'((r,0),(r,a)) = \\ &= (d(c),0) + (r,a) = (d(c)+r, d(c)\cdot a). \end{aligned}$$

Thus, for each $r \in C_0$, $c \in \operatorname{Ker} d_0$ and $a \in \operatorname{Ker}\pi$

$$e'((r,c),(r,a)) = (d(c)+r, a),$$
$$d(c)\cdot a = a.$$

Consider the following equality:

$$e'\big(((r,c),(r,a)) * ((r',c'),(r',a'))\big) = e'((r,c),(r,a)) * e'((r',c'),(r',a')).$$

Direct computations give

$$e'\big((r*r', c*c' + r*c' + c*r'), (r*r', r*a' + a*r')\big) = (d(c)+r, a) * (d(c')+r', a'),$$
$$\big(d(c*c') + d(r*c') + d(c*r') + r, r*a' + a*r'\big) =$$
$$= \big(d(c)*d(c') + r*d(c') + d(c)*r' + r*r', d(c)*a' + r*a' + a*d(c') + a*r'\big).$$

For the case where $a' = 0$, we obtain

$$d(c) * a = 0,$$

for each $a \in \operatorname{Ker}\pi$, $c \in \operatorname{Ker} d_0$ and $* \in \Omega_2'$.

Thus we can conclude that $\operatorname{Ker}\pi$ is a $\operatorname{Coker} d$-module in the sense of [76] (see Definition 3.1.5 and Sec. 3.3).

2.2. The Standard Complex and the Cohomology

Let C$=(\mathrm{C}_0,\mathrm{C}_1,d_0,d_1,i,m)$ be an internal category in $\mathbf{C}$, and $A=(A_0,\pi,e,\eta,\mu)$ an abelian group in the category $\mathbf{C}^{\mathrm{C}}$ of internal diagrams on C. We construct the standard complex $\{K^*(\mathrm{C},A),\partial^*\}$ in analogy with the definition of the cohomology groups of ordinary categories [24, 25]:

$$\begin{aligned} K^0(\mathrm{C},A) &= \{f\in\mathbf{C}(\mathrm{C}_0,A_0)\mid \pi f=1_{\mathrm{C}_0}\},\\ K^1(\mathrm{C},A) &= \{\varphi\in\mathbf{C}(\mathrm{C}_1,A_0)\mid \pi\varphi=d_1\},\\ K^n(\mathrm{C},A) &= \Big\{\varphi\in\mathbf{C}(\underbrace{\mathrm{C}_1\times_{\mathrm{C}_0}\cdots\times_{\mathrm{C}_0}\mathrm{C}_1}_{n},A_0)\mid \pi\varphi=d_1\pi_1\Big\}, \end{aligned}$$

for $n>1$; here π_1 denotes the first projection (i.e., $\pi_1(x_n,\dots,x_1)=x_n$). For each $f\in K^0(\mathrm{C},A)$, $\varphi\in K^n(\mathrm{C},A)$, $x\in\mathrm{C}_1$, and $(x_{n+1},\dots,x_1)\in\mathrm{C}_1\times_{\mathrm{C}_0}\cdots\times_{\mathrm{C}_0}\mathrm{C}_1$ $(n>0)$, the differentials are defined by

$$\begin{aligned} \partial^0(f)(x) &= e(x,fd_0(x))-fd_1(x),\\ \partial^n(\varphi)(x_{n+1},\dots,x_1) &= e\big(x_{n+1},\varphi(x_n,\dots,x_1)\big)\\ &\quad+\sum_{i=1}^{n}(-1)^i\varphi\big(x_{n+1},\dots,m(x_{i+1},x_i),\dots,x_1\big)+(-1)^{n+1}\varphi(x_{n+1},\dots,x_2). \end{aligned}$$

We set $\mathrm{H}^n(\mathrm{C},A)=\mathrm{H}^n\{K^*(\mathrm{C},A),\partial^*\}$ for $n\geq 0$.

Now we shall give another (semi-trivial extension) form to this complex, which will be isomorphic to the previous one. By the diagrams

$$\begin{array}{ccc} \mathrm{C}_0 & \xrightarrow{\;f\;} & A_0\\ \| & & \downarrow\approx\\ \mathrm{C}_0 & \overset{f'}{\dashrightarrow} & \mathrm{C}_0\ltimes\operatorname{Ker}\pi \end{array},$$

$$\begin{array}{ccc} \underbrace{\mathrm{C}_1\times_{\mathrm{C}_0}\cdots\times_{\mathrm{C}_0}\mathrm{C}_1}_{n} & \xrightarrow{\;\varphi\;} & A_0\\ \approx\downarrow & & \downarrow\approx\\ \underbrace{(\mathrm{C}_0\ltimes\operatorname{Ker}d_0)\times_{\mathrm{C}_0}\cdots\times_{\mathrm{C}_0}(\mathrm{C}_0\ltimes\operatorname{Ker}d_0)}_{n} & \overset{\varphi'}{\dashrightarrow} & \mathrm{C}_0\ltimes\operatorname{Ker}\pi \end{array}$$

we define

$$\begin{aligned} &K^0(\mathrm{C}_0\ltimes\operatorname{Ker}d_0,\mathrm{C}_0\ltimes\operatorname{Ker}\pi)=\Big\{f'\in\mathbf{C}(\mathrm{C}_0,\mathrm{C}_0\ltimes\operatorname{Ker}\pi)\mid \pi'f'=1_{\mathrm{C}_0}\Big\},\\ &K^1(\mathrm{C}_0\ltimes\operatorname{Ker}d_0,\mathrm{C}_0\ltimes\operatorname{Ker}\pi)=\Big\{\varphi'\in\mathbf{C}(\mathrm{C}_0\ltimes\operatorname{Ker}d_0,\mathrm{C}_0\ltimes\operatorname{Ker}\pi)\mid \pi'\varphi'=d_1'\Big\},\\ &K^n(\mathrm{C}_0\ltimes\operatorname{Ker}d_0,\mathrm{C}_0\ltimes\operatorname{Ker}\pi)\\ &\qquad=\Big\{\varphi'\in\mathbf{C}\big(\underbrace{(\mathrm{C}_0\ltimes\operatorname{Ker}d_0)\times_{\mathrm{C}_0}\cdots\times_{\mathrm{C}_0}(\mathrm{C}_0\ltimes\operatorname{Ker}d_0)}_{n},\mathrm{C}_0\ltimes\operatorname{Ker}\pi'\big)\mid \pi'\varphi'=d_1'\pi_1\Big\}\quad\text{for}\quad n>1. \end{aligned}$$

For the differentials we have the following equalities:

$$\begin{aligned} {\partial'}^0(f')(r,c) &= e'((r,c),f'd_0(r,c))-f'(d_1'(r,c))=(d(c)+r,\pi_2f'(r))-f'(d(c)+r)\\ &=(d(c)+r,\pi_2f'(r))-(d(c)+r,\pi_2f'(d(c)+r)) \end{aligned}$$

$$
\begin{aligned}
&= (d(c)+r, \pi_2 f'(r) - \pi_2 f' d(c) - \pi_2 f'(r)) = (d(c)+r, -\pi_2 f' d(c)),\\
\partial'^{\,1}(\varphi')((d(c_1)+r, c_2),(r,c_1)) &= e'((d(c_1)+r,c_2), \varphi'(r,c_1))\\
&\quad - \varphi'(m'((d(c_1)+r,c_2),(r,c_1))) + \varphi'(d(c_1)+r, c_2)\\
&= (d(c_2)+d(c_1)+r, \pi_2\varphi'(r,c_1)) - \varphi'(r, c_2+c_1) + \varphi'(d(c_1)+r, c_2)\\
&= (d(c_2)+d(c_1)+r, \pi_2\varphi'(r,c_1)) - \big(d(c_2)+d(c_1)+r, \pi_2\varphi'(r,c_2+c_1)\big)\\
&\quad + \big(d(c_2)+d(c_1)+r, \pi_2\varphi'(d(c_1)+r, c_2)\big) \qquad (2.2.1)\\
&= \big(d(c_2)+d(c_1)+r, \pi_2\varphi'(r,c_1) - \pi_2\varphi'(r,c_2+c_1) + \pi_2\varphi'(d(c_1)+r,c_2)\big);\\
\partial'^{\,n}(\varphi')\big((d(c_n)+\cdots+d(c_1)+r, c_{n+1}),\ldots,&(d(c_1)+r,c_2),(r,c_1)\big)\\
&= \big(d(c_{n+1})+\cdots+d(c_1)+r, \pi_2\varphi'((d(c_{n-1})+\cdots+d(c_1)+r,c_n),\ldots,(r,c_1))\big)\\
&\quad + \sum_{i=1}^{n}(-1)^i \pi_2\varphi'\big((d(c_n)+\cdots+d(c_1)+r, c_{n+1}),\ldots\\
&\qquad\qquad \ldots,(d(c_{i-1})+\cdots+d(c_1)+r, c_{i+1}+c_i),\ldots,(r,c_1)\big)\\
&\quad + (-1)^{n+1}\pi_2\varphi'\big((d(c_n)+\cdots+d(c_1)+r,c_{n+1}),\ldots,(d(c_1)+r,c_2)\big),
\end{aligned}
$$

for $n > 1$. For $f' \in K^0(\mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \mathrm{C}_0 \ltimes \operatorname{Ker}\pi)$ we can write $f'(r) = (r, \pi_2 f'(r))$. Since $\varphi' \in K^1(\mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \mathrm{C}_0 \ltimes \operatorname{Ker}\pi)$ is a morphism of $\mathbf{C}$, we can write

$$
\begin{aligned}
\varphi'(r,c) &= \varphi'((0,c)+(r,0)) = \varphi'(0,c) + \varphi'(r,0)\\
&= (d(c), \pi_2\varphi'(0,c)) + (r, \pi_2\varphi'(r,0)) = (d(c)+r, \pi_2\varphi'(0,c) + \pi_2\varphi'(r,0)).
\end{aligned}
$$

Further, since we have

$$
\begin{aligned}
(d(c_1)+r, c_2) &= (0,c_2) + (d(c_1),0) + (r,0),\\
(r,c_1) &= (0,c_1) + (r,0),
\end{aligned}
$$

for $\varphi' \in K^2(\mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \mathrm{C}_0 \ltimes \operatorname{Ker}\pi)$, we obtain

$$
\begin{aligned}
\varphi'((d(c_1)+r,c_2),(r,c_1)) &= \varphi'(((0,c_2),(0,0)) + ((d(c_1),0),(0,c_1)) + ((r,0),(r,0)))\\
&= \varphi'((0,c_2),(0,0)) + \varphi'((d(c_1),0),(0,c_1)) + \varphi'((r,0),(r,0))\\
&= \big(d(c_2)+d(c_1)+r, \pi_2\varphi'((0,c_2),(0,0))\big) + \pi_2\varphi'\big((d(c_1),(0,c_1)) + \pi_2\varphi'((r,0),(r,0))\big).
\end{aligned}
$$

Similarly, since

$$
\begin{aligned}
&\big((d(c_{n-1})+\cdots+d(c_1)+r, c_n),\ldots,(d(c_1)+r,c_2),(r,c_1)\big)\\
&\quad= ((0,c_n),(0,0),\ldots,(0,0)) + ((d(c_{n-1}),0),(0,c_{n-1}),(0,0),\ldots\\
&\qquad \ldots,(0,0)) + \cdots + ((\underbrace{d(c_k),0),\ldots,(d(c_k),0}_{n-k}),(0,c_k),\\
&\qquad \underbrace{(0,0),\ldots,(0,0)}_{k-1}) + \cdots + ((d(c_1),0),\ldots,(d(c_1),0),(0,c_1)) + ((r,0),\ldots,(r,0)),
\end{aligned}
$$

for $n > 1$ and $\varphi' \in K^n(\mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \mathrm{C}_0 \ltimes \operatorname{Ker}\pi)$, we obtain

$$
\begin{aligned}
&\varphi'((d(c_{n-1})+\cdots+d(c_1)+r,c_n),\ldots,(d(c_1)+r,c_2),(r,c_1))\\
&\quad= \big(d(c_n)+d(c_{n-1})+\cdots+d(c_1)+r, \pi_2\varphi'((0,c_n),(0,0),\ldots,(0,0))\big)\\
&\qquad + \pi_2\varphi'\big((d(c_{n-1}),0),(0,c_{n-1}),(0,0),\ldots,(0,0)\big) + \ldots
\end{aligned}
$$

$$+\pi_2\varphi'\big(\underbrace{(d(c_k),0),\dots,(d(c_k),0)}_{n-k},(0,c_k),\underbrace{(0,0),\dots,(0,0)}_{k-1}\big)+\dots$$
$$+\pi_2\varphi'\big((d(c_1),0),\dots,(d(c_1),0),(0,c_1)\big)+\pi_2\varphi'((r,0),\dots,(r,0)).$$

Denote for $n>0$ and $\varphi'\in K^n(\mathrm{C}_0\ltimes \operatorname{Ker} d_0, \mathrm{C}_0\ltimes \operatorname{Ker}\pi)$

$$\begin{aligned}
&\varphi_0(r)=\pi_2\varphi'\big(\underbrace{(r,0),\dots,(r,0)}_{n}\big) \quad \text{for each} \quad r\in \mathrm{C}_0;\\
&\varphi_1(c)=\pi_2\varphi'\big((d(c),0),\dots,(d(c),0),(0,c)\big) \quad \text{for each} \quad c\in \operatorname{Ker} d_0;\\
&\varphi_k(c)=\pi_2\varphi'\big(\underbrace{(d(c),0),\dots,(d(c),0)}_{n-k}),(0,c),\underbrace{(0,0),\dots,(0,0)}_{k-1})\\
&\qquad\qquad \text{for each} \quad 1\le k\le n, \quad c\in \operatorname{Ker} d_0.
\end{aligned} \tag{2.2.2}$$

So we have

$$\begin{aligned}
&\varphi'(r,c)=(d(c)+r,\varphi_1(c)+\varphi_0(r)) \quad \text{for} \quad \varphi'\in K^1(\mathrm{C}_0\ltimes \operatorname{Ker} d_0, \mathrm{C}_0\ltimes \operatorname{Ker}\pi);\\
&\varphi'((d(c_1)+r,c_2),(r,c_1))=(d(c_2)+d(c_1)+r,\varphi_2(c_2)+\varphi_1(c_1)+\varphi_0(r))\\
&\qquad \text{for} \quad \varphi'\in K^2(\mathrm{C}_0\ltimes \operatorname{Ker} d_0, \mathrm{C}_0\ltimes \operatorname{Ker}\pi);\\
&\varphi'((d(c_{n-1})+\dots+d(c_1)+r,c_n),\dots,(d(c_1)+r,c_2),(r,c_1))=\\
&\quad =(d(c_n)+\dots+d(c_1)+r,\varphi_n(c_n)+\dots+\varphi_1(c_1)+\varphi_0(r))\\
&\qquad \text{for} \quad n>1, \quad \varphi'\in K^n(\mathrm{C}_0\ltimes \operatorname{Ker} d_0, \mathrm{C}_0\ltimes \operatorname{Ker}\pi).
\end{aligned} \tag{2.2.3}$$

The functions $\varphi_0,\varphi_1,\dots,\varphi_n$ are completely determined by $\varphi'\in K^n(\mathrm{C}_0\ltimes \operatorname{Ker} d_0, \mathrm{C}_0\ltimes \operatorname{Ker}\pi)$. Again, from the fact that φ' is a morphism in $\mathbf{C}$, we obtain the following identities for $\varphi_0,\varphi_1,\dots,\varphi_n$, for each $r,r'\in \mathrm{C}_0$, $c,c'\in \operatorname{Ker} d_0$:

$$\begin{aligned}
&\varphi_0(\omega(r))=\omega(\varphi_0(r)) \quad \text{for each} \quad \omega\in\Omega_1',\\
&\varphi_0(r+r')=\varphi_0(r)+r\cdot\varphi_0(r'),\\
&\varphi_0(r*r')=r*\varphi_0(r')+\varphi_0(r)*r' \quad \text{for each} \quad *\in\Omega_2';
\end{aligned} \tag{2.2.4}$$

for $1\le k\le n$;

$$\begin{aligned}
&\varphi_k(\omega(c))=\omega(\varphi_k(c)) \quad \text{for each} \quad \omega\in\Omega_1,\\
&\varphi_k(c+c')=\varphi_k(c)+\varphi_k(c'),\\
&\varphi_k(r*c)=r*\varphi_k(c) \quad \text{for each} \quad *\in\Omega_2',\\
&\varphi_k(c*c')=0 \quad \text{for each} \quad *\in\Omega_2',\\
&\varphi_k(r\cdot c)=r\cdot\varphi_k(c).
\end{aligned}$$

Maps satisfying conditions (2.2.4) are called derivations [76] (see Sec. 3.2), and the case of groups is a special case of a crossed homomorphism defined by S. MacLane [69] and J. H. C. Whitehead [89].

It is easy to see that $pr_2 f'\big|_{\mathrm{C}_0}:\mathrm{C}_0\longrightarrow \operatorname{Ker}\pi$ for $f'\in K^0(\mathrm{C}_0\ltimes \operatorname{Ker} d_0, \mathrm{C}_0\ltimes \operatorname{Ker}\pi)$ also satisfies conditions (2.2.4). Denote

$$\begin{aligned}
K^0&=K^0(\operatorname{Ker} d_0,\operatorname{Ker}\pi)=\Big\{\varphi_0 \mid \varphi_0:\mathrm{C}_0\longrightarrow \operatorname{Ker}\pi \text{ satisfies conditions (2.2.4)}\Big\}\\
&=\operatorname{Der}(\mathrm{C}_0,\operatorname{Ker}\pi),\\
K^1&=\Big\{\varphi_1\in \mathbf{C}(\operatorname{Ker} d_0,\operatorname{Ker}\pi) \mid \varphi_1(r\cdot c)=r\cdot\varphi_1(c),\\
&\qquad\qquad \varphi_1(r*c)=r*\varphi_1(c) \quad \text{for each } r\in\mathrm{C}_0,\ c\in\operatorname{Ker} d_0,\ *\in\Omega_2'\Big\},
\end{aligned}$$

$$K^k = \underbrace{K^1 \times \cdots \times K^1}_{k} \quad \text{for} \quad k > 0,$$

$$K^n(\operatorname{Ker} d_0, \operatorname{Ker}\pi) = \big\{(\varphi_0, \dots, \varphi_n) \mid \ \varphi_0 \in K^0,\ (\varphi_1, \dots, \varphi_n) \in K^n\big\} \quad \text{for} \quad n > 0.$$

The correspondence

$$\begin{aligned} \varphi &\longmapsto f' : r \longmapsto (r, \varphi_0(r)), \\ (\varphi_0, \varphi_1) &\longmapsto \varphi' : (r, c) \longmapsto \big(d(c) + r, \varphi_1(c) + \varphi_0(r)\big), \\ (\varphi_0, \dots, \varphi_n) &\longmapsto \varphi' \quad \text{defined by (2.2.3) for} \quad n > 1, \end{aligned}$$

gives an isomorphism of Abelian groups

$$K^n(\mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \mathrm{C}_0 \ltimes \operatorname{Ker}\pi) \approx K^n(\operatorname{Ker} d_0, \operatorname{Ker}\pi) \quad \text{for} \quad n \geq 0.$$

Differentials for the complex $K^*(\operatorname{Ker} d_0, \operatorname{Ker}\pi)$ are obtained from the diagrams

$$\begin{array}{ccc} K^n(\mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \mathrm{C}_0 \ltimes \operatorname{Ker}\pi) & \xrightarrow{\partial^n} & K^{n+1}(\mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \mathrm{C}_0 \ltimes \operatorname{Ker}\pi \\ \approx \downarrow & & \downarrow \approx \\ K^n(\operatorname{Ker} d_0, \operatorname{Ker}\pi) & \overset{\overline{\partial}^n}{\dashrightarrow} & K^{n+1}(\operatorname{Ker} d_0, \operatorname{Ker}\pi) \end{array}$$

for all $n \geq 0$. For the case $n = 0$ we have the following situation: Let $\varphi_0 \in K^0(\operatorname{Ker} d_0, \operatorname{Ker}\pi)$; the vertical isomorphism carries φ_0 to the homomorphism $f' : \mathrm{C}_0 \longrightarrow \mathrm{C}_0 \ltimes \operatorname{Ker}\pi$ defined by $f'(r) = (r, \varphi_0(r))$ and also $\partial^0(f')(r, c) = (d(c) + r, -\pi_2 f' d(c))$; so from (2.2.1) and (2.2.2) we obtain

$\overline{\partial}^0(\varphi_0) = (\overline{\varphi}_0, \overline{\varphi}_1)$, where

$$\overline{\varphi}_0(r) = \pi_2(\partial^0(f'))(r, 0) = \pi_2(r, 0) = 0 \quad \text{for each} \quad r \in \mathrm{C}_0;$$

$$\overline{\varphi}_1(c) = \pi_2(\partial^0(f'))(0, c) = \pi_2(d(c), -\pi_2 f' d(c)) = -\pi_2(d(c), \varphi_0 d(c)) = -\varphi_0 d(c) \quad \text{for each} \quad c \in \operatorname{Ker} d_0;$$

$\overline{\partial}^1(\varphi_0, \varphi_1) = (\overline{\varphi}_0, \overline{\varphi}_1, \overline{\varphi}_2)$, where

$$\begin{aligned} \overline{\varphi}_0(r) &= \pi_2(\partial'^{\,1}(\varphi'))((r, 0), (r, 0)) = \\ &= \pi_2\varphi'(r, 0) - \pi_2\varphi'(r, 0) + \pi_2\varphi'(r, 0) = \pi_2\varphi'(r, 0) = \varphi_0(r) \quad \text{for each} \quad r \in \mathrm{C}_0; \\ \overline{\varphi}_1(c) &= \pi_2(\partial'^{\,1}(\varphi')((d(c), 0)(0, c)) = \\ &= \pi_2\varphi'(0, c) - \pi_2\varphi'(0, c) + \pi_2\varphi'(d(c), 0) = \varphi_0 d(c) \quad \text{for each} \quad c \in \operatorname{Ker} d_0; \\ \overline{\varphi}_2(c) &= \pi_2(\partial'^{\,1}(\varphi'))((0, c), (0, 0)) = -\pi_2\varphi'(0, c) + \pi_2\varphi'(0, c) = 0, \end{aligned}$$

for each $c \in \operatorname{Ker} d_0$, where φ' denotes the element of $K^1(\mathrm{C}_0 \ltimes \operatorname{Ker} d_0, \mathrm{C}_0 \ltimes \operatorname{Ker}\pi)$ corresponding to $(\varphi_0, \varphi_1) \in K^1(\operatorname{Ker} d_0, \operatorname{Ker}\pi)$. So we have

$$\begin{aligned} \overline{\partial}^0(\varphi_0) &= (0, -\varphi_0 d), \\ \overline{\partial}^1(\varphi_0, \varphi_1) &= (\varphi_0, \varphi_0 d, 0). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \overline{\partial}^2(\varphi_0, \varphi_1, \varphi_2) &= (0, \varphi_1 - \varphi_0 d, 0, -\varphi_2), \\ \overline{\partial}^3(\varphi_0, \varphi_1, \varphi_2, \varphi_3) &= (\varphi_0, \varphi_0 d, \varphi_2, \varphi_2, 0), \\ \overline{\partial}^4(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4) &= (0, \varphi_1 - \varphi_0 d, 0, \varphi_3 - \varphi_2, 0, -\varphi_4), \\ \overline{\partial}^5(\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) &= (\varphi_0, \varphi_0 d, \varphi_2, \varphi_2, \varphi_4, \varphi_4, 0), \end{aligned}$$

and generally for $n \geq 2$

$$\overline{\partial}^{2n}(\varphi_0,\ldots,\varphi_{2n}) = (0,\varphi_1-\varphi_0 d, 0, \varphi_3-\varphi_2,\ldots,0,\varphi_{2n-1}-\varphi_{2n-2},0,-\varphi_{2n}),$$
$$\overline{\partial}^{2n+1}(\varphi_0,\ldots,\varphi_{2n+1}) = (\varphi_0,\varphi_0 d,\varphi_2,\varphi_2,\ldots,\varphi_{2n},\varphi_{2n},0).$$

Note that the complex $\{K^*(\operatorname{Ker} d_0, \operatorname{Ker}\pi), \overline{\partial}^*\}$ depends only on K^0, K^1 and d.

So we have $\mathrm{H}^n(C,A) \approx \mathrm{H}^n\{K^*(\operatorname{Ker} d_0,\operatorname{Ker}\pi),\overline{\partial}^*\}$ for $n \geq 0$. Direct computations give for the kernels and images of $\overline{\partial}$ the following equalities:

$$\operatorname{Ker}\overline{\partial}^0 = \{f \in K^0 \mid fd = 0\},$$
$$\operatorname{Ker}\overline{\partial}^1 = \{(0,\varphi_1) \mid \varphi_1 \in K^1\} \approx K^1,$$
$$\operatorname{Ker}\overline{\partial}^2 = \{(\varphi_0,\varphi_0 d, 0) \mid \varphi_0 \in K^0\} \approx K^0,$$
$$\operatorname{Ker}\overline{\partial}^3 = \{0,\varphi_1,0,\varphi_2) \mid (\varphi_1,\varphi_3) \in K^2\} \approx K^2,$$

for $n \geq 2$

$$\operatorname{Ker}\overline{\partial}^{2n} = \Big\{(\varphi_0,\varphi_0 d,\varphi_2,\varphi_2,\ldots,\varphi_{2n-2},\varphi_{2n-2},0) \mid \varphi_0 \in K^0,\ (\varphi_2,\ldots,\varphi_{2n-2}) \in K^{n-1}\Big\} \approx K^0 \times K^{n-1},$$
$$\operatorname{Ker}\overline{\partial}^{2n+1} = \Big\{(0,\varphi_1,0,\varphi_3,\ldots,0,\varphi_{2n-1},0,\varphi_{2n+1}) \mid (\varphi_1,\varphi_3,\ldots,\varphi_{2n+1}) \in K^{n+1}\Big\} \approx K^{n+1};$$
$$\operatorname{Im}\overline{\partial}^0 = \{(0,fd) \mid f \in K^0\},$$
$$\operatorname{Im}\overline{\partial}^1 = \{(\varphi_0,\varphi_0 d,0) \mid \varphi_0 \in K^0\} \approx K^0,$$
$$\operatorname{Im}\overline{\partial}^2 = \Big\{(0,\varphi_1-\varphi_0 d,0,\varphi_2) \mid \varphi_0 \in K^0 (\varphi_1,\varphi_2) \in K^2\Big\} \approx K^2,$$
$$\operatorname{Im}\overline{\partial}^4 = \Big\{(0,\varphi_1-\varphi_0 d,0,\varphi_3-\varphi_2,0,\varphi_4) \mid \varphi_0 \in K^0,\ (\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in K^4\Big\} \approx K^3,$$

for $n \geq 2$

$$\operatorname{Im}\overline{\partial}^{2n} = \Big\{(0,\varphi_1-\varphi_0 d,0,\varphi_3-\varphi_2,\ldots,0,\varphi_{2n-1}-\varphi_{2n-2},0,\varphi_{2n}) \mid \varphi_0 \in K^0,\ (\varphi_1,\ldots,\varphi_{2n}) \in K^{2n}\Big\} \approx K^{n+1},$$
$$\operatorname{Im}\overline{\partial}^{2n+1} = \Big\{(\varphi_0,\varphi_0 d,\varphi_2,\varphi_2,\varphi_4,\varphi_4,\ldots,\varphi_{2n},\varphi_{2n},0) \mid \varphi_0 \in K^0,\ (\varphi_2,\varphi_4,\ldots,\varphi_{2n}) \in K^n\Big\} \approx K^0 \times K^n.$$

This proves the following theorem:

Theorem 2.2.1. *Let* C *be an internal category in* $\mathbf{C}$. *For each abelian group* A *in* $\mathbf{C}^{\mathrm{C}}$ *we have:*

(i) $\mathrm{H}^0(\mathrm{C},A) \approx \{f \in K^0 \mid fd = 0\}$;
$\mathrm{H}^1(\mathrm{C},A) \approx K^1/\{fd \mid f \in K^0\}$;
$\mathrm{H}^n(\mathrm{C},A) = 0$ *for* $n \geq 2$;

(ii) *for* $n \geq 2$ *the exact sequence*

$$0 \longrightarrow \operatorname{Ker}\partial^n \longrightarrow K^n(\mathrm{C},A) \xrightarrow{\partial^n} \operatorname{Ker}\partial^{n+1} \longrightarrow 0$$

is split; $\operatorname{Ker}\partial^{2n+1} \approx K^{n+1}$ *for* $n \geq 0$, $\operatorname{Ker}\partial^2 \approx K^0$ *and* $\operatorname{Ker}\partial^{2n} \approx K^0 \times K^{n-1}$ *for* $n > 1$.

We will see in Chap. 3 that $\mathrm{H}^0(\mathrm{C},A) \approx \operatorname{Der}(\operatorname{Coker} d, \ker\pi)$.

Chapter 3

SOME PROPERTIES OF INTERNAL CATEGORY COHOMOLOGY AND COHOMOLOGICALLY TRIVIAL INTERNAL CATEGORIES

In this chapter we study the functorial properties of internal category cohomology and the relations between homological, internal category, and cohomological equivalences. We relate internal category cohomology with the cohomology of crossed modules in groups defined by G. J. Ellis [42]. A natural next step would be to study the cohomological dimension of internal categories in $\mathbf{C}$. However, on one hand $\mathrm{H}^n(\mathrm{C}, -) = 0$, for all $n \geq 2$, and on the other hand $\mathrm{H}^0(\mathrm{C}, -) = 0$ does not imply $\mathrm{H}^1(\mathrm{C}, -) = 0$ (see Example 3, Sec. 3.4). So the subject of investigation is to characterize those internal categories C for which $\mathrm{H}^0(\mathrm{C}, -) = 0$, and separately, those C for which $\mathrm{H}^1(\mathrm{C}, -) = 0$. Our characterizations become very simple in the case where $\mathbf{C}$ is the category of groups: in that case $\mathrm{H}^0(\mathrm{C}, -) = 0$ if and only if C is a connected category, and $\mathrm{H}^1(\mathrm{C}, -) = 0$ if and only if the certain abelianization of C is internally equivalent to a discrete category.

3.1. Extensions in Categories of Groups with Operations

This section contains preliminary results on the extensions in categories of groups with operations that are essentially known [1, 2, 40, 80, 84]. Our purpose is to present them in the form convenient for us.

Let $\mathbf{C}$ be a category of groups with operations with a set of operations Ω and with a set of identities $\mathbf{E}$ (see Sec. 1.1 for the definition).

We formulate two more axioms on $\mathbf{C}$ (Axiom (7) and Axiom (8) of [76]).

If C is an object of $\mathbf{C}$ and $x_1, x_2, x_3 \in \mathrm{C}$:

Axiom 1. $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$ for each $* \in \Omega_2'$.

Axiom 2. For each ordered pair $(*, \overline{*}) \in \Omega_2' \times \Omega_2'$ there is a word W such that

$$(x_1 * x_2)\overline{*}x_3 = W\big(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2\big),$$

where each juxtaposition represents an operation in Ω_2'.

A category of groups with operations satisfying Axiom 1 and Axiom 2 is called a category of interest by Orzech [76] (see also [78]). All examples of a category of groups with operations, given in Sec. 1.1, can be interpreted as categories of interest.

As we have noted in Chap. 1, in categories of groups with operations, from the equalities

$$(x + y) * (z + t) = x * z + x * t + y * z + y * t = x * z + y * z + x * t + y * t$$

it follows that $x * t + y * z = y * z + x * t$ for $* \in \Omega_2'$, $x, y, z, t \in \mathrm{C}$, $\mathrm{C} \in \mathbf{C}$.

Denote by $\mathbf{E}_G$ a subset of identities of $\mathbf{E}$ which includes the group laws and the identities (c) and (d) from the definition of a category of groups with operations. We denote by $\mathbf{C}_G$ the corresponding category of groups with operations. Thus we have $\mathbf{E}_G \hookrightarrow \mathbf{E}$, $\mathbf{C} = (\Omega, \mathbf{E})$, $\mathbf{C}_G = (\Omega, \mathbf{E}_G)$ and there is a full inclusion functor $E : \mathbf{C} \hookrightarrow \mathbf{C}_G$. Let Q be the left adjoint to the E. Thus $Q(\mathrm{C})$ is the greatest quotient of C from $\mathbf{C}_G$ such that $Q(\mathrm{C}) \in \mathbf{C}$.

We shall denote by $\mathbf{C}$ a category of groups with operations and it will be mentioned when it is a category of interest.

Let $\mathrm{E} : 0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0$ be an extension of B by A in $\mathbf{C}$. This means that p is surjective and i is the kernel of p. An object A is called singular if it is an abelian group and $a_1 * a_2 = 0$

for all $a_1, a_2 \in A$, $* \in \Omega_2'$. An extension $0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0$ is called singular [76] if A is singular, and it is called split if there is a morphism $s : B \longrightarrow E$ such that $ps = 1_B$.

Let E be a split extension. We shall identify $a \in A$ with its image $i(a)$.

We have induced operations in $B \times A$:

$$\begin{aligned} \omega(b, a) &= (\omega(b), \omega(a)) \quad \text{for each} \quad \omega \in \Omega_1', \\ (b', a') + (b, a) &= (b' + b, a' + s(b') + a - s(b')), \\ (b', a') * (b, a) &= \big(b' * b, a' * a + a' * s(b) + s(b') * a\big) \quad \text{for each} \quad * \in \Omega_2'. \end{aligned}$$

The set $B \times A$ with the above structure is an object of $\mathbf{C}$; denote it by $B \ltimes A$; we have an isomorphism $E \approx B \ltimes A$. We shall use the following notations as in [76] and the previous chapters:

$$\begin{aligned} b \cdot a &= s(b) + a - s(b), \\ b * a &= s(b) * a, \end{aligned}$$

for each $a \in A$, $b \in B$ and $* \in \Omega_2'$. Thus, a split extension induces actions of B on A corresponding to each operation in $\mathbf{C}$. These actions are called derived actions of B on A [76]. We shall call them split derived actions.

Proposition 3.1.1. *A set of actions in $\mathbf{C}_G$ is a set of split derived actions if and only if it satisfies the following conditions:*

1. $0 \cdot a = a$,
2. $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$,
3. $(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$,
4. $b * (a_1 + a_2) = b * a_1 + b * a_2$,
5. $(b_1 + b_2) * a = b_1 * a + b_2 * a$,
6. $(b_1 * b_2) \cdot (a_1 * a_2) = a_1 * a_2$,
7. $(b_1 * b_2) \cdot (a * b) = a * b$,
8. $a_1 * (b \cdot a_2) = a_1 * a_2$,
9. $b * (b_1 \cdot a) = b * a$,
10. $\omega(b \cdot a) = \omega(b) \cdot \omega(a)$,
11. $\omega(a * b) = \omega(a) * b = a * \omega(b)$,
12. $x * y + z * t = z * t + x * y$,

for each $\omega \in \Omega_1'$, $* \in \Omega_2'$, $b, b_1, b_2 \in B$, $a, a_1, a_2 \in A$ and for $x, y, z, t \in A \cup B$ whenever each side of 12 *has a sense.*

The proof is based on the construction of the object $B \ltimes A \in \mathbf{C}_G$ and the corresponding split extension $0 \longrightarrow A \longrightarrow B \ltimes A \longrightarrow B \longrightarrow 0$ which induces the given set of actions as split derived actions. It is similar to the cases of groups [71] and is left to the reader.

Note that in the formulation of Proposition 1.1 in [32] we mean that the set of identities of the category of groups with operations contains only identities from $\mathbf{E}_G$, but it is not mentioned there. The same concerns some other statements of [32], which we formulate and prove here with corresponding corrections. In $\mathbf{C}$ the conditions 1-12 of Proposition 3.1.1 are the necessary conditions for the split derived actions. Of course according to the identities included in $\mathbf{E}$ we can add the corresponding conditions to 1–12 and obtain the necessary and sufficient conditions of split derived actions in $\mathbf{C}$.

We have analogous results for categories of interest. In this case the set of split derived actions satisfies conditions 1–5, 8–11; conditions 6 and 7 are replaced by

$$\begin{aligned} b \cdot (a_1 * a_2) &= a_1 * a_2, \\ b \cdot (a * b_1) &= a * b_1, \\ (b_1 * b_2) \cdot a &= a; \end{aligned}$$

condition 12 is replaced by

$$x + y * z = y * z + x \quad \text{for each} \quad x, y \in A, \quad z \in B;$$

we also have an additional condition: for each ordered pair $(*,\bar{*}) \in \Omega_2' \times \Omega_2'$ there is a word (from Axiom 2) with $(x_1 * x_2)\bar{*}x_3 = W(\ )$ for each $x_1, x_2, x_3 \in A \cup B$.

Let $0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0$ be an extension of B by A. For each element $b \in B$ choose a representative $u(b)$, with $pu(b) = b$ and $u(0) = 0$. It induces a set of actions

$$\begin{aligned} b \cdot a &= u(b) + a - u(b), \\ b * a &= u(b) * a, \end{aligned}$$

which we shall call derived actions. As for the case of groups or rings [71], [70], we have a family of factor systems $\{f, (g_*)_{*\in\Omega_2'} \mid f, g_* : B \times B \longrightarrow A\}$ corresponding to each operation in Ω_2:

$$\begin{aligned} u(b) + u(b') &= f(b, b') + u(b + b'), \\ u(b) * u(b') &= g_*(b, b') + u(b * b'), \quad * \in \Omega_2'. \end{aligned} \tag{3.1.1}$$

The associative law for the addition and the distributive law give

$$f(b_1, b_2) + f(b_1 + b_2, b_3) = b_1 \cdot f(b_2, b_3) + f(b_1, b_2 + b_3), \tag{3.1.2}$$

$$f(b_1 \cdot b_2) * b_3 + g_*(b_1 + b_2, b_3) = g_*(b_1, b_3) + (b_1 * b_3) \cdot g_*(b_2, b_3) + f(b_1 * b_3, b_2 * b_3); \tag{3.1.3}$$

also

$$\begin{aligned} f(b, 0) &= f(0, b) = 0, \\ g_*(b, 0) &= g_*(0, b) = 0, \end{aligned}$$

for each $b, b_1, b_2, b_3 \in B$ and $* \in \Omega_2'$.

From (3.1.1) we obtain that the set of derived actions of B on A obtained from nonsplit extensions satisfies conditions 1–12 with

3′. $b_1 \cdot (b_2 \cdot a) = f(b_1, b_2) + (b_1 + b_2) \cdot a - f(b_1, b_2)$,

5′. $b_1 * a + b_2 * a = f(b_1, b_2) * a + (b_1 + b_2) * a$

instead of conditions 3 and 5. Note that if A is singular, then conditions 3′ and 5′ coincide with conditions 3 and 5, respectively.

Remark. According to other identities included in **E** we will obtain the corresponding identities for derived actions in **C**. If **C** is a category of interest, then the factor systems satisfy additional conditions

$$\overline{W}(b_1, b_2, b_3, f, g_*, g_{\bar{*}}) = 0, \tag{3.1.4}$$

$$b_1 \cdot g_*(b_2, b_3) + f(b_1, b_2 * b_3) = g_*(b_2, b_3) + f(b_2 * b_3, b_1), \tag{3.1.5}$$

where condition (3.1.4) follows from Axiom 2, $\overline{W}$ is a word for each ordered pair $(*, \bar{*}) \in \Omega_2' \times \Omega_2'$, corresponding to W, and (3.1.5) follows from Axiom 1. For example, if in Axiom 2 W has the form

$$(b_1 * b_2)\bar{*}b_3 = b_1 * (b_2\bar{*}b_3) + (b_1\bar{*}b_3) * b_2,$$

then (3.1.4) is of the following form:

$$\begin{aligned} &b_1 * g_{\bar{*}}(b_2, b_3) + g_{\bar{*}}(b_1, b_3) * b_2 + g_*(b_1, b_2\bar{*}b_3) + \\ &\quad + (b_1 * (b_2\bar{*}b_3)) \cdot g_*(b_1\bar{*}b_3, b_2) + f(b_1 * (b_2\bar{*}b_3), (b_1\bar{*}b_3) * b_2) = g_*(b_1, b_2)\bar{*}b_3 + g_{\bar{*}}(b_1 + b_2, b_3). \end{aligned}$$

The following two lemmas are well known for the case of groups $\mathbf{C} = \mathbb{G}\mathrm{r}$ [71].

Lemma 3.1.2. *Let A and B be groups with operations, and let B act on A such that a set of actions satisfies conditions* 1, 2, 3′, 4, 5′, 6–12, *where $\{f, (g_*)_{*\in\Omega_2'}\}$ is a family of functions from $B \times B$ to*

A satisfying conditions (3.1.2), (3.1.3) *for each* $* \in \Omega_2'$*. Then the set* $B \ltimes_{\{f,(g_*)\}} A$ *of all pairs* (b,a)*,* $b \in B$*,* $a \in A$*, with operations*

$$\begin{aligned} \omega(b,a) &= (\omega(b),\omega(a)) \quad \textit{for each} \quad \omega \in \Omega_1', \\ (b,a)+(b',a') &= \big(b+b', a+b\cdot a' + f(b,b')\big), \\ (b,a)*(b',a') &= \big(b*b', a*a' + a*b' + b*a' + g_*(b,b')\big) \end{aligned}$$

is an object of $\mathbf{C}_G$*. The homomorphisms* $a \longmapsto (a,0)$*,* $(a,b) \longmapsto b$ *define the extension*

$$0 \longrightarrow A \xrightarrow{i_1} B \ltimes_{\{f,(g_*)\}} A \xrightarrow{p_1} B \longrightarrow 0$$

of B by A.

Proof. Straightforward verification. □

From Lemma 3.1.2 and the observations before, we have the following assertion.

Lemma 3.1.3. *For each extension*

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0$$

in $\mathbf{C}_G$ *there is an object* $B \ltimes_{\{f,(g_*)\}} A$ *such that* $E \approx B \ltimes_{\{f,(g_*)\}} A$*, and we have the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{p} & B & \longrightarrow & 0 \\ & & \Vert & & \downarrow \approx & & \Vert & & \\ 0 & \longrightarrow & A & \xrightarrow[i_1]{} & B \ltimes_{\{f,(g_*)\}} A & \xrightarrow[p_1]{} & B & \longrightarrow & 0 \end{array} .$$

Note that Lemma 3.1.3 holds also for categories of interest.

Corollary 3.1.4. *A set of actions in* $\mathbf{C}_G$ *is a set of derived actions if and only if it satisfies conditions* 1, 2, 3′, 4, 5′, 6–12, *where* $\{f, (g_*)_{*\in\Omega_1}\}$ *is a family of functions satisfying conditions* (3.1.2) *and* (3.1.3)*.*

According to the above remark concerning the set of identities for the set of actions in $\mathbf{C}$, we can prove statements analogous to 3.1.2, 3.1.3 and 3.1.4 for the category $\mathbf{C}$.

Definition 3.1.5. Let $\mathbf{C}$ be a category of groups with operations or a category of interest and A and $B \in \mathbf{C}$. We say that A has a B-structure if there is an extension $0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$. We say that A has a split B-structure if the above extension is split; we say that A is a B-module if A is singular and has a split B-structure.

Let $0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$ be a singular extension; then the action of B on A does not depend on the choice of representatives of the elements of B in E. Let $\{f, (g_*)_{*\in\Omega_2'}\}$ and $\{f', (g_*')_{*\in\Omega_2'}\}$ be two different families of factor systems corresponding to two different choices of representatives of elements of B in E, u and u' respectively. Then we have

$$\begin{aligned} u(b)+u(b') &= f(b,b') + u(b+b'), \\ u'(b)+u'(b') &= f'(b,b') + u'(b+b'), \\ u(b)+u(b')-u'(b')-u'(b) &= f(b,b') + u(b+b') - u'(b+b') - f'(b,b'), \\ b\cdot(u(b')-u'(b')) + u(b) - u'(b) &= u(b+b') - u'(b+b') + f(b,b') - f'(b,b'). \end{aligned}$$

Define $\psi(b) = u(b) - u'(b)$, for each $b \in B$; then we obtain

$$f(b,b') - f'(b,b') = b\cdot\psi(b') - \psi(b+b') + \psi(b). \tag{3.1.6}$$

Similarly, for each $* \in \Omega_2'$ we have

$$b * \psi(b') + \psi(b) * b' - \psi(b * b') = g_*(b, b') - g_*'(b, b'). \tag{3.1.7}$$

We state without proof the following proposition since it is similar to the case of groups and it will be cited in the following section.

Proposition 3.1.6. *A singular extension*

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0$$

is split if and only if there is a function $\psi : B \longrightarrow A$ such that the family of factor systems $\{f, (g_*)_{*\in\Omega_2'}\}$ of the given extension satisfies the conditions

$$f(b, b') = b \cdot \psi(b') - \psi(b + b') + \psi(b),$$
$$g_*(b, b') = b * \psi(b') - \psi(b * b') + \psi(b) * b',$$

for each $* \in \Omega_2'$ and $(b, b') \in B \times B$.

Let A be an object of $\mathbf{C}$. Denote by $S(A)$ the greatest singular quotient of A. Then S is an abelianization functor from $\mathbf{C}$ to the category of abelian groups $\mathbb{A}$b.

Lemma 3.1.7. *Let $\mathbf{C}$ be a category of interest and $A, B \in \mathbf{C}$. If A has a split B-structure, then $S(A)$ is a B-module and the natural homomorphism $A \xrightarrow{\tau_A} S(A)$ is a homomorphism of split B-structures.*

Proof. $S(A)$ has an induced B-structure defined by $b \cdot \mathrm{cl}\, a = \mathrm{cl}(b \cdot a)$ and $b * \mathrm{cl}\, a = \mathrm{cl}(b * a)$ for each $* \in \Omega_2'$. From Axiom 1 and Axiom 2 it follows that these actions are defined correctly and τ_A is a homomorphism of B-structures. It is also easy to check that $S(A)$ satisfies conditions analogous to 1–12 for categories of interest and thus has a split B-structure; it is also abelian by definition, which proves that $S(A)$ is a B-module. □

3.2. Derivations

Let $\mathbf{C}$ be a category of groups with operations and $B \in \mathbf{C}$. Consider categories B-mod($\mathbf{C}_G$) and B-mod($\mathbf{C}$), B-modules in $\mathbf{C}_G$ and in $\mathbf{C}$ respectively. We have the full inclusion functor $\mathcal{E} : B$-mod($\mathbf{C}$) $\longrightarrow B$-mod($\mathbf{C}_G$). For any $C \in B$-mod($\mathbf{C}_G$) by definition we have the split exact sequence in $\mathbf{C}_G$

$$0 \longrightarrow \mathrm{C} \longrightarrow B \ltimes \mathrm{C} \underset{\longleftarrow}{\overset{\tau}{\longrightarrow}} B \longrightarrow 0\,;$$

Denote $\mathcal{S}(\mathrm{C}) = \operatorname{Ker} Q(\tau)$; we will have a surjection $\mathrm{C} \longrightarrow \mathcal{S}(\mathrm{C})$. Actually it defines a functor B-mod($\mathbf{C}_G$) $\longrightarrow B$-mod($\mathbf{C}$). It is easy to check that $\mathcal{S}$ is a left adjoint to $\mathcal{E}$; thus for any $A \in B$-mod($\mathbf{C}$) and $\mathrm{C} \in B$-mod($\mathbf{C}_G$) we have the natural isomorphism

$$B\text{-mod}(\mathbf{C}_G)(\mathrm{C}, \mathcal{E}(A)) \approx B\text{-mod}(\mathbf{C})(\mathcal{S}(C), A).$$

Definition 3.2.1 ([76])**.** Let A be a B-module. A map $\delta : B \longrightarrow A$ is called a derivation if

$$\delta(\omega(b)) = \omega(\delta(b)),$$
$$\delta(b + b') = \delta(b) + b \cdot \delta(b'),$$
$$\delta(b * b') = \delta(b) * b' + b * \delta(b'),$$

for each $\omega \in \Omega_1'$, $* \in \Omega_2'$ and $b, b' \in B$.

As in Chap. 2 derivations from B to A will be denoted by $\mathrm{Der}(B, A)$. Let

$$D = \Big\{0_1 0_2 \dots 0_n \partial b \mid \; 0_i \in \Omega_1' \cup \{b' * - \mid b' \in B,\; * \in \Omega_2'\} \cup$$
$$\cup \{ - * b' \mid b' \in B,\; * \in \Omega_2'\} \cup \{b' \cdot - \mid b' \in B\},\; i = 1, \dots, n,\; b \in B\Big\}.$$

So D is the set of all words of the form $\mathrm{w} = 0_1 0_2 \dots 0_n \partial b$. Let $\mathbb{I}(B)$ be the free abelian group generated by D modulo:

1. $\omega(\mathrm{w} + \mathrm{w}') = \omega\,\mathrm{w} + \omega\,\mathrm{w}'$,
2. $\partial 0 = 0$,
3. $0 \cdot \mathrm{w} = \mathrm{w}$,
4. $b \cdot (\mathrm{w} + \mathrm{w}') = b \cdot \mathrm{w} + b \cdot \mathrm{w}'$,
5. $(b + b') \cdot \mathrm{w} = b \cdot (b' \cdot \mathrm{w})$,
6. $b * (\mathrm{w} + \mathrm{w}') = b * \mathrm{w} + b * \mathrm{w}'$,
7. $(b + b') * \mathrm{w} = b * \mathrm{w} + b' * \mathrm{w}$,
8. $b * \mathrm{w} = \mathrm{w} *^{\circ} b$,
9. $(b * b') \cdot (\mathrm{w} * b) = \mathrm{w} * b$,
10. $b * (b' \cdot \mathrm{w}) = b * \mathrm{w}$,
11. $\omega(b \cdot \mathrm{w}) = \omega(b) \cdot (\omega\,\mathrm{w})$,
12. $\omega(\mathrm{w} * b) = (\omega\,\mathrm{w}) * b = \mathrm{w} * \omega(b)$,
13. $x * y + z * t = z * t + x * y$,
    $x, y, z, t \in B \cup D$,
    whenever each side has a sense,
14. $\partial(b + b') = \partial b + b \cdot \partial b'$,
15. $\partial(b * b') = b * \partial b' + (\partial b) * b'$,
16. $\partial(\omega b) = \omega\, \partial b$,

for each $b, b' \in B$, $\mathrm{w}, \mathrm{w}' \in D$, $\omega \in \Omega_1'$, $* \in \Omega_2'$.

Denote $\mathbf{I}(B) = \mathcal{S}\mathbb{I}(B)$.

Note that with some modifications of the above construction, $\mathbf{I}(B)$ can be constructed in the categories of interest.

It is easy to check that the map $\partial_G : B \longrightarrow \mathbb{I}(B)$ defined by $\partial_G(b) = \partial b$ is a derivation, which defines the derivation $\partial : B \longrightarrow \mathbf{I}(B)$.

Proposition 3.2.2. *Let $B \in \mathbf{C}$ and A be a B-module in $\mathbf{C}$. For any derivation $\delta : B \longrightarrow A$ there is a unique homomorphism of B-modules $\overline{\delta} : \mathbf{I}(B) \longrightarrow A$ such that the diagram*

$$\begin{array}{ccc} B & \xrightarrow{\;\delta\;} & A \\ {\scriptstyle\partial}\big\downarrow & \nearrow_{\overline{\delta}} & \\ \mathbf{I}(B) & & \end{array}$$

is commutative.

Proof. Let $\mathrm{w} = 0_1 0_2 \dots 0_n \partial b$ be an element in $\mathbb{I}(B)$; define $\overline{\delta_G}(\mathrm{cl}\,\mathrm{w}) = 0_1 0_2 \dots 0_n \delta b$. It is easy to show that $\overline{\delta_G}$ is defined correctly and it is a homomorphism of B-modules in $\mathbf{C}_G$. By adjunction of $\mathcal{E}$ and $\mathcal{S}$, $\overline{\delta_G}$ defines a unique homomorphism $\overline{\delta}$ of B-modules in $\mathbf{C}$ which makes the diagram commutative. □

Proposition 3.2.3. *Let $0 \longrightarrow A \xrightarrow{\;i\;} E \xrightarrow{\;p\;} B \longrightarrow 0$ be an extension of B by A, A' be a B-module, and $\tau : A \longrightarrow A'$ be a homomorphism of B-structures. Let $u(b)$ be a representative for each $b \in B$ in E, which induces a B-structure on A; consider E-structures on A and A' due to p. Then there exists a derivation $\delta : E \longrightarrow A'$ with $\delta i = \tau$ if and only if the extension of B by A' obtained from the given extension by τ is split.*

Proof. Let $\{f, (g_*)_{* \in \Omega_2'}\}$ be a family of factor systems of the given extension, corresponding to the map u. Then the extension of B by A' corresponds to the given B-structures on A', and the family of factor systems $\{\tau f, (\tau g_*)_{*\in\Omega_2'}\}$ is the extension obtained from the given one by τ. Suppose there is a derivation $\delta : E \longrightarrow A'$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & E \\ {\scriptstyle\tau}\big\downarrow & \swarrow{\scriptstyle\delta} & \\ A' & & \end{array}$$

is commutative. By Lemma 3.1.3 we have $E \approx B \ltimes_{\{f,(g_*)\}} A$. We shall identify E with $B \ltimes_{\{f,(g_*)\}} A$. So we obtain

$$\delta(b,a) = \delta((0,a)+(b,0)) = \delta(0,a) + (0,a)\cdot\delta(b,0) = \delta(0,a) + \delta(b,0) = \tau(a) + \psi(b),$$

where ψ denotes the function $B \longrightarrow A'$ defined by $\psi(b) = \delta(b,0)$. From the fact that δ is a derivation, we obtain that the factor systems $\{\tau f, (\tau g_*)_{*\in\Omega_2'}\}$ satisfy the conditions of Proposition 3.1.6 and the corresponding extension is split.

Suppose that the extension

$$0 \longrightarrow A' \xrightarrow{\;i'\;} B \ltimes_{\{\tau f,(\tau g_*)\}} A' \xrightarrow{\;p'\;} B \longrightarrow 0$$

is split. Then there is a function $\psi : B \longrightarrow A'$ satisfying the conditions of Proposition 3.1.6. Define a function $\delta : B \ltimes_{\{f,(g_*)\}} A \longrightarrow A'$ by $\delta(b,a) = \tau(a) + \psi(b)$. δ satisfies the condition $\delta i = \tau$, and it is easy to check that δ is a derivation, which proves the proposition. $\square$

For the complete solution of an analogous question in the case of groups, see [69].

3.3. Functorial Properties of the Cohomology, Internal Category Equivalence, and Homological and Cohomological Equivalences, Relation with Ellis's Cohomology of Crossed Modules

Let $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ be an internal category in $\mathbf{C}$ and $A = (A_0, \pi, e, \eta, \mu)$ be an abelian group in the category $\mathbf{C}^{\mathrm{C}}$ of internal diagrams on C. Recall from Sec. 2.1 that by Lemma 1.1.1 applied to A, $\operatorname{Ker}\pi$ is an abelian group and $a * a' = 0$ for each $a, a' \in \operatorname{Ker}\pi$; So $\operatorname{Ker}\pi$ is a C_0-module in the sense of [76] (see Sec. 3.1, Definition 3.1.5). Moreover, we proved in Sec. 2.1 that $\operatorname{Im} d$ acts trivially on $\operatorname{Ker}\pi$, where $d = d_1|_{\operatorname{Ker} d_0}$; $\operatorname{Im} d$ is an ideal of C_0 in the sense of [76] (see Chap. 5, Definition 5.1.1), thus $\operatorname{Coker} d \in \mathbf{C}$. As we have noted in Sec. 2.1 we can consider $\operatorname{Ker}\pi$ as a $\operatorname{Coker} d$-module. Conversely, for each $\operatorname{Coker} d$-module L we can construct an abelian group in $\mathbf{C}^{\mathrm{C}}$, and these two processes are converse to each other. Thus for any $\operatorname{Coker} d$-module L we can speak of cohomologies $\mathrm{H}^i(\mathrm{C}, L)$, $i \geq 0$, and $\mathrm{H}^i(\mathrm{C}, -)$ is a functor $\operatorname{Coker} d\text{-mod} \longrightarrow \mathbb{A}\mathrm{b}$. The properties obtained for internal category cohomology give the corresponding properties of the crossed module cohomology.

Proposition 3.3.1. *Let* $\mathrm{C} \in \operatorname{Cat}(\mathbf{C})$*, and*

$$0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0 \tag{3.3.1}$$

be an exact sequence of $\operatorname{Coker} d$*-modules.*

(a) *Then we have the exact sequences of cohomology groups*

$$0 \longrightarrow \mathrm{H}^0(\mathrm{C}, L') \longrightarrow \mathrm{H}^0(\mathrm{C}, L) \longrightarrow \mathrm{H}^0(\mathrm{C}, L''),$$

$$\mathrm{H}^1(\mathrm{C}, L') \longrightarrow \mathrm{H}^1(\mathrm{C}, L) \longrightarrow \mathrm{H}^1(\mathrm{C}, L'').$$

(b) *If exact sequence* (3.3.1) *is split, then it induces the exact sequence of cohomology groups*

$$0 \longrightarrow \mathrm{H}^0(\mathrm{C},L') \longrightarrow \mathrm{H}^0(\mathrm{C},L) \longrightarrow \mathrm{H}^0(\mathrm{C},L'') \longrightarrow$$
$$\longrightarrow \mathrm{H}^1(\mathrm{C},L') \longrightarrow \mathrm{H}^1(\mathrm{C},L) \longrightarrow \mathrm{H}^1(\mathrm{C},L'') \longrightarrow 0.$$

Proof. Both statements follow from Theorem 2.2.1 and the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathrm{C}_0-\mathrm{str}}(\mathrm{Ker}\, d_0, L') & \longrightarrow & \mathrm{Hom}_{\mathrm{C}_0-\mathrm{str}}(\mathrm{Ker}\, d_0, L) & \longrightarrow & \mathrm{Hom}_{\mathrm{C}_0-\mathrm{str}}(\mathrm{Ker}\, d_0, L'') \\ & \uparrow & & \uparrow & & \uparrow \\ 0 \longrightarrow & \mathrm{Der}(\mathrm{C}_0, L') & \longrightarrow & \mathrm{Der}(\mathrm{C}_0, L) & \xrightarrow[\tau]{} & \mathrm{Der}(\mathrm{C}_0, L'') \end{array}$$

with exact rows, where the vertical homomorphisms are induced by d. Note that τ is surjective if (3.3.1) is split. □

Let **C** be a category of groups with operations, and $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ be an internal category in **C**. We have the split exact sequence

$$0 \longrightarrow \mathrm{Ker}\, d_0 \longrightarrow \mathrm{C}_1 \underset{i}{\overset{d_0}{\rightleftarrows}} \mathrm{C}_0 \longrightarrow 0, \tag{3.3.2}$$

where $d_0 i = 1$. Consider the group with operations $\mathrm{C}_0 \ltimes \mathrm{Ker}\, d_0$. Recall that (see Sec. 1.1)

$$(r',c') + (r,c) = (r'+r, c'+r'\cdot c),$$
$$(r',c') * (r,c) = (r' * r, c' * c + c' * r + r' * c),$$

where $r \cdot c = i(r) + c - i(r)$, $r * c = i(r) * c$, $* \in \Omega_2'$ and $\mathrm{C}_1 \approx \mathrm{C}_0 \ltimes \mathrm{Ker}\, d_0$;

We can define a derivation $\delta : \mathrm{C}_0 \longrightarrow L$, for $L \in \mathrm{Coker}\, d$-mod, as a map satisfying the conditions of Definition 3.2.1 (L has a C_0-module structure due to the natural homomorphism $\mathrm{C}_0 \longrightarrow \mathrm{Coker}\, d$).

As we have mentioned above, the split extension (3.3.2) induces a split C_0-structure on $\mathrm{Ker}\, d_0$, which by Lemma 3.1.7 induces a C_0-module structure on $S(\mathrm{Ker}\, d_0)$. Also $\mathrm{Im}\, d$ acts trivially on $S(\mathrm{Ker}\, d_0)$, and we can define $\mathrm{Coker}\, d$-module structure on $S(\mathrm{Ker}\, d_0)$ by

$$\mathrm{cl}\, r \cdot \mathrm{cl}\, c = \mathrm{cl}(i(r) + c - i(r)) \quad \text{and} \quad \mathrm{cl}\, r * \mathrm{cl}\, c = \mathrm{cl}(r * c),$$

for each $r \in \mathrm{C}_0$, $c \in \mathrm{Ker}\, d_0$ and $* \in \Omega_2'$.

We have a split C_0-structure on $\mathrm{Im}\, d$: $r \cdot dc = r + dc - r$, $r * dc = r * dc$, which also induces a $\mathrm{Coker}\, d$-module structure on $S(\mathrm{Im}\, d)$:

$$\mathrm{cl}\, r \cdot \mathrm{cl}\, dc = \mathrm{cl}(r + dc - r),$$
$$\mathrm{cl}\, r * \mathrm{cl}\, dc = \mathrm{cl}(r * dc),$$

for each $r \in \mathrm{C}_0$, $c \in \mathrm{Ker}\, d_0$ and $* \in \Omega_2'$. By Lemma 3.1.7 the natural homomorphisms $\tau_0 : \mathrm{Ker}\, d_0 \longrightarrow S(\mathrm{Ker}\, d_0)$ and $\tau : \mathrm{Im}\, d \longrightarrow S(\mathrm{Im}\, d)$ are C_0-structure homomorphisms. The homomorphism d is a split C_0-structure homomorphism:

$$d(r \cdot c) = r + dc - r = r \cdot dc,$$
$$d(r * c) = r * dc.$$

It induces a $\mathrm{Coker}\, d$-module homomorphism $S(\mathrm{Ker}\, d) \xrightarrow{S(\overline{d})} S(\mathrm{Im}\, d)$ where $\overline{d}$ is defined from the decomposition of

$$d : \mathrm{Ker}\, d_0 \xrightarrow{\overline{d}} \mathrm{Im}\, d \xrightarrow{i_1} \mathrm{C}_0 \,.$$

Consider the extension

$$0 \longrightarrow \operatorname{Im} d \xrightarrow{i_1} \mathrm{C}_0 \xrightarrow{p_1} \operatorname{Coker} d \longrightarrow 0$$

for the internal category C in $\mathbf{C}$. Let $\{f, (g_*)_{*\in\Omega_2'}\}$ be one of the families of its factor systems. We have $\mathrm{C}_0 \approx \operatorname{Coker} d \ltimes_{\{f,(g_*)\}} \operatorname{Im} d$. As we have seen in the proof of Proposition 3.2.3, for an abelian group (in $\mathbf{C}^{\mathrm{C}}$) $A = (A_0, \pi, e, \eta, \mu)$, a derivation $\delta : \mathrm{C}_0 \longrightarrow \operatorname{Ker}\pi$, and each element $(\mathrm{cl}, dc) \in \operatorname{Coker} d \ltimes_{\{f,(g_*)\}} \operatorname{Im} d$, we have

$$\delta(\mathrm{cl}\, r, dc) = \delta(0, dc) + \delta(\mathrm{cl}\, r, 0) = \delta i_1(dc) + \delta(\mathrm{cl}\, r, 0).$$

Denote $\delta i_1(dc) = \tau(dc)$, $\delta(\mathrm{cl}\, r, 0) = \psi(\mathrm{cl}\, r)$. It is easy to check that $\tau : \operatorname{Im} d \longrightarrow \operatorname{Ker}\pi$ is a $\operatorname{Coker} d$-structure homomorphism ($\operatorname{Im} d$ has a $\operatorname{Coker} d$-structure from the above extension and C_0-structure is due to p_1), and $\psi : \operatorname{Coker} d \longrightarrow \operatorname{Ker}\pi$ is a function satisfying the conditions

$$\tau f(\mathrm{cl}\, r, \mathrm{cl}\, r') = \mathrm{cl}\, r \cdot \psi(\mathrm{cl}\, r') - \psi(\mathrm{cl}\, r + \mathrm{cl}\, r') + \psi(\mathrm{cl}\, r), \tag{3.3.3}$$

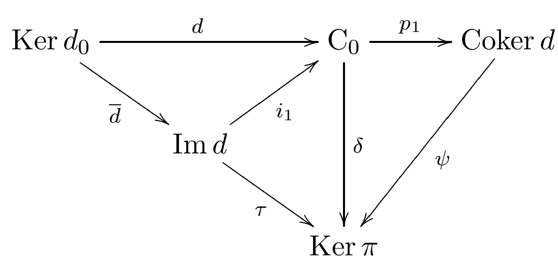
$$\tau g_*(\mathrm{cl}\, r, \mathrm{cl}\, r') = \mathrm{cl}\, r * \psi(\mathrm{cl}\, r') - \psi(\mathrm{cl}\, r * \mathrm{cl}\, r') + \psi(\mathrm{cl}\, r) * \mathrm{cl}\, r'. \tag{3.3.4}$$

From this and from the proof of Proposition 3.2.3 we conclude that there is a one-to-one correspondence: $\{\delta \mid \delta \in \operatorname{Der}(\mathrm{C}_0 \operatorname{Ker}\pi)\} \longleftrightarrow \{(\tau, \psi) \mid \tau : \operatorname{Im} d \longrightarrow \operatorname{Ker}\pi$ is a C_0-structure homomorphism and $\psi : \operatorname{Coker} d \longrightarrow \operatorname{Ker}\pi$ is a function satisfying conditions (3.3.3), (3.3.4)$\}$.

Note that since $\operatorname{Im} d$ has C_0-structure due to p_1,

$$\operatorname{Hom}_{\mathrm{C}_0\text{-str}}(\operatorname{Im} d, \operatorname{Ker}\pi) \approx \operatorname{Hom}_{\operatorname{Coker} d\text{-str}}(\operatorname{Im} d, \operatorname{Ker}\pi).$$

From the condition $\delta i_1 = 0$ we have that $\tau = 0$ and ψ is a derivation. The picture is the following:



Thus from Theorem 2.2.1 (i) we obtain

$$\begin{aligned} \mathrm{H}^0(\mathrm{C}, A) &= \operatorname{Der}(\operatorname{Coker} d, \operatorname{Ker}\pi) \\ &\approx \operatorname{Hom}_{\operatorname{Coker} d\text{-mod}}(\mathbf{I}(\operatorname{Coker} d), \operatorname{Ker}\pi). \end{aligned} \tag{3.3.5}$$

Note that actually we proved the exactness of the sequence

$$0 \longrightarrow \operatorname{Der}(\operatorname{Coker} d, \operatorname{Ker}\pi) \longrightarrow \operatorname{Der}(\mathrm{C}_0, \operatorname{Ker}\pi) \longrightarrow \operatorname{Der}(\operatorname{Im} d, \operatorname{Ker}\pi)$$

for any $L \in \mathrm{C}_0$-mod, such that $\operatorname{Im} d$ acts trivially on L; this could be done directly, but we will use the argument in extensions terminology below for H^1 too.

By Proposition 3.2.3 and Proposition 3.1.6 we have

$$\{\delta d \mid \delta \in \operatorname{Der}(\mathrm{C}_0, \operatorname{Ker}\pi)\} \approx \{\tau \in \operatorname{Hom}_{\operatorname{Coker} d\text{-str}}(\operatorname{Im} d, \operatorname{Ker}\pi) \mid$$

an extension of $\operatorname{Coker} d$ by $\operatorname{Ker}\pi$ corresponding to the factor systems $\{\tau f, (\tau g_*)_{*\in\Omega_2'}\}$ is split$\}$.

If d is a surjective homomorphism, then internal category C is connected (i.e., for each two objects r and r' there is a morphism $r \longrightarrow r'$). In this case $\operatorname{Coker} d = 0$, and we have $f = g_* = 0$ for each $* \in \Omega_2'$. Thus we obtain

$$\big\{\delta d \mid \delta \in \operatorname{Der}(\mathrm{C}_0, \operatorname{Ker}\pi)\big\} \approx \operatorname{Hom}_{\mathrm{C}_0\text{-str}}(\mathrm{C}_0, \operatorname{Ker}\pi).$$

From Theorem 2.2.1 (i) for a connected internal category C in the category of interest $\mathbf{C}$, we have the following isomorphisms:

$$\begin{aligned}\mathrm{H}^1(\mathrm{C}, A) &\approx \mathrm{Hom}_{\mathrm{C}_0\text{-str}}(\mathrm{Ker}\, d_0, \mathrm{Ker}\, \pi)/\, \mathrm{Hom}_{\mathrm{C}_0\text{-str}}(\mathrm{C}_0, \mathrm{Ker}\, \pi)\\ &\approx \mathrm{Coker}\, \mathrm{Hom}_{\mathrm{C}_0\text{-str}}(d, \mathrm{Ker}\, \pi)\\ &\approx \mathrm{Coker}\, \mathrm{Hom}_{\mathrm{C}_0\text{-mod}}(S(d), \mathrm{Ker}\, \pi).\end{aligned} \tag{3.3.6}$$

Let C and C′ be internal categories in $\mathbf{C}$;

$$\mathrm{C}_* : 0 \longrightarrow \mathrm{Ker}\, d_0 \xrightarrow{d} \mathrm{C}_0 \longrightarrow 0 \quad \text{and} \quad \mathrm{C}'_* : 0 \longrightarrow \mathrm{Ker}\, d_0 \xrightarrow{d'} \mathrm{C}'_0 \longrightarrow 0$$

be the corresponding crossed modules considered as complexes. Denote by $\mathrm{H}^i(\mathrm{C}_*)$ and $\mathrm{H}^i(\mathrm{C}, A)$, $i = 0, 1$, respectively the homology of the complex C_* and the cohomology of the internal category C with the coefficient in A, $A \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}})$. Note that in the case $\mathbf{C} = Ab$, $\mathrm{H}^i(\mathrm{C}, A) = \mathrm{H}^i(\mathrm{C}_*, A)$, where the right side denotes the cohomology $\mathrm{H}^i(\mathrm{Hom}(\mathrm{C}_*, A))$ of the complex C_* with coefficient in A.

Definition 3.3.2. We shall say that internal categories C and C′ in $\mathbf{C}$ are homologically equivalent if there are internal functors $\mathrm{C} \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{C}'$ that induce the isomorphisms $H_i(\mathrm{C}_*) \approx H_i(\mathrm{C}'_*)$, $i = 0, 1$.

Definition 3.3.3. We shall say that internal categories C and C′ in $\mathbf{C}$ are cohomologically equivalent if there are internal functors $\mathrm{C} \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{C}'$ that induce isomorphisms $\mathrm{H}^i(\mathrm{C}, A) \approx \mathrm{H}^i(\mathrm{C}', A)$, $\mathrm{H}^i(\mathrm{C}, A') \approx \mathrm{H}^i(\mathrm{C}', A')$, $i = 0, 1$ for each $A \in \mathbb{A}\mathrm{b}(\mathrm{C}^{\mathrm{C}})$ and $A' \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}'})$, where A has C′-module structure due to S and A' has C-module structure due to T.

Theorem 3.3.4. *If internal categories* C *and* C′ *in* $\mathbf{C}$ *are equivalent, then they are homologically and cohomologically equivalent.*

Proof. By Proposition 1.3.6 the equivalence of internal categories

$$\mathrm{C} : \mathrm{Ker}\, d_0 \xrightarrow{d} \mathrm{C} \quad \text{and} \quad \mathrm{C}' : \mathrm{Ker}\, d'_0 \xrightarrow{d'} \mathrm{C}'$$

means that there are internal functors $(T_0, T_1) : \mathrm{C} \longrightarrow \mathrm{C}'$, $(S_0, S_1) : \mathrm{C}' \longrightarrow \mathrm{C}$ and maps $\varphi : \mathrm{C}'_0 \longrightarrow \mathrm{Ker}\, d'_0$, $\psi : \mathrm{C}_0 \longrightarrow \mathrm{Ker}\, d_0$ satisfying conditions (1.3.1) and (1.3.2). To prove that C and C′ are homologically equivalent, we must show that (T_0, T_1) induces the isomorphisms $\mathrm{Ker}\, d \approx \mathrm{Ker}\, d'$ and $\mathrm{Coker}\, d \approx \mathrm{Coker}\, d'$. From (1.3.1) and (1.3.2) we have

$$\begin{aligned}\varphi d'(c') &= c' - T_1 S_1(c'),\\ \psi d(c) &= S_1 T_1(c) - c,\end{aligned}$$

for each $c \in \mathrm{Ker}\, d_0$ and $c' \in \mathrm{Ker}\, d'_0$. Then for each $c \in \mathrm{Ker}\, d$ and $c' \in \mathrm{Ker}\, d'$ we obtain

$$S_1 T_1(c) = c, \quad T_1 S_1(c') = c'.$$

Thus $T_1\big|_{\mathrm{Ker}\, d} \cdot S_1\big|_{\mathrm{Ker}\, d'} = 1_{\mathrm{Ker}\, d}$, and $S_1\big|_{\mathrm{Ker}\, d} \cdot T_1\big|_{\mathrm{Ker}\, d} = 1_{\mathrm{Ker}\, d}$.

We have the following diagram:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \operatorname{Ker} d & \longrightarrow & \operatorname{Ker} d_0 & \xrightarrow{d} & \mathrm{C}_0 & \xrightarrow{\tau} & \operatorname{Coker} d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T_1|_{\operatorname{Ker} d}} & & \downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle \widetilde{T}_0} & & \\
0 & \longrightarrow & \operatorname{Ker} d' & \longrightarrow & \operatorname{Ker} d'_0 & \xrightarrow{d'} & \mathrm{C}'_0 & \xrightarrow{\tau'} & \operatorname{Coker}' d' & \longrightarrow & 0\ , \\
 & & \downarrow{\scriptstyle S_1|_{\operatorname{Ker} d'}} & & \downarrow{\scriptstyle S_1} & & \downarrow{\scriptstyle S_0} & & \downarrow{\scriptstyle \widetilde{S}_0} & & \\
0 & \longrightarrow & \operatorname{Ker} d & \longrightarrow & \operatorname{Ker} d_0 & \xrightarrow[d]{} & \mathrm{C}_0 & \xrightarrow[\tau]{} & \operatorname{Coker} d & \longrightarrow & 0
\end{array}
\tag{3.3.7}
$$

where τ and τ' denote the natural morphisms, and $\widetilde{T}_0$, $\widetilde{S}_0$ are induced by T_0 and S_0, respectively.

We shall now show that $\operatorname{Coker} d \approx \operatorname{Coker} d'$. Again from (1.3.1) we have $d'\varphi(r') = r' - T_0S_0(r')$, from which we obtain $\tau' d'\varphi(r') = \tau(r') - \tau T_0 S_0(r')$. Since $\tau' d'\varphi(r') = 0$, we conclude that $\tau' T_0 S_0(r') = \tau'(r')$ and $\widetilde{T}_0\widetilde{S}_0 = 1_{\operatorname{Coker} d'}$. Similarly, from $d\psi(r) = S_0T_0(r) - r$ we obtain that $\widetilde{S}_0\widetilde{T}_0 = 1_{\operatorname{Coker} d}$.

To prove that the internal category equivalence implies cohomological equivalence, consider C-module A, i.e., $A \in \operatorname{Coker} d$-mod. A can be considered as a $\operatorname{Coker} d'$-module due to $\widetilde{S}_0$, because $\widetilde{S}_0$ is a homomorphism in $\mathbf{C}$.

From (3.3.5) we have ($\operatorname{Ker} \pi$ is denoted by A):

$$
\begin{aligned}
\mathrm{H}^0(\mathrm{C}, A) &= \big\{\delta \in \operatorname{Der}(\mathrm{C}_0, A) \mid \ \delta d = 0\big\} = \operatorname{Der}(\operatorname{Coker} d, A), \\
\mathrm{H}^0(\mathrm{C}', A) &= \big\{\delta' \in \operatorname{Der}(\mathrm{C}'_0, A) \mid \ \delta' d' = 0\big\} = \operatorname{Der}(\operatorname{Coker} d', A).
\end{aligned}
$$

As we have proved above, $\operatorname{Coker} d \approx \operatorname{Coker} d'$; analogously, as will be shown below, for T_0, S_0, we have $\delta'\widetilde{T}_0 \in \operatorname{Der}(\operatorname{Coker} d, A)$ and $\delta\widetilde{S}_0 \in \operatorname{Der}(\operatorname{Coker} d', A)$, from which it follows that $\mathrm{H}^0(\mathrm{C}, A) \approx \mathrm{H}^0(\mathrm{C}', A)$.

To prove that $\mathrm{H}^1(\mathrm{C}, A) \approx \mathrm{H}^1(\mathrm{C}', A)$, first we shall show that if $(T_0, T_1) : \mathrm{C} \longrightarrow \mathrm{C}'$ is an equivalence of internal categories, then T induces a homomorphism of abelian groups $\operatorname{Der}(T_0, A) : \operatorname{Der}(\mathrm{C}'_0, A) \longrightarrow \operatorname{Der}(\mathrm{C}_0, A)$ defined by $\delta' \longmapsto \delta' T_0$. We must show that $\delta' T_0 \in \operatorname{Der}(\mathrm{C}_0, A)$. For this we shall prove that $\delta' T_0$ satisfies conditions of Definition 3.2.1.

1. $\delta' T_0(\omega(r)) = \omega(\delta' T_0(r))$;

2. To prove the second condition, recall that $r' \cdot a = S_0(r') \cdot a$ for each $r' \in \mathrm{C}'_0$ and $a \in A$. We have

$$
\delta' T_0(r_1 + r_2) = \delta'(T_0(r_1) + T_0(r_2)) = \delta' T_0(r_1) + T_0(r_1) \cdot \delta' T_0(r_2) = \delta' T_0(r_1) + S_0 T_0(r_1) \cdot \delta' T_0(r_2).
$$

By (1.3.2) $S_0T_0(r) = d\psi(r) + r$ and from Sec. 2.2 we know that $d(c) \cdot a = a$ for each $c \in \operatorname{Ker} d_0$ and $a \in A$. Thus $\delta' T_0(r_1 + r_2) = \delta' T_0(r_1) + r_1 \cdot \delta' T_0(r_2)$.

3. Since $d(c) * a = 0$ (see Sec. 2.2) and hence $S_0T_0(r) * a = (d\psi(r) + r) * a$, for each $r \in \mathrm{C}_0$, $a \in A$ we have

$$
\begin{aligned}
\delta' T_0(r_1 * r_2) &= \delta'\big(T_0(r_1) * T_0(r_2)\big) \\
&= (\delta' T_0(r_1)) * T_0(r_2) + T_0(r_1) * \delta' T_0(r_2) = (\delta' T_0)(r_1)) * S_0 T_0(r_2) + T_0(r_1) * \delta' T_0(r_2) \\
&= \delta' T_0(r_1) * S_0 T_0(r_2) + S_0 T_0(r_1) * \delta' T_0(r_2) = \delta' T_0(r_1) * r_2 + r_1 * \delta' T_0(r_2).
\end{aligned}
$$

Similarly, S_0 induces the homomorphism $\operatorname{Der}(\mathrm{C}_0, A) \longrightarrow \operatorname{Der}(\mathrm{C}'_0, A)$. Now we shall show that T induces the homomorphism of abelian groups

$$
\operatorname{Hom}(T_1, A) : \operatorname{Hom}_{\mathrm{C}'_0 - str}(\operatorname{Ker} d', A) \longrightarrow \operatorname{Hom}_{\mathrm{C}_0 - str}(\operatorname{Ker} d, A)
$$

defined by

$$
\alpha' \longrightarrow \alpha' T_1.
$$

Here

$$\mathrm{Hom}_{\mathrm{C}_0-str}(\mathrm{Ker}\, d_0, A) = \Big\{\alpha \in \mathbf{C}(\mathrm{Ker}\, d_0, A) \;\Big|\; \alpha(r\cdot c) = r\cdot\alpha(c), \alpha(r * c) = r * \alpha(c) \text{ for each binary operation } * \text{ in } \mathbf{C} \text{ except the addition, for each } r \in \mathrm{C}_0,\ c \in \mathrm{Ker}\, d_0\Big\}.$$

The following equalities show that $\alpha' T_1$ is a C_0-structure homomorphism:

$$\begin{aligned}\alpha' T_1(r\cdot c) &= \alpha'(T_0(r)\cdot T_1(c)) = T_0(r)\cdot\alpha' T_1(c) = S_0T_0(r)\cdot\alpha' T_1(c)\\ &= (d\psi(r)+r)\cdot\alpha' T_1(c) = d\psi(r)\cdot(r\cdot\alpha' T_1(c)) = r\cdot\alpha' T_1(c);\\ \alpha' T_1(r * c) &= \alpha'(T_0(r) * T_1(c)) = T_0(r) * \alpha' T_1(c) = S_0T_0(r) * \alpha' T_1(c)\\ &= (d\psi(r)+r) * \alpha' T_1(c) = r * \alpha' T_1(c).\end{aligned}$$

Similarly, S_1 induces the homomorphism

$$\mathrm{Hom}(S_1, A) : \mathrm{Hom}_{\mathrm{C}_0-str}(\mathrm{Ker}\, d_0, A) \longrightarrow \mathrm{Hom}_{\mathrm{C}'_0-str}(\mathrm{Ker}\, d'_0, A).$$

By the definition of the cohomologies of internal categories, we have the commutative diagram

$$\begin{array}{ccccc}\mathrm{Der}(\mathrm{C}'_0, A) & \xrightarrow{\widetilde{d'}} & \mathrm{Hom}_{\mathrm{C}'_0-\mathrm{str}}(\mathrm{Ker}\, d'_0, A) & \twoheadrightarrow & \mathrm{H}^1(\mathrm{C}', A)\\ \downarrow{\scriptstyle \mathrm{Der}(T_0,A)} & & \downarrow{\scriptstyle \mathrm{Hom}(T_1,A)} & & \downarrow{\scriptstyle \mathrm{H}^1(T,A)}\\ \mathrm{Der}(\mathrm{C}_0, A) & \xrightarrow{\widetilde{d}} & \mathrm{Hom}_{\mathrm{C}_0-\mathrm{str}}(\mathrm{Ker}\, d_0, A) & \twoheadrightarrow & \mathrm{H}^1(\mathrm{C}, A)\\ \downarrow{\scriptstyle \mathrm{Der}(S_0,A)} & & \downarrow{\scriptstyle \mathrm{Hom}(S_1,A)} & & \downarrow{\scriptstyle \mathrm{H}^1(S,A)}\\ \mathrm{Der}(\mathrm{C}'_0, A) & \xrightarrow{\widetilde{d'}} & \mathrm{Hom}_{\mathrm{C}'_0-\mathrm{str}}(\mathrm{Ker}\, d'_0, A) & \twoheadrightarrow & \mathrm{H}^1(\mathrm{C}', A)\end{array} \quad . \tag{3.3.8}$$

Here the homomorphisms $\mathrm{H}^1(T, A)$ and $\mathrm{H}^1(S, A)$ are induced by $\mathrm{Hom}(T_1, A)$ and $\mathrm{Hom}(S_1, A)$, respectively, and $\widetilde{d}$ denotes the homomorphism defined by $\widetilde{d}(\delta) = \delta d$. From the commutativity of (3.3.8) for each $\mathrm{cl}\,\alpha' \in \mathrm{H}^1(\mathrm{C}', A)$, $\alpha' \in \mathrm{Hom}_{\mathrm{C}'_0-str}(\mathrm{Ker}\, d'_0, A)$ we obtain

$$\mathrm{H}^1(TS, A)(\mathrm{cl}\,\alpha') = \mathrm{cl}(\alpha' T_1 S_1).$$

From ($1.3.1_2$), we have $\alpha' T_1 S_1 = \alpha'(-\varphi d' + 1) = -\alpha'\varphi d' + \alpha'$. But $\alpha'\varphi d' \in \mathrm{Der}(\mathrm{C}'_0, A)$:

$$\alpha'\varphi(\omega(r')) = \omega(\alpha'\varphi(r'));$$

$$\begin{aligned}\alpha'\varphi(r'_1 + r'_2) &= \alpha'(\varphi(r'_1) + T_0S_0(r'_1)\cdot\varphi(r'_2)) = \alpha'\varphi(r'_1) + T_0S_0(r'_1)\cdot\alpha'\varphi(r'_2)\\ &= \alpha'\varphi(r'_1) + (-d'\varphi(r'_1) + r'_1)\cdot\alpha'\varphi(r'_2) = \alpha'\varphi(r'_1) + r'_1\cdot\alpha'\varphi(r'_2).\end{aligned}$$

The corresponding condition for the binary operation $*$ of $\mathbf{C}$ is proved analogously. Thus, $\alpha'\varphi d'$ is a derivation for each α' and $\mathrm{cl}(\alpha' T_1 S_1) = \mathrm{cl}\,\alpha'$. In the same way we prove that $\mathrm{H}^1(ST, A) = 1$ and so $\mathrm{H}^1(\mathrm{C}, A) \approx \mathrm{H}^1(\mathrm{C}', A)$ for each $A \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}})$. Similarly, we can prove that $\mathrm{H}^1(\mathrm{C}, A') \approx \mathrm{H}^1(\mathrm{C}', A')$ for each $A' \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}'})$. This completes the proof. □

Consider the following conditions on internal categories C and C′:

(i) C and C′ are homologically equivalent;

(ii) C and C′ are equivalent;

(iii) C and C′ are cohomologically equivalent.

Proposition 3.3.5. *Let* C *and* C′ *be internal categories in the category of abelian groups. Then we have the following implications* (i) $\Longleftarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

Proof. By Theorem 3.3.4 it suffices to show that (iii) $\Longrightarrow$ (i). From (3.3.5) we know that

$$\begin{aligned} \mathrm{H}^0(\mathrm{C}, A) &= \mathrm{Der}(\mathrm{Coker}\, d, A) \approx \mathrm{Hom}_{\mathrm{Coker}\, d\text{-mod}}(\mathbf{I}(\mathrm{Coker}\, d), A), \\ \mathrm{H}^0(\mathrm{C}', A') &= \mathrm{Hom}_{\mathrm{Coker}\, d'\text{-mod}}(\mathbf{I}(\mathrm{Coker}\, d'), A'), \end{aligned}$$

for each $A \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}})$ and $A' \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}'})$. In the case of abelian groups ($\mathbf{C} = \mathbb{A}\mathrm{b}$) we have

$$\begin{aligned} \mathrm{H}^0(\mathrm{C}, A) &= \mathrm{Hom}(\mathrm{Coker}\, d, A), \\ \mathrm{H}^0(\mathrm{C}', A) &= \mathrm{Hom}(\mathrm{Coker}\, d', A), \end{aligned}$$

for each abelian group A. Here and below Hom denotes $\mathrm{Hom}_{\mathbb{A}\mathrm{b}}$. By (iii) we have $\mathrm{Hom}(\mathrm{Coker}\, d, A) \approx \mathrm{Hom}(\mathrm{Coker}\, d', A)$ for any A, which implies an isomorphism $\mathrm{Coker}\, d \approx \mathrm{Coker}\, d'$.

To show the isomorphism $\mathrm{Ker}\, d \approx \mathrm{Ker}\, d'$, apply the functor $\mathrm{Hom}(-, A)$ to the diagram (3.3.7); we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Hom}(\mathrm{Coker}\, d, A) & \longrightarrow & \mathrm{Hom}(\mathrm{C}_0, A) & \xrightarrow{\mathrm{Hom}(d,A)} & \mathrm{Hom}(\mathrm{Ker}\, d_0, A) & \longrightarrow & \mathrm{H}^1(\mathrm{C}, A) & \longrightarrow 0 \\ & \uparrow \approx & & \uparrow {\scriptstyle \mathrm{Hom}(d',A)} & & \uparrow {\scriptstyle \mathrm{Hom}(T_1,A)} & & \uparrow {\scriptstyle \mathrm{H}^1(T,A)} & \\ 0 \longrightarrow & \mathrm{Hom}(\mathrm{Coker}\, d', A) & \longrightarrow & \mathrm{Hom}(\mathrm{C}'_0, A) & \xrightarrow{\mathrm{Hom}(d',A)} & \mathrm{Hom}(\mathrm{Ker}\, d'_0, A) & \longrightarrow & \mathrm{H}^1(\mathrm{C}', A) & \longrightarrow 0 \\ & \uparrow \approx & & \uparrow {\scriptstyle \mathrm{Hom}(d,A)} & & \uparrow {\scriptstyle \mathrm{Hom}(S_1,A)} & & \uparrow {\scriptstyle \mathrm{H}^1(S,A)} & \\ 0 \longrightarrow & \mathrm{Hom}(\mathrm{Coker}\, d, A) & \longrightarrow & \mathrm{Hom}(\mathrm{C}_0, A) & \xrightarrow{\mathrm{Hom}(d,A)} & \mathrm{Hom}(\mathrm{Ker}\, d_0, A) & \longrightarrow & \mathrm{H}^1(\mathrm{C}, A) & \longrightarrow 0 \end{array} \cdot \tag{3.3.9}$$

Take $A = \mathrm{Ker}\, d_0$; from the right-hand side of the diagram (3.3.9) for elements, we have

$$\begin{array}{ccc} S_1T_1 & \longmapsto & \mathrm{cl}(S_1T_1) \\ \uparrow & & \uparrow \\ 1_{\mathrm{Ker}\, d_0} & \longmapsto & \mathrm{cl}\ 1_{\mathrm{Ker}\, d_0} \end{array}$$

By (iii), $\mathrm{H}^1(ST, \mathrm{Ker}\, d_0)(\mathrm{cl}\, 1_{\mathrm{Ker}\, d_0}) = \mathrm{cl}\, 1_{\mathrm{Ker}\, d_0}$; thus $\mathrm{cl}(S_1T_1) = \mathrm{cl}\, 1_{\mathrm{Ker}\, d_0}$. From this it follows that there is an element $\alpha \in \mathrm{Hom}(\mathrm{C}_0, \mathrm{Ker}\, d_0)$ such that $\alpha d = S_1T_1 - 1_{\mathrm{Ker}\, d_0}$. Thus, if $c \in \mathrm{Ker}\, d$, then $c - S_1T_1(c) = 0$ and $c = S_1T_1(c)$. In the same way we show that $c' = T_1S_1(c')$, for each $c' \in \mathrm{Ker}\, d'$, which proves that $\mathrm{Ker}\, d \approx \mathrm{Ker}\, d'$. $\square$

Proposition 3.3.6. *If* C *and* C′ *are internal categories in the category of vector spaces* Vect_k *over a field* k*, then* (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii).

Proof. It suffices to show that (i) $\Longrightarrow$ (ii). Consider the exact sequence

$$0 \longrightarrow \mathrm{Im}\, d \longrightarrow \mathrm{C}_0 \longrightarrow \mathrm{Coker}\, d \longrightarrow 0,$$

which is split as we are in the category of vector spaces. Thus $\mathrm{C}_0 \approx \mathrm{Coker}\, d \times \mathrm{Im}\, d$, and each element $r \in \mathrm{C}_0$ can be viewed as a pair $r = (clr, dc)$ for some $c \in \mathrm{Ker}\, d_0$. Define the map $\psi : \mathrm{C}_0 \longrightarrow \mathrm{Ker}\, d_0$ by

$$\psi(\mathrm{cl}\, r, dc) = S_1T_1(c) - c.$$

We must show that ψ is defined correctly and that it does not depend on the choice of c. Let $dc = dc_1$, then $c_1 - c \in \mathrm{Ker}\, d$. We have

$$\begin{aligned} S_1T_1(c) - c - (S_1T_1(c_1) - c_1) &= S_1T_1(c) - S_1T_1(c_1) - c + c_1 \\ &= S_1T_1(c - c_1) - c + c_1 = c - c_1 - c + c_1 = 0, \end{aligned}$$

as $S_1T_1\big|_{\mathrm{Ker}\, d}$ is an identity morphism by (i).

Now we shall prove that ψ is a homomorphism of vector spaces, $d\psi(r) = S_0T_0(r) - r$ and $\psi d(c) = S_1T_1(c) - c$.

$$\begin{aligned}\psi(kr) &= \psi(k\,\mathrm{cl}\, r, kdc) = \psi(\mathrm{cl}\, kr, dkc) = S_1T_1(kc) = kS_1T_1(c) = k\psi(c) \quad \text{for each} \quad k \in K,\\ \psi(r + r_1) &= \psi((\mathrm{cl}\, r, dc) + (\mathrm{cl}\, r_1, dc_1)) = \psi(\mathrm{cl}(r + r_1), d(c + c_1))\\ &= S_1T_1(c + c_1) - c - c_1 = S_1T_1(c) + S_1T_1(c_1) - c - c_1;\\ \psi(r) + \psi(r_1) &= \psi(\mathrm{cl}\, r, dc) + \psi(\mathrm{cl}\, r_1, dc_1) = S_1T_1(c) - c + S_1T_1(c_1) - c_1\\ &= S_1T_1(c) + S_1T_1(c_1) - c - c_1.\end{aligned}$$

From the definition of ψ we obtain

$$d\psi(\mathrm{cl}\, r, dc) = d(0, S_1T_1(c) - c) = (0, dS_1T_1(c) - dc) = (0, S_0T_0d(c) - d(c)). \tag{3.3.10}$$

On the other hand, as $\widetilde{S}\widetilde{T} = 1_{\mathrm{Coker}\, d}$ by (i), we have (see diagram (3.3.7)):

$$\begin{aligned}S_0T_0(\mathrm{cl}\, r, dc) - (\mathrm{cl}\, r, dc) &= \big(\widetilde{S}_0\widetilde{T}_0(\mathrm{cl}\, r), S_0T_0d(c)\big) - (\mathrm{cl}\, r, dc)\\ &= (\mathrm{cl}\, r, S_0T_0d(c)) - (\mathrm{cl}\, r, dc) = (0, S_0T_0d(c) - dc). \quad (3.3.11)\end{aligned}$$

(3.3.10) and (3.3.11) prove that $d\psi(r) = S_0T_0(r) - r$,

Again by definition we have

$$\psi d(c) = \psi(0, dc) = S_1T_1(c) - c$$

which completes the proof of Proposition 3.3.6. □

In [42] G. J. Ellis, following [9], regards the (co)homology of the classifying space $B(M \to G)$ of a crossed module $\partial : M \longrightarrow G$ in groups as the (co)homology of the crossed module $\partial : M \to G$.

Let $\partial : M \longrightarrow G$ be a crossed module in $\mathbb{G}\mathrm{r}$ and A be a $\mathrm{Coker}\,\partial$-module. Suppose $\partial' : \mathcal{B} \longrightarrow F$ is a crossed module with F a free group, and crossed modules ∂ and ∂' are weakly equivalent in the sense of [42], which means that there is a morphism between these crossed modules, which induces isomorphisms $\mathrm{Ker}\,\partial \approx \mathrm{Ker}\,\partial'$, $\mathrm{Coker}\,\partial \approx \mathrm{Coker}\,\partial'$. Denote by $\mathrm{H}^2(\partial : M \longrightarrow G, A)$ the cohomology defined in [42]. Theorem 6 of [42] states that there is an isomorphism

$$\mathrm{H}^2(\partial : M \longrightarrow G, A) \approx \mathrm{Coker}\,\big(\mathrm{Der}(F, A) \xrightarrow{\partial^*} \mathrm{Hom}_{F-\mathrm{str}}(\mathcal{B}, A)\big).$$

Applying this result, from Theorem 2.2.1 we obtain the following assertion.

Proposition 3.3.7. *There is an isomorphism*

$$\mathrm{H}^2(\partial : M \longrightarrow G, A) \approx \mathrm{H}^1(\mathrm{C}, \bar{A}),$$

where C *denotes the internal category in* $\mathbb{G}\mathrm{r}$ *associated to* $\partial' : \mathcal{B} \longrightarrow F$, *and* $\bar{A}$ *is an object of* $\mathbb{A}\mathrm{b}(\mathbb{G}\mathrm{r}^{\mathrm{C}})$ *associated with* $\mathrm{Coker}\,\partial$*-mod* A.

3.4. Relations $\mathrm{H}^0(\mathrm{C}, -) = 0$, $\mathrm{H}^1(\mathrm{C}, -) = 0$

Theorem 3.4.1. *Let* $\mathbf{C}$ *be a category of groups with operations, and* $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ *be an internal category in* $\mathbf{C}$. $\mathrm{H}^0(\mathrm{C}, A) = 0$ *for each* $A \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}})$ *if and only if* $\mathbf{I}(\mathrm{Coker}\, d) = 0$.

Proof. From (3.3.5) we obtain

$$\mathrm{H}^0(\mathrm{C}, -) = 0 \Longleftrightarrow \mathrm{Hom}_{\mathrm{Coker}\, d\text{-mod}}(\mathbf{I}(\mathrm{Coker}\, d), -) = 0 \Longleftrightarrow \mathbf{I}(\mathrm{Coker}\, d) = 0. \quad □$$

Corollary 3.4.2. *Let* C *be an internal category in the category of groups*; *then* $\mathrm{H}^0(\mathrm{C}, -) = 0$ *if and only if* C *is a connected category.*

Proof. In the category of groups $\mathbf{I}(\mathrm{Coker}\, d) = 0 \Longleftrightarrow \mathrm{Coker}\, d = 0 \Longleftrightarrow d$ is a surjective homomorphism $\Longleftrightarrow$ C is a connected internal category. □

Consider the case $\mathrm{H}^1(\mathrm{C}, -) = 0$. If an internal category C in the category of interest $\mathbf{C}$ is connected, then d is surjective and from (3.3.6) we obtain

$$\mathrm{H}^1(\mathrm{C}, -) = 0 \Longleftrightarrow \operatorname{Coker} \operatorname{Hom}_{\mathrm{C}_0 - \mathrm{mod}}(S(d), -) = 0 \Longleftrightarrow \operatorname{Hom}_{\mathrm{C}_0 - \mathrm{mod}}(S(d), -)$$
$$\text{is an epimorphism} \Longleftrightarrow S(d) \text{ is a split monomorphism} \Longleftrightarrow S(d) \text{ is an isomorphism.}$$

Let $\mathbf{C}$ be a category of interest and $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ be an internal category in $\mathbf{C}$. Denote by $S(\mathrm{C})$ an internal category represented as a crossed module in the following form:

$$S(\operatorname{Ker} d_0) \xrightarrow{S(\overline{d})} \operatorname{Coker} d \ltimes_{\{\tau f, (\tau g_*)\}} S(\operatorname{Im} d),$$

where $d = d_1\big|_{\operatorname{Ker} d_0}$, $\{f, (g_*)_{* \in \Omega_2'}\}$ is one of the families of factor systems of the extension

$$0 \longrightarrow \operatorname{Im} d \xrightarrow{i_1} \mathrm{C}_0 \xrightarrow{p_1} \operatorname{Coker} d \longrightarrow 0,$$

$\overline{d} : \operatorname{Ker} d_0 \longrightarrow \operatorname{Im} d$ is defined by d ($d = i_1 \overline{d}$), and $\tau : \operatorname{Im} d \longrightarrow S(\operatorname{Im} d)$ is a natural homomorphism.

For ordinary internal categories we have the following theorem.

Theorem 3.4.3. *Let* $\mathbf{C}$ *be a category of interest and* $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ *be an internal category in* $\mathbf{C}$, $d = d_1\big|_{\operatorname{Ker} d_0}$, $S : \mathbf{C} \longrightarrow \mathbb{A}\mathrm{b}$ *be an abelianization functor, and* $\tau : \operatorname{Im} d \longrightarrow S(\operatorname{Im} d)$ *be a natural homomorpshism of* Coker d*-structures. The following conditions are equivalent:*

(i) $\mathrm{H}^1(\mathrm{C}, A) = 0$ *for each* $A \in \mathbb{A}\mathrm{b}(\mathbf{C}^{\mathrm{C}})$*;*

(ii) d *induces an isomorphism of* Coker d*-modules* $S(\overline{d}) : S(\operatorname{Ker} d_0) \xrightarrow{\approx} S(\operatorname{Im} d)$ ($\overline{d}$ *is defined by* d) *and the extension of* Coker d *by* $S(\operatorname{Im} d)$ *obtained from*

$$0 \longrightarrow \operatorname{Im} d \xrightarrow{i_1} \mathrm{C}_0 \xrightarrow{p_1} \operatorname{Coker} d \longrightarrow 0$$

by τ *is split*;

(iii) $S(\mathrm{C})$ *is equivalent to the discrete category* $(\operatorname{Coker} d, \operatorname{Coker} d, 1, 1, 1, 1)$.

Proof. (i) $\Longrightarrow$ (ii). Let $\tau_0 : \operatorname{Ker} d_0 \longrightarrow S(\operatorname{Ker} d_0)$ be a natural split C_0-structure homomorpshism; then

$$\tau \overline{d} = S(\overline{d}) \tau_0. \tag{3.4.1}$$

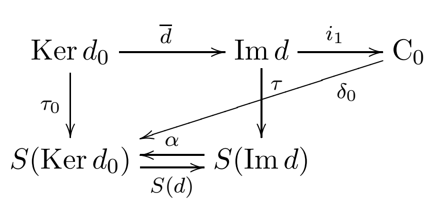
As we know from Sec. 3.3, $S(\operatorname{Ker} d_0)$ is a Coker d-module. So if $\mathrm{H}^1(\mathrm{C}, -) = 0$, then in particular $\mathrm{H}^1(\mathrm{C}, \mathrm{C}_0 \ltimes S(\operatorname{Ker} d_0)) = 0$ and from Theorem 2.12.1 (i) we conclude that for the homomorphism τ_0 there is a derivation $\delta_0 : \mathrm{C}_0 \longrightarrow S(\operatorname{Ker} d_0)$ with $\delta_0 d = \tau_0$; since $d = i_1 \overline{d}$, we have

$$\delta_0 i_1 \overline{d} = \tau_0. \tag{3.4.2}$$

The composite $i_1 \delta_0$ is a C_0-structure homomorphism, so there is a unique Coker d-module homomorphism $\alpha : S(\operatorname{Im} d) \longrightarrow S(\operatorname{Ker} d_0)$, such that

$$\alpha \tau = \delta_0 i_1. \tag{3.4.3}$$

The picture is as follows:



From (3.4.1), (3.4.2) and (3.4.3) we obtain

$$\alpha S(\overline{d}_0)\tau_0 = \alpha\tau\overline{d} = \delta_0 i_1 \overline{d} = \tau_0,$$

from which it follows that $\alpha S(\overline{d}) = 1$; so $S(\overline{d})$ is a monomorphism. But it is also an epimorphism. Thus we obtain that $S(\overline{d})$ is an isomorphism.

Again, since $\mathrm{H}^1(\mathrm{C}, \mathrm{C}_0 \ltimes S(\operatorname{Im} d)) = 0$, for C_0-structure homomorphism $\overline{d}\tau$ there is a derivation $\delta : \mathrm{C}_0 \longrightarrow S(\operatorname{Im} d)$ with $\delta d = \tau\overline{d}$. Since $d = i_1\overline{d}$ and $\overline{d}$ is an epimorphism, we obtain that $\delta i_1 = \tau$. By Proposition 3.2.3 we conclude that an extension of $\operatorname{Coker} d$ by $S(\operatorname{Im} d)$ obtained from the extension $0 \longrightarrow \operatorname{Im} d \xrightarrow{i_1} \mathrm{C}_0 \xrightarrow{p_1} \operatorname{Coker} d \longrightarrow 0$ by τ is split.

(ii) $\Longrightarrow$ (i). Now let $U \in \operatorname{Coker} d$-mod, and $\phi : \operatorname{Ker} d_0 \longrightarrow U$ be an arbitrary C_0-structure homomorphism. We shall show that there is a derivation $\delta_1 : \mathrm{C}_0 \longrightarrow U$ such that the diagram

$$\begin{array}{ccc} \operatorname{Ker} d_0 & \xrightarrow{d} & \mathrm{C}_0 \\ {\scriptstyle\phi}\downarrow & \swarrow {\scriptstyle\delta_1} & \\ U & & \end{array}$$

is commutative. By Proposition 3.2.3 there is a derivation $\delta : \mathrm{C}_0 \longrightarrow S(\operatorname{Im} d)$ with $\delta i_1 = \tau$. $\delta_0 = S(\overline{d})^{-1}\delta$ is also a derivation. We have

$$\delta_0 i_1 \overline{d} = S(\overline{d})^{-1}\delta i_1 \overline{d} = S(\overline{d})^{-1}\tau\overline{d} = S(\overline{d})^{-1}S(\overline{d})\tau_0 = \tau_0. \tag{3.4.4}$$

ϕ induces a $\operatorname{Coker} d$-module homomorpshism $\overline{\phi} : S(\operatorname{Ker} d_0) \longrightarrow U$ such that $\overline{\phi}\tau_0 = \phi$. Take $\delta_1 = \overline{\phi}\delta_0$. Applying (3.4.4) we obtain

$$\delta_1 d = \overline{\phi}\delta_0 d = \overline{\phi}\delta_0 i_1 \overline{d} = \overline{\phi}\tau_0 = \phi.$$

(ii) $\Longleftrightarrow$ (iii) is obvious by the definition of $S(\mathrm{C})$ and Proposition 1.3.14. □

Corollary 3.4.4. *Let* $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_1, d_0, d_1, i, m)$ *be an internal category in the category of abelian groups,* $d = d_1|_{\operatorname{Ker} d_0}$. *The following conditions are equivalent:*

(i) $\mathrm{H}^1(\mathrm{C}, -) = 0$*;*

(ii) d is a split monomorphism;

(iii) C *is equivalent to the discrete category* $(\operatorname{Coker} d, \operatorname{Coker} d, 1, 1, 1, 1)$.

Examples.

1. C is a discrete internal category in **C**. We have $\mathrm{C} = (\mathrm{C}_0, \mathrm{C}_0, 1_{\mathrm{C}_0}, 1_{\mathrm{C}_0}, 1_{\mathrm{C}_0}, 1_{\mathrm{C}_0})$, $d = 0$ and $\operatorname{Coker} d = \mathrm{C}_0$. From (3.3.5) we obtain

$$\mathrm{H}^0(\mathrm{C}, A) = \operatorname{Hom}_{\mathrm{C}_0\text{-mod}}(\mathbf{I}(\mathrm{C}_0), \operatorname{Ker}\pi).$$

Thus

$$\mathrm{H}^0(\mathrm{C}, -) = 0 \Longleftrightarrow \mathbf{I}(\mathrm{C}_0) = 0.$$

Since $\operatorname{Ker} d_0 = 0$, we have

$$\mathrm{H}^1(\mathrm{C}, -) = 0.$$

In the case where **C** is the category of groups,

$$\mathrm{H}^0(\mathrm{C}, -) = 0 \Longleftrightarrow \mathrm{C}_0 = 0.$$

2. C is an antidiscrete category; as a crossed module it has the form $\mathrm{C}_0 \xrightarrow{1_{\mathrm{C}_0}} \mathrm{C}_0$. Since $\operatorname{Coker} d = 0$ we have

$$\mathrm{H}^0(\mathrm{C}, -) = \operatorname{Der}(\operatorname{Coker} d, -) = 0.$$

From Theorem 3.4.3 it follows that

$$\mathrm{H}^1(\mathrm{C}, -) = 0.$$

Note that this can be obtained easily by direct computation.

3. An internal category as a crossed module is of the form $\mathrm{C}_1 \xrightarrow{0} 0$. So we have $\mathrm{C}_0 = 0$, $\operatorname{Ker} d_0 = \mathrm{C}_1$, $d = 0$, from which we obtain that C_1 is a singular object and

$$\mathrm{H}^0(\mathrm{C}, -) = 0,$$
$$\mathrm{H}^1(\mathrm{C}, -) = \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(\mathrm{C}_1, -).$$

Thus $\mathrm{H}^1(\mathrm{C}, -) = 0 \Longleftrightarrow \mathrm{C}_1 = 0$.

Chapter 4

KAN EXTENSIONS OF INTERNAL FUNCTORS

We consider the notion of the Kan extension for internal functors in the category of groups $\mathbb{G}\mathrm{r}$. According to the equivalence of categories $\mathrm{Cat}(\mathbb{G}\mathrm{r}) \cong X\,\mathrm{Mod}(\mathbb{G}\mathrm{r})$ [78], the internal nature of categories enables us to consider them as crossed modules and to think of the problem of necessary and sufficient conditions for the existence of internal Kan extensions, which is not known for the case of ordinary categories. Thus we follow the algebraic approach to this problem, use homological algebra methods, and under certain assumptions establish the necessary and sufficient conditions for the existence of internal Kan extensions. Questions related to this problem are also discussed.

Due to MacLane–Whitehead's well-known classification of connected cell complexes according to their 3-type [73], we can also consider the topological approach to this question; it will be the subject of the forthcoming paper (see [34]).

Since every internal category in $\mathbb{G}\mathrm{r}$ is a groupoid, this kind of questions can be treated by means of category theory (groupoid) methods. Note that in our case the groupoid approach did not give desirable result.

4.1. Extensions in the Category $\mathrm{Cat}(\mathbb{A}\mathrm{b})$

Let $\mathrm{Cat}(\mathbb{A}\mathrm{b})$ denote the category of internal categories in the category of abelian groups $\mathbb{A}\mathrm{b}$. By the equivalence of categories $\mathrm{Cat}(\mathbb{A}\mathrm{b}) \cong X\,\mathrm{Mod}(\mathbb{A}\mathrm{b})$, where $X\,\mathrm{Mod}(\mathbb{A}\mathrm{b})$ is a category of crossed modules in $\mathbb{A}\mathrm{b}$, we can consider an internal category in $\mathbb{A}\mathrm{b}$ as a pair of abelian groups (A_0, A_1) together with a homomorphism between them:

$$A : A_1 \xrightarrow{d^A} A_0\,, \quad A_0, A_1 \in Ob(\mathbb{A}\mathrm{b}), \quad d^A \in \mathrm{Mor}(\mathbb{A}\mathrm{b}).$$

In what follows, an internal category $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ will be denoted by $A = (A_0, A_1)$; this means that there is also a homomorphism $d^A : A_1 \longrightarrow A_0$.

Definition 4.1.1. Let $A : A_1 \xrightarrow{d^A} A_0$, $C : C_1 \xrightarrow{d^C} C_0 \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$. Define $\mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A)$ as the pullback of the diagram

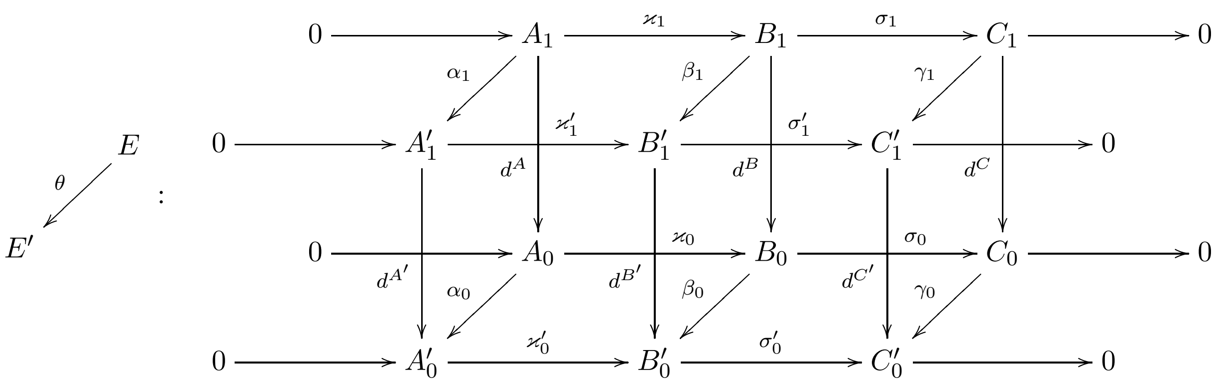
$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_1, A_1) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Ext}^1(C_1, d^A)} \\ \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_0, A_0) & \xrightarrow[\mathrm{Ext}^1(d^C, A_0)]{} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_1, A_0) \end{array}.$$

Definition 4.1.2. An extension of $C = (C_0, C_1)$ by $A = (A_0, A_1)$ is a commutative diagram of the type

$$E:\quad \begin{array}{ccccccccc} E_1 : 0 & \longrightarrow & A_1 & \xrightarrow{\varkappa_1} & B_1 & \xrightarrow{\sigma_1} & C_1 & \longrightarrow & 0 \\ \Gamma\downarrow & & \downarrow d^A & & \downarrow d^B & & \downarrow d^C & & \\ E_0 : 0 & \longrightarrow & A_0 & \xrightarrow[\varkappa_0]{} & B_0 & \xrightarrow[\sigma_0]{} & C_0 & \longrightarrow & 0 \end{array}$$

with exact rows; thus $E_i \in \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_i, A_i)$, $i = 0, 1$.

Denote by $\mathcal{E}\mathrm{xt}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A)$ the set of all extensions of C by A. By a homomorphism $\theta : E \longrightarrow E'$ we mean a pair $\theta = (\theta_0, \theta_1)$ of triples $\theta_0 = (\alpha_0, \beta_0, \gamma_0)$, $\theta_1 = (\alpha_1, \beta_1, \gamma_1)$ such that the diagram



is commutative. It can be written in short from as follows:

$$\begin{array}{ccc} E_1 & \xrightarrow{\theta_1} & E_1' \\ \Gamma\downarrow & & \downarrow\Gamma' \\ E_1 & \xrightarrow[\theta_0]{} & E_1' \end{array} .$$

Let $A = A'$ and $C = C'$. We shall say that two extensions E and E' of C by A are congruent $E \equiv E'$ if there is a morphism $\theta = (\theta_0, \theta_1) : E \longrightarrow E'$, where $\theta_0 = (1_{A_0}, \beta_0, 1_{C_0})$ and $\theta_1 = (1_{A_1}, \beta_1, 1_{C_1})$. As is well known, in this case β_0 and β_1 are isomorphisms, and so $\beta = (\beta_0, \beta_1) : B \longrightarrow B'$ is an isomorphism of internal categories in $\mathbb{A}\mathrm{b}$.

If $E' : E_1' \xrightarrow{\Gamma'} E_0'$ and $E'' : E_1'' \xrightarrow{\Gamma''} E_0''$ are two extensions and $E_0' = E_1''$, we can define the composition of these extensions $E : E_1' \xrightarrow{\Gamma} E_0''$, where $\Gamma = \Gamma'' \circ \Gamma'$.

Thus E is of the form

$$E:\quad \begin{array}{ccccccccc} E_1' : & \longrightarrow & A_1 & \longrightarrow & B_1' & \longrightarrow & C_1' & \longrightarrow & 0 \\ \downarrow\Gamma & & \downarrow d^{A''}\circ d^{A'} & & \downarrow d^{B''}\circ d^{B'} & & \downarrow d^{C''}\circ d^{C'} & & \\ E_0'' : 0 & \longrightarrow & A_0'' & \longrightarrow & B_0'' & \longrightarrow & C_0'' & \longrightarrow & 0 \end{array} .$$

Now we can define a larger congruence relation in $\mathcal{E}\mathrm{xt}(C, A)$. If $E = E^n \circ E^{n-1} \circ \cdots \circ E^1$, then $E \equiv E'$ if and only if E' is obtained from E by replacing E^i by its congruent extension $E^{i\prime} \equiv E^i$, for any i, $i = 1, \dots, n$. Let R be the equivalence relation generated by this congruence relation. It is easy to see that

$$\mathcal{E}\mathrm{xt}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A)/R \approx \mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A).$$

It is straightforward to verify that Cat($\mathbb{A}$b) admits limits and colimits. Thus in a natural way we can define homomorphisms $\mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(\varphi, A)$ and $\mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, \psi)$, where φ and ψ are morphisms in Cat($\mathbb{A}$b), Also they can be defined directly by Definition 4.1.1. Thus $\mathrm{Ext}_{\mathrm{Cat}(\ )}(\cdot\,,\,\cdot)$ is a bifunctor.

Let $\mathbb{A}\mathrm{b}_d$ denote the category whose objects are the diagrams of the form

$$d: \quad \begin{array}{ccc} M & & \\ & \searrow & \\ & & K \\ & \nearrow & \\ N & & \end{array}$$

in $\mathbb{A}$b, and morphisms $d \longrightarrow d'$ are the morphisms of diagrams, i.e., the triples

$$(\mu, \nu, \varkappa): \quad \begin{array}{ccccc} M & \xrightarrow{\mu} & M' & & \\ & \searrow & & \searrow & \\ & & K & \xrightarrow{\varkappa} & K' \\ & \nearrow & & \nearrow & \\ N & \xrightarrow{\nu} & N' & & \end{array}$$

such that the above diagram commutes. The exactness in $\mathbb{A}\mathrm{b}_d$ means the exactness of each row in the corresponding diagram. Let

$$d\ \mathrm{Ext}^1 : \mathrm{Cat}(\mathbb{A}\mathrm{b})^0 \times \mathrm{Cat}(\mathbb{A}\mathrm{b}) \longrightarrow \mathbb{A}\mathrm{b}_d$$

denote the functor defined by

$$d\ \mathrm{Ext}^1(A, B): \quad \begin{array}{ccc} \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, B_1) & & \\ & \searrow^{\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, d^B)} & \\ & & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, B_0) \\ & \nearrow_{\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(d^A, B_1)} & \\ \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_0, B_0) & & \end{array}$$

on objects, and by the commutative diagram

$$d\ \mathrm{Ext}^1(A, \beta): \quad \begin{array}{ccccc} \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, B_1) & \xrightarrow{\mathrm{Ext}^1(A_1, \beta_1)} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, B'_1) & & \\ & \searrow & & \searrow & \\ & & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, B_0) & \xrightarrow{\mathrm{Ext}^1(A_1, \beta_0)} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, B'_0) \\ & \nearrow & & \nearrow & \\ \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_0, B_0) & \xrightarrow{\mathrm{Ext}^1(A_0, \beta_0)} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_0, B'_0) & & \end{array}$$

for the morphism $\beta = (\beta_0, \beta_1) : B \longrightarrow B'$ in Cat($\mathbb{A}$b).

Similarly, for $\alpha = (\alpha_0, \alpha_1) : A \longrightarrow A'$ in $\mathrm{Cat}(\mathbb{A}\mathrm{b})^0$, $d\ \mathrm{Ext}^1(\alpha, B)$ is defined by the commutative diagram

$$
d\ \mathrm{Ext}^1(\alpha, B) : \quad
\begin{array}{ccccc}
\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, B_1) & \xrightarrow{\mathrm{Ext}^1(\alpha_1, B_1)} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1', B_1) & & \\
& \searrow & & \searrow & \\
& & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1, B_0) & \xrightarrow{\mathrm{Ext}^1(\alpha_1, B_0)} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1', B_0) \\
& \nearrow & & \nearrow & \\
\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_0, B_0) & \xrightarrow{\mathrm{Ext}^1(\alpha_0, B_0)} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_0', B_0) & &
\end{array}
$$

For the morphism $(\alpha, \beta) : (A, B) \longrightarrow (A', B') \in \mathrm{Cat}(\mathbb{A}\mathrm{b})^0 \times \mathrm{Cat}(\mathbb{A}\mathrm{b})$, $d\ \mathrm{Ext}^1(\alpha, \beta)$ is defined by the composition

$$d\ \mathrm{Ext}^1(A', \beta) \circ d\ \mathrm{Ext}^1(\alpha, B) = d\ \mathrm{Ext}^1(\alpha, B') \circ d\ \mathrm{Ext}^1(A, \beta).$$

For any object $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ we have a functor $(A, \cdot\,) : \mathrm{Cat}(\mathbb{A}\mathrm{b}) \longrightarrow \mathrm{Cat}(\mathbb{A}\mathrm{b})^0 \times \mathrm{Cat}(\mathbb{A}\mathrm{b})$ defined by $B \longmapsto (A, B)$ and $\beta \longmapsto (1_A, \beta)$. The functor $(\,\cdot\,, B)$ is defined similarly. Also, there is a pullback functor $\varprojlim : \mathbb{A}\mathrm{b}_d \longrightarrow \mathbb{A}\mathrm{b}$, which to each diagram $\begin{array}{ccc} M & & \\ & \searrow & \\ & & K \\ & \nearrow & \\ N & & \end{array}$ assigns the pullback object P of this diagram

$$
\begin{array}{ccccc}
& & M & & \\
& \nearrow & & \searrow & \\
P & & & & K\ . \\
& \searrow & & \nearrow & \\
& & N & &
\end{array}
$$

It is easy to see that

$$\mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A, B) = \varprojlim \circ d\ \mathrm{Ext}^1 \circ (A, \,\cdot\,)(B) = \varprojlim \circ d\ \mathrm{Ext}^1 \circ (\,\cdot\,, B)(A).$$

4.2. Kan Extensions of Internal Functors in Cat($\mathbb{G}$r)

(a) The Notion of an Internal Kan Extension.

For the definition of the Kan extension of a functor for ordinary categories, see [72].

Let $A, C, M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$ be internal categories in the category of groups $\mathbb{G}\mathrm{r}$. Proceeding from the definition of the Kan extension and our aim, to give its internal analogy, according to the equivalence of categories $\mathrm{Cat}(\mathbb{G}\mathrm{r}) \cong X\,\mathrm{Mod}(\mathbb{G}\mathrm{r})$ (here $X\,\mathrm{Mod}(\mathbb{G}\mathrm{r})$ denotes the category of crossed modules in $\mathbb{G}\mathrm{r}$), we consider internal categories as crossed modules and denote them as follows: $A : A_1 \xrightarrow{d^A} A_0$, $M : M_1 \xrightarrow{d^M} M_0$, $C : C_1 \xrightarrow{d^C} C_0$. Let $T : M \longrightarrow A$ and $K : M \longrightarrow C$ be internal functors. From Sec. 1.1 we have $T = (T_0, T_1)$, $K = (K_0, K_1)$, where $T_0 : M_0 \longrightarrow A_0$ and $K_0 : M_0 \longrightarrow C_0$ are homomorphisms of groups, $T_1 : M_1 \longrightarrow A_1$ and $K_1 : M_1 \longrightarrow C_1$ are structural maps, i.e., they are homomorphisms of groups and satisfy the conditions

$$
\begin{aligned}
K_1(r \cdot m) &= K_0(r) \cdot K_1(m), \\
T_1(r \cdot m) &= T_0(r) \cdot T_1(m)
\end{aligned}
\tag{4.2.1}
$$

for $r \in M_0$, $m \in M_1$, and the following diagrams are commutative:

$$\begin{array}{ccc} M_1 & \xrightarrow{d^M} & M_0 \\ {\scriptstyle T_1}\downarrow & & \downarrow{\scriptstyle T_0} \\ A_1 & \xrightarrow[d^A]{} & A_0 \end{array}, \qquad \begin{array}{ccc} M_1 & \xrightarrow{d^M} & M_0 \\ {\scriptstyle K_1}\downarrow & & \downarrow{\scriptstyle K_0} \\ C_1 & \xrightarrow[d^A]{} & C_0 \end{array}.$$

We write $T_1 \in \mathrm{Hom}_{oph}(M_1, A_1)$, $K_1 \in \mathrm{Hom}_{oph}(M_1, C_1)$.

Since internal categories in $\mathbb{G}\mathrm{r}$ are groupoids, every morphism between internal functors is an isomorphism (see also Proposition 1.2.1).

Let $S = (S_0, S_1)$, $R = (R_0, R_1) : C \longrightarrow A$ be internal functors and $\sigma : S \longrightarrow R$ and $\varepsilon : RK \longrightarrow T$ morphisms of internal functors. Recall (see Sec. 1.2) that we can consider these morphisms as maps $\sigma : C_0 \longrightarrow A_1$, $\quad \varepsilon : M_0 \longrightarrow A_1$, which satisfy the conditions

$$d^A\sigma = R_0 - S_0, \quad \sigma d^C = R_1 - S_1, \tag{4.2.2}$$
$$\sigma(r + r') = \sigma(r) + S_0(r) \cdot \sigma(r'), \quad r, r' \in C_0;$$
$$d^A\varepsilon = T_0 - R_0K_0, \quad \varepsilon d^M = T_1 - R_1K_1, \tag{4.2.3}$$
$$\varepsilon(m + m') = \varepsilon(m) + R_0K_0(m) \cdot \varepsilon(m'), \quad m, m' \in M_0.$$

Note that if $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$, then the action of A_0 on A_1 is trivial, so that σ and ε are group homomorphisms satisfying conditions (4.2.2) and (4.2.3). In that case,

$$\mathrm{Hom}(SK, T) = \left\{ \beta \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1) \;\middle|\; \begin{array}{l} d^A\beta = T_0 - S_0K_0 \\ \beta d^C = T_1 - S_1K_1 \end{array} \right\}.$$

From Chap. 2, if α and α' are morphisms of internal functors

$$F \xrightarrow{\alpha} F' \xrightarrow{\alpha'} F'', \qquad F, F', F'' : C \longrightarrow C' \in \mathrm{Cat}(\mathbb{G}\mathrm{r}),$$

then the composite $\alpha'\alpha$ can be considered as a map

$$\alpha' + \alpha : C_0 \longrightarrow C_1,$$

satisfying the corresponding conditions.

As for the case of abelian groups, an internal category $A \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$ will be denoted for simplicity as a pair (A_0, A_1); this means that there is a group homomorphism $d^A : A_1 \longrightarrow A_0$ satisfying the usual conditions (see [78], [31]).

Definition 4.2.1. Let $A = (A_0, A_1)$, $C = (C_0, C_1)$, $M = (M_0, M_1)$ be internal categories in the category $\mathbb{G}\mathrm{r}$, and suppose $K = (K_0, K_1) : M \longrightarrow C$ and $T = (T_0, T_1) : M \longrightarrow A$ are internal functors. An internal right Kan extension of T along K is a pair $(R = (R_0, R_1), \varepsilon)$, where R is an internal functor $C \longrightarrow A$ and $\varepsilon : RK \longrightarrow T$ a morphism of internal functors such that for each internal functor $S = (S_0, S_1) : C \longrightarrow A$ and a morphism $\alpha : SK \longrightarrow T$ there is a unique morphism of internal functors $\sigma : S \longrightarrow R$ with $\alpha = \varepsilon + \sigma K$.

Here σK means the morphism $SK \xrightarrow{\sigma K} RK$, i.e., the composite of the maps

$$M_0 \xrightarrow{K_0} C_0 \xrightarrow{\sigma} A_1,$$

satisfying the conditions

$$d^A \cdot \sigma K = R_0K_0 - S_0K_0,$$
$$\sigma K \cdot d^M = R_1K_1 - S_1K_1,$$

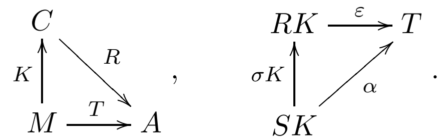
$$\sigma K_0(m+m') = \sigma K_0(m) + S_0K_0(m)\cdot\sigma K_0(m'), \quad m, m' \in M_0.$$

Also, by the observation given before Definition 4.2.1, ε is a map $M_0 \longrightarrow A_1$ satisfying conditions (4.2.3); α is a map $M_0 \longrightarrow A_1$ satisfying the conditions

$$d^A\alpha = T_0 - S_0K_0,$$
$$\alpha d^M = T_1 - S_1K_1;$$

σ is a map $C_0 \xrightarrow{\sigma} A_1$ satisfying conditions (4.2.2).

The diagram is



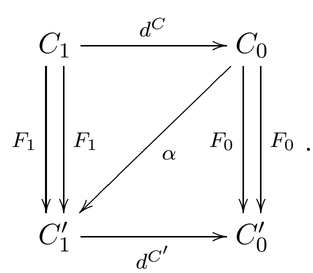
(4.2.4)

Note that, as in the case of groupoids, the notions of internal right and left Kan extensions are equivalent. Thus in what follows we can omit the word "right."

Recall that an internal category C is called connected if for its two objects r, r' there exists a morphism $r \longrightarrow r'$, which in the language of crossed modules means that d^C is a surjective homomorphism for $C_1 \xrightarrow{d^C} C_0$.

Lemma 4.2.2. *Let $C, C' \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$ and C be a connected internal category. Then for any internal functor $F = (F_0, F_1) : C \longrightarrow C'$ there is only one endomorphism $F \longrightarrow F$, which is the identity morphism.*

Proof. Let $\alpha : F \longrightarrow F$ be an endomorphism; thus $\alpha : C_0 \longrightarrow C'_1$ is a map satisfying the conditions $\alpha d^C = 0$, $d^{C'}\alpha = 0$, and $\alpha(r + r') = \alpha(r) + F_0(r)\cdot\alpha(r')$, $r, r' \in C_0$. The diagram is



Since C is connected, d^C is surjective; so it follows that $\alpha = 0$, which means that $\alpha : F \longrightarrow F$ is the identity morphism (see Chap. 1). ☐

Corollary 4.2.3. *Let $C, C' \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, C be a connected internal category, and $F, G : C \longrightarrow C'$ be internal functors. If $\sigma : F \longrightarrow G$ is a morphism of internal functors, then it is a unique isomorphism.*

Proof. Since C and C' are groupoids, σ is an isomorphism (see Chap. 1). Thus we have a bijection

$$\mathrm{Hom}(F,F) \underset{\mathrm{Hom}(F,\sigma^{-1})}{\overset{\mathrm{Hom}(F,\sigma)}{\rightleftarrows}} \mathrm{Hom}(F,G)$$

which by Lemma 4.2.2 implies that σ is a unique isomorphism. ☐

Remark. Corollary 4.2.3 can be proved directly, and the statement of Lemma 4.2.2 can obviously be obtained as its special case. Both statements are true also in the case where C is an arbitrary internal category and $d^{C'}$ is a monomorphism. When $C' \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$, we obtain a more general condition $\mathrm{Hom}(\mathrm{Coker}\, d^C, \mathrm{Ker}\, d^{C'}) = 0$ (see Lemma 4.2.7).

Proposition 4.2.4. *Let C and M be connected internal categories in* Cat($\mathbb{G}$r)*, and* $A \in$ Cat($\mathbb{G}$r)*; (R, ε) is an internal Kan extension of an internal functor $T : M \longrightarrow A$ along $K : M \longrightarrow C$ if and only if R is a unique (up to isomorphism) internal functor $R : C \longrightarrow A$ with the property that there is an isomorphism $\varepsilon : RK \approx T$.*

Proof. Let (R, ε) be an internal Kan extension of T along K. Then from Definition 4.2.1 it follows that (R, ε) satisfies the conditions of the proposition. Conversely, let (R, ε) be a pair satisfying the conditions of the proposition. We shall show that (R, ε) is an internal Kan extension. Let $S : C \longrightarrow A$ be an internal functor and $\alpha : SK \longrightarrow T$ a morphism of internal functors. Then α is an isomorphism; by the condition, R is a unique (up to isomorphism) functor with this property. Thus there is an isomorphism $\sigma : S \cong R$. This isomorphism is unique by Corollary 4.2.3 and $\varepsilon \cdot \sigma K = \alpha$, since by the same Corollary 4.2.3, $\mathrm{Hom}(SK, T)$ consists of only one element, which proves the proposition. □

Note that if $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$, then Proposition 4.2.4 is true under the more general condition: $\mathrm{Hom}(\mathrm{Coker}\, d^C, \mathrm{Ker}\, d^A) = 0$ and $\mathrm{Hom}(\mathrm{Coker}\, d^M, \mathrm{Ker}\, d^A) = 0$.

Proposition 4.2.5. *Let* $A, C, M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$*, $K : M \longrightarrow C$, $T : M \longrightarrow A$, $R : C \longrightarrow A$ be internal functors, and $\varepsilon : RK \xrightarrow{\approx} T$ be an isomorphism. (R, ε) is an internal Kan extension of T along K if and only if for each internal functor $S : C \longrightarrow A$ the assignment $\sigma \longmapsto \sigma K_0$ $(K = (K_0, K_1))$ determines a bijection*

$$\mathrm{Hom}(S, R) \longrightarrow \mathrm{Hom}(SK, RK).$$

Proof. It follows from the composite

$$\mathrm{Hom}(S, R) \longrightarrow \mathrm{Hom}(SK, RK) \xrightarrow{\mathrm{Hom}(SK,\varepsilon)} \mathrm{Hom}(SK, T),$$

where $\mathrm{Hom}(SK, \varepsilon)$ is a bijection. □

(b) On $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M, A)$, $M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$.

Let $M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$ and $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ be internal categories and $T : M \longrightarrow A$ an internal functor. So $T_0 \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_0)$, $T_1 \in \mathrm{Hom}_{\mathrm{oph}}(M_1, A_1)$ and $T_0 d^M = d^A T_1$. We can express $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M, A)$ as a pullback of the diagram in $\mathbb{A}\mathrm{b}$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M, A) & \longrightarrow & \mathrm{Hom}_{\mathrm{oph}}(M_1, A_1) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(M_1, d^A)} \\ \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_0) & \xrightarrow{\mathrm{Hom}(d^M, A_0)} & \mathrm{Hom}_{\mathrm{oph}}(M_1, A_0) \end{array} . \tag{4.2.5}$$

Thus $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M, A)$ is an abelian group. Denote by $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M, A)$ the set of all isomorphic classes of internal functors from M to A. Obviously, this set has an abelian group structure; note that the action of A_0 on A_1 is trivial, since $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$. So if $\alpha : T \longrightarrow T'$ is a morphism of internal functors, $T = (T_0, T_1)$, $T' = (T'_0, T'_1) : M \longrightarrow A$, then α is a group homomorphism $M_0 \xrightarrow{\alpha} A_1$ satisfying the conditions

$$d^A \alpha = T'_0 - T_0,$$
$$\alpha d^M = T'_1 - T_1.$$

For each $\alpha \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$ we have $d^A \alpha \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_0)$, $\alpha d^M \in \mathrm{Hom}_{\mathrm{Str}}(M_1, A_1)$. The first is obvious, and the second is obtained from the following equalities for each $m_0 \in M_0$, $m \in M$:

$$\alpha d^M(m_0 \cdot m) = \alpha(m_0 + d^M(m) - m_0) = \alpha(m_0) + \alpha d^M(m) - \alpha(m_0) = \alpha d^M(m)$$

(here we apply the fact that $d^A : M_1 \longrightarrow M_0$ is a crossed module);

$$d^A\alpha(m_0) \cdot \alpha d^M(m) = \alpha d^M(m)$$

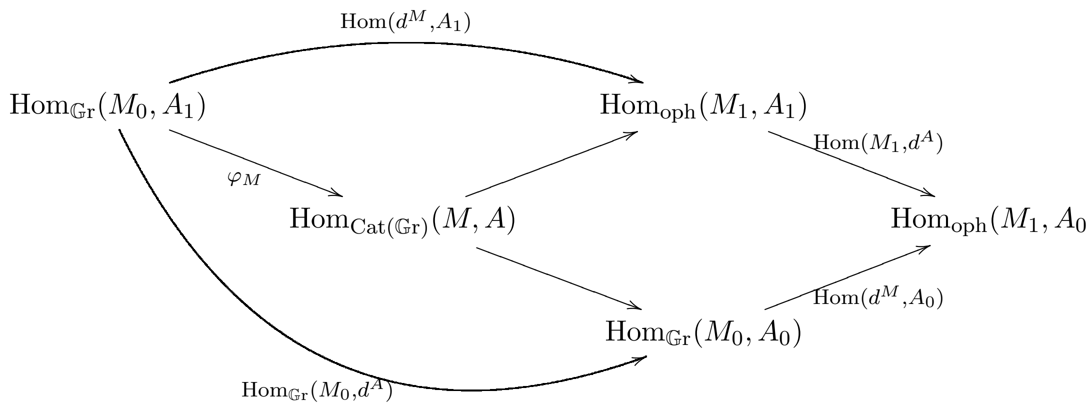
since $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$, and A_0 acts trivially on A_1.

Thus $\alpha d^M(m_0 \cdot m) = d^A\alpha(m_0) \cdot \alpha d^M(m)$.

For each $\alpha \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$ we have

$$\mathrm{Hom}(M_1, d^A) \cdot \mathrm{Hom}(d^M, A_0)(\alpha) = \mathrm{Hom}(d^M, A_0) \cdot \mathrm{Hom}(M_0, d^A)(\alpha).$$

Since $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M, A)$ is a pullback of diagram (4.2.5), this means that $\mathrm{Hom}(d^M, A_1)$ and $\mathrm{Hom}(M_0, d^A)$ induce homomorphism of abelian groups φ_M defined by $\varphi_M(\alpha) = (d^A\alpha, \alpha d^M)$, for $\alpha \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$; the diagram looks as follows:



Note that since for $\alpha \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$, $(d^A\alpha, \alpha d^M)$ is an internal functor $M \longrightarrow A$, any $\alpha \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$ can be considered as a morphism of internal functors $(d^A\alpha, \alpha d^M) \xrightarrow{\alpha} (0, 0)$, and the diagram is

$$\begin{array}{ccc} M_1 & \xrightarrow{d^M} & M_0 \\ {\scriptstyle \alpha d^M}\downarrow\downarrow{\scriptstyle 0} & & {\scriptstyle d^A\alpha}\downarrow\downarrow{\scriptstyle 0} \\ A_1 & \xrightarrow[d^A]{} & A_0 \end{array} .$$

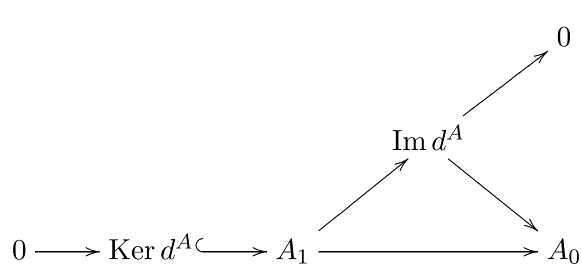
Thus $\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$ is an abelian group of all morphisms between the internal functors $M \longrightarrow A$. Now one can easily prove the following

Proposition 4.2.6. *For* $M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ *we have*

$$\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M, A) = \mathrm{Coker}\,\varphi_M.$$

Lemma 4.2.7. $\mathrm{Ker}\,\varphi_M = \mathrm{Hom}(\mathrm{Coker}\,d^M, \mathrm{Ker}\,d^A)$.

Proof. We have $\mathrm{Ker}\,\varphi_M = \mathrm{Ker}\,\mathrm{Hom}(d^M, A_1) \cap \mathrm{Ker}\,\mathrm{Hom}(M_0, d^A)$. From the exact sequence



we obtain the exact sequence

$$\begin{array}{ccccccc}
 & & & & & & \mathrm{Hom}(M_0, \mathrm{Im}\, d^A) \\
 & & & & & \nearrow & \big\downarrow \\
0 \longrightarrow & \mathrm{Hom}_{\mathbb{Gr}}(M_0, \mathrm{Ker}\, d^A) & \longrightarrow & \mathrm{Hom}_{\mathbb{Gr}}(M_0, A_1) & \xrightarrow{\mathrm{Hom}(M_0, d^A)} & & \mathrm{Hom}_{\mathbb{Gr}}(M_0, A_0)
\end{array}$$

which implies

$$\mathrm{Ker}\,\mathrm{Hom}(M_0, d^A) = \mathrm{Hom}_{\mathbb{Gr}}(M_0, \mathrm{Ker}\, d^A).$$

Similarly, we have a sequence

$$M_1 \longrightarrow \mathrm{Im}\, d^M \hookrightarrow M_0 \longrightarrow \mathrm{Coker}\, d^M \longrightarrow 0 \qquad (d^M : M_1 \to M_0)$$

from which we obtain the sequence

$$\begin{array}{ccccccc}
 & & & & & & \mathrm{Hom}(M_1, A_1) \\
 & & & & & \overset{\mathrm{Hom}(d^M, A_1)}{\nearrow} & \big\uparrow \\
0 \longrightarrow & \mathrm{Hom}(\mathrm{Coker}\, d^M, A_1) & \longrightarrow & \mathrm{Hom}(M_0, A_1) & \longrightarrow & & \mathrm{Hom}(\mathrm{Im}\, d^M, A_1) \\
 & & & & & & \big\uparrow \\
 & & & & & & 0
\end{array}$$

and $\mathrm{Ker}\,\mathrm{Hom}(d^M, A_1) = \mathrm{Hom}(\mathrm{Coker}\, d^M, A_1)$. Now it is easy to see that $\mathrm{Ker}\,\varphi_M$ is a pullback of the diagram

$$\begin{array}{ccc}
\mathrm{Ker}\,\varphi_M & \longrightarrow & \mathrm{Hom}_{\mathbb{Gr}}(M_0, \mathrm{Ker}\, d^A) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Hom}_{\mathbb{Gr}}(\mathrm{Coker}\, d^M, A_1) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Gr}}(M_0, A_1)
\end{array}.$$

From the diagram

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \big\downarrow \\
 & & & & \mathrm{Ker}\, d^A \\
 & & & \nearrow & \big\downarrow \\
M_0 \twoheadrightarrow & \mathrm{Coker}\, d^M & \longrightarrow & & A_1
\end{array}$$

we see that $\mathrm{Ker}\,\varphi_M = \mathrm{Hom}_{\mathbb{Gr}}(\mathrm{Coker}\, d^M, \mathrm{Ker}\, d^A)$. □

Using the notation of Sec. 4.1, we can define the functor

$$d\ \mathrm{Hom} : \mathrm{Cat}(\mathbb{Ab})^0 \times \mathrm{Cat}(\mathbb{Ab}) \longrightarrow \mathbb{Ab}_d$$

by

$$d\ \mathrm{Hom}(A,B):\qquad \begin{array}{ccc} \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_1,B_1) & & \\ & \searrow^{\mathrm{Hom}(A_1,d^B)} & \\ & & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_1,B_0)\ . \\ & \nearrow_{\mathrm{Hom}_{\mathbb{A}\mathrm{b}}(d^A,B_0)} & \\ \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_0,B_0) & & \end{array}$$

The functor on homomorphisms is defined obviously. It is easy to see that

$$\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,B) = \varprojlim \circ d\ \mathrm{Hom}(A,\,\cdot\,)(B) = \varprojlim \circ d\ \mathrm{Hom}(\,\cdot\,,B)(A).$$

Note that analogous equalities are true for the case $A \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, $B \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$. The functors $(A,\,\cdot\,)$, $(\,\cdot\,,B)$, $d\,\mathrm{Hom}$, $\varprojlim$ are left exact, so the functors $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,\,\cdot\,)$, $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(\,\cdot\,,B)$ are also left exact; the same is true for $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(A,\,\cdot\,) : \mathrm{Cat}(\mathbb{A}\mathrm{b}) \longrightarrow \mathbb{A}\mathrm{b}$. This can also be proved directly.

Lemma 4.2.8. *For the exact sequence*

$$0 \longrightarrow B' \xrightarrow{\beta'} B \xrightarrow{\beta''} B'' \longrightarrow 0$$

in $\mathrm{Cat}(\mathbb{A}\mathrm{b})$ *and an internal category* $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ *we have a complex of abelian groups*

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,B') \xrightarrow{\mathrm{Hom}(A,\beta')} \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,B) \xrightarrow{\mathrm{Hom}(A,\beta'')} \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,B'') \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,B') \xrightarrow{\mathrm{Ext}^1(A,\beta')} \mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,B) \xrightarrow{\mathrm{Ext}^1(A,\beta'')} \mathrm{Ext}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,B'')\ ,$$

where $\mathrm{Hom}(A,\beta')$ *is a monomorphism, and we have the exactness in* $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,B)$*; thus* $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,\,\cdot\,)$ *is a left exact functor.*

Proof. The given exact sequence induces an exact sequence in $\mathbb{A}\mathrm{b}_d$:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_1,B_1') & \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_1,B_1) & \longrightarrow & & \\ & \searrow^{\mathrm{Hom}(A_1,d^{B'})} & & \searrow^{\mathrm{Hom}(A_1,d^B)} & & & \\ 0 \longrightarrow & & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_1,B_0') & \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_1,B_0) & \longrightarrow & \\ & \nearrow_{\mathrm{Hom}_{\mathbb{A}\mathrm{b}}(d^A,B_0')} & & \nearrow_{\mathrm{Hom}_{\mathbb{A}\mathrm{b}}(d^A,B_0)} & & & \\ 0 \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_0,B_0') & \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_0,B_0) & \longrightarrow & & \end{array}$$

$$\begin{array}{ccccccc} \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_1,B_1'') & \longrightarrow & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1,B_1') & \longrightarrow & & \\ & \searrow^{\mathrm{Hom}(A_1,d^{B''})} & & \searrow & & & \\ \longrightarrow & & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_1,B_0'') & \longrightarrow & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1,B_0') & \longrightarrow & \\ & \nearrow_{\mathrm{Hom}_{\mathbb{A}\mathrm{b}}(d^A,B_0'')} & & \nearrow & & & \\ \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_0,B_0'') & \longrightarrow & \mathrm{Ext}^2_{\mathbb{A}\mathrm{b}}(A_0,B_0') & \longrightarrow & & \end{array}$$

$$
\begin{array}{ccccccc}
\longrightarrow & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1,B_1) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1,B_1'') & & & \\
& & \searrow & & \searrow & & \\
& & \longrightarrow & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1,B_0) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(A_1,B_0'') & . \\
& & \nearrow & & \nearrow & & \\
\longrightarrow & \mathrm{Ext}^2_{\mathbb{A}\mathrm{b}}(A_0,B_0) & \longrightarrow & \mathrm{Ext}^2_{\mathbb{A}\mathrm{b}}(A_0,B_0'') & & &
\end{array}
\tag{4.2.6}
$$

Applying the functor $\varprojlim : \mathbb{A}\mathrm{b}_d \longrightarrow \mathbb{A}\mathrm{b}$ to (4.2.6), we obtain the desirable complex. $\square$

Note that an analogous statement is true for the first argument.

Let $B \xrightarrow{J} C \underset{S}{\overset{T}{\rightrightarrows}} D$ be the diagram in $\mathrm{Cat}(\mathbb{G}\mathrm{r})$, where J is a surjective internal functor (i.e., for $J = (J_0, J_1)$, J_0 and J_1 are surjective homomorphisms). It is obvious that if $T \approx S$, then $TJ \approx SJ$. But as we shall see below (Lemmas 4.2.9 and 4.2.10), the converse statement is not always true.

Consider an exact sequence of internal categories in $\mathbb{A}\mathrm{b}$:

$$
0 \longrightarrow A \overset{I}{\hookrightarrow} B \overset{J}{\twoheadrightarrow} C \longrightarrow 0 \ .
$$

For any $D = (D_0, D_1) \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ it induces the commutative diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(C_0,D_1) & \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(B_0,D_1) & \longrightarrow & & \\
& \downarrow{\scriptstyle \varphi_C} & & \downarrow{\scriptstyle \varphi_B} & & & \\
0 \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C,D) & \xrightarrow{\mathrm{Hom}(J,D)} & \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(B,D) & \xrightarrow{\mathrm{Hom}(I,D)} & & \\
& & & & & & \\
\longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(A_0,D_1) & \xrightarrow{d} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_0,D_1) & & & \\
& \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \zeta} & & , & \\
\longrightarrow & \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,D) & \xrightarrow{\delta} & \mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C,D) & & &
\end{array}
\tag{4.2.7}
$$

where φ_A, φ_B, φ_C are the morphisms defined above, and ζ is induced by the pair $(\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_0, d^B), \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(d^C, D))$. By Lemma 4.2.8 the second row in (4.2.7) is a complex, where $\mathrm{Hom}(J, D)$ is a monomorphism, and we have the exactness in $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(B, D)$.

Denote $\widetilde{\mathrm{Ext}}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, D) = \mathrm{Coker}\,\zeta$; then δ induces the homomorphism

$$
\widetilde{\delta} : \widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A, D) \longrightarrow \widetilde{\mathrm{Ext}}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, D).
$$

From (4.2.7) we obtain the sequence

$$
\begin{array}{l}
\mathrm{Ker}\, d \cap \mathrm{Ker}\, \varphi_A \xrightarrow{\varkappa} \widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C,D) \xrightarrow{\widetilde{\mathrm{Hom}}(J,D)} \widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(B,D) \xrightarrow{\widetilde{\mathrm{Hom}}(I,D)} \\
\qquad\qquad \longrightarrow \widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A,D) \xrightarrow{\widetilde{\delta}} \widetilde{\mathrm{Ext}}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C,D) \ .
\end{array}
\tag{4.2.8}
$$

Lemma 4.2.9. *In* (4.2.8) *we have:*

(i) (4.2.8) *is a complex;*

(ii) *if* φ_B *is a monomorphism, then so is* $\varkappa$*;*

(iii) (4.2.8) *is exact in* $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\ )}(C, D)$*;*

(iv) *if* ζ *is a monomorphism, then* (4.2.8) *is exact in* $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\ )}(B, D)$*;*

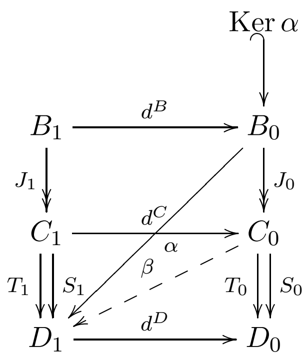
(v) *if* φ_A *is a monomorphism, then so is* $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\ )}(J, D)$*.*

The proof is easy by diagram (4.2.7) and is left to the reader.

Note that if A is a connected internal category, or D has no parallel morphisms, then φ_A is a monomorphism, and by (v) $\widetilde{\mathrm{Hom}}(J, D)$ is also a monomorphism; thus, in this case, $TJ \approx SJ$ implies the isomorphism $T \approx S$ for the surjective internal functor $J \in \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(B, C)$, and any $T, S \in \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, D)$.

Lemma 4.2.10. *Let* B*,* C*,* D *be internal categories in* $\mathbb{A}\mathrm{b}$*, and* $J = (J_0, J_1) : B \longrightarrow C$ *be a surjective internal functor. For internal functors* $T, S : C \rightrightarrows D$ *the isomorphism* $TJ \approx SJ$ *implies the isomorphism* $T \approx S$ *if and only if there is an isomorphism* $TJ \xrightarrow{\approx} SJ$ *given by the homomorphism* $\alpha : B_0 \longrightarrow D_1$*, with* $\mathrm{Ker}\, J_0 \hookrightarrow \mathrm{Ker}\, \alpha$*.*

Proof. It follows from diagrams (4.2.7) and (4.2.8); the picture is



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and $\beta : T \longrightarrow S$ is defined by $\alpha : \beta J_0 = \alpha$. ∎

Note that when all categories are internal in the category of vector spaces over a field, J_0 and J_1 have sections; if these sections give the section of internal functor J (i.e., the corresponding diagram for the sections commutes), then we can construct β without any additional conditions on α and consequently in that case $TJ \approx SJ$ implies $T \approx S$.

4.3. The Existence of Internal Kan Extensions

Let $K : M \longrightarrow C$ be an internal functor in $\mathbb{A}\mathrm{b}$. Define the internal categories $\mathrm{Im}\, K$ and $\mathrm{Ker}\, K$ in a natural way. If K has the form

$$\begin{array}{ccc} M_1 & \xrightarrow{d^M} & M_0 \\ {\scriptstyle K_1}\downarrow & & \downarrow{\scriptstyle K_0} \\ C_1 & \xrightarrow[d^C]{} & C_0 \end{array} ,$$

then these internal categories as crossed modules have the form

$$\mathrm{Im}\, K_1 \xrightarrow{d^C|_{\mathrm{Im}\, K_1}} \mathrm{Im}\, K_0 ,$$

$$\mathrm{Ker}\, K_1 \xrightarrow{d^M|_{\mathrm{Ker}\, K_1}} \mathrm{Ker}\, K_0 .$$

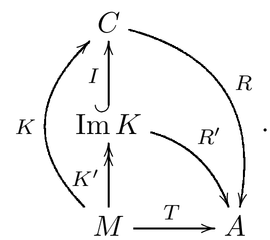
Let $K' : M \longrightarrow \operatorname{Im} K$ be an internal functor defined by K. Then K has a decomposition

$$M \overset{K'}{\twoheadrightarrow} \operatorname{Im} K \overset{I}{\hookrightarrow} C \ .$$

Here for simplicity we consider the case of abelian groups.

Proposition 4.3.1. *Let $A, C, M \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$; $K : M \longrightarrow C$ and $T : M \longrightarrow A$ be internal functors. Suppose that C and M are connected. If R' is an internal Kan extension of T along K', and R is an internal Kan extension of R' along I, where K' and I are defined by the decomposition $K = IK'$, then R is an internal Kan extension of T along K.*

Proof. The picture is



(4.3.1)

Since M is connected, $\operatorname{Im} K$ is also a connected internal category.

By Proposition 4.2.4, $R'K' \approx T$, R' is a unique functor up to isomorphism with this property, $RI \approx R'$, and R is also unique up to isomorphism. Hence we have $RK = R(IK') \approx (RI)K' \approx R'K' \approx T$.

We shall show that R is unique up to isomorphism with this property. Let $\overline{R} : C \longrightarrow A$ be an internal functor with $\overline{R}K \approx T$. We have $R'K' \approx T \approx \overline{R}K \approx \overline{R}(IK') = (\overline{R}I)K'$.

By the uniqueness of R' it follows that $R' \approx \overline{R}I$; but $R' \approx RI$, and again by the uniqueness of R we obtain $R \approx \overline{R}$. □

Is the statement converse to Proposition 4.3.1 true? The answer is negative in general. Let R be an internal Kan extension of T along K. Then in (4.3.1) R is a Kan extension of RI along I, but, in general, RI is not a Kan extension of T along K'. Indeed, we have

$$RK \approx T,$$

and R is unique up to isomorphism with this property. Suppose that there is an internal functor $\overline{R} : C \longrightarrow A$ with $\overline{R}I \approx RI$. Then

$$\overline{R}IK' \approx RIK' \Longrightarrow \overline{R}K \approx RK \approx T \Longrightarrow \overline{R} \approx R. \qquad \square$$

Now we shall show that in the above situation RI is not an internal Kan extension of T along K'. Let $R' : \operatorname{Im} K \longrightarrow A$ be an internal functor with $R'K' \approx T$; by $RK \approx T$ we have $RIK' \approx R'K'$, but this does not always imply the isomorphism $RI \approx R'$ (see Lemma 4.2.10).

We shall investigate the necessary and sufficient conditions for the existence of internal Kan extensions for the case where K is a surjective and K is an injective internal functor (i.e., K_0 and K_1 are injective homomorphisms). For the above observation, these results, in general, do not provide an answer for arbitrary K (in this way we obtain only sufficient conditions for arbitrary K, but not necessary).

Theorem 4.3.2. *Let $C, M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$ and $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$. Suppose that M is a connected internal category. Let $K = (K_0, K_1) : M \longrightarrow C$ be a surjective internal functor such that $\operatorname{Ker} K_0 \hookrightarrow M_0$ has a retraction. There exists a Kan extension of $T = (T_0, T_1) : M \longrightarrow A$ along K if and only if $T\big|_{\operatorname{Ker} K} \approx 0$ and $\operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^{A}) = 0$.*

Proof. From the conditions it follows that C is also connected. For the exact sequence of internal categories

$$0 \longrightarrow \operatorname{Ker} K \hookrightarrow M \xrightarrow{K} C \longrightarrow 0$$

we have the commutative diagram shown in Fig. 1.

By Lemma 4.2.7, $\operatorname{Ker} \varphi_{\operatorname{Ker} K} = \operatorname{Hom}_{Gr}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A)$ and, since C and M are connected, $\operatorname{Coker} d^C = \operatorname{Coker} d^M = 0$ so that $\operatorname{Ker} \varphi_C = \operatorname{Ker} \varphi_M = 0$. The second row is exact, since $\operatorname{Ker} K_0 \hookrightarrow M_0$ has a retraction. The third row is also exact, since $\operatorname{Hom}(K_1, A_0)$ is a monomorphism. Applying the Snake lemma to the diagram from Fig. (1), we obtain the exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Gr}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) \xrightarrow{\eta} \widetilde{\operatorname{Hom}}_{\operatorname{Cat}(\mathbb{Gr})}(C, A) \longrightarrow$$
$$\longrightarrow \widetilde{\operatorname{Hom}}_{\operatorname{Cat}(\mathbb{Gr})}(M, A) \longrightarrow \widetilde{\operatorname{Hom}}_{\operatorname{Cat}(\mathbb{Gr})}(\operatorname{Ker} K, A) \ . \quad (4.3.2)$$

We have $\operatorname{cl} T \in \widetilde{\operatorname{Hom}}(M, A)$. By the exactness of (4.3.2), there exists $\operatorname{cl} R \in \widetilde{\operatorname{Hom}}_{\operatorname{Cat}(\mathbb{Gr})}(C, A)$ such that $\operatorname{cl}(RT) = \operatorname{cl} T$ if and only if $\operatorname{cl} T\big|_{\operatorname{Ker} K} = 0$, i.e., $T\big|_{\operatorname{Ker} K} \approx 0$, and it is unique if and only if $\operatorname{Hom}_{\mathbb{Gr}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) = 0$, which proves the theorem. ☐

Corollary 4.3.3. *Let A, C, M be internal categories in the category of vector spaces over some field* $\mathtt{k}$*. Suppose that M is connected. If $K : M \longrightarrow C$ is a surjective internal functor, then a Kan extension of $T : M \longrightarrow A$ along K exists if and only if $T\big|_{\operatorname{Ker} K} \approx 0$, and either A has no parallel morphisms, or* $\operatorname{Ker} K$ *is a connected internal category.*

Proof. The result follows from the fact that in the category of vector spaces K_0 has a section and therefore $\operatorname{Ker} K_0 \hookrightarrow M_0$ is always split. The condition $\operatorname{Hom}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) = 0$ is equivalent to the condition: either $\operatorname{Coker} d^{\operatorname{Ker} K} = 0$ or $\operatorname{Ker} d^A = 0$. $\operatorname{Coker} d^{\operatorname{Ker} K} = 0$ means that $\operatorname{Ker} K$ is connected and it is easy to verify that $\operatorname{Ker} d^A = 0$ means that A has no parallel morphisms. ☐

Remark. The statement of Theorem 4.3.2 and therefore of Corollary 4.3.3 holds also for arbitrary M under the same condition that $\operatorname{Ker} K_0 \hookrightarrow M_0$ has a retraction; for the proof we have to apply the general Definition 4.2.1 of internal Kan extension (see Sec. 4.4).

Consider now the case of abelian groups.

Theorem 4.3.4. *Let $A = (A_0, A_1)$, $C = (C_0, C_1)$, $M = (M_0, M_1) \in \operatorname{Cat}(\mathbb{Ab})$; suppose that M is connected and $K : M \longrightarrow C$ is a surjective internal functor. If one of the following conditions holds:*

(i) *A_1 is injective in $\mathbb{Ab}$;*

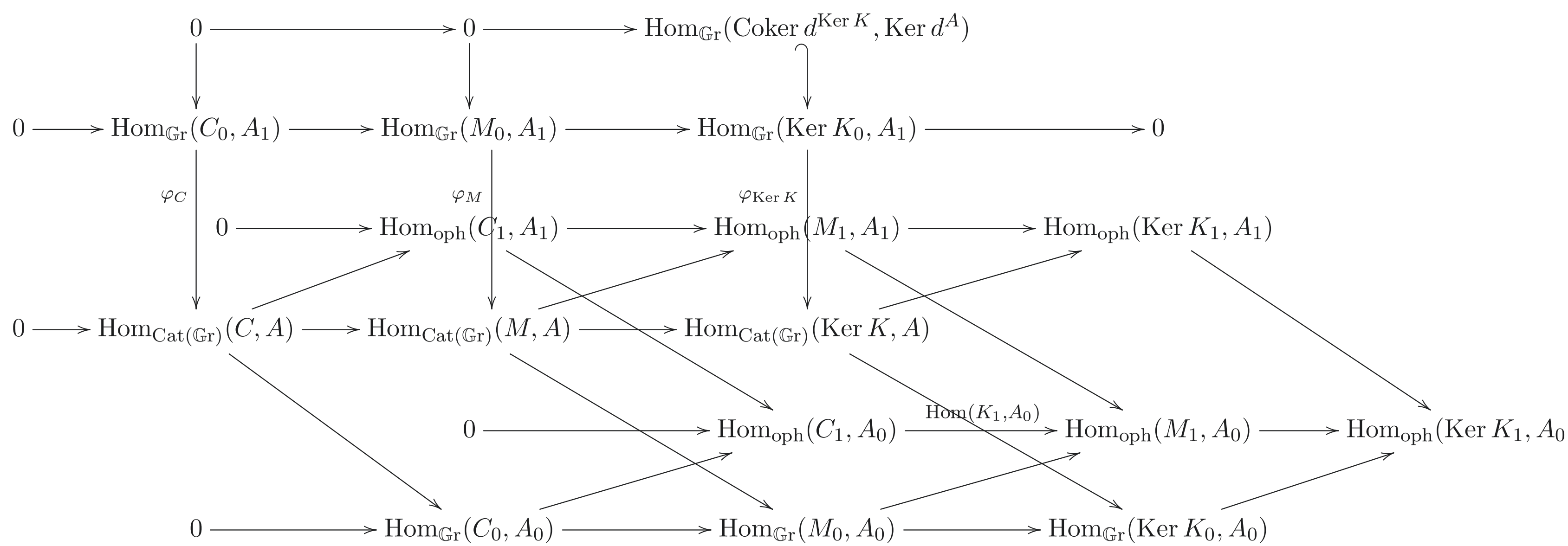
(ii) *C_0 is projective in $\mathbb{Ab}$;*

(iii) *d^C is a split epimorphism;*

(iv) *d^A is a split monomorphism,*

then a Kan extension of T along K exists if and only if

$$T\big|_{\operatorname{Ker} K} \approx 0 \quad \text{and} \quad \operatorname{Hom}_{\mathbb{Ab}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) = 0.$$



0
0
$\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Coker}\, d^{\mathrm{Ker}\, K}, \mathrm{Ker}\, d^A)$
0
$\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(C_0, A_1)$
$\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$
$\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Ker}\, K_0, A_1)$
0
φ_C
φ_M
$\varphi_{\mathrm{Ker}\, K}$
0
$\mathrm{Hom}_{\mathrm{oph}}(C_1, A_1)$
$\mathrm{Hom}_{\mathrm{oph}}(M_1, A_1)$
$\mathrm{Hom}_{\mathrm{oph}}(\mathrm{Ker}\, K_1, A_1)$
0
$\mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(C, A)$
$\mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(M, A)$
$\mathrm{Hom}_{\mathrm{Cat}(\mathbb{G}\mathrm{r})}(\mathrm{Ker}\, K, A)$
0
$\mathrm{Hom}_{\mathrm{oph}}(C_1, A_0)$
$\mathrm{Hom}(K_1, A_0)$
$\mathrm{Hom}_{\mathrm{oph}}(M_1, A_0)$
$\mathrm{Hom}_{\mathrm{oph}}(\mathrm{Ker}\, K_1, A_0)$
0
$\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(C_0, A_0)$
$\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_0)$
$\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Ker}\, K_0, A_0)$

Fig. 1.

Proof. It follows from the conditions that C is also connected. From the results of Sec. 4.2 we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(C_0, A_1) & \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(M_0, A_1) & \longrightarrow & \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(\mathrm{Ker}\, K_0, A_1) & \xrightarrow{d} & \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_0, A_1) \\ & & \downarrow{\scriptstyle \varphi_C} & & \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_{\mathrm{Ker}\, K}} & & \downarrow{\scriptstyle \zeta} \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A) & \xrightarrow{\mathrm{Hom}(K,A)} & \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A) & \longrightarrow & \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(\mathrm{Ker}\, K, A) & \xrightarrow{\delta} & \mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A) \end{array} \tag{4.3.3}$$

from which we obtain the complex of abelian groups

$$\mathrm{Ker}\, d \cap \mathrm{Ker}\, \varphi_{\mathrm{Ker}\, K} \xrightarrow{\varkappa} \widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A) \xrightarrow{\widetilde{\mathrm{Hom}}(K,A)} \widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A) \longrightarrow$$
$$\longrightarrow \widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(\mathrm{Ker}\, K, A). \tag{4.3.4}$$

Since M is connected, φ_M is a monomorphism, and from Lemma 4.2.9 it follows that $\varkappa$ is a monomorphism. Recall that ζ is induced by the homomorphisms $\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(d^C, A_1)$ and $\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_0, d^A)$; thus under the conditions of the theorem it follows that ζ is either zero or a monomorphism in (4.3.3). Again by Lemma 4.2.9, (4.3.4) is an exact sequence. We have $\mathrm{cl}\, T \in \mathrm{Hom}(M, A)$; from the exactness of (4.3.4) there exists $R : C \longrightarrow A$ with $RK \approx T$ if and only if $T\big|_{\mathrm{Ker}\, K} \approx 0$, and R is unique up to isomorphism with this property if and only if $\widetilde{\mathrm{Hom}}(K, A)$ is a monomorphism, which is equivalent to the condition $\mathrm{Ker}\, d \cap \mathrm{Ker}\, \varphi_{\mathrm{Ker}\, K} = 0$. Since ζ is a monomorphism, we have $\mathrm{Ker}\, \varphi_{\mathrm{Ker}\, K} \hookrightarrow \mathrm{Ker}\, d$, and so we obtain the condition $\mathrm{Ker}\, \varphi_{\mathrm{Ker}\, K} = 0$, which is equivalent to the condition $\mathrm{Hom}_{\mathbb{A}\mathrm{b}}(\mathrm{Coker}\, d^{\mathrm{Ker}\, K}, \mathrm{Ker}\, d^A) = 0$. □

Note that if condition (iv) holds, then the condition $\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Coker}\, d^{\mathrm{Ker}\, K}, \mathrm{Ker}\, d^A) = 0$ is automatically satisfied. See also the remark at the end of the proof of Proposition 4.3.6.

Now we shall consider the case where K is an injective homomorphism. This is more complicated, and we have to establish the necessary and sufficient conditions under stronger restrictions than for the case of an epimorphism.

Definition 4.3.5. Let $F : A \longrightarrow B$ be an internal functor. We shall say that F is a contractible functor if $F \approx 0$.

Theorem 4.3.6. *Let A, C, M be internal categories in* $\mathbb{A}\mathrm{b}$ *and $K : M \longrightarrow C$ an injective internal functor. Suppose that C and M are connected and the following conditions hold:*

(i) *d^C has a section;*

(ii) *$d^{C/M}$ has a section;*

(iii) *we have an inclusion* $\mathrm{Ker}\, d^C \hookrightarrow M_1$.

There exists a Kan extension of $T : M \longrightarrow A$ along K if and only if $\widetilde{\delta}(T) = 0$ ($\widetilde{\delta} : \widetilde{\mathrm{Hom}}(M, A) \longrightarrow \widetilde{\mathrm{Ext}}^1(C/M, A)$), and every internal functor $C/M \longrightarrow A$ is contractible.

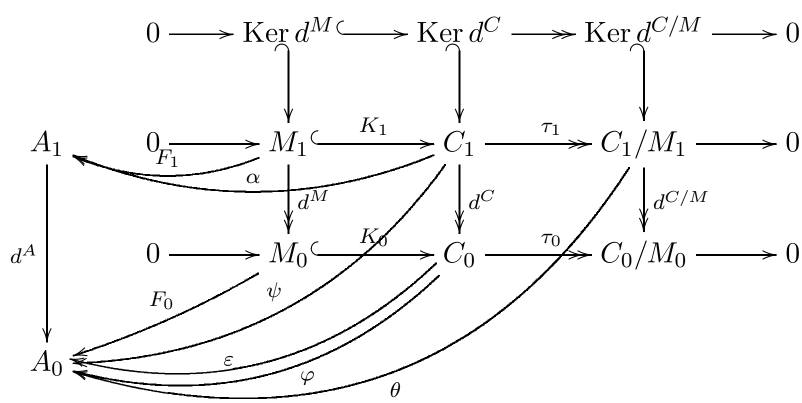
Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{K_1} & C_1 & \xrightarrow{\tau_1} & C_1/M_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^M} & & \downarrow{\scriptstyle d^C} & & \downarrow{\scriptstyle d^{C/M}} & & \\ 0 & \longrightarrow & M_0 & \xrightarrow[K_0]{} & C_0 & \xrightarrow[\tau_0]{} & C_0/M_0 & \longrightarrow & 0 \end{array} \tag{4.3.5}$$

with exact rows. We see that if C is connected, then the quotient category C/M is also a connected category. (4.3.5) induces the commutative diagram shown in Fig. 2.

Here the first row is exact; by Lemma 4.2.8 the second row is a complex and we have exactness in $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C/M, A)$ and $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A)$ in general.

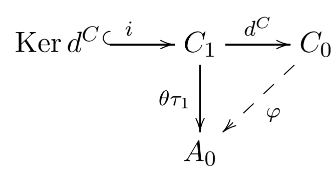
We shall show that under the conditions of the theorem, the second row is also exact in $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A)$. Let $F \in \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A)$ and $\delta(F) = 0$. Then there exist $F_1 \in \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(M_1, A_1)$ and $F_0 \in \mathrm{Hom}(M_0, A_0)$ such that $\partial_1(F_1) = 0$ and $\partial_0(F_0) = 0$ and $\mathrm{Hom}(M_1, d^A)(F_1) = \mathrm{Hom}(d^M, A_0)(F_0)$. From the exactness in $\mathrm{Hom}(M_1, A_1)$ and $\mathrm{Hom}(M_0, A_0)$, there exist $\alpha \in \mathrm{Hom}(C_1, A_1)$ and $\varepsilon \in \mathrm{Hom}(C_0, A_0)$ such that $\alpha \longmapsto F_1$ and $\varepsilon \longmapsto F_0$; but $\mathrm{Hom}(C_1, d^A)(\alpha) \neq \mathrm{Hom}(d^C, A_0)(\varepsilon)$ in general. Taking the difference $\psi = d^A\alpha - \varepsilon d^C$, we have $\mathrm{Hom}(K_1, A_0)(\psi) = 0$, and by the exactness of the corresponding row there exists $\theta \in \mathrm{Hom}(C_1/M_1, A_0)$ with $\mathrm{Hom}(\tau_1, A_0)(\theta) = \psi$, where τ_1 is the natural epimorphism from (4.3.5). The diagram is



(4.3.6)

We have $\theta\tau_1 = d^A\alpha - \varepsilon d^C$ and $\theta\tau_1\big|_{M_1} = 0$.

Since, from condition (iii) of the theorem, $\mathrm{Ker}\, d^C \hookrightarrow M_1$, we obtain $\theta\tau_1\big|_{Ker d^C} = 0$, and in the diagram



(4.3.7)

there exists $\varphi : C_0 \longrightarrow A_0$ with $\varphi d^C = \theta\tau_1 = \psi$. Taking $\varphi + \varepsilon : C_0 \longrightarrow A_0$, we have

$$(\varphi + \varepsilon)d^C = \varphi d^C + \varepsilon d^C = \psi + \varepsilon d^C = d^A\alpha - \varepsilon d^C + \varepsilon d^C = d^A\alpha.$$

Now it remains to show that $(\varphi + \varepsilon)K_0 = F_0$. Since $\varepsilon K_0 = F_0$, it is sufficient to show that $\varphi K_0 = 0$. Consider the composite $\varphi K_0 d^M$, where d^M is an epimorphism; by (4.3.6) and (4.3.7) we have

$$\varphi K_0 d^M = \varphi d^C K_1 = (\theta\tau_1)K_1 = 0,$$

since $\tau_1 K_1 = 0$. Thus $\varphi K_0 = 0$. This completes the proof of the exactness in $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A)$.

In (4.3.7), by definition,

$$\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\ )}(*, A) = \mathrm{Coker}\, \varphi_*, \quad \widetilde{\mathrm{Ext}}_{\mathrm{Cat}(\ )}(*, A) = \mathrm{Coker}\, \zeta_*.$$

By Lemma 4.2.9 (v), since φ_M is a monomorphism, $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(\tau, A)$ is also a monomorphism. From the conditions of the theorem it follows that $\zeta_{C/M}$ is a monomorphism. It is easy to check that we have the exactness in $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A)$.

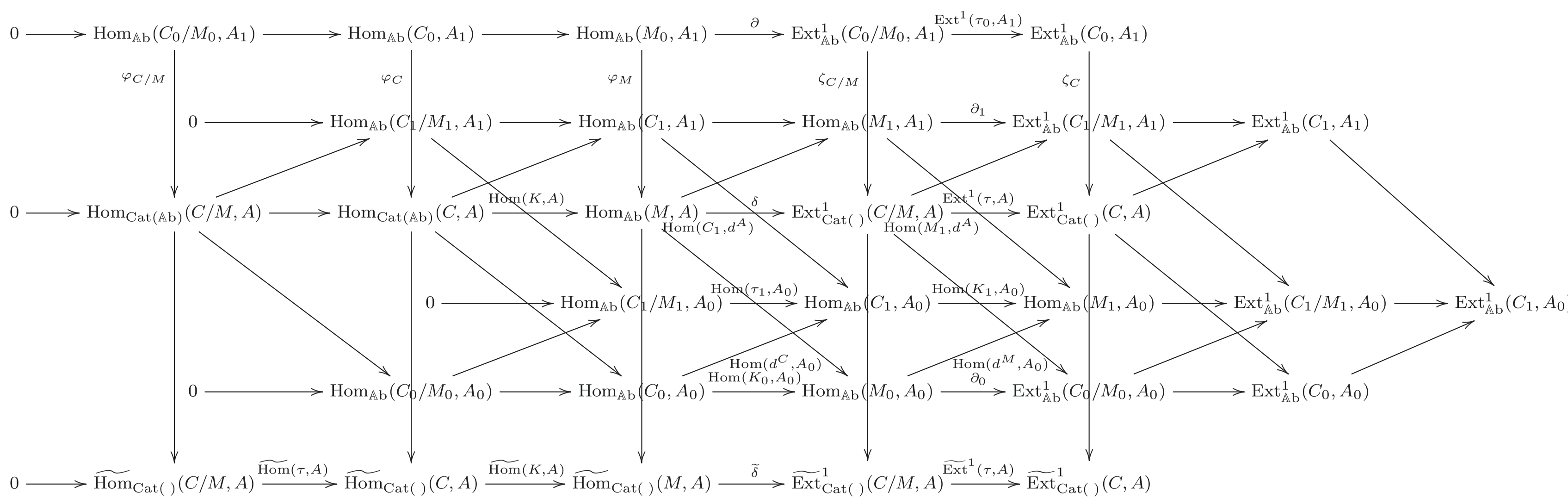

0
Hom_Ab(C_0/M_0, A_1)
Hom_Ab(C_0, A_1)
Hom_Ab(M_0, A_1)
∂
Ext^1_Ab(C_0/M_0, A_1)
Ext^1(τ_0, A_1)
Ext^1_Ab(C_0, A_1)
φ_{C/M}
φ_C
φ_M
ζ_{C/M}
ζ_C
0
Hom_Ab(C_1/M_1, A_1)
Hom_Ab(C_1, A_1)
Hom_Ab(M_1, A_1)
∂_1
Ext^1_Ab(C_1/M_1, A_1)
Ext^1_Ab(C_1, A_1)
0
Hom_Cat(Ab)(C/M, A)
Hom_Cat(Ab)(C, A)
Hom(K, A)
Hom_Ab(M, A)
δ
Hom(C_1, d^A)
Ext^1_Cat( )(C/M, A)
Ext^1(τ, A)
Hom(M_1, d^A)
Ext^1_Cat( )(C, A)
0
Hom_Ab(C_1/M_1, A_0)
Hom(τ_1, A_0)
Hom_Ab(C_1, A_0)
Hom(K_1, A_0)
Hom_Ab(M_1, A_0)
Ext^1_Ab(C_1/M_1, A_0)
Ext^1_Ab(C_1, A_0)
0
Hom_Ab(C_0/M_0, A_0)
Hom_Ab(C_0, A_0)
Hom(d^C, A_0)
Hom(K_0, A_0)
Hom_Ab(M_0, A_0)
Hom(d^M, A_0)
∂_0
Ext^1_Ab(C_0/M_0, A_0)
Ext^1_Ab(C_0, A_0)
0
~Hom_Cat( )(C/M, A)
~Hom(τ, A)
~Hom_Cat( )(C, A)
~Hom(K, A)
~Hom_Cat( )(M, A)
~δ
~Ext^1_Cat( )(C/M, A)
~Ext^1(τ, A)
~Ext^1_Cat( )(C, A)

Fig. 2.

We shall show that under the conditions of the theorem we have the exactness in $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A)$. It is obvious from diagram (4.3.7) that

$$\widetilde{\delta} \cdot \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(K, A) = 0.$$

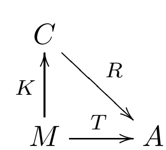
Let $\mathrm{cl}\, F \in \widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\ )}(M, A)$ and $\widetilde{\delta}(\mathrm{cl}\, F) = 0$. Then for $F \in \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A)$ there exists $y \in \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(C_0/M_0, A_1)$ such that $\delta(F) = \zeta_{C/M}(y)$. Now we have

$$\zeta_C \,\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(\tau_0, A_1)(y) = \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(\tau, A)\zeta_{C/M}(y) = \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(\tau, A)\delta(F) = 0.$$

From the conditions of the theorem it follows that ζ_C is a monomorphism and therefore $\mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(\tau_0, A_1)(\psi) = 0$. By the exactness of the first row, there exists an element $z \in \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(M_0, A_1)$ such that $\partial(z) = y$ and $\delta\varphi_M(z) = \zeta_{C/M}(y) = \delta(F)$. Thus $\delta(F - \varphi_M(z)) = 0$. By the exactness of the second row in $\mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A)$ it follows that there is an element $L \in \mathrm{Hom}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(C, A)$ such that $\mathrm{Hom}_{\mathrm{Cat}}(K, A)(L) = F - \varphi_M(z)$, and from the commutativity of diagram (4.3.7) we obtain $\widetilde{\mathrm{Hom}}(K, A)(\mathrm{cl}\, L) = \mathrm{cl}\, F$, which proves the exactness in $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(M, A)$ and completes the proof of the theorem. □

Remark. From the proof of Theorem 4.3.6 it is not difficult to see that conditions (i) and (ii) in the theorem can be replaced by the condition: d^A is a split monomorphism, which is equivalent to the condition that A is (internally) equivalent to the discrete internal category (Serction 1.3., Proposition 1.3.14). Also, condition (iii) can be replaced by: d^A is a split epimorphism. It can be proved that this condition means that A is equivalent to the one-object internal category $\mathrm{Ker}\, d^A \longrightarrow 0$, i.e., to the abelian group $\mathrm{Ker}\, d^A$ considered as an internal category. This remark concerns also conditions (iii) and (iv) of Theorem 4.3.4.

Remark. Suppose that all categories A, C, M are connected internal categories with only one object in the category of groups; i.e., $d^A = d^C = d^M = 0$ and $A_0 = C_0 = M_0 = 0$. These conditions imply that A_1, C_1, and M_1 are abelian groups, internal functors are abelian group homomorphisms, and an isomorphism between internal functors is an equality. In this case the notion of an internal Kan extension (Definition 4.2.1) reduces to the notion of a unique extension of a homomorphism in the category $\mathbb{A}\mathrm{b}$



.

For the case where K is surjective, we find that there exists a unique homomorphism $R : C \longrightarrow A$ with $RK = T$ if and only if $T\big|_{\mathrm{Ker}\, K} = 0$. We obtain the same condition from Theorem 4.3.4, where in our case condition (iii) (d^C is a split epimorphism) is automatically satisfied. Note that we have $\mathrm{Coker}\, d^{\mathrm{Ker}\, K} = 0$, since $d^{\mathrm{Ker}\, K} = 0$ and consequently $\mathrm{Hom}_{\mathbb{A}\mathrm{b}}(\mathrm{Coker}\, d^{\mathrm{Ker}\, K}, \mathrm{Ker}\, d^A) = 0$ always.

Now suppose that K is an injective homomorphism. For the case of abelian groups we find that there exists a unique homomorphism $R : C \longrightarrow A$ satisfying the condition $RK = T$ if and only if $\delta(T) = 0$ ($\delta : \mathrm{Hom}_{\mathbb{A}\mathrm{b}}(M, A) \longrightarrow \mathrm{Ext}^1_{\mathbb{A}\mathrm{b}}(\mathrm{Coker}\, K, A)$) and $\mathrm{Hom}_{\mathbb{A}\mathrm{b}}(\mathrm{Coker}\, K, A) = 0$. As we have mentioned (see Remark above), condition (iii) in Theorem 4.3.6 can be replaced by the condition: d^A is a split epimorphism. Thus in our case all the conditions of Theorem 4.3.6 are satisfied, and we obtain the same necessary and sufficient conditions which we have for the unique extension of a homomorphism between abelian groups.

4.4. Nonconnected Case

In this section, under the same assumptions as for the connected case, we give the necessary and sufficient conditions for the existence of internal Kan extensions in the case where M is any internal category in $\mathbb{G}\mathrm{r}$ and K in the diagram (4.2.4) is a surjective internal functor.

Let $K = (K_0, K_1)$ be an internal functor $M \longrightarrow C$. As in Sec. 4.2. we denote by $\operatorname{Ker} K$ the internal category $d^{\operatorname{Ker} K} : \operatorname{Ker} K_1 \longrightarrow \operatorname{Ker} K_0$, where $d^{\operatorname{Ker} K} = d^M\big|_{\operatorname{Ker} K_1}$.

Theorem 4.4.1. *Let $C = (C_0, C_1)$, $M = (M_0, M_1) \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, $A = (A_0, A_1) \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$, $K = (K_0, K_1) : M \longrightarrow C$ be a surjective internal functor such that the injection $\operatorname{Ker} K_0 \longrightarrow M_0$ has a retraction. There exists a Kan extension of $T = (T_0, T_1) : M \longrightarrow A$ along $K : M \longrightarrow C$ if and only if $T\big|_{\operatorname{Ker} K} \approx 0$ and*

$$\operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) = 0.$$

Proof. Similarly to the proof of Theorem 4.3.2, we obtain in $\mathbb{A}\mathrm{b}$ the commutative diagram shown in Fig. 3.

Since $I_0 : \operatorname{Ker} K_0 \longmapsto M_0$ has a retraction, $\operatorname{Hom}(I_0, A_1)$ is an epimorphism. Each row and column in diagram shown in Fig. 3 is exact. Applying the Snake lemma to this diagram and Proposition 4.2.6, we obtain the exact sequence of abelian groups

$$\operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^M, \operatorname{Ker} d^A) \xrightarrow{\varphi} \operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) \xrightarrow{\psi} \widetilde{\operatorname{Hom}}_{\mathrm{Cat}}(C, A)$$

$$\xrightarrow{\widetilde{\operatorname{Hom}}(K,A)} \widetilde{\operatorname{Hom}}_{\mathrm{Cat}}(M, A) \xrightarrow{\widetilde{\operatorname{Hom}}(I,A)} \widetilde{\operatorname{Hom}}_{\mathrm{Cat}}(\operatorname{Ker} K, A)\ . \qquad (4.4.1)$$

Suppose that the conditions of the theorem hold. We shall show that there exists a Kan extension of T along K.

Since $T\big|_{\operatorname{Ker} K} \approx 0$, this means that $\widetilde{\operatorname{Hom}}(I, A)(\operatorname{cl} T) = 0$. By the exactness of (4.4.1), there exists an internal functor $R \in \operatorname{Hom}_{\mathrm{Cat}}(C, A)$ such that $\widetilde{\operatorname{Hom}}(K, A)(\operatorname{cl} R) = \operatorname{cl} T$, which is equivalent to the condition that there exists an isomorphism $\varepsilon : RK \xrightarrow{\approx} T$. Suppose that there is an internal functor $S : C \longrightarrow A$ with $\alpha : SK \xrightarrow{\approx} T$. This gives an equality $\operatorname{cl} SK = \operatorname{cl} RK = \operatorname{cl} T$. Since $\operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) = 0$, $\widetilde{\operatorname{Hom}}(K, A)$ in (4.4.1) is a monomorphism. Thus we have an isomorphism $S \approx R$. We have to show that there exists a unique isomorphism $\sigma : S \longrightarrow R$, with $\sigma K_0 = -\varepsilon + \alpha$. From the diagram shown in Fig. 3, we have the following commutative diagram:

$$\begin{array}{ccccc} & & -\varepsilon + \alpha & \xmapsto{\operatorname{Hom}(I_0, A_1)} & \varkappa \\ & & \downarrow & & \downarrow \varphi_{\operatorname{Ker} K} \\ R - S & \xmapsto{\operatorname{Hom}(K,A)} & RK - SK & \xmapsto{\operatorname{Hom}(I,A)} & 0 \end{array} \qquad (4.4.2)$$

Since $\operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) = 0$, $\varphi_{\operatorname{Ker} K}$ is a monomorphism. So in (4.4.2) $\varkappa = 0$. From the exactness of the corresponding row of the diagram shown in Fig. 3, we conclude that there exists a unique $\sigma \in \operatorname{Hom}_{Gr}(C_0, A_1)$ such that $\sigma \xmapsto{K_0} -\varepsilon + \alpha$.

Since the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbb{G}\mathrm{r}}(C_0, A) & \xrightarrow{\operatorname{Hom}(K_0, A_1)} & \operatorname{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1) \\ \varphi_C \downarrow & & \downarrow \varphi_M \\ \operatorname{Hom}_{\mathrm{Cat}}(C, A) & \xrightarrow[\operatorname{Hom}(K,A)]{} & \operatorname{Hom}_{\mathrm{Cat}}(M, A) \end{array}$$

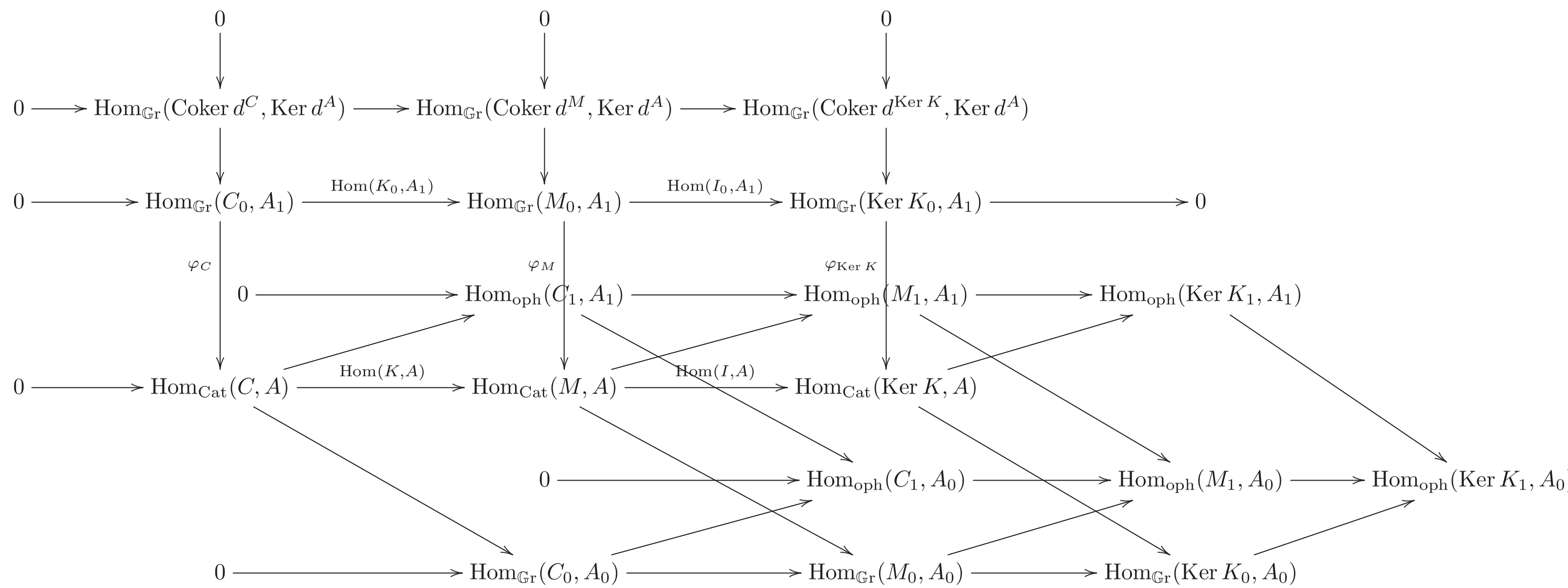

0
0
0
0
$\mathrm{Hom}_{\mathbb{Gr}}(\mathrm{Coker}\, d^C, \mathrm{Ker}\, d^A)$
$\mathrm{Hom}_{\mathbb{Gr}}(\mathrm{Coker}\, d^M, \mathrm{Ker}\, d^A)$
$\mathrm{Hom}_{\mathbb{Gr}}(\mathrm{Coker}\, d^{\mathrm{Ker}\, K}, \mathrm{Ker}\, d^A)$
0
$\mathrm{Hom}_{\mathbb{Gr}}(C_0, A_1)$
$\mathrm{Hom}(K_0, A_1)$
$\mathrm{Hom}_{\mathbb{Gr}}(M_0, A_1)$
$\mathrm{Hom}(I_0, A_1)$
$\mathrm{Hom}_{\mathbb{Gr}}(\mathrm{Ker}\, K_0, A_1)$
0
φ_C
φ_M
$\varphi_{\mathrm{Ker}\, K}$
0
$\mathrm{Hom}_{\mathrm{oph}}(C_1, A_1)$
$\mathrm{Hom}_{\mathrm{oph}}(M_1, A_1)$
$\mathrm{Hom}_{\mathrm{oph}}(\mathrm{Ker}\, K_1, A_1)$
0
$\mathrm{Hom}_{\mathrm{Cat}}(C, A)$
$\mathrm{Hom}(K, A)$
$\mathrm{Hom}_{\mathrm{Cat}}(M, A)$
$\mathrm{Hom}(I, A)$
$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Ker}\, K, A)$
0
$\mathrm{Hom}_{\mathrm{oph}}(C_1, A_0)$
$\mathrm{Hom}_{\mathrm{oph}}(M_1, A_0)$
$\mathrm{Hom}_{\mathrm{oph}}(\mathrm{Ker}\, K_1, A_0)$
0
$\mathrm{Hom}_{\mathbb{Gr}}(C_0, A_0)$
$\mathrm{Hom}_{\mathbb{Gr}}(M_0, A_0)$
$\mathrm{Hom}_{\mathbb{Gr}}(\mathrm{Ker}\, K_0, A_0)$

Fig. 3.

is commutative and $\mathrm{Hom}(K, A)$ is a monomorphism, we obtain $\varphi_C(\sigma) = R - S$, which means that σ is a morphism of internal functors $\sigma : S \longrightarrow R$; this proves that (R, ε) is a Kan extension.

Now suppose that there exists a Kan extension (R, ε) of T along K. Then from the diagram shown in Fig. 3 we have $\mathrm{Hom}(I, A)(T) \approx 0$, which means that $T_{\mathrm{Ker}\,K} \approx 0$. Since R is unique up to isomorphism with the property $RK \xrightarrow{\approx} T$, from the exact sequence (4.4.1) we obtain that $\widetilde{\mathrm{Hom}}(K, A)$ is a monomorphism so that $\psi = 0$ in (4.4.1). Next we shall show that $\varphi = 0$ in (4.4.1), from which it follows that $\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Coker}\, d^{\mathrm{Ker}\,K}, \mathrm{Ker}\, d^A) = 0$. Let $\beta \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$ and $\varphi_M(\beta) = 0$; therefore we can consider β as a morphism of internal functors $\beta : RK \longrightarrow RK$. By Proposition 4.2.5 we have the bijection $\mathrm{Hom}(R, R) \longrightarrow \mathrm{Hom}(RK, RK)$. Thus there is a morphism $\gamma : R \longrightarrow R$ with $\gamma K_0 = \beta$. In the diagram shown in Fig. 3, we have

$$\gamma \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(C_0, A_1), \quad \mathrm{Hom}(K_0, A_1)(\gamma) = \beta.$$

From this it follows that $\varphi = 0$, which completes the proof of the theorem. □

In the case, where $A, C, M \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$, the condition "the injection $I_0 : \mathrm{Ker}\, K_0 \longmapsto M_0$ has a retraction" can be replaced by one of the conditions (i)–(iv) of Theorem 4.3.4. We do not give here the proof of this theorem, since it is based on arguments analogous to those given for the connected case in Sec. 4.3 (Theorem 4.3.4).

Chapter 5

ACTORS IN CATEGORIES OF INTEREST

This chapter is dedicated to questions of the definition, the existence, and the construction of an actor for the objects in categories of interest. For an object A of a category of interest $\mathbf{C}$ we construct the group with operations $\mathfrak{B}(A)$ and the semidirect product $\mathfrak{B}(A) \ltimes A$ and prove that there exists an actor of A in $\mathbf{C}$ if and only if $\mathfrak{B}(A) \ltimes A \in \mathbf{C}$. The examples of groups, associative, Lie, Leibniz and alternative algebras, modules over some ring, crossed modules and precrossed modules in the category of groups are discussed.

5.1. Preliminary Definitions and Results

This section contains well-known definitions and results that will be used in what follows.

Let $\mathbf{C}$ be a category of interest with a set of operations Ω and with a set of identities $\mathbf{E}$ (see Sec. 3.1 for the definition).

We will write the right side of Axiom 2 in the definition of a category of interest as $W(x_1, x_2; x_3; *, \overline{*})$.

As in Sec. 3.1 we denote by $\mathbf{E}_G$ the subset of identities of $\mathbf{E}$, which includes the group laws and the identities (c) and (d) from the definition of a category of groups with operations (see Sec. 1.1). We denote by $\mathbf{C}_G$ the corresponding category of groups with operations. Thus we have $\mathbf{E}_G \hookrightarrow \mathbf{E}$, $\mathbf{C} = (\Omega, \mathbf{E})$, $\mathbf{C}_G = (\Omega, \mathbf{E}_G)$, and there is a full inclusion functor $\mathbf{C} \hookrightarrow \mathbf{C}_G$.

In the case of associative algebras with multiplication represented by $*$, we have $\Omega_2' = \{*, *^\circ\}$. For Lie algebras take $\Omega_2' = ([\ ,\ ], [\ ,\ ]^\circ)$ (where $[a, b]^\circ = [b, a] = -[a, b]$). For Leibniz algebras (see the definition below), take $\Omega_2' = ([\ ,\ ], [\ ,\ ]^\circ)$ (here $[a, b]^\circ = [b, a]$). It is easy to see that all these algebras are categories of interest. In the example of groups, $\Omega_2' = \varnothing$.

We recall the following definitions and the results from [76].

Definition 5.1.1 ([76])**.** Let $C \in \mathbf{C}$. A subobject of C is called an ideal if it is the kernel of some morphism.

Theorem 5.1.2 ([76]). *Let A be a subobject of B in $\mathbf{C}$. Then A is an ideal of B if and only if the following conditions hold;*

(i) *A is a normal subgroup of B;*

(ii) *For any $a \in A$, $b \in B$ and $* \in \Omega'_2$, we have $a * b \in A$.*

For the definition of a split extension and B-structure on A, $A, B \in \mathbf{C}$ we refer the reader to Sec. 3.1.

As in [76] and Chap. 3, for $A, B \in \mathbf{C}$ we will say we have "a set of actions of B on A", whenever there is a set of maps $f_* : B \times A \longrightarrow A$, for each $* \in \Omega_2$.

A B-structure induces a set of actions of B on A corresponding to the operations in $\mathbf{C}$. If

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0 \tag{5.1.1}$$

is a split extension in $\mathbf{C}$, with the section $s : B \longrightarrow E$, $ps = 1_B$, then for $b \in B$, $a \in A$, and $* \in {\Omega_2}'$ we have

$$b \cdot a = s(b) + a - s(b), \tag{5.1.2}$$

$$b * a = s(b) * a. \tag{5.1.3}$$

(5.1.2) and (5.1.3) are called derived actions of B on A in [76] and split derived actions in Chap. 3, since we considered there the actions derived from nonsplit extensions too when A is a singular object.

Given a set of actions of B on A (one for each operation in Ω_2), let $B \ltimes A$ be a universal algebra whose underlying set is $B \times A$ and whose operations are

$$(b', a') + (b, a) = (b' + b, a' + b' \cdot a),$$
$$(b', a') * (b, a) = (b' * b, a' * a + a' * b + b' * a).$$

Theorem 5.1.3 ([76]). *A set of actions of B on A is a set of derived actions if and only if $B \ltimes A$ is an object of $\mathbf{C}$.*

Together with the condition on the set of derived actions given in the theorem above, we will need Proposition 3.1.1 of Chap. 3, where the identities are given which satisfy the set of derived actions in the case $A, B \in \mathbf{C}_G$ and which guarantee that the set of actions is a set of derived actions in $\mathbf{C}_G$.

As we remarked in Sec. 3.1, if we are in the category $\mathbf{C}$ with the set of identities $\mathbf{E}$, conditions 1–12 of the Proposition are necessary conditions. In every concrete case it is possible, according to other identities included in $\mathbf{E}$, to write the corresponding conditions for derived actions that will be necessary and sufficient for the set of actions to be a set of derived actions (i.e. for $B \ltimes A \in \mathbf{C}$). Denote all these identities of derived actions by $\widetilde{\mathbf{E}}_G$ and $\widetilde{\mathbf{E}}$ respectively. If the addition is commutative in $\mathbf{C}$, then $\widetilde{\mathbf{E}}$ (resp. $\widetilde{\mathbf{E}}_G$) consists of the same kind of identities that we have in $\mathbf{E}$ (resp. in $\mathbf{E}_G$), written down for the elements from the set $A \cup B$, whenever each identity has a sense. We will denote by $\widetilde{\text{Axiom 2}}$ the identities for the action in $\mathbf{C}$, which correspond to Axiom 2 (see Chap. 3). In the category of groups, Lie, associative, and Leibniz algebras derived actions are called simply actions. We will use this terminology in these special cases; we will also say "an action in $\mathbf{C}$" if it is a derived action, and we will say a set of actions is not an action in $\mathbf{C}$ if this set is not a set of derived actions. Recall that a left action of a group B on A is a map $\varepsilon : B \times A \longrightarrow A$, which we denote by $\varepsilon(b, a) = b \cdot a$, with the conditions

$$(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a),$$
$$0 \cdot a = a,$$
$$b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2.$$

The right action is defined in an analogous way.

All algebras below are considered over a commutative ring k with unit.

In the case of associative algebras, an action of B on A is a pair of bilinear maps

$$B \times A \longrightarrow A, \quad A \times B \longrightarrow A, \tag{5.1.4}$$

which we denote respectively by $(b, a) \longmapsto b * a$, $(a, b) \longmapsto a * b$, with conditions

$$\begin{aligned}
(b_1 * b_2) * a &= b_1 * (b_2 * a), \\
a * (b_1 * b_2) &= (a * b_1) * b_2, \\
(b_1 * a) * b_2 &= b_1 * (a * b_2), \\
b * (a_1 * a_2) &= (b * a_1) * a_2, \\
(a_1 * a_2) * b &= a_1 * (a_2 * b), \\
a_1 * (b * a_2) &= (a_1 * b) * a_2.
\end{aligned}$$

Here the associative algebra operation is denoted by $*$ (resp. $a_1 * a_2$), and the corresponding action by the same sign $*$ (respectively, $b * a$).

Recall that a Lie algebra $(A, (\ ,\ ))$ over k is given by a k-module A and a k-module homomorphism $(\ ,\ ) : A \otimes_k A \longrightarrow A$, called a round bracket, such that the equation

$$(x, x) = 0$$

and Jacobi identity

$$((x, y), z) + ((y, z), x) + ((z, x), y) = 0 \tag{5.1.5}$$

hold for $x, y, z \in A$.

For Lie algebras an action of B on A is a bilinear map $B \times A \longrightarrow A$, where the result of action is denoted by (b, a), which satisfies the conditions

$$\begin{aligned}
((b_1, b_2), a) &= (b_1, (b_2, a)) - (b_2, (b_1, a)), \\
((b, (a_1, a_2)) &= (a_1, (b, a_2)) + ((b, a_1), a_2).
\end{aligned}$$

Note that we actually have above again two bilinear maps (5.1.4): $b, a \longmapsto (b, a)$, $a, b \longmapsto (a, b)$ with the conditions

$$\begin{aligned}
(b, a) &= -(a, b), \\
((b_1, b_2), a) + ((b_2, a), b_1) + ((a, b_1), b_2) &= 0, \\
((b, a_2), a_1) + ((a_2, a_1), b) + ((a_1, b), a_2) &= 0.
\end{aligned}$$

Recall from [62] that a Leibniz algebra L over a commutative ring k with unit is a k-module equipped with a bilinear map $[-, -] : \mathrm{L} \times \mathrm{L} \to \mathrm{L}$ which satisfies the following identity, called the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for all $x, y, z \in \mathrm{L}$.

Obviously, when $[x, x] = 0$ for all $x \in \mathrm{L}$, the Leibniz bracket is skew-symmetric; therefore the Leibniz identity comes down to the Jacobi identity, and a Leibniz algebra is then just a Lie algebra.

For Leibniz algebras, an action of B on A is a pair of bilinear maps (5.1.4), which we denote by $b, a \longmapsto [b, a]$, $a, b \longmapsto [a, b]$ with the conditions

$$\begin{aligned}
[a_1, [a_2, b]] &= [[a_1, a_2], b] - [[a_1, b], a_2], \\
[a_1, [b, a_2]] &= [[a_1, b], a_2] - [[a_1, a_2], b], \\
[b, [a_1, a_2]] &= [[b, a_1], a_2] - [[b, a_2], a_1], \\
[a, [b_1, b_2]] &= [[a, b_1], b_2] - [[a, b_2], b_1],
\end{aligned}$$

$$[b_1,[a,b_2]] = [[b_1,a],b_2] - [[b_1,b_2],a],$$
$$[b_1,[b_2,a]] = [[b_1,b_2],a] - [[b_1,a],b_2].$$

Recall [83] that a derivation for an algebra A over a ring k is a k-linear map $D : A \longrightarrow A$ with

$$D(a_1,a_2) = (D(a_1),a_2) + (a_1,D(a_2)).$$

The set of all derivations $\mathrm{Der}(A)$ of A with the operation defined by

$$(D,D') = DD' - D'D$$

is a Lie algebra.

We recall the construction of the k-algebra $\mathrm{Bim}(A)$ of bimultipliers of an associative k-algebra A (called multiplications in [50] and bimultiplications in [70]). An element of $\mathrm{Bim}(A)$ is a pair $f = (f*,*f)$ of k-linear maps from A to A with

$$f*(a*a') = (f*a)*a',$$
$$(a*a')*f = a*(a'*f),$$
$$a*(f*a') = (a*f)*a'.$$

We prefer to use the notation $*f$ instead of $f*^\circ$. We denote by $f*a$ (resp. $a*f$) the value $f*(a)$ (resp. $*f(a)$). $\mathrm{Bim}(A)$ is a k-module in an obvious way. The operation in $\mathrm{Bim}(A)$ is defined by

$$f*f' = (f*f'*,*f*f'),$$

and $\mathrm{Bim}(A)$ becomes a k-algebra. Note that here we use notations different from those in [57, 70]. Here, as above, $*$ denotes an operation in associative algebra, and $f*f'*$, $*f*f'$ denote the composites of maps. Thus

$$(f*f'*)(a) = f*(f'*a),$$
$$(*f*f')(a) = (a*f)*f'.$$

For the addition we have

$$f+f' = ((f*)+f'*,*f+(*f')),$$

where

$$((f*)+f'*)(a) = f*a+f'*a,$$
$$(*f+(*f'))(a) = a*f+a*f'.$$

For a Leibniz k-algebra A we define the k-algebra $\mathrm{Bider}(A)$ of biderivations in the following way. An element of $\mathrm{Bider}(A)$ is a pair $\varphi = ([\ ,\varphi],[\varphi,\ ])$ of k-linear maps $A \longrightarrow A$ with

$$\big[[a_1,a_2],\varphi\big] = \big[a_1,[a_2,\varphi]\big] + \big[[a_1,\varphi],a_2\big],$$
$$\big[\varphi,[a_1,a_2]\big] = \big[[\varphi,a_1],a_2\big] - \big[[\varphi,a_2],a_1\big],$$
$$\big[a_1,[a_2,\varphi]\big] = -\big[a_1,[\varphi,a_2]\big].$$

We used above the notation $[\varphi,\ ](a) = [\varphi,a], [\ ,\varphi](a) = [a,\varphi]$. Biderivations were defined by Loday in [62], where another notation is used; biderivation is a pair (d,D), where, according to our definition, $[\varphi,\ ] = D$, $[\ ,\varphi] = -d$, and instead of the third condition we have in [62] $[a_1,d(a_2)] = [a_1,D(a_2)]$.

The operation in $\mathrm{Bider}(A)$ is defined by

$$[\varphi,\varphi'] = \big([\ ,[\varphi,\varphi']],[[\varphi,\varphi'],\ ]\big),$$

where

$$\big[a,[\varphi,\varphi']\big] = \big[[a,\varphi],\varphi'\big] - \big[[a,\varphi'],\varphi\big], \tag{5.1.6$_1$}$$

$$[[\varphi, \varphi'], a] = [\varphi, [\varphi', a]] + [[\varphi, a], \varphi']. \tag{5.1.6$_2$}$$

Note that we can define $[[\varphi, \varphi'], \ ]$ by

$$[[\varphi, \varphi'], a] = -[\varphi, [a, \varphi']] + [[\varphi, a], \varphi']. \tag{5.1.6$'_2$}$$

To avoid confusion, we disregard $*^\circ$ in special cases, e.g., for the $[\ ,\ ]$ operation. Both above given operations define a Leibniz algebra structure on $\mathrm{Bider}(A)$. It is easy to see that the second definition (5.1.6$_1$), (5.1.6$'_2$) gives the algebra which is isomorphic to the biderivation algebra defined in [62]; according to this definition $[(d, D), (d', D')] = (dd' - d'd, Dd' - d'D)$.

We have a set of actions of $\mathrm{Der}(A)$, $\mathrm{Bim}(A)$ and $\mathrm{Bider}(A)$ on A. These actions are defined by

$$\begin{aligned}
&[D, a] = D(a),\\
&f * a = f * (a),\\
&a * f = *f(a),\\
&[\varphi, a] = [\varphi, \ ](a), \quad [a, \varphi] = [\ , \varphi](a),
\end{aligned}$$

where $a \in A$, $D \in \mathrm{Der}(A)$, $f = (f*, *f) \in \mathrm{Bim}(A)$, $\varphi = ([\ , \varphi], [\varphi, \ ]) \in \mathrm{Bider}(A)$ and A is a Lie algebra, an associative algebra, and a Leibniz algebra respectively.

In the case of Lie algebras the action of $\mathrm{Der}(A)$ on A is a set of derived actions; thus this action satisfies the corresponding conditions of an action in $\mathbb{L}\mathrm{ie}$, but for the case of associative and Leibniz algebras these actions do not satisfy all the conditions given above respectively for the action in $\mathbb{A}\mathrm{ss}$ and $\mathbb{L}\mathrm{eibniz}$. Note that for the case of Leibniz algebras if $[\varphi, [\varphi', a]] = -[\varphi, [a, \varphi']]$ for any $a \in A$ and $\varphi, \varphi' \in \mathrm{Bider}(A)$, then the above two ways of defining operations in $\mathrm{Bider}(A)$ are equal, and the action of $\mathrm{Bider}(A)$ becomes a derived action (see below Proposition 5.3.8).

We have an analogous situation for associative algebras. The action of $\mathrm{Bim}(A)$ on A is not a derived action because the condition

$$(f * a) * f' = f * (a * f') \tag{5.1.7}$$

fails. So if we have the condition for associative algebra A that for any two bimultipliers is fulfilled (5.1.7), then the action of $\mathrm{Bim}(A)$ on A defined above is a set of derived actions on A (see below Proposition 5.3.7).

An *alternative algebra* A over a field F is an algebra that satisfies the identities

$$x^2 y = x(xy)$$

and

$$yx^2 = (yx)x$$

for all $x, y \in A$. These identities are known respectively as the left and right alternative laws. We denote the corresponding category of alternative algebras by $\mathbb{A}\mathrm{lt}$. Clearly any associative algebra is alternative. The class of 8-dimensional Cayley algebras is an important class of alternative algebras that are not associative [81].

The axioms above for alternative algebras are equivalent to the following:

$$x(yz) = (xy)z + (yx)z - y(xz)$$

and

$$(xy)z = x(yz) - (xz)y + x(zy)$$

We consider these conditions as Axiom 2, and consequently alternative algebras can be interpreted as a category of interest.

For alternative algebras over a field F, an action of B on A is a pair of bilinear maps (5.1.4), which we denote again by $(b, a) \longmapsto ba$, $(a, b) \longmapsto ab$ with the conditions

$$\begin{aligned}
b(a_1a_2) &= (ba_1)a_2 + (a_1b)a_2 - a_1(ba_2),\\
(a_1a_2)b &= a_1(a_2b) - (a_1b)a_2 + a_1(ba_2),\\
(ba_1)a_2 &= b(a_1a_2) - (ba_2)a_1 + b(a_2a_1),\\
a_1(a_2b) &= (a_1a_2)b + (a_2a_1)b - a_2(a_1b),\\
(b_1b_2)a &= b_1(b_2a) - (b_1a)b_2 + b_1(ab_2),\\
a(b_1b_2) &= (ab_1)b_2 + (b_1a)b_2 - b_1(ab_2),\\
(ab_1)b_2 &= a(b_1b_2) - (ab_2)b_1 + a(b_2b_1),\\
b_1(b_2a) &= (b_1b_2)a + (b_2b_1)a - b_2(b_1a).
\end{aligned}$$

For the definition of a crossed module in the categories of interest, we refer the reader to Sec. 1.1, i.e., the definition is analogous to that of for categories of groups with operations.

Definition 5.1.4. For any object A in $\mathbf{C}$, an actor of A is a crossed module $\partial : A \longrightarrow \mathrm{Actor}(A)$, such that for any object C of $\mathbf{C}$ and an action of C on A there is a unique morphism $\varphi : C \longrightarrow \mathrm{Actor}(A)$ with $c \cdot a = \varphi(c) \cdot a$, $c * a = \varphi(c) * a$ for any $* \in {\Omega_2}'$, $a \in A$, and $c \in C$.

See the equivalent Definition 5.2.9 in Sec. 5.2.

From this definition it follows that an actor $\mathrm{Actor}(A)$, for the object $A \in \mathbf{C}$, with these properties is a unique object up to an isomorphism in $\mathbf{C}$.

Note that according to the universal property of an actor object, for any two elements x, y in $\mathrm{Actor}(A)$ from $x \dot{*} a = y \dot{*} a$ (here we mean equalities for the dot action and the action $*$, for any $* \in {\Omega_2}'$ and any $a \in A$) and $(w_1 \cdots w_n x) \cdot a = (w_1 \cdots w_n y) \cdot a, w_1 \cdots w_n \in \Omega_1'$, it follows that $x = y$.

It is well known that for the case of groups $\mathrm{Actor}(G) = \mathrm{Aut}(G)$; the corresponding crossed module is $\partial : G \longrightarrow \mathrm{Aut}(G)$, where ∂ sends any $g \in G$ to the inner automorphism of G defined by g (i.e. $\partial(g)(g') = g + g' - g$, $g' \in G$). For the case of Lie algebras, $\mathrm{Actor}(A) = \mathrm{Der}(A)$, $A \in \mathbb{L}\mathrm{ie}$, and the operator homomorphism $\partial : A \longrightarrow \mathrm{Der}(A)$ is defined by $\partial(a) = [a,\ ]$, so $\partial(a)(a') = [a, a']$.

As we have seen above, in general, in $\mathbb{A}\mathrm{ss}$ and $\mathbb{L}\mathrm{eibniz}$ the objects $\mathrm{Bim}(A)$ and $\mathrm{Bider}(A)$ do not have derived actions on A in the corresponding categories. So the obvious homomorphisms $A \longrightarrow \mathrm{Bim}(A)$, $A \longrightarrow \mathrm{Bider}(A)$ do not define crossed modules in $\mathbb{A}\mathrm{ss}$ and $\mathbb{L}\mathrm{eibniz}$ for any A from $\mathbb{A}\mathrm{ss}$ and $\mathbb{L}\mathrm{eibniz}$, respectively.

It is well known [75] that for the case of groups if N is a normal subgroup of G and $\tau : N \longrightarrow \mathrm{Inn}(N)$ is the homomorphism sending any element n to the corresponding inner automorphism $(\tau(n)(n') = n + n' - n)$, since G acts on N by conjugation, we have a unique homomorphism $\theta : G \longrightarrow \mathrm{Actor}(N)$, with $\theta(g) \cdot n = g \cdot n$. $\mathrm{Inn}(N)$ is a normal subgroup of $\mathrm{Actor}(N)$, θ extends τ, and we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & G/N & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\theta} & & \vdots & & \\
0 & \longrightarrow & \mathrm{Inn}(N) & \longrightarrow & \mathrm{Actor}(N) & \longrightarrow & \mathrm{Out}(N) & \longrightarrow & 0.
\end{array} \tag{5.1.8}$$

According to the work of R. Lavendhomme and Th. Lucas [57] in the categories $\mathbb{G}r$, $\mathbb{L}\mathrm{ie}$ the actor crossed modules $A \longrightarrow \mathrm{Actor}(A)$ are terminal objects in the categories of crossed modules under A. If $\mathrm{Ann}(A) = (0)$ or $A^2 = A$, then $\mathrm{Bim}(A)$ acts on A, and the corresponding crossed module $A \longrightarrow \mathrm{Bim}(A)$ is a terminal object in the category of crossed modules under A. It is easy to see that in this case

$$\mathrm{Bim}(A) = \mathrm{Actor}(A).$$

Definition 5.1.5. A general actor object GActor(A) for A, $A \in \mathbf{C}$, is an object from $\mathbf{C}_G$, which has a set of actions on A, which is a set of split derived actions in $\mathbf{C}_G$, i.e. satisfies the conditions of Proposition 3.1.1; there is a morphism $d : A \longrightarrow \mathrm{GActor}\,(A)$ in $\mathbf{C}_G$ that defines a crossed module in $\mathbf{C}_G$, and for any object $C \in \mathbf{C}$ and a split derived action of C on A, there exists in $\mathbf{C}_G$ a unique morphism $\varphi : C \longrightarrow \mathrm{GActor}\,(A)$ such that $c \,\dot{*}\, a = \varphi(c) \,\dot{*}\, a$ for any $c \in C$, $a \in A$, $* \in {\Omega_2}'$.

It is easy to see that Bim(A) and Bider(A) are general actor objects for $A \in \mathbb{A}\mathrm{ss}$, $A \in \mathbb{L}\mathrm{eibniz}$ respectively. These constructions satisfy the existence of a commutative diagram like (5.1.8).

5.2. The Main Construction

In this section $\mathbf{C}$ is a category of interest. Let $A \in \mathbf{C}$; consider all split extensions of A in $\mathbf{C}$

$$E_j : 0 \longrightarrow A \xrightarrow{i_j} C_j \xrightarrow{p_j} B_j \longrightarrow 0, \quad j \in \mathbb{J}.$$

Note that it may happen that $B_j = B_k = B$, for $j \neq k$; then these extensions will correspondent to different actions of B on A. Let $\{b_j\cdot, b_j* \mid\ b_j \in B_j,\ * \in \Omega'_2\}$ be the corresponding set of derived actions for $j \in \mathbb{J}$. For any element $b_j \in B_j$, denote $\mathbf{b}_j = \{b_j\cdot, b_j*,\ * \in \Omega'_2\}$. Let $\mathbb{B} = \{\mathbf{b}_j \mid\ b_j \in B_j,\ j \in \mathbb{J}\}$.

Thus each element $\mathbf{b}_j \in \mathbb{B}$, $j \in \mathbb{J}$ is a special type of function $\mathbf{b}_j : \Omega_2 \longrightarrow \mathrm{Maps}(A \longrightarrow A)$, $\mathbf{b}_j(*) = b_j * - : A \longrightarrow A$.

According to Axiom 2, from the definition of a category of interest, we define the $*$ operation, $\mathbf{b}_i * \mathbf{b}_k, * \in \Omega'_2$, for the elements of $\mathbb{B}$ by the equalities

$$(\mathbf{b}_i * \mathbf{b}_k)\overline{*}\,(a) = W(b_i, b_k; a; *, \overline{*}),$$
$$(\mathbf{b}_i * \mathbf{b}_k) \cdot (a) = a.$$

We define the operation of addition by

$$(\mathbf{b}_i + \mathbf{b}_k) \cdot (a) = b_i \cdot (b_k \cdot a),$$
$$(\mathbf{b}_i + \mathbf{b}_k) * (a) = b_i * a + b_k * a.$$

For a unary operation $\omega \in \Omega'_1$ we define

$$\begin{aligned}
\omega(\mathbf{b}_k) \cdot (a) &= \omega(b_k) \cdot (a),\\
\omega(\mathbf{b}_k) * (a) &= \omega(b_k) * (a),\\
\omega(b * b') &= \omega(b) * b' \text{ and we will have } \omega(b) * b' = b * \omega(b'),\\
\omega(b_1 + \cdots + b_n) &= \omega(b_1) + \cdots + \omega(b_n),\\
(-\mathbf{b}_k) \cdot (a) &= (-b_k) \cdot a,\\
(-b) \cdot (a) &= a,\\
(-\mathbf{b}_k) * (a) &= -(b_k * a),\\
(-b) * (a) &= -(b * (a)),\\
-(b_1 + \cdots + b_n) &= -b_n - \cdots - b_1,
\end{aligned}$$

where $b, b', b_1, \ldots, b_n$ are certain combinations of star operations on the elements of $\mathbb{B}$, i.e. the elements of the type $\mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$, $n > 1$.

We do not know if the new functions defined by us are again in $\mathbb{B}$. Denote by $\mathfrak{B}(A)$ the set of functions $(\Omega_2 \longrightarrow \mathrm{Maps}(A \longrightarrow A))$ obtained by performing all kinds of the above-defined operations on elements of $\mathbb{B}$ and new obtained elements as the result of operations. Note that $b = b'$ in $\mathfrak{B}(A)$

means that $b \dot{*} a = b' \dot{*} a$, $w_1 \dots w_n b \cdot a = w_1 \dots w_n b' \cdot a$ for any $a \in A$, $* \in \Omega_2'$, $w_1 \dots w_n \in \Omega_1'$ and for any n. It is an equivalence relation and by $\mathfrak{B}(A)$ we mean the corresponding quotient object.

Proposition 5.2.1. *$\mathfrak{B}(A)$ is an object of $\mathbf{C}_G$.*

Proof. Direct easy checking of the identities. □

As above, we will write for simplicity $b \cdot (a)$ and $b * (a)$ instead of $(b(+))(a)$ and $(b(*))(a)$ for $b \in \mathfrak{B}(A)$ and $a \in A$. Define the set of actions of $\mathfrak{B}(A)$ on A in a natural way. For $b \in \mathfrak{B}(A)$ we define $b \cdot a = b \cdot (a)$, $b * a = b * (a)$, $* \in \Omega_2'$. Thus if $b = \mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$, where we mean certain brackets, we have

$$b \,\overline{*}\, a = (\mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}) \,\overline{*}\, (a),$$
$$b \cdot a = a.$$

The right side of the equality is defined inductively according to Axiom 2. For $b_k \in B_k$, $k \in \mathbb{J}$, we have

$$\mathbf{b}_k * a = \mathbf{b}_k * (a) = b_k * a,$$
$$\mathbf{b}_k \cdot a = \mathbf{b}_k \cdot (a) = b_k \cdot a.$$

Also

$$(b_1 + b_2 + \cdots + b_n) * a = b_1 * (a) + \cdots + b_n * (a), \quad \text{for} \quad b_i \in \mathfrak{B}(A), \quad i = 1, \cdots, n$$
$$(b_1 + b_2 + \cdots + b_n) \cdot a = b_1 \cdot (b_2 \cdots (b_n \cdot (a)) \cdots), \quad b_i \in \mathfrak{B}(A), \quad i = 1, \cdots, n$$
$$\omega(b) \cdot a = a \quad \text{if} \quad b = b_1 * \cdots * b_n, \quad b_i \in \mathfrak{B}(A), \quad i = 1, \cdots, n$$
$$\omega(\mathbf{b}_k) \cdot a = \omega(b_k) \cdot a, \quad k \in \mathbb{J}, \quad b_k \in B_k.$$

Proposition 5.2.2. *The set of actions of $\mathfrak{B}(A)$ on A is a set of derived actions in $\mathbf{C}_G$.*

Proof. The checking shows that the set of actions of $\mathfrak{B}(A)$ on A satisfies the conditions of Proposition 3.1.1, which proves that it is a set of derived actions in $\mathbf{C}_G$. □

Define the map $d : A \longrightarrow \mathfrak{B}(A)$ by $d(a) = \mathfrak{a}$, where $\mathfrak{a} = \{a\cdot, a*, * \in \Omega_2'\}$. Thus we have by definition

$$d(a) \cdot a' = a + a' - a,$$
$$d(a) * a' = a * a', \quad \forall a, a' \in A, \quad * \in \Omega_2'.$$

Lemma 5.2.3. *d is a homomorphism in $\mathbf{C}_G$.*

Proof. We have to show that $d(\omega a) = \omega d(a)$ for any $\omega \in \Omega_1'$. For this we need to show that

$$d(\omega a) \cdot (a') = (\omega d(a)) \cdot (a')$$
$$\omega'(d(\omega a)) \cdot a' = \omega'(\omega d(a)) \cdot a' \quad \text{for any } \omega' \in \Omega_1'$$
$$d(\omega a) * (a') = (\omega d(a)) * (a') \quad \text{for any} \quad * \in \Omega_2'.$$

We have

$$d(\omega a) \cdot a' = \omega a + a' - \omega a,$$
$$\omega d(a) \cdot a' = \omega(\mathfrak{a}) \cdot a' = \omega a + a' - \omega a,$$

The second equality follows form the first one. For the third equality we have

$$d(\omega a) * a' = (\omega a) * a',$$
$$(\omega d(a)) * a' = \omega(\mathfrak{a}) * a' = \omega(a) * a'$$

for $\omega = -$ we have to show that $d(-a)\cdot(a') = (-(da))\cdot a'$ and $(d(-a)) * a' = (-d(a)) * a'$. The checking of these equalities is an easy exercise.

Now we will show that $d(a_1 + a_2) = d(a_1) + d(a_2)$. Direct computation of both sides for each $a \in A$ gives

$$d(a_1 + a_2)\cdot(a) = a_1 + a_2 + a - a_2 - a_1,$$
$$\big(d(a_1) + d(a_2)\big)\cdot(a) = d(a_1)\cdot\big(d(a_2)\cdot a\big),$$

which shows that the desired equality holds for the dot action. The proof of $\omega(d(a_1 + a_2))\cdot a = \omega(d(a_1) + d(a_2))\cdot a$ is based on the first equality, the property of unary operations with respect to the addition, and the fact that d commutes with unary operations.

For any $* \in \Omega_2'$ we shall show that

$$d(a_1 + a_2) * (a) = \big(d(a_1) + d(a_2)\big) * (a).$$

We have

$$d(a_1 + a_2) * (a) = (a_1 + a_2) * a = a_1 * a + a_2 * a,$$
$$\big(d(a_1) + d(a_2)\big) * (a) = d(a_1) * a + d(a_2) * a = a_1 * a + a_2 * a$$

which proves the equality.

The next equality we have to prove is $d(a_1 * a_2) = d(a_1) * d(a_2)$. For this we need to show that $d(a_1 * a_2)\cdot(a) = (d(a_1) * d(a_2))\cdot(a)$, $\omega(d(a_1 * a_2))\cdot a = \omega(d(a_1) * d(a_2))\cdot a$, and $d(a_1 * a_2)\overline{*}(a) = (d(a_1) * d(a_2))\overline{*}(a)$ for any $\overline{*} \in \Omega_2'$.

We have $d(a_1 * a_2)\cdot a = a_1 * a_2 + a - a_1 * a_2 = a$, since $A \in \mathbf{C}$ and therefore it satisfies Axiom 1. $(d(a_1) * d(a_2))\cdot a = a$, by the definition of the action of $\mathfrak{B}(A)$ on A. The next equality is proved in a similar way by applying the fact that d commutes with ω and $\omega(a_1 * a_2) = \omega(a_1) * a_2$.

For the next above given identity we have the following computations:

$$d(a_1 * a_2)\overline{*}(a) = (a_1 * a_2)\overline{*}a = W(a_1, a_2; a; *, \overline{*}),$$
$$\big(d(a_1) * d(a_2)\big)\overline{*}(a) = W\big(d(a_1), d(a_2); a; *, \overline{*}\big).$$

These two expressions on the right sides of above equalities are equal, by the type of the word W in Axiom 2 and the definition of d. ☐

Proposition 5.2.4. *$d : A \longrightarrow \mathfrak{B}(A)$ is a crossed module in $\mathbf{C}_G$.*

Proof. We have to check conditions (i)–(iv) from the definition of a crossed module given in Sec. 1.1.

Condition (i) states that $d(b\cdot a) = b + d(a) - b$ for $a \in A$, $b \in \mathfrak{B}(A)$; so we have to show that $d(b\cdot a)\,\dot{*}\,a' = (b + da - b)\,\dot{*}\,a'$ and $\omega_1 \dots \omega_n(d(b\cdot a))\cdot a' = \omega_1 \dots \omega_n(b + da - b)\cdot a'$. Below we compute each side for the dot action of the first equality:

$$d(b\cdot a)\cdot a' = b\cdot a + a' - b\cdot a,$$
$$(b + d(a) - b)\cdot a' = b\cdot(d(a)\cdot(-b\cdot a')) = b\cdot(a - b\cdot a' - a) = b\cdot a + a' - b\cdot a.$$

The second equality is proved in a similar way. Now we compute each side of the first equality for the $*$ action. $d(b\cdot a) * a' = (b\cdot a) * a' = a * a'$ by Proposition 3.1.1; $(b + da - b) * a' = b * a' + d(a) * a' - b * a' = b * a' + a * a' - b * a' = a * a'$; here we apply Axiom 1, that $\overline{a} + a * a' = a * a' + \overline{a}$, for any element $\overline{a}$ of A.

We have to show: (ii) $d(a_1)\cdot a_2 = a_1 + a_2 - a_1$, (iii) $d(a_1) * a_2 = a_1 * a_2$; both (ii) and (iii) are true by the definition of d. Note that $a_1 * (d(a_2)) = d(a_2) *^{\circ} a_1 = a_2 *^{\circ} a_1 = a_1 * a_2$.

The first condition of (iv) states that

$$d(b * a) = b * d(a) \quad \text{for any} \quad b \in \mathfrak{B}(A), \quad a \in A, * \in \Omega_2'.$$

Thus we have to show

$$d(b * a)\dot{\overline{*}}\, a' = (b * d(a))\dot{\overline{*}}\, a', \quad \omega(db * a) \cdot a' = \omega(b * da) \cdot a' \quad \text{for} \quad \omega \in \Omega_1', \quad \overline{*} \in \Omega_2'. \tag{5.2.1}$$

First we show (5.2.1) for the dot operation. The second equality for the dot operation is proved similarly by applying the properties of unary operations. The right side of (5.2.1) in this case is equal to a'. For the left side we obtain

$$d(b * a) \cdot a' = b * a + a' - b * a.$$

If $b = \mathbf{b}_i$, then $b * a = b_i * a$ and since $B_i \in \mathbf{C}$ and B_i acts on A (action is in $\mathbf{C}$), by Axiom 1 for the action of B_j on A we shall have $b * a + a' = a' + b * a$, and so $d(b * a) \cdot a' = a'$.

If $b = \mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$ then, by the definition of the $*$ operation in $\mathfrak{B}(A)$, $b * a$ is the sum of the elements of the type $b_{i_t} * \overline{a}_t$ for certain i_t, and the element $\overline{a}_t \in A$; this kind of element again commutes with any element of A. So, $d(b * a) \cdot a' = a'$. We will have the same result if b is the sum of the elements of the type $\mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$.

Now we shall show (5.2.1) for the $*$ operation. By the definition of d we have

$$d(b * a)\overline{*}\, a' = (b * a)\overline{*}\, a'.$$

In the case $b = \mathbf{b}_i$, $i \in \mathbb{J}$, $b * a = \mathbf{b}_i * a = b_i * a$, so we obtain

$$d(b * a)\overline{*}\, a' = (b_i * a)\overline{*}\, a = W(b_i, a; a'; *, \overline{*}).$$

We have the last equality according to the properties of an action in $\mathbf{C}$, which correspond to Axiom 2. For the right side of (5.2.1) in the case $b = \mathbf{b}_i$ we have

$$(b * d(a))\overline{*}\, a' = (\mathbf{b}_i * \mathfrak{a})\overline{*}\, a' = W(b_i, a; a'; *, \overline{*}).$$

Suppose $b = \mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$; then in the same way as in the previous proof, we have that $b * a$ is the sum of the elements of the type $b_{i_t} * \overline{a}_t$, and $(b * a)\overline{*}\, a'$ is the sum of the elements of the type $(b_{i_t} * \overline{a}_t)\overline{*}\, a'$. The element from the right side of (5.2.1) will be the same type as the sum of the elements $(b_{i_t} * \overline{a}_t) * a'$. Applying Axiom 2 to the element $(b_{i_t} * \overline{a}_t)\overline{*}\, a'$, by the definition of the operation for the elements of $\mathfrak{B}(A)$ (for the element $(\mathbf{b}_{i_t} * \overline{\mathfrak{a}}_t) * a'$) and from the facts that $\mathbf{b}_{i_t} * a = b_{i_t} * a$, $\overline{\mathfrak{a}}_t * a = \overline{a}_t * a$, we will have the desired equality (5.2.1). In an analogous way we will prove (5.2.1) for the $*$ operation in the case where b is a sum of the elements of the form $\mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$. The second condition of (iv) can be proved in a similar way. □

Proposition 5.2.5. *If A has an actor in $\mathbf{C}$, then $\mathfrak{B}(A) = \mathrm{Actor}(A)$.*

Proof. From the existence of $\mathrm{Actor}(A)$ it follows that $\mathrm{Actor}(A)$ is one of the objects B_i, which acts on A. We have a natural homomorphism $e : \mathrm{Actor}(A) \longrightarrow \mathfrak{B}(A)$ in $\mathbf{C}_G$ sending b_i to $\mathbf{b}_i$, $b_i \in B_i$. According to the note made in Sec. 5.1, if $b_i \neq b_i'$ in $\mathrm{Actor}(A)$, then $\mathbf{b}_i \neq \mathbf{b}_i'$; thus e is an injective homomorphism. Let $\varphi_j : B_j \longrightarrow \mathrm{Actor}(A)$ be a unique morphism with $\varphi(b_j)\dot{*}a = b_j\dot{*}a$, $b_j \in B_j$, $j \in \mathbb{J}$, $a \in A$; e is a surjective homomorphism, since for any element $\mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$ of $\mathfrak{B}(A)$ there exists the element $\varphi_{i_1}(b_{i_1}) *_1 \cdots *_{n-1} \varphi_{i_n}(b_{i_n})$ in $\mathrm{Actor}(A)$ with $e(\varphi_{i_1}(b_{i_1}) *_1 \cdots *_{n-1} \varphi_{i_n}(b_{i_n})) = \mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_n$, which ends the proof. □

Theorem 5.2.6. *Let $\mathbf{C}$ be a category of interest and $A \in \mathbf{C}$; A has an actor if and only if $\mathfrak{B}(A) \ltimes A \in \mathbf{C}$. If it is the case, then $\mathrm{Actor}(A) = \mathfrak{B}(A)$.*

Proof. From Proposition 5.2.5 it follows that if A has an actor, then $\mathfrak{B}(A) \in \mathbf{C}$ and $\mathfrak{B}(A)$ has a derived action on A. By the theorem of Orzech [76] (see Sec. 5.1, Theorem 5.1.3) we will have $\mathfrak{B}(A) \ltimes A \in \mathbf{C}$. The converse is also easy to prove. Since $\mathfrak{B}(A) \ltimes A \in \mathbf{C}$, from the split exact sequence $0 \longrightarrow A \xrightarrow{i} \mathfrak{B}(A) \ltimes A \longrightarrow \mathfrak{B}(A) \longrightarrow 0$ in $\mathbf{C}_G$, $\mathfrak{B}(A) = \mathrm{Coker}\, i$, and thus it is an object of $\mathbf{C}$;

again by Theorem 5.1.3, $\mathfrak{B}(A)$ has a derived action on A in $\mathbf{C}$ (it is the action we have defined). By Proposition 5.2.4, $d : A \longrightarrow \mathfrak{B}(A)$ is a crossed module in $\mathbf{C}_G$; since $\mathfrak{B}(A) \in \mathbf{C}$ and the action of $\mathfrak{B}(A)$ on A is a derived action in $\mathbf{C}$, it follows that $d : A \longrightarrow \mathfrak{B}(A)$ is a crossed module in $\mathbf{C}$. Now we have to show the universal property of this crossed module. For any action of B_k on A, $k \in \mathbb{J}$, we define $\varphi_k : B_k \longrightarrow \mathfrak{B}(A)$ by $\varphi_k(b_k) = \mathbf{b}_k$, for any $b_k \in B_k$, where $\mathbf{b}_k \in \mathbb{B}$. By the definition of $\mathbb{B}$, $\mathbf{b}_k \dot{*} a = b_k * a$, $* \in \Omega_2'$, and we obtain $\varphi_k(b_k) \dot{*} a = b_k * a$; φ_k is a homomorphism in $\mathbf{C}$. For another homomorphism φ_k' we would have $\varphi_k'(b_k) \dot{*} a = b_k * a = \varphi_k(b_k) \dot{*} a$, $\omega(\varphi_k'(b_k)) \cdot a = \varphi_k'(\omega b_k) \cdot a = (\omega b_k) \cdot a = \omega(\varphi(b_k)) \cdot a$, for any $b_k \in B_k$, $a \in A$, $\omega \in \Omega_1'$, and $* \in \Omega_2'$, which means that $\varphi_k(b_k) = \varphi_k'(b_k)$, for any $b_k \in B_k$; this gives the equality $\varphi_k = \varphi_k'$, which proves the theorem. ☐

Theorem 5.2.7. *Let $\mathbf{C}$ be a category of interest. For any $A \in \mathbf{C}$, $\mathfrak{B}(A) = \mathrm{GActor}(A)$.*

Proof. By Propositions 5.2.2 and 5.2.4 and Lemma 5.2.3 we have the crossed module $d : A \longrightarrow \mathfrak{B}(A)$ in $\mathbf{C}_G$. For any object $C \in \mathbf{C}$ which has a derived action on A we construct the homomorphism $\varphi : C \longrightarrow \mathfrak{B}(A)$ in $\mathbf{C}_G$ with the property $c \dot{*} a = \varphi(c) \dot{*} a$ and show that φ is unique with this property in the same way as we have done for φ_k in the proof of Theorem 5.2.6. ☐

Suppose I is an ideal of C in $\mathbf{C}$ and $\mathrm{Actor}(I)$ exists. Thus we have the crossed module $d : I \longrightarrow \mathrm{Actor}(I)$. Denote $\mathrm{Im}\ d = \mathrm{Inn}(I)$. Thus we have

$$\mathrm{Inn}(I) = \{\mathfrak{a} \in \mathrm{Actor}(I) \mid \ a \in I\}.$$

Recall that by definition of d, $d(a) = \mathfrak{a}$, and $\mathfrak{a}$ is defined by

$$\mathfrak{a} \cdot (a') = a + a' - a,$$
$$\mathfrak{a} * (a') = a * a'.$$

It is easy to see that $\mathrm{Inn}(I)$ is an ideal of $\mathrm{Actor}(I)$. this follows from the fact that $d : I \longrightarrow \mathrm{Actor}(I)$ is a crossed module, and it can also be checked directly. Since I is an ideal of C, we have an action of C on I, defined by $c \cdot a = c + a - c$, $c * a = c * a$, $* \in \Omega_2'$. It is a derived action. Thus there exists a unique homomorphism $\theta : C \longrightarrow \mathrm{Actor}(I)$, such that

$$\theta(c) \dot{*} a = c \dot{*} a, \quad a \in I, \quad c \in C, \quad * \in \Omega_2'.$$

Let $\tau : I \longrightarrow \mathrm{Inn}(I)$ be a homomorphism defined by d; then θ induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & C & \longrightarrow & C/I & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Inn}(I) & \longrightarrow & \mathrm{Actor}(I) & \longrightarrow & \mathrm{Out}(I) & \longrightarrow & 0,
\end{array}$$

which is well known for the case of groups [75] (see Sec. 5.1).

For any object $C \in \mathbf{C}$ there is an action of A on itself defined by $a \cdot a' = a + a' - a; a * a' = a * a'$, for any $a, a' \in A, * \in \Omega_2'$, where $*$ on the left side denotes the action and on the right side the operation in A. We call this action the conjugation.

Let $E_A :\ 0 \longrightarrow A \longrightarrow A \ltimes A \rightleftarrows A \longrightarrow 0$ be the split extension which corresponds to the action of A on itself by conjugation. Consider the category of all split extensions with fixed A; thus the objects are $0 \longrightarrow A \longrightarrow C \rightleftarrows C' \longrightarrow 0$, and the arrows are triples $(1_A, \gamma, \gamma')$ between extensions which commute with section homomorphism too.

Proposition 5.2.8. *If $E_t :\ 0 \longrightarrow A \longrightarrow C \rightleftarrows B \longrightarrow 0$ is a terminal object in the category of split extensions with fixed A, then the unique arrow $(1, \gamma, \beta) : E_A \longrightarrow E_t$ defines a crossed module $\beta : A \longrightarrow B$, which is an actor of A.*

Proof. The proof is similar to that of Proposition 5.2.4. It is obvious that B has the universal property of an actor. We have to prove that $\beta : A \longrightarrow B$ is a crossed module; thus we shall show the following identities:

$$\begin{aligned} \beta(a) \cdot a' &= a + a' - a, \\ \beta(b \cdot a) &= b + \beta(a) - b, \\ \beta(a) * a' &= a * a' \\ \beta(b * a) &= b * \beta(a). \end{aligned}$$

for any $a \in A$, $b \in B$, $* \in \Omega_2'$. We have the commutative diagram

$$\begin{array}{lccccccccc} E_A: & 0 & \longrightarrow & A & \longrightarrow & A \ltimes A & \rightleftarrows & A & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \\ E_t: & 0 & \longrightarrow & A & \longrightarrow & C & \rightleftarrows & B & \longrightarrow & 0 \end{array}$$

from which we obtain $\beta(a) \cdot a' = a + a' - a$ and $\beta(a) * a' = a * a'$ for any $a, a' \in A, * \in \Omega_2'$, which proves the first and third equalities. Since E_t is a terminal extension, it has the following property: if for $b, b' \in B$ we have $b \dot{*} a = b' \dot{*} a$, $\omega_1 \cdots \omega_n(b) \dot{*} a = \omega_1 \cdots \omega_n(b') \dot{*} a$ for any $a \in A$ and any unary operations $\omega_1, \cdots, \omega_n \in \Omega_1', n \in \mathbb{N}$, then $b = b'$.

For the second equality we have

$$\begin{gathered} (\beta(b \cdot a)) \cdot a' = b \cdot a + a' - b \cdot a, \\ (b + \beta(a) - b) \cdot a' = b \cdot (\beta(a) \cdot (-b \cdot a')) = b \cdot (a - b \cdot a' - a) = b \cdot a + a' - b \cdot a, \\ (\beta(b \cdot a)) * a' = (b \cdot a) * a' = a * a' \end{gathered}$$

by condition 8 of Proposition 3.1.1.

For the fourth equality we have

$$\beta(b * a) \cdot a' = (b * a) \cdot a' = a';$$

it follows from the property of the derived action in the categories of interest as a result of Axiom 1 (Proposition 3.1.1). The same property gives

$$(b * \beta(a)) \cdot a' = a'.$$

For a star operation we have

$$\begin{gathered} \beta(b * a) * a' = (b * a) * a', \\ (b * \beta(a)) * a' = (b * a) * a', \end{gathered}$$

here we apply $\widetilde{\text{Axiom2}}$ for the set $(A \cup B)$ and the fact $\beta(a) * a' = a * a'$. For any unary operation $\omega \in \Omega_1'$,

$$\omega(\beta(b \cdot a)) = \beta(\omega(b \cdot a)) = \beta(\omega(b) \cdot \omega(a));$$

here we apply condition 10 of Proposition 3.1.1:

$$\omega(b + \beta(a) - b) = \omega(b) + \beta(\omega(a)) - \omega(b).$$

As we have proved above, these elements are equal.

Below we apply condition 11 of Proposition 3.1.1 and obtain

$$\begin{gathered} \omega(\beta(b * a)) = \beta(\omega(b * a)) = \beta(\omega(b) * a), \\ \omega(b * \beta(a)) = \omega(b) * \omega(a). \end{gathered}$$

As we have shown above, these elements are equal. For $\omega_1, \dots, \omega_n$ the corresponding equalities are obtained similarly. □

By Proposition 5.2.8, Definition 5.1.4 is equivalent to the following one.

Definition 5.2.9. For any object A in $\mathbf{C}$ an actor of A is an object $\mathrm{Actor}(A)$ with (split derived) action on A, such that for any object C of $\mathbf{C}$ and an action of C on A there is a unique morphism $\varphi : C \longrightarrow \mathrm{Actor}(A)$ with $c \cdot a = \varphi(c) \cdot a$, $c * a = \varphi(c) * a$ for any $* \in \Omega_2{}'$, $a \in A$ and $c \in C$.

It is a well-known fact that the category of crossed modules in the category of groups $X\,\mathrm{Mod}(\mathbb{G}r)$ is equivalent to the category $\mathbb{G}$ with objects groups with the additional two unary operations $\omega_0, \omega_1 : G \longrightarrow G$, $G \in \mathbb{G}r$, which are group homomorphisms satisfying the conditions

(1) $\omega_0\omega_1 = \omega_1$, $\omega_1\omega_0 = \omega_0$,

(2) $\omega_1(x) + y - \omega_1(x) = x + y - x$, $x, y \in \mathrm{Ker}\,\omega_0$.

This category is a category of interest. The computations and properties of actions in this category and the direct checking of identities (1), (2) show that $\mathfrak{B}(A)$ is an actor of $A \in \mathbb{G}$. Thus the same is true for the category of crossed modules $X\,\mathrm{Mod}(\mathbb{G}r)$. From the results of Norrie [75] it follows that the object $A(T, G, \partial)$ constructed by her, for any crossed module (T, G, ∂), is an actor in the sense of Definition 5.1.4. Thus it follows that in the category of interest $\mathbb{G}$ there exists an actor for any $A \in \mathbb{G}$. By Proposition 5.2.5 it follows that $\mathfrak{B}(A)$ is an actor for any $A \in \mathbb{G}$. This is another way of proving that $\mathfrak{B}(A) = \mathrm{Actor}(A)$ in $\mathbb{G}$.

The category of precrossed modules is equivalent to the category of interest $\bar{\mathbb{G}}$, whose objects are groups with additional two unary operations ω_0, ω_1, which are group homomorphisms satisfying identity (1). By Theorem 5.2.7, $\mathfrak{B}(A) = \mathrm{GActor}(A)$, for any $A \in \bar{\mathbb{G}}$. It is easy to check that $\mathfrak{B}(A)$ satisfies identity (1) and thus $\mathfrak{B}(A) \in \bar{\mathbb{G}}$; therefore $\mathfrak{B}(A) = \mathrm{Actor}(A)$. From this we conclude that in the category of precrossed modules always exists an actor.

Internal object actions were studied recently by F. Borceux, G. Janelidze, and G. M. Kelly [11], where the authors introduce a new notion of representable action. From Theorem 6.3 of [11], applying Proposition 5.2.8 it follows that in the case of the category of interest $\mathbf{C}$ the existence of representable object actions is equivalent to the existence of an $\mathrm{Actor}(A)$ for any $A \in \mathbf{C}$ in the sense of the Definition 5.1.4. Thus by Theorem 5.2.6, $\mathbf{C}$ has representable object actions if and only if $\mathfrak{B}(A) \ltimes A \in \mathbf{C}$, for any $A \in \mathbf{C}$, and if it is the case, the corresponding representing objects are $\mathfrak{B}(A), A \in \mathbf{C}$. For the categorical approach to the question of an actor, see also [12, 17].

5.3. The Case $\Omega_2 = \{+, *, *^\circ\}$

It is interesting to know in which kind of categories of interest $\mathbf{C}$ there exists $\mathrm{Actor}(A)$ for any object $A \in \mathbf{C}$; or what the sufficient conditions for the existence of $\mathrm{Actor}(A)$ for a certain $A \in \mathbf{C}$ are. In the case of groups ($\Omega_2 = \{+\}$), a direct check shows that $\mathfrak{B}(A) \in \mathrm{Gr}$, and the action of $\mathfrak{B}(A)$ on A is a derived action. This follows also from Propositions 5.2.1 and 5.2.2; thus $\mathfrak{B}(A)$ is an actor of A by the Theorem 5.2.6. This fact is also a consequence of Proposition 5.2.5 since it is well known that $\mathrm{Aut}(A)$ is an actor of A in Gr; thus $\mathfrak{B}(A) \approx \mathrm{Aut}(A)$. In the case of Lie algebras ($\Omega_2 = \{+, [\ ,\ ]\}$), the object $\mathfrak{B}(A) \in \mathbb{L}\mathrm{ie}$ and the action of $\mathfrak{B}(A)$ on A is a derived action, so $\mathfrak{B}(A)$ is an actor again in $\mathbb{L}\mathrm{ie}$ and therefore $\mathfrak{B}(A) \approx \mathrm{Der}(A)$.

Consider the case of Leibniz algebras. In this case we can define the bracket operation for the elements of $\mathbb{B}$ in two ways (see Sec. 5.1 for the definition of the set $\mathbb{B}$).

Definition 5.3.1.

$$\big[\, a, [\mathbf{b}_i, \mathbf{b}_j] \,\big] = \big[\, [a, b_i], b_j \,\big] - \big[\, [a, b_j], b_i \,\big],$$

$$\big[\, [\mathbf{b}_i, \mathbf{b}_j], a \,\big] = \big[\, b_i, [b_j, a] \,\big] + \big[\, [b_i, a], b_j \,\big].$$

Definition 5.3.2.

$$\big[\, a, [\mathbf{b}_i, \mathbf{b}_j] \,\big] = \big[\, [a, b_i], b_j \,\big] - \big[\, [a, b_j], b_i \,\big],$$
$$\big[\, [\mathbf{b}_i, \mathbf{b}_j], a \,\big] = -\big[\, b_i, [a, b_j] \,\big] + \big[\, [b_i, a], b_j \,\big].$$

The bracket operation $[b, b']$ for any b, b', which are the results of bracket operations itself, is defined according to the above formulas.

The addition is defined by

$$[\mathbf{b}_i + \mathbf{b}_j, a] = [b_i, a] + [b_j, a],$$
$$[a, \mathbf{b}_i + \mathbf{b}_j] = [a, b_i] + [a, b_j].$$

For any $b, b' \in \mathfrak{B}(A)$, $b + b'$ is defined by the same formulas.

The action of $\mathfrak{B}(A)$ on A is defined according to Definition 5.3.1 or 5.3.2. So we have two different ways of definition of an action. It is easy to check that none of them is the derived action in $\mathbb{L}$eibniz.

The algebras $\mathfrak{B}(A)$ defined by Definitions 5.3.1 and 5.3.2 are not isomorphic.

Condition 1. For $A \in \mathbb{L}$eibniz, and any two objects $B, C \in \mathbb{L}$eibniz, which act on A, we have

$$\big[\, c, [a, b] \,\big] = -\big[\, c, [b, a] \,\big],$$

$a \in A$, $b \in B$, $c \in C$.

Note that, in this condition, by action we mean the derived action.

Example. If $\mathrm{Ann}(A) = (0)$ or $[A, A] = A$, then A satisfies Condition 1.

Proposition 5.3.3. *For any object $A \in \mathbb{L}$eibniz, the Definitions 5.3.1 and 5.3.2 give the same algebras if A satisfies Condition 1.*

The proof follows directly from the definitions of operations in $\mathfrak{B}(A)$ and Condition 1.

Below we mean that $\mathfrak{B}(A)$ is defined in one of the ways.

Proposition 5.3.4. *For any $A \in \mathbb{L}$eibniz, $\mathfrak{B}(A)$ is a Leibniz algebra. The set of actions of $\mathfrak{B}(A)$ on A is a set of derived actions if and only if A satisfies Condition 1.*

Proof. The computation shows that if Condition 1 holds, then the same kind of condition is fulfilled for $b, c \in \mathfrak{B}(A)$, from which follows the result. □

Proposition 5.3.5. *For a Leibniz algebra A there exists an actor if and only if A satisfies Condition 1. If it is the case, then $\mathfrak{B}(A) = \mathrm{Actor(A)}$.*

Proof. By Proposition 5.3.3, $\mathfrak{B}(A)$ is always a Leibniz algebra, and by Theorem 5.2.7, $\mathfrak{B}(A) = G\mathrm{Actor}(A)$. If A satisfies Condition 1, by Proposition 5.3.4, $\mathfrak{B}(A)$ has a derived action on A and thus $\mathfrak{B}(A) = \mathrm{Actor}(A)$. Conversely, if A has an actor, then $\mathfrak{B}(A) = \mathrm{Actor}(A)$ by Proposition 5.2.5, and so the action of $\mathfrak{B}(A)$ on A is a derived action; thus we have, for any $a \in A$, $b_i \in B_i$, $b_j \in B_j$, $i, j \in \mathbb{J}$, the following equalities:

$$\big[\, b_i, [a, b_j] \,\big] = \big[\, [b_i, a], b_j \,\big] - \big[\, [b_i, b_j], a \,\big],$$
$$\big[\, b_i, [b_j, a] \,\big] = \big[\, [b_i, b_j], a \,\big] - \big[\, [b_i, a], b_j \,\big],$$

from which follows Condition 1, which proves the theorem. □

We have an analogous picture for associative algebras. The operations for the elements of $\mathbb{B}$ (see Sec. 5.1 for the notation) in this category are given by

$$\begin{aligned}(\mathbf{b}_i * \mathbf{b}_j) * (a) &= b_i * (b_j * a),\\ *(\mathbf{b}_i * \mathbf{b}_j)(a) &= (a * b_i) * b_j,\\ (\mathbf{b}_i + \mathbf{b}_j) * (a) &= b_i * a + b_j * a,\\ *(\mathbf{b}_i + \mathbf{b}_j)(a) &= a * b_i + a * b_j.\end{aligned} \tag{5.3.1}$$

The set of actions of $\mathfrak{B}(A)$ on A is defined according to (5.3.1).

Condition 2. For $A \in \mathbb{A}\text{ss}$ and any two objects B and C from $\mathbb{A}\text{ss}$ which have derived actions on A, we have

$$c * (a * b) = (c * a) * b,$$

for any $a \in A$, $b \in B$, $c \in C$.

Example. If $\text{Ann}(A) = (0)$ or $A^2 = A$, then A satisfies Condition 2. For this kind of associative algebras it is proved in [57] that $A \longrightarrow \text{Bim}(A)$ is a terminal object in the category of crossed modules under A.

Proposition 5.3.6. *For $A \in \mathbb{A}\text{ss}$, the algebra $\mathfrak{B}(A)$ is an associative algebra and the set of actions of $\mathfrak{B}(A)$ on A defined according to* (5.3.1) *is the set of derived actions in* $\mathbb{A}\text{ss}$ *if and only if A satisfies Condition* 2. *If it is the case,* $\mathfrak{B}(A) = \text{Actor}(A)$.

The proof contains analogous arguments as for the case of Leibniz algebras and is left to the reader.
It is easy to see that in $\mathbb{A}\text{ss}$ and $\mathbb{L}\text{eibniz}$ generally we have the injections

$$\mathfrak{B}(A) \longrightarrow \text{Bim}(A) \quad \text{and} \quad \mathfrak{B}(A) \longrightarrow \text{Bider}(A)$$

which are homomorphisms in $\mathbb{A}\text{ss}$ and $\mathbb{L}\text{eibniz}$ respectively.

Proposition 5.3.7. *Let A be an associative algebra with the condition* $\text{Ann}(A) = 0$ *or* $A^2 = A$. *Then* $\mathfrak{B}(A) \approx \text{Bim}(A) = \text{Actor}(A)$.

Proof. It is well known that $\text{Bim}(A)$ is an associative algebra [70]. The action of $\text{Bim}(A)$ on A (see Sec. 5.1) is not a derived action in general, and the condition

$$f * (a * f') = (f * a) * f' \tag{5.3.2}$$

fails for any $f = (f*, *f)$ and $f' = (f'*, *f')$ from $\text{Bim}(A)$. A direct check shows that in the case $\text{Ann}(A) = (0)$ or $A^2 = A$, identity (5.3.2) holds for the action [57]. For any action of the object B on A, $B \in \mathbb{A}\text{ss}$, we define $\varphi : B \longrightarrow \text{Bim}(A)$ by $\varphi(b) = (b*, *b)$, which is a unique homomorphism with the property that $\varphi(b) * a = b * a$, $* \in \Omega_2'$, since in $\text{Bim}(A)$ for any two elements $f, f' \in \text{Bim}(A)$ from $f = f'$ it follows that $f* = f'*$, $*f = *f'$. Thus $\text{Bim}(A)$ is an actor of A in $\mathbb{A}\text{ss}$, and the isomorphism $\mathfrak{B}(A) \approx \text{Bim}(A)$ follows from Proposition 5.2.5. □

We have the analogous result for Leibniz algebras.

Proposition 5.3.8. *Let* $A \in \mathbb{L}\text{eibniz}$ *and* $\text{Ann}(A) = (0)$ *or* $[A, A] = A$. *Then* $\mathfrak{B}(A) \approx \text{Bider}(A) = \text{Actor}(A)$.

Proof. We will follow the first definition of the bracket operation in $\text{Bider}(A)$ (see Sec. 5.1, $(5.1.6_1)$, $(5.1.6_2)$). A direct check shows that $\text{Bider}(A)$ is a Leibniz algebra (see Remark below and cf. [62]). The action of $\text{Bider}(A)$ on A is not a derived action, and the following condition fails:

$$\big[\,\varphi, [a, \varphi']\,\big] = \big[\,[\varphi, a], \varphi'\,\big] - \big[\,[\varphi, \varphi'], a\,\big], \tag{5.3.3}$$

where $\varphi = [[\ , \varphi], [\varphi,\ ]]$ and $\varphi' = [[\ , \varphi'], [\varphi',\ ]] \in \text{Bider}(A)$.

From $(5.1.6_2)$ we have

$$\big[\,\varphi,[\varphi',a]\,\big]=\big[\,[\varphi,\varphi'],a\,\big]-\big[\,[\varphi,a],\varphi'\,\big]. \tag{5.3.4}$$

We shall show that if $\mathrm{Ann}(A)=(0)$, then $[\varphi,[\varphi',a]]=-[\varphi,[a,\varphi']]$, and from (5.3.4) will follow (5.3.3).

For any $a'\in A$ we have the following equalities:

$$\big[\,a',[\varphi,[\varphi',a]]\,\big]=-\big[\,a',[[\varphi',a],\varphi]\,\big]=-\big[\,[a',[\varphi',a]],\varphi\,\big]+\big[\,[a',\varphi],[\varphi',a]\,\big],$$

$$\big[\,a',[\varphi,[a,\varphi']]\,\big]=-\big[\,[a',[a,\varphi']],\varphi\,\big]+\big[\,[a',\varphi],[a,\varphi']\,\big]=\big[\,[a',[\varphi',a]],\varphi\,\big]-\big[\,[a',\varphi],[\varphi',a]\,\big].$$

Thus we obtain that for $a'\in A$

$$\big[\,a',[\varphi,[\varphi',a]]+[\varphi,[a,\varphi']]\,\big]=0.$$

In analogous way we show that

$$\big[\,[\varphi,[\varphi',a]]+[\varphi,[a,\varphi']],a'\,\big]=0.$$

From which we conclude that

$$\big[\,\varphi,[\varphi',a]\,\big]+\big[\,\varphi,[a,\varphi']\,\big]=0.$$

The case $[A,A]=A$ can be proved analogously. Thus we have a derived action of $\mathrm{Bider}(A)$ on A and the crossed module $A\longrightarrow \mathrm{Bider}(A)$ $(a\longmapsto([\ ,a],[a,\ ])$ has the universal property of the actor object. By Proposition 5.2.5 $\mathfrak{B}(A)\approx\mathrm{Bider}(A)$, which ends the proof. ☐

Remark. As we have also mentioned in Sec. 5.1, if $[\varphi,[\varphi',a]]=-[\varphi,[a,\varphi']]$ for any $\varphi=([\ ,\varphi],[\varphi,\ ])$ and $\varphi'=([\ ,\varphi'],[\varphi',\ ])$ from $\mathrm{Bider}(A)$, then the two definitions of $\mathrm{Bider}(A)$ according to $(5.1.6_1)$, $(5.1.6_2)$ and $(5.1.6_1)$, $(5.1.6'_2)$ coincide, and this algebra is isomorphic to the Leibniz algebra of biderivations defined by Loday [62].

In the category of R-modules over a ring R, it is obvious that $\mathrm{Actor}(A)=0$ for any A since every action is trivial in this category. The same result gives our construction, $\mathfrak{B}(A)=0$, for any R-module A.

As in the case of associative algebras, in the category of commutative associative algebras the condition for the action $(b_1a)b_2=b_1(ab_2)$ fails; also in this category we must have $ba=ab$, for $b\in\mathfrak{B}(A)$, and $b_1b_2=b_2b_1$ for $b_1,b_2\in\mathfrak{B}(A)$. All these conditions are satisfied and we have $\mathfrak{B}(A)=$Actor(A) in commutative associative algebras if and only if A satisfies Condition 2. If $\mathrm{Ann}(A)=(0)$ or $A^2=A$, then A satisfies Condition 2. For this kind of commutative algebras, $\mathrm{Actor}(A)=\mathrm{Bim}(A)=\mathrm{M}(A)$, where $\mathrm{M}(A)$ is the set of multiplications (or multipliers) of A [59],[57], i.e., k-linear maps $f:A\longrightarrow A$ with $f(aa')=f(a)a'$.

In the category of alternative algebras, $\mathrm{Actor}(A)$ does not exist for any A. The existence of an actor in $\mathbb{A}\mathrm{lt}$ will be studied in the future.

Chapter 6

NONCOMMUTATIVE LEIBNIZ–POISSON ALGEBRAS

In this chapter we study one of the generalizations of the classical Poisson algebras. This kind of studies was begun in [22]. Recall that a Poisson algebra is an associative commutative algebra A equipped with a binary bracket operation $[-,-]:A\otimes A\longrightarrow A$ such that $(A,[-,-])$ is a Lie algebra and the following condition holds:

$$[a\cdot b,c]=a\cdot[b,c]+[a,c]\cdot b$$

for all $a,b,c\in A$.

Here we consider the case where algebras are not commutative and the bracket operation defines the Leibniz algebra structure (see below Definition 6.1.1). This kind of algebras we call noncommutative Leibniz–Poisson algebras and denote the corresponding category by **NLP**. We give a construction of free NLP-algebras and define actions, representations, and crossed modules, which are special cases of the corresponding notions for the category of groups with operations [76] (see Secs. 1.1 and 3.1). We define the cohomology of a NLP-algebras P (over a field k) with coefficients in a representation M over P and study its properties, in particular, the relation with extensions of NLP-algebras and with Hochshild and AWB (algebras with bracket [22]) cohomologies. Algebras over the dual operad (in the sense of [63]) of NLP-algebras are considrered. The construction of free objects in this category and their relation with certain types of planar binary rooted trees are studied.

6.1. Noncommutative Leibniz–Poisson Algebras

6.1.1. Preliminaries. Let k be a commutative ring with unit. All modules are taken over k. In what follows Hom and $\otimes$ mean Hom_k and $\otimes_k$ respectively. Associative algebras considered in this work are in general without unit.

For the definition of a Leibniz algebra, we refer the reader to Sec. 5.1.

Definition 6.1.1. A noncommutative Leibniz–Poisson algebra (for short, NLP-algebra) is an associative algebra P equipped with a k-module homomorphism $[-,-] : P\otimes P \longrightarrow P$, such that $(P, [-,-])$ is a Leibniz algebra and the following identity holds:

$$[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b \tag{6.1.1}$$

for all $a, b, c \in P$. In other words, a NLP-algebra is an AWB [22] P such that the bracket satisfies the Leibniz identity.

A morphism between NLP-algebras is a homomorphism of associative algebras which respects the bracket operation. We shall denote the category of NLP-algebras by **NLP**.

Examples 6.1.2.

1. Poisson algebras.
2. Any Leibniz algebra is a NLP-algebra with trivial associative product ($a \cdot b = 0$). On the other hand, any associative algebra is a NLP-algebra with the usual bracket $[a, b] = ab - ba$.
3. Any associative dialgebra [63] is a NLP-algebra with respect to the operations $ab = a \vdash b$; $[a, b] = a \dashv b - b \vdash a$.
4. If P_1 and P_2 are NLP-algebras, then the k-module $P_1 \otimes P_2$ endowed with the operations
$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (a_1b_1) \otimes (a_2b_2),$$
$$[a_1 \otimes a_2, b_1 \otimes b_2] = \big[\, a_1, [b_1, b_2]\,\big] \otimes a_2 + a_1 \otimes \big[\, a_2, [b_1, b_2]\,\big]$$
is a NLP-algebra.
5. For the example of a graded version of NLP-algebra coming from physics, the reader is referred to [53].

Lemma 6.1.3. *In any* NLP*-algebra the following identity holds:*

$$[a, b[c, d]] + [a, [b, d]c] = [[a, bc], d] - [[a, d], bc]. \tag{6.1.2}$$

Proof. One easily sees that both sides of the identity equal $[a, [bc, d]]$. □

Remark. Any other two different decompositions according to Leibniz identity and (6.1.1) do not give a new identity.

Definition 6.1.4. Let $P \in \mathbf{NLP}$. A subalgebra of P is an associative and Leibniz subalgebra of P. A subalgebra R of P is called a two-sided ideal if $a \cdot r, r \cdot a, [a, r], [r, a] \in R$, for all $a \in P, r \in R$.

The inclusion functor $inc : \mathbf{Poiss} \longrightarrow \mathbf{NLP}$ from the category of Poisson algebras to the category of noncommutative Leibniz–Poisson algebras has a left adjoint $(-)_{Poiss} : \mathbf{NLP} \longrightarrow \mathbf{Poiss}$, which assigns to a NLP-algebra P the Poisson algebra obtained by the quotient of P with the smallest two-sided ideal spanned by the elements $[x, x]$ and $xy - yx$, for all $x, y \in P$. On the other hand, the Liezation functor [65] assigns to any NLP-algebra P the noncommutative Poisson algebra $P_{Lie} = P/\langle\{[x, x] \mid x \in P\}\rangle$.

6.1.2. Free NLP-algebras. For any set X we shall build an associative k-algebra with the additional binary bracket operation satisfying the Leibniz identity and condition (6.1.1).

We consider those formal combinations (words) of two operations $(\cdot, [-, -])$ with elements from X which have a sense and do not contain elements of the form $[a, [b, c]]$, $[a \cdot b, c]$, $[[a, b],\ c \cdot d]$ in their combination, where a, b, c are from X or are combinations of elements in X and dot and bracket operations. Denote by $W(X)$ the set that contains X and all above described type of words. Let $W_n(X)$ be the subset of those words of $W(X)$ that contain n elements of X, i.e. the number of both operations together is $n - 1$; we say that this word is of length n. Obviously, $W(X) = \bigcup_{n \geq 1} W_n(X)$. We define the following maps:

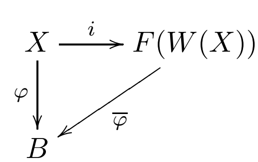
$$\alpha_{n,m}, \beta_{n,m} : W_n(X) \times W_m(X) \longrightarrow W_{n+m}(X)$$

$\alpha_{n,m}$ is defined for any pair $(a, b) \in W_n(X) \times W_m(X)$ by $\alpha_{n,m}(a, b) = a \cdot b$, where the right side denotes the word from $W_{n+m}(X)$, which is defined uniquely; $\beta_{n,m}$ is defined only on those pairs (a, b), for which the word $[a, b] \in W_{n+m}(X)$, and by definition $\beta_{n,m}(a, b) = [a, b]$. In the case $[a, b] \notin W_{n+m}(X)$, $\beta_{n,m}$ is not defined.

Let $F(W(X))$ be the free k-module generated by the set $W(X)$. The dot operation on $F(W(X))$ is defined as a linear extension of $\alpha_{n,m}$ on the whole $F(W(X))$. The bracket operation is also a linear extension on $F(W(X))$ of $\beta_{n,m}$ for those words on which $\beta_{n,m}$ is defined. If $[a, b] \notin W_{n+m}(X)$, for $a \in W_n(X), b \in W_m(X)$, we decompose $[a, b]$ according to the Leibniz identity and the identities (6.1.1), (6.1.2), until we obtain the sum of bracket operations on such pairs of words on which β is defined. Acting on every step in such a way we will obtain the sum $c_1 + \cdots + c_n$ with $c_i \in F_{n+m}(W(X))$ and, by definition, $[a, b] = c_1 + \cdots + c_n$. According to Remark 6.1.1, any two different decompositions give the same elements of $F(W(X))$, and $[a, b]$ is uniquely defined.

By construction, $F(W(X))$ has a structure of an NLP-algebra. Let $i : X \longrightarrow FW(X)$ be the natural inclusion of sets.

Proposition 6.1.5. *For any* NLP*-algebra B and a map $\varphi : X \longrightarrow B$, there exists a unique* NLP*-algebra homomorphism $\overline{\varphi} : F(W(X)) \longrightarrow B$ such that the following diagram commutes:*



Proof. For any element $w_F(x_1, \ldots, x_n) \in F(W(X))$ we define

$$\overline{\varphi}(w_F(x_1, \ldots, x_n)) = w_B(\varphi(x_1), \ldots, \varphi(x_n)),$$

where $w_B(-, \ldots, -)$ denotes the corresponding element of B. It is obvious that $\overline{\varphi}$ is an NLP-algebra homomorphism, $\overline{\varphi} i = \varphi$, and it is a unique homomorphism with this property. □

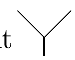
Actually our construction defines the functor F from **Set** to **NLP**, $F(X) = F(W(X))$, which is a left adjoint to the underlying functor

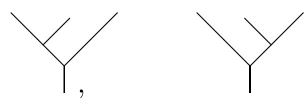
$$\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{NLP}\ .$$

If X is a one element set, $X = \{e\}$, the free NLP-algebra construction on $\{e\}$ has an interesting interpretation in terms of planar binary rooted trees.

By a *tree* we mean in this paper a planar binary rooted tree. We let $\mathfrak{T}$ be the set of trees. It is a graded set

$$\mathfrak{T} = \coprod_{n \geq 1} \mathfrak{T}_n,$$

where $\mathfrak{T}_n$ is the set of trees with n leaves. So $\mathfrak{T}_1$ has one element  denoted by e, and $\mathfrak{T}_2$ has two elements



while $\mathfrak{T}_3$ has five elements. The number of elements of $\mathfrak{T}_n$ is known as Catalan numbers, and they are $c_n = \frac{(2n)!}{n!(n+1)!}$ [63].

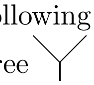
Let us also recall that on trees there exists an operation is called *grafting*. The grafting defines a map

$$\mathsf{gr} : \mathfrak{T}_{n_1} \times \cdots \times \mathfrak{T}_{n_k} \longrightarrow \mathfrak{T}_n, \quad n = n_1 + \cdots + n_k.$$

The grafting of a tree t and a tree s is obtained by joining the roots of t and s and creating a new root from that vertex.

We shall define two operations on trees, dot and bracket operations; the second is a partial operation, i.e., defined not for every type of trees. These operations will enable us to "read" trees. By "reading" we mean: to correspond to each tree a unique word, which is a formal combination of the element e and operations $\cdot$ and $[-,-]$, whenever they have a sense. We shall define operations on trees in such a way that all words obtained by "reading" trees according to our laws will belong to $W(\{e\})$ and it will be a one-to-one correspondence between the set $\mathfrak{T}$ of all planar binary rooted trees and $W(\{e\})$.

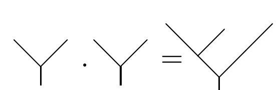
The left side of a tree is "generally" for the dot operation, and the right side is for the bracket operation if it is not a special case explained below. We denote by s^r and s^l the right and the left side trees respectively of a tree s.

The associative dot operation is defined for a pair of any type of trees t and s by the following formulas in terms of the grafting operation according to [77]. As above we shall denote the tree  by e:

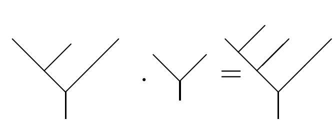
$$t \cdot s = gr(t, s^r) \quad \text{for} \quad s = e,$$

$$t \cdot s = gr(t \cdot s^l, s^r) \quad \text{for any } t \text{ and } s.$$

Thus



and for any tree t, $t \cdot e$ is the tree obtained by "adding" on the left from the vertex of e the tree t. For instance,

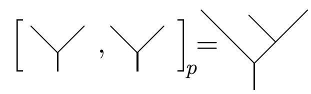


We shall define two preliminary bracket operations on every type of trees (the second operation is defined by means of the first one). The general bracket operation $[-,-]$ is a partial operation; it is defined only for special type of trees by means of the two preliminary bracket operations depending on the case.

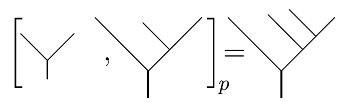
The first preliminary bracket operation we denote by $[-,-]_p$; it is analogous to the dot operation, but the "reading" law for $[-,-]_p$ is not associative. In terms of the grafting operation, we have the following formula:

$$[t,s]_p = gr\big(t^l, [t^r, s]_p\big)$$

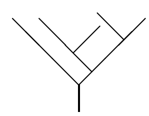
having in mind that



and for any tree t, $[e,t]_p$ is the tree obtained by adding on the right from the vertex of e the tree t. For instance,



and the tree

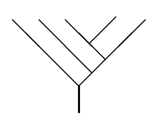


is $[[e, e\cdot e]_p, e]_p$.

We will apply this operation as a final result of the bracket operation only for a special kind of trees. The list will be given below.

The second bracket operation $SR[-,-]_p$ we will use for another type of trees. As indicated in the notation for this operation, it is defined by means of $[-,-]_p$. For a pair of trees (t,s) we first apply $[-,-]_p$; thus, we take $[t,s]_p$, then we perform the 90° rotation procedure of s to the left and then take a symmetric picture of s to give a normal look to the tree.

Thus, for instance,



is $SR[e, e\cdot[e,e]]_p$.

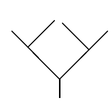
We have the following rules for performing the partial bracket operation for certain types of trees.

1. $[t,e] = [t,e]_p$ for any type of tree, but not a dot product (i.e., $t \neq a\cdot b$).

2. $[e,s] = [e,s]_p$, when s is any dot product $s = a_1 \cdot \dots \cdot a_{n-1}\cdot a_n, n \geq 1$, with $a_n = e$.

3. $[e,s] = SR[e,s]_p$, when s is a dot product $s = b_1\cdot\dots\cdot b_{n-1}\cdot b_n, n > 1$, where $b_n \neq d_1\cdot\dots\cdot d_k\cdot e, k \geq 0$.

We shall "read" the tree



as $e\cdot[e,e]$. All other "reading" rules for trees follow from the definition of a dot product and 1–3 for the bracket operation. One can see that we are considering two classes of trees. Any tree belongs to one of the class of trees, and each tree from each class can be read only in a unique way and gives a word from $W(\{e\})$. Conversely, to every word we can correspond a unique tree according to the same rules, and this correspondence is one to one.

Let $V_A : \mathbf{NLP} \to \mathbb{A}\text{ss}$ and $V_L : \mathbf{NLP} \to \mathbb{L}\text{eibniz}$ be the forgetful functors.

Proposition 6.1.6. *If P is a free* NLP*-algebra, then $V_A(P)$ and $V_L(P)$ are free associative and free Leibniz algebras, respectively.*

Proof. Let P be the free NLP-algebra on the set X. Denote by X' the set of all kinds of those words of the type $[\cdots, \cdots]$, which does not contain words of the form

$$[a \cdot b, c],\ [a, [b, c]],\ [[a, b],\ c \cdot d],\ a,\ b,\ c,\ d \in P. \tag{6.1.3}$$

Let X'' be the set of all kind of words of the types $a_1 \cdots a_n$ and $a_1 \cdots a_n \cdot [\ldots, \ldots]$, where $a_1, \ldots, a_n \in X$, $n \geq 1$ and the bracket $[\cdots, \cdots]$ does not contain words of the form (6.1.3). Let $X_1 = X \cup X'$ and $X_2 = X \cup X''$. It is easy to show that $V_A(P)$ is the free associative algebra on the set X_1 and $V_L(P)$ is the free Leibniz algebra on the set X_2. □

6.1.3. Representations of NLP-algebras. Let $P \in \mathbf{NLP}$. In particular, P is an associative algebra and a Leibniz algebra, so we can speak of P-P-bimodules and Leibniz representations over P (see [65]).

Definition 6.1.7. A representation over P is a P-P-bimodule M together with two k-module homomorphisms

$$[-,-] : P \otimes M \longrightarrow M, [-,-] : M \otimes P \longrightarrow P$$

such that the following identities hold:

$$\begin{aligned}
\big[\,p_1, [p_2, m]\,\big] &= \big[\,[p_1, p_2], m\,\big] - \big[\,[p_1, m], p_2\,\big],\\
\big[\,p_1, [m, p_2]\,\big] &= \big[\,[p_1, m], p_2\,\big] - \big[\,[p_1, p_2], m\,\big],\\
\big[\,m, [p_1, p_2]\,\big] &= \big[\,[m, p_1], p_2\,\big] - \big[\,[m, p_2], p_1\,\big],\\
[p_1 m, p_2] &= p_1[m, p_2] + [p_1, p_2]m,\\
[m p_1, p_2] &= m[p_1, p_2] + [m, p_2]p_1,\\
[p_1 p_2, m] &= p_1[p_2, m] + [p_1, m]p_2
\end{aligned}$$

for all $m \in M$, $p_1, p_2 \in P$.

Let us observe that the first three axioms mean that M is a representation over P as Leibniz algebras. In the case of Poisson algebras, this definition gives the well-known definition of a Poisson representation in [44]. Note that a representation over P is a P-module in the sense of Definition 3.1.5, for the case $\mathbf{C} = \mathbf{NLP}$.

Examples 6.1.8.

1. Let R be a two-sided ideal of a NLP-algebra P; then R is a representation over P operating on R. In particular, if $R = P$, then P is a P-representation.
2. Let $\varphi : P \longrightarrow Q$ be a homomorphism of NLP-algebras; then Q is a representation over P with the operations $pq = \varphi(p)q$; $qp = q\varphi(p)$; $[p, q] = [\varphi(p), q]$; $[q, p] = [q, \varphi(p)]$, $p \in P$, $q \in Q$.

Definition 6.1.9. A homomorphism of representations over P is a linear map $f : M \longrightarrow M'$ satisfying

$$f(pm) = pf(m), \quad f(mp) = f(m)p, \quad f[p, m] = [p, f(m)], \quad f[m, p] = [f(m), p],$$

$p \in P$, $m \in M$.

Definition 6.1.10. Let M be a representation over P. We define the semidirect product $M \rtimes P$ as the NLP-algebra with underlying k-module $M \oplus P$ and operations defined by

$$(m_1, p_1) \cdot (m_2, p_2) = (p_1 m_2 + m_1 p_2, p_1 p_2),$$

$$\big[(m_1,p_1),(m_2,p_2)\big] = \big([p_1,m_2]+[m_1,p_2],[p_1,p_2]\big).$$

Note that here we follow the usual notation of semidirect products in algebras with bracket and denote the operating algebra from the right side.

The following definition is a special case of the definition given in [76] (see Definition 3.2.1).

Definition 6.1.11. Let $P \in \mathbf{NLP}$, and M be a representation over P. A derivation from P to M is a linear map $d : P \longrightarrow M$ satisfying

$$\begin{aligned} d(p_1p_2) &= d(p_1)p_2 + p_1d(p_2),\\ d[p_1,p_2] &= [d(p_1),p_2] + [p_1,d(p_2)]. \end{aligned}$$

We denote by $\mathrm{Der}_{\mathrm{NLP}}(P,M)$ the k-module of such derivations.

Lemma 6.1.12. *Let $P \in \mathbf{NLP}$, and M be a representation over P. Then there is a one-to-one correspondence between the derivations from P to M and the sections of the projection $pr : M \rtimes P \longrightarrow P$.*

Definition 6.1.13. Let $P, M \in \mathbf{NLP}$. An abelian extension of P by M is a short exact sequence

$$E : 0 \longrightarrow M \xrightarrow{\ i\ } Q \xrightarrow{\ j\ } P \longrightarrow 0,$$

where $Q \in \mathbf{NLP}$ and M is abelian (i.e., $mm' = [m,m'] = 0$, $m, m' \in M$).

Any abelian extension defines a unique representation on M over P in such a way that

$$\begin{aligned} i(j(q)m) = qi(m), &\quad i(mj(q)) = i(m)q,\\ i[j(q),m] = [q,i(m)], &\quad i([m,j(q)]) = [i(m),q] \end{aligned}$$

for any $m \in M$, $q \in Q$.

Two abelian extensions E and E' are called equivalent if there exists a homomorphism of NLP-algebras $f : Q \longrightarrow Q'$ inducing the identity morphisms on M and P. Note that in this case f is an isomorphism. Let M be any representation over P. Denote by $\mathrm{Ext}_{\mathrm{NLP}}(P,M)$ the set of all equivalence classes of those abelian extensions of P by M that induce the given representation on M over P.

6.1.4. Actions and Crossed Modules in NLP.

Since **NLP** is a category of groups with operations, according to the general definition of an action (called split derived actions in Chap. 3) of one object on another in this category, we obtain the corresponding definition for NLP-algebras.

First recall that an action of P on M for associative algebras is given by two k-module homomorphisms $\cdot : P \otimes M \longrightarrow M$, $\cdot : M \otimes P \longrightarrow M$ with the conditions

$$\begin{aligned} p(m_1m_2) = (pm_1)m_2, &\quad m_1(pm_2) = (m_1p)m_2,\\ (m_1m_2)p = m_1(m_2p), &\quad p_1(p_2m) = (p_1p_2)m,\\ p_1(mp_2) = (p_1m)p_2, &\quad m(p_1p_2) = (mp_1)p_2. \end{aligned}$$

An action of P on M for Leibniz algebras is given by two k-module homomorphisms $[-,-] : P \otimes M \longrightarrow M$, $[-,-] : M \otimes P \longrightarrow M$ with the conditions

$$\begin{aligned} \big[p,[m_1,m_2]\big] = \big[[p,m_1],m_2\big] - \big[[p,m_2],m_1\big], &\quad \big[m_1,[p,m_2]\big] = \big[[m_1,p],m_2\big] - \big[[m_1,m_2],p\big],\\ \big[m_1,[m_2,p]\big] = \big[[m_1,m_2],p\big] - \big[[m_1,p],m_2\big], &\quad \big[p_1,[p_2,m]\big] = \big[[p_1,p_2],m\big] - \big[[p_1,m],p_2\big],\\ \big[p_1,[m,p_2]\big] = \big[[p_1,m],p_2\big] - \big[[p_1,p_2],m\big], &\quad \big[m,[p_1,p_2]\big] = \big[[m,p_1],p_2\big] - \big[[m,p_2],p_1\big]. \end{aligned}$$

Definition 6.1.14. Let $M, P \in \mathbf{NLP}$. We say that P acts on M if we have an action of P on M as associative and Leibniz algebras given, respectively, by the k-module homomorphisms

$$\cdot, [-,-] : P \otimes M \longrightarrow M, \quad \cdot, [-,-] : M \otimes P \longrightarrow M$$

and the following conditions hold:

$$[p_1p_2, m] = p_1[p_2, m] + [p_1, m]p_2, \quad [p_1m, p_2] = p_1[m, p_2] + [p_1, p_2]m,$$
$$[mp_1, p_2] = m[p_1, p_2] + [m, p_2]p_1, \quad [m_1m_2, p] = m_1[m_2, p] + [m_1, p]m_2,$$
$$[m_1p, m_2] = m_1[p, m_2] + [m_1, m_2]p, \quad [pm_1, m_2] = p[m_1, m_2] + [p, m_2]m_1$$

for all $m, m_1, m_2 \in M$, $p, p_1, p_2 \in P$.

It is easy to verify that having a P-action on M, we can construct the semi-direct product $M \rtimes P$ with the usual operations; we will have $M \rtimes P \in \mathbf{NLP}$, and the corresponding natural extension $0 \longrightarrow M \longrightarrow M \rtimes P \longrightarrow P \longrightarrow 0$ will split. And, conversely, every split extension of M by P in $\mathbf{NLP}$ induces a set of actions, i.e., bilinear maps satisfying conditions of Definition 6.1.14. For the general case of groups with operations see [76] (see Sec. 3.1).

Let us observe that when M is an abelian NLP-algebra, that is, $M \cdot M = 0 = [M, M]$, then the last definition gives the axioms of representation from Definition 6.1.7. It is easy to verify that in this case the semidirect product agrees with Definition 6.1.10.

Definition 6.1.15. Let $M, P \in \mathbf{NLP}$ with an action of P on M. A crossed module is a morphism $\mu : M \longrightarrow P$ in $\mathbf{NLP}$ satisfying the following axioms:

$$\mu(pm) = p\mu(m), \quad \mu(mp) = \mu(m)p,$$
$$\mu[p, m] = [p, \mu(m)], \quad \mu[m, p] = [\mu(m), p],$$
$$\mu(m)m' = mm' = m\mu(m'),$$
$$[\mu(m), m'] = [m, m'] = [m, \mu(m')].$$

A homomorphism of crossed modules is a pair $(\alpha, \beta) : (M, P, \mu) \longrightarrow (M', P', \mu')$, where α, β are morphisms in $\mathbf{NLP}$ such that

$$\beta\mu = \mu'\alpha \quad \text{and} \quad \alpha(pm) = \beta(p)\alpha(m);$$
$$\alpha(mp) = \alpha(m)\beta(p);$$
$$\alpha[p, m] = [\beta(p), \alpha(m)];$$
$$\alpha[m, p] = [\alpha(m), \beta(p)] \quad \text{for all} \quad p \in P, \quad m \in M.$$

Thus the crossed module notion in $\mathbf{NLP}$ is a special case of the corresponding notion in categories of groups with operations [78] (see Sec. 1.1).

Examples 6.1.16.

1. Let $f : P \longrightarrow P'$ be a homomorphism in $\mathbf{NLP}$; then $i : \operatorname{Ker} f \hookrightarrow P$ is a crossed module.
2. Let R be a two-sided ideal of P; then $i : R \longrightarrow P$ is a crossed module. In particular, (P, P, Id) is a crossed module.
3. Let M be a representation over P, then the homomorphism $0 : M \longrightarrow P$ is a crossed module.

Note that example 1 follows from example 2 since Ker f is an ideal.

Let $\operatorname{Cat}(\mathbf{NLP})$ be the category of internal categories in $\mathbf{NLP}$, and let $X\operatorname{Mod}(\mathbf{NLP})$ be the category of crossed modules in $\mathbf{NLP}$. As a special case of the result of [78] we obtain

Proposition 6.1.17. *There is an equivalence of categories*

$$\operatorname{Cat}(\mathbf{NLP}) \simeq X\operatorname{Mod}(\mathbf{NLP}).$$

6.2. Cohomology of NLP-Algebras

Through this section we consider NLP-algebras over a field k. Let P be a NLP-algebra over k, and M a representation over P. In particular, P is an associative algebra and M is a P-P-bimodule and, on the other hand, P is a Leibniz algebra and M is a representation over P. Let $C^*_H(P,M)$ be the Hochschild complex, and $C^*_L(P,M)$ the Leibniz complex. We recall that for $n \geq 0$

$$C^n_H(P,M) = C^n_L(P,M) = \mathrm{Hom}(P^{\otimes n}, M)$$

and coboundary maps ∂^n_H and ∂^n_L are given by

$$\partial^n_H(f)(p_1,\dots,p_{n+1}) = (-1)^{n+1}\Big\{p_1 f(p_2,\dots,p_{n+1}) + \\ + \sum_{i=1}^{n}(-1)^i f(p_1,\dots,p_ip_{i+1},\dots,p_{n+1}) + (-1)^{n+1} f(p_1,\dots,p_n)p_{n+1}\Big\},$$

$$\partial^n_L(f)(p_1,\dots,p_{n+1}) = \big[p_1, f(p_2,\dots,p_{n+1})\big] + \sum_{i=2}^{n+1}(-1)^i\big[f(p_1,\dots,\widehat{p_i},\dots,p_{n+1}),p_i\big] + \\ + \sum_{1\leq i<j\leq n+1}(-1)^{j+1} f\Big(p_1,\dots,p_{i-1},[p_i,p_j],p_{i+1},\dots,\widehat{p_j},\dots,p_{n+1}\Big).$$

Thus $C^n_H(P,M)$ and $C^n_L(P,M)$ are k-vector spaces complexes.

We will need below the P-P-bimodule M^e, defined by $M^e = \mathrm{Hom}(P,M)$ as a k-vector space, and a bimodule structure is given by $(p_1 f)(p_2) = p_1 f(p_2)$; $(fp_1)(p_2) = f(p_2)p_1$. On the other hand, M^e has the structure of a P-representation by means of $[p_1,f](p_2) = [p_1, f(p_2)]$; $[f,p_1](p_2) = [f(p_2),p_1]$. We have an isomorphism of k-vector spaces $\theta_n : C^{n+1}_H(P,M) \longrightarrow C^n_H(P,M^e)$, $n \geq 1$. We denote the coboundary maps of the complex $C^*_H(P,M^e)$ by $\partial^{e,*}_H$. Thus we can define the homomorphism

$$\beta^n : C^n_L(P,M) \longrightarrow C^n_H(P,M^e), \quad n \geq 1$$

by

$$\beta^{2k+1} = \theta_{2k+1}\partial^{2k+1}_L, \quad k \geq 0,$$
$$\beta^{2k} = \partial^{e,2k-1}_H \theta_{2k-1}, \quad k \geq 1.$$

It is easy to see that β^* is a homomorphism between the cochain complexes $\overline{C}^*_L(P,M) = (C^n_L(P,M), \partial^n_L, n \geq 1)$ and $\overline{C}^*_H(P,M^e) = (C^n_H(P,M^e), \partial^{e,n}_H, n \geq 1)$. There is also a homomorphism of cochain complexes

$$\alpha^* : \overline{C}^*_H(P,M) = \big(C^n_H(P,M), \partial^n_H, n \geq 1\big) \longrightarrow \overline{C}^*_H(P,M^e)$$

defined in [22] by

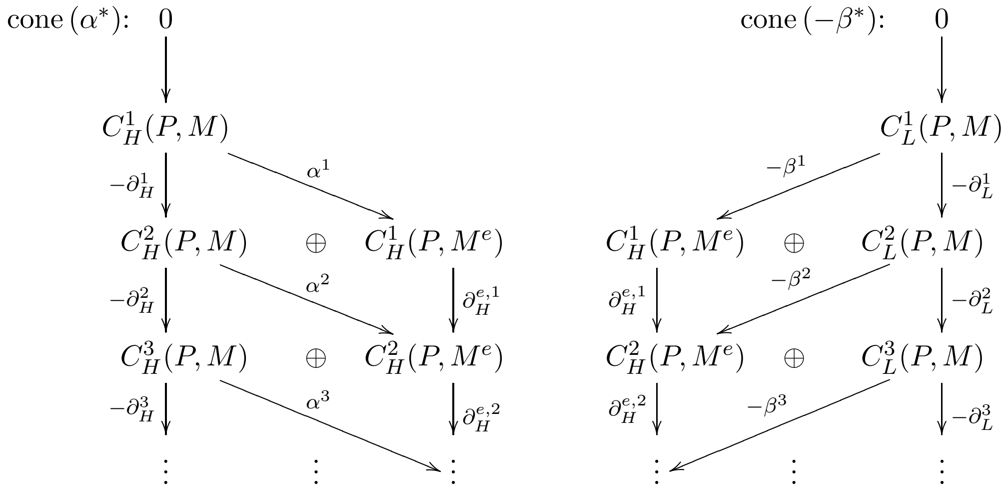
$$\alpha^1(f)(p_1)(p_2) = [p_1, f(p_2)] + [f(p_1),p_2] - f([p_1,p_2])$$

and for $n > 1$ by

$$\alpha^n(f)(p_1,\dots,p_n)(p_{n+1}) = \big[f(p_1,\dots,p_n),p_{n+1}\big] - \\ - f\big([p_1,p_{n+1}],p_2,\dots,p_n\big) - f\big(p_1,[p_2,p_{n+1}],\dots,p_n\big) - \cdots - f\big(p_1,\dots,p_{n-1},[p_n,p_{n+1}]\big).$$

Note that $\alpha_1 = \beta_1$.

Take $\overline{C}^n_H(P,M) = 0$, $\overline{C}^n_H(P,M^e) = 0$, $\overline{C}^n_L(P,M) = 0$, $\alpha^n = \beta^n = 0$, for $n \leq 0$ and consider the mapping cones: cone (α^*) and cone $(-\beta^*)$. By definition we have the complexes

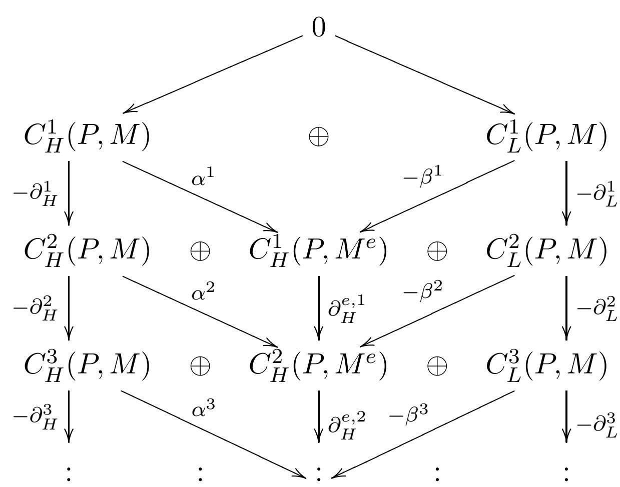


Let i_1 and i_2 be the following injections of complexes

$$\operatorname{cone}(\alpha^*) \xleftarrow{\;i_1\;} \overline{C}_H^{*-1}(P, M^e) \xrightarrow{\;i_2\;} \operatorname{cone}(-\beta^*).$$

Consider the pushout $C^*(P,M) = \operatorname{cone}(\alpha^*) \bigsqcup_{(i_1,i_2)} \operatorname{cone}(-\beta^*)$. Thus we have the complex

$(C^*(P,M), \partial^*)$:



Take $i = (i_1, -i_2)$; then the following sequence is exact:

$$0 \longrightarrow \overline{C}_H^{*-1}(P, M^e) \xrightarrow{\;i\;} \operatorname{cone}(\alpha^*) \sqcup \operatorname{cone}(-\beta^*) \longrightarrow C^*(P,M) \longrightarrow 0\ . \qquad (6.2.1)$$

From (6.2.1) we obtain the long exact sequence of cohomologies

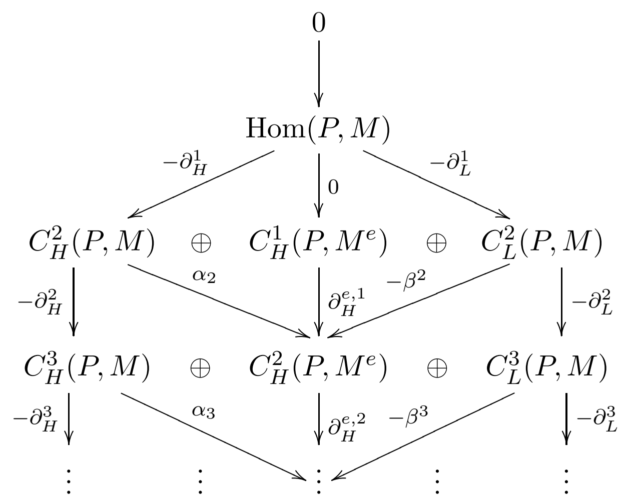
$$\begin{aligned} &0 \longrightarrow H^1(\operatorname{cone}(\alpha^*) \sqcup \operatorname{cone}(-\beta^*)) \xrightarrow{\;\varepsilon\;} H^1C^*(P,M) \longrightarrow \\ &\longrightarrow H^1\overline{C}_H^*(P, M^e) \xrightarrow{\;\eta\;} H^2(\operatorname{cone}(\alpha^*) \sqcup \operatorname{cone}(-\beta^*)) \longrightarrow H^2C^*(P,M) \longrightarrow \\ &\longrightarrow H^2\overline{C}_H^*(P, M^e) \longrightarrow H^3(\operatorname{cone}(\alpha^*) \sqcup \operatorname{cone}(-\beta^*)) \longrightarrow H^3C^*(P,M) \longrightarrow \cdots\ . \end{aligned} \qquad (6.2.2)$$

Note that ε is an isomorphism in (6.2.2), which implies that η is a monomorphism.

Define

$$C^0_{\mathrm{NLP}}(P,M)=0, \quad C^1_{\mathrm{NLP}}(P,M)=\mathrm{Hom}(P,M),$$
$$C^n_{\mathrm{NLP}}(P,M)=C^n(P,M), \quad n\geq 2;$$
$$\partial^0_{\mathrm{NLP}}=0, \quad \partial^1_{\mathrm{NLP}}=(\partial^1_H,0,\partial^1_L),$$
$$\partial^n_{\mathrm{NLP}}=\partial^n, \quad n\geq 2.$$

We have $\partial^{n+1}_{\mathrm{NLP}}\,\partial^n_{\mathrm{NLP}}=0$, $n\geq 0$, so $\{C^n_{\mathrm{NLP}}(P,M),\partial^n_{\mathrm{NLP}}, n\geq 0\}$ is a complex that has the form



The cohomology groups $H^n_{\mathrm{NLP}}(P,M)$, $n\geq 0$, of an NLP-algebra P with coefficients in the representation M over P are defined by

$$H^n_{\mathrm{NLP}}(P,M)=H^n(C^*_{\mathrm{NLP}}(P,M),\partial^n_{\mathrm{NLP}}), \quad n\geq 0.$$

We have $H^k_{\mathrm{NLP}}(P,M)=H^k(C^*(P,M),\partial^*)$ for $k>2$. According to the definition of AWB cohomology given in [23], we have $H^n(\mathrm{cone}(\alpha^*))=H^{n-1}_{\mathrm{AWB}}(P,M)$, where P is considered as an algebra with bracket and exact sequence (6.2.2) gives the corresponding exact sequence for cohomologies of an NLP-algebra P in the dimensions >2.

Proposition 6.2.1. *The following sequence is exact*:

$$H^2_H(P,M^e)\longrightarrow H^2_{AWB}(P,M)\oplus H^3(\mathrm{cone}(-\beta^*))\longrightarrow H^3_{NLP}(P,M)\longrightarrow$$
$$\longrightarrow H^3_H(P,M^e)\longrightarrow H^3_{AWB}(P,M)\oplus H^4(\mathrm{cone}(-\beta^*))\longrightarrow H^4_{NLP}(P,M)\longrightarrow\cdots.$$

As is well known from the general results on mapping cones, the short exact sequence

$$0\longrightarrow \overline{C}^{*-1}_H(P,M^e)\xrightarrow{i_2}\mathrm{cone}(-\beta^*)\longrightarrow(\overline{C}^*_L(P,M),-\partial^*_L)\longrightarrow 0$$

yields the long exact sequence

$$0\longrightarrow H^1(cone(-\beta^*))\longrightarrow \mathrm{Der}_L(P,\mathrm{M})\xrightarrow{\delta^1}\mathrm{Der}_H(P,M^e)\longrightarrow$$
$$\longrightarrow H^2(\mathrm{cone}(-\beta^*))\longrightarrow H^2_L(P,\mathrm{M})\xrightarrow{\delta^2}H^2_H(P,M^e)\longrightarrow\cdots$$

relating $H^*(\mathrm{cone}(-\beta^*))$ with the Hochschild and Leibniz cohomologies, where the connecting homomorphism δ^j is induced by β^j, $j\geq 1$.

We have the natural injection $C^2_H(P,M)\oplus C^2_L(P,M)\longrightarrow C^2_{\mathrm{NLP}}(P,M)$, which image we denote again by the sum $C^2_H(P,M)\oplus C^2_L(P,M)$. Consider the restriction

$$d^2_{\mathrm{NLP}}=\partial^2_{\mathrm{NLP}}\Big|_{C^2_H()\oplus C^2_L()}.$$

We define the 2-dimensional restricted cohomology of NLP-algebra P with coefficients in M by

$$\mathbb{H}^2_{\mathrm{NLP}}(P,M)=\operatorname{Ker} d^2_{\mathrm{NLP}}/\operatorname{Im}\partial^1_{\mathrm{NLP}}.$$

We have an obvious injection

$$\kappa:\operatorname{Ker} d^2_{\mathrm{NLP}}\longrightarrow \operatorname{Ker}\partial^2_{\mathrm{NLP}}$$

which induces the injection of the corresponding cohomologies

$$\chi:\mathbb{H}^2_{\mathrm{NLP}}(P,M)\longrightarrow H^2_{\mathrm{NLP}}(P,M).$$

From the definition of $C^*_{\mathrm{NLP}}(P,M)$ we have

Lemma 6.2.2.

$$H^0_{\mathrm{NLP}}(P,M)=0,$$
$$H^1_{\mathrm{NLP}}(P,M)=\mathrm{Der}_{\mathrm{NLP}}(P,M).$$

Proof. The proof follows directly from the fact that $C^0_{\mathrm{NLP}}(P,M)=0$ and from the definition of $\partial^1_{\mathrm{NLP}}$ and the Definition 6.1.11. □

Theorem 6.2.3. $\mathbb{H}^2_{\mathrm{NLP}}(P,M)\cong \mathrm{Ext}_{\mathrm{NLP}}(P,M)$.

Proof. Let $(f_\cdot,0,f_{[]})$ be a restricted 2-cocycle in $C^2_{\mathrm{NLP}}(P,M)$. Thus we have

$$-p_1f_\cdot(p_2,p_3)+f_\cdot(p_1p_2,p_3)-f_\cdot(p_1,p_2p_3)+f_\cdot(p_1,p_2)p_3=0,$$
$$\big[p_1,f_{[]}(p_2,p_3)\big]+\big[f_{[]}(p_1,p_3),p_2\big]-\big[f_{[]}(p_1,p_2),p_3\big]-$$
$$-f_{[]}([p_1,p_2],p_3)+f_{[]}(p_1,[p_2,p_3])+f_{[]}([p_1,p_3],p_2)=0,$$
$$\big[f_\cdot(p_1,p_2),p_3\big]-f_\cdot([p_1,p_3],p_2)-f_\cdot(p_1,[p_2,p_3])=p_1f_{[]}(p_2,p_3)-f_{[]}(p_1p_2,p_3)+f_{[]}(p_1,p_3)p_2,$$

$p_1,p_2,p_3\in P$. Let $Q=M\oplus P$ be a k-vector space. We define the operations on Q in the following usual way:

$$(m_1,p_1)\cdot(m_2,p_2)=\big(p_1m_2+m_1p_2+f_\cdot(p_1,p_2),p_1p_2\big),$$
$$\big[(m_1,p_1),(m_2,p_2)\big]=\Big([p_1,m_2]+[m_1,p_2]+f_{[]}(p_1,p_2),[p_1,p_2]\Big).$$

A straightforward verification shows that Q is a NLP-algebra and we have the abelian extension $E:0\longrightarrow M\xrightarrow{\ i\ }Q\xrightarrow{\ j\ }P\longrightarrow 0$ with $i(m)=(m,0)$, $j(m,p)=p$, and the induced P-representation structure on M is the given one. It is easy to show that if $(f'_\cdot,0,f'_{[]})$ is a 2-cocycle from the same class of 2-cohomology, then the extension E' defined by the pair $(f'_\cdot,f'_{[]})$ is isomorphic to E.

Given any class of extensions $E:0\longrightarrow M\xrightarrow{\ i\ }Q\xrightarrow{\ j\ }P\longrightarrow 0$ from $\mathrm{Ext}_{\mathrm{NLP}}(P,M)$, we choose a k-linear section u of j, $uj=1$, defining a 2-cocycle $(f_\cdot,0,f_{[]})$ and show that the class of $(f_\cdot,0,f_{[]})$ in $\mathbb{H}^2_{\mathrm{NLP}}(P,M)$ does not depend on the choice of a section of j, which ends the proof. □

Note that, in the above proved bijection, to a split extension corresponds the 2-cocycle $(f_\cdot,0,f_{[]})$ for which there exists a k-linear map $g:P\longrightarrow M$ such that

$$f_\cdot(p,p')=pg(p')+g(p)p'-g(pp'),$$
$$f_{[]}(p,p')=[p,g(p')]+[g(p),p']-g[p,p']$$

for all $p, p' \in P$.

Corollary 6.2.4. *If P is a free* NLP*-algebra, then*

$$\mathbb{H}^2_{\mathrm{NLP}}(P, -) = 0$$

and

$$H^n_{\mathrm{NLP}}(P, -) = 0$$

for $n > 2$.

Proof. Since every extension $0 \longrightarrow M \xrightarrow{i} Q \xrightarrow{j} P \longrightarrow 0$ splits for a free algebra P, for $n = 2$ the fact follows from Theorem 6.2.3. Let $n > 2$. From Proposition 6.1.6 it follows that $V_A(P)$ is a free associative algebra and $V_L(P)$ is a free Leibniz algebra. It is well known that cohomologies of free associative algebras and free Leibniz algebras vanish in the dimensions ≥ 2 [65]. Thus we have $H^n_{\mathrm{H}}(P, -) = 0$ and $H^n_{\mathrm{L}}(P, -) = 0$ for $n \geq 2$. Applying this and the fact that α and β are homomorphisms of cochain complexes, one easily shows that $C^*_{\mathrm{NLP}}(P, M)$ is exact in dimensions > 2 for a free NLP-algebra P, which ends the proof. □

Corollary 6.2.5. *If P is a free* NLP*-algebra, then for any representation M over P we have*

$$H^2_{\mathrm{NLP}}(P, M) \approx \mathrm{Ker}\partial^2_{\mathrm{NLP}} / \mathrm{Ker} d^2_{\mathrm{NLP}}.$$

Proof. It follows from Corollary 6.2.4 and the facts that we have the injection

$$\chi : \mathbb{H}^2_{\mathrm{NLP}}(P, M) \longrightarrow H^2_{\mathrm{NLP}}(P, M)$$

defined above and the isomorphism

$$\mathrm{Coker}\chi \approx \mathrm{Coker}\kappa$$

for any NLP-algebra P. □

6.2.1. Relative cohomology of NLP**-algebras and 3-fold crossed sequences.** As for the case of Lie algebras (see [54]), we consider the relative cohomology of NLP-algebras and its relation with 3-fold crossed sequences of the special type.

Consider an exact sequence

$$E : 0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \xrightarrow{\nu} P \longrightarrow 0 \tag{6.2.3}$$

in **NLP**, where N acts on M and $\mu : M \longrightarrow N$ is a crossed module in **NLP**.

From this it follows that $\lambda(L)$ is in the center of M. The sequence (6.2.3) uniquely determines an action of P on L by

$$p \cdot l = \lambda^{-1}(n \cdot \lambda(l)), \quad l \cdot p = \lambda^{-1}(\lambda(l) \cdot n),$$
$$[p, l] = \lambda^{-1}[n, \lambda(l)], \quad [l, p] = \lambda^{-1}[\lambda(l), n],$$

where $p \in P$, $l \in L$, $n \in N$, $\nu(n) = p$; here we use the fact that $\mu : M \longrightarrow N$ is a crossed module and $\mu(n \cdot \lambda(l)) = n \cdot \mu\lambda(l) = 0$, $\mu(\lambda(l) \cdot n) = \mu([n, \lambda(l)]) = \mu([\lambda(l), n]) = 0$. This action does not depend on the choice of the element n, with $\nu(n) = p$, since $\lambda(l)$ is in the center of M. In particular, L is an abelian object and we have a P-representation structure on it.

Let L be a representation over P. Now we fix N and a surjective homomorphism $\nu : N \longrightarrow P$ and consider all kinds of the above defined crossed extensions, which induce the given representation structure on L. A morphism between two such crossed extensions $E \longrightarrow E'$ is a morphism of extensions $(1_L, \varphi, 1_N, 1_P)$, which respects the action. It is easy to see that if there exists a morphism $(1_L, \varphi, 1_N, 1_P)$, then it is an isomorphism of crossed modules

$$(M, N, \mu) \xrightarrow{(\varphi, 1)} (M', N, \mu') \ .$$

We shall say that two such crossed extensions are equivalent $E \sim E'$ if there exists a morphism $(1_L, \varphi, 1_N, 1_P) : E \longrightarrow E'$.

Let $\mathrm{CExt}_{\mathrm{NLP}}(P, N; L)$ denote the set of all equivalence classes of such crossed extensions.

Let $P, N \in \mathbf{NLP}$, $\nu : N \longrightarrow P$ be a fixed surjective homomorphism, and L be a representation over P; ν induces a N representation structure on L, and the homomorphism of cochain complexes $\nu^* : C^*_{\mathrm{NLP}}(P, L) \longrightarrow C^*_{\mathrm{NLP}}(N, L)$. We define $C^*_{\mathrm{NLP}}(P, N; L)$ as the cokernel of ν^*. Thus we have the exact sequence of complexes

$$0 \longrightarrow C^*_{\mathrm{NLP}}(P, L) \xrightarrow{\nu^*} C^*_{\mathrm{NLP}}(N, L) \xrightarrow{\sigma^*} C^*_{\mathrm{NLP}}(P, N; L) \longrightarrow 0 \ .$$

Denote $H^n_{\mathrm{NLP}}(P, N; L) = H^n(C^*_{\mathrm{NLP}}(P, N; L))$ for $n \geq 0$, and $\mathbb{H}^2_{\mathrm{NLP}}(P, N; L) = \mathbb{H}^2(C^*_{\mathrm{NLP}}(P, N; L))$, where $\mathbb{H}^2(C^*_{\mathrm{NLP}}(P, N; L))$ has the obvious meaning; i.e., we consider the second restricted coboundary homomorphisms in $C^*_{\mathrm{NLP}}(P, L)$ and in $C^*_{\mathrm{NLP}}(N, L)$, which give the corresponding restricted coboundary homomorphism in $C^*_{\mathrm{NLP}}(P, N; L)$ and define the second relative restricted cohomology.

Theorem 6.2.6. *There is a bijection*

$$\mathrm{CExt}_{\mathrm{NLP}}(P, N; L) \xrightarrow{\sim} \mathbb{H}^2_{\mathrm{NLP}}(P, N; L) \ .$$

Proof. Let E be an extension (6.2.3). Denote $R = \mathrm{Ker}\,\nu$; let $\tau : M \twoheadrightarrow R$ and $\kappa : R \hookrightarrow N$ be the canonical surjective homomorphism and the inclusion respectively. Let u be a linear section of τ. Thus M is isomorphic to $L \oplus R$ as a vector space and the isomorphism is given by

$$\begin{aligned} m &\longmapsto (\lambda^{-1}(m - u\tau(m)), \tau(m)), \\ (l, r) &\longmapsto \lambda(l) + u(r). \end{aligned}$$

The action of N on M defines an action of N on $L \oplus R$ given by

$$n \cdot \big(\lambda^{-1}(m - u\tau(m)), \tau(m)\big) = \Big(\nu(n) \cdot \lambda^{-1}(m - u\tau(m)) + f_.(n, \tau(m)), n \cdot \tau(m)\Big).$$

Thus the left dot action defines a k-linear map $f_. : N \otimes R \longrightarrow L$. In the same way, for the right dot action and actions by bracket, we will have k-linear maps $f'_., f'_{[]} : R \otimes N \longrightarrow L$ and $f_{[]} : N \otimes R \longrightarrow L$. Moreover, $f_. \mid_{R\otimes R} = f'_. \mid_{R\otimes R}, f_{[]} \mid_{R\otimes R} = f'_{[]} \mid_{R\otimes R}$.

Let $\overline{f} = (\overline{f_.}, 0, \overline{f_{[]}})$ be an element from $C^2_{\mathrm{NLP}}(N, L)$ such that

$$\begin{aligned} \overline{f_.}(n_1 \otimes n_2) = f_.(n_1 \otimes n_2), \overline{f_{[]}}(n_1 \otimes n_2) = f_{[]}(n_1 \otimes n_2), &\quad \text{if} \quad n_2 \in R, \\ \overline{f_.}(n_1 \otimes n_2) = f'_.(n_1 \otimes n_2), \overline{f_{[]}}(n_1 \otimes n_2) = f'_{[]}(n_1 \otimes n_2), &\quad \text{if} \quad n_1 \in R. \end{aligned} \tag{6.2.4}$$

Such a pair exists, we can take, e.g., $\overline{f_.}(p_1 \otimes p_2) = \overline{f_{[]}}(p_1 \otimes p_2) = 0$, $p_1, p_2 \in P$, and define by (6.2.4) other types of elements of $N \otimes N$. Here we have in mind that as a vector space $N \approx R \oplus P$. From the properties of the action we obtain $d^2(\overline{f})(n_1 \otimes n_2 \otimes n_3) = 0$ if at least one of the n_i belongs to R; from the fact that $\mu : M \longrightarrow N$ is a crossed module, we obtain that $(\overline{f_.}, \overline{f_{[]}}) \mid_{R\otimes R}$ is a factor system of the extension $0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\tau} R \longrightarrow 0$. From this it follows that there exists $k_3 \in C^3_{\mathrm{NLP}}(P, L)$ such that $\nu^3(k_3) = d^2(\overline{f})$; then we have $\sigma^3 d^2(\overline{f}) = 0$, which gives that $\sigma^2(\overline{f})$ is a 2-dimensional cocycle in $C^2_{\mathrm{NLP}}(P, N; L)$. It is easy to see that $\sigma^2(\overline{f})$ does not depend on the choice of $\overline{f}$ in $C^2_{\mathrm{NLP}}(N, L)$.

For another linear section u' of τ, we will have functions $\varphi_.$, $\varphi'_.$, $\varphi_{[]}$, $\varphi'_{[]}$, which will give

$$\begin{aligned} &\nu(n) \cdot \lambda^{-1}(m - u\tau(m)) + f_.(n, \tau(m)) + u(n \cdot \tau(m)) \\ &\qquad = \nu(n) \cdot \lambda^{-1}(m - u'\tau(m)) + \varphi_.(n, \tau(m)) + u'(n \cdot \tau(m)). \end{aligned} \tag{6.2.5}$$

Analogous equalities exist for pairs $(f'_., \varphi'_.)$, $(f_{[]}, \varphi_{[]})$, $(f'_{[]}, \varphi'_{[]})$. $u - u'$ defines a function $R \longrightarrow L$ which can be extended up to $N \longrightarrow L$ (taking e.g. $0 : P \longrightarrow L$), denote one of such extensions by $f^1 : N \longrightarrow L$. From (6.2.5) we obtain

$$f.(n, \tau(m)) - \varphi.(n, \tau(m)) = \nu(n) \cdot f^1(\kappa\tau(m)) - f^1(n \cdot \kappa\tau(m)) = \partial^1_H(f^1)(n, \kappa\tau(m));$$

here we use the fact that $f^1(n) \cdot (\nu\kappa\tau(m)) = 0$ and that N acts on L due to ν. Analogous formulas exist for $(f_{[]}, \varphi_{[]})$, $(f., \varphi.)$, $(f'_{[]}, \varphi'_{[]})$.

From this we conclude that $\sigma^2(\overline{\varphi}., \overline{\varphi}_{[]}) \in C^2_{\mathrm{NLP}}(P, N; L)$ defined by $(\varphi., \varphi'_., \varphi_{[]}, \varphi'_{[]})$ is in the same cohomology class in $\mathbb{H}^2_{\mathrm{NLP}}(P, N; L)$ as $\sigma^2(\overline{f}., \overline{f}_{[]})$ and one can easily check that this procedure does not depend on the choice of the extension map f^1.

Thus we showed that each extension (6.2.3) uniquely defines a determined cohomology class and actually at the same time we proved that to isomorphic extensions corresponds the same class in the relative cohomology.

The second part of the proof is analogous to the one given in [54] for Lie algebras. Let $clf = cl(f., 0, f_{[]}) \in C^2_{\mathrm{NLP}}(P, N; L)$ be a 2-cocycle. Choose any cochain $f = (f., 0, f_{[]}) \in C^2_{\mathrm{NLP}}(N, L)$ as a representative of this class. Since clf is a cocycle, there exists $k \in C^3_{\mathrm{NLP}}(P, L)$ with $d^2(f) = \nu^3(k)$. The diagram is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^2_{\mathrm{NLP}}(P,L) & \longrightarrow & C^2_{\mathrm{NLP}}(N,L) & \longrightarrow & C^2_{\mathrm{NLP}}(P,N;L) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d^2} & & \downarrow{\scriptstyle d^2} & & \downarrow{\scriptstyle d^2_{rel}} & & \\ 0 & \longrightarrow & C^3_{\mathrm{NLP}}(P,L) & \longrightarrow & C^3_{\mathrm{NLP}}(N,L) & \longrightarrow & C^3_{\mathrm{NLP}}(P,N;L) & \longrightarrow & 0 \end{array}.$$

From this it follows that the restriction of f on $R \otimes R$ is a cocycle $R \otimes R \longrightarrow L$, and moreover $d^2 f(n_1 \otimes n_2 \otimes n_3) = 0$ if at least one of the n_i belongs to R, $i = 1, 2, 3$. Note that by the restriction of f we mean the corresponding restrictions of $f.$ and $f_{[]}$, and similarly for $d^2 f(n_1 \otimes n_2 \otimes n_3)$. We take $M = L \oplus R$ as a vector space and define operations on M by

$$(l, r) \cdot (l', r') = (f.(r, r'), r \cdot r'),$$
$$\big[(l, r), (l', r')\big] = \big(f_{[]}(r, r'), [r, r']\big).$$

The actions of N on M are defined according to the following formulas:

$$n \cdot (l, r) = \big(\nu(n) \cdot l + f.(n, r), n \cdot r\big),$$
$$[n, (l, r)] = \big([\nu(n), l] + f_{[]}(n, r), [n, r]\big),$$
$$(l, r) \cdot n = \big(l \cdot \nu(n) + f.(r, n), r \cdot n\big),$$
$$[(l, r), n] = \big([l, \nu(n)] + f_{[]}(r, n), [r, n]\big).$$

A straightforward verification shows that M is an NLP-algebra and the structure defined on M does not depend on the choice of representatives for clf in $C^2_{\mathrm{NLP}}(N, L)$; λ and ν are defined in the obvious way: $\lambda(l) = (l, 0), \mu(l, r) = r$ and $\mu : M \longrightarrow N$ is a crossed module in **NLP**. It is easy to show that to the cocycles of the same cohomology class correspond isomorphic extensions, and we have a one-to-one correspondence between isomorphic classes of extensions and cohomology $\mathbb{H}^2_{\mathrm{NLP}}(P, N; L)$, which ends the proof. ∎

6.3. $\mathrm{NLP}^!$-Algebras

Let k be a field. The operad theory gives rise to a duality for quadratic operads [45]. The following description of $\mathrm{NLP}^!$-algebras is due to T. Pirashvili and J. M. Casas [23]; it follows from Proposition B3 of the Appendix B to [63].

An $\mathrm{NLP}^!$-algebra is an associative algebra A equipped with a bilinear binary operation $\star : A\otimes A \longrightarrow A$ such that the following identities hold:

(1) $(a \cdot b) \star c = a \cdot (b \star c)$;

(2) $(a \cdot b) \star c + (a \star c) \cdot b = 0$;

(3) $(a \star b) \star c = a \star (b \star c) + a \star (c \star b)$;

(4) $a \star (b \cdot c) = 0$.

From (1) and (2) it follows that

$2'$. $(a \star c) \cdot b = -a \cdot (b \star c)$.

For any set X, consider the set

$$W(X) = \Big\{x_1 \dots x_k\big(x_{k+1} \star (x_{k+2} \star (\cdots \star (x_{n-1} \star x_n) \dots))\big),\ 0 \le k \le n,\ n \ge 1,\ x_i \in X,\ i = 1, \dots, n\Big\}.$$

Thus the elements of $W(X)$ are certain type of symbolic words from the elements of X, dot, and $\star$ operations. Note that if $n-k=1$, then the word has the form $x_1 \cdots x_k \cdot x_{k+1}$; it is clear that $W(X)$ contains X.

Let $F(W(X))$ be the k-vector space with basis $W(X)$. We can define operations $\cdot, \star : W(X) \times W(X) \longrightarrow W(X)$ by gluing the words due to dot and $\star$ symbols when such a word exists in $W(X)$. For instance,

$$\big((a \cdot b), (c \star d)\big) \stackrel{\cdot}{\longmapsto} a \cdot b \cdot (c \star d) \ ,$$

$$(a, b \star c) \stackrel{\star}{\longmapsto} a \star (b \star c) \ .$$

If such a word does not exist in $W(X)$, then we perform $\cdot$ and $\star$ operations

$$W(X) \times W(X) \stackrel{\cdot,\star}{\longmapsto} F(W(X))$$

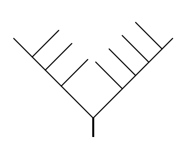
according to the identities (1), ($2'$), (3) and (4). We extend bilinearly this operations on $F(W(X))$, and it is easy to see that $F(W(X))$ has the structure of an $\mathrm{NLP}^!$-algebra.

Proposition 6.3.1. *For any set X, $F(W(X))$ is a free $\mathrm{NLP}^!$-algebra generated by X and we have the pair of adjoint functors*

$$\mathbf{NLP}^! \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Set}$$

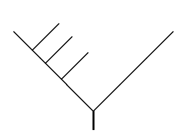
Proof. A straightforward verification. $\square$

Consider the case where $X = \{e\}$. As for the case of NLP-algebras, we have the description of $W(\{e\})$ in terms of certain types of trees. Consider trees of the following simple type:

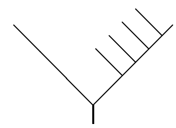


(6.3.1)

We denote the set of this kind of trees with n leaves by T_n. The left side of the tree we shall use for the dot operation, and the right side for the $\star$ operation. The tree



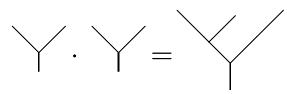
we shall "read" as $e \cdot e \cdot e \cdot e$, and we denote it by $t^{\cdot}$. The tree



we shall "read" as $e \star (e \star (e \star (e \star e)))$, and we denote it by $t^\star$.

The tree (6.3.1) we "read" as $e \cdot e \cdot e \cdot (e \star (e \star (e \star (e \star e))))$.

Let $T = \bigcup\limits_{n \geq 1} T_n$. Thus, for certain types of trees of T we have defined the dot and $\star$ operations. These operations can be expressed in terms of grafting in the following way:

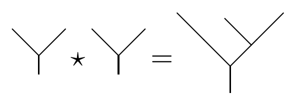


Denote  by e. We have defined

$$e \cdot t^{\cdot} = gr(e \cdot t^l, t^r)$$

and if $s^{\cdot} = e \overset{n}{\cdots} e$, $n > 1$, then $s^{\cdot} \cdot t^{\cdot} = e \cdot ((e \overset{n-1}{\cdots} e) \cdot t^{\cdot})$.

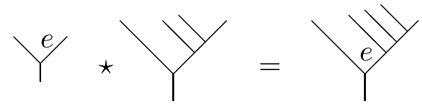
For the $\star$ operation we have the following rules:



and for any $t^\star$ we have

$$e \star t^\star = gr(e^l, t^\star).$$

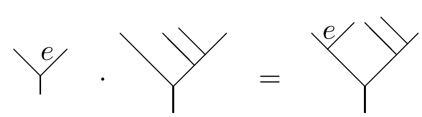
Thus



We have defined

$$e \cdot t^\star = gr(e, t^r). \tag{6.3.2}$$

Thus



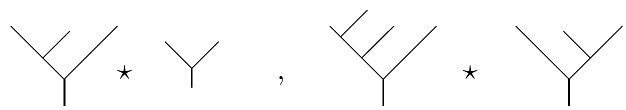
For any $t \in T$, such that $t^l \neq |$, we have defined

$$e \cdot t = gr(e \cdot t^l, t^r);$$

here we can have in mind that $e \cdot | = e$ in the case $t^l = |$, and in this case (6.3.2) follows from this formula. In general, for $s^{\cdot}$ and any t we will have

$$s^{\cdot} \cdot t = gr(s^{\cdot} \cdot t^l, t^r).$$

Note that products of the kind



are not defined in T.

According to the above rules, each tree from T corresponds to a word of one of the following types: $a_1 \dots a_n$, $a_1 \star \cdots \star a_n$ or $a_1 \cdots a_k \cdot (a_{k+1} \star (a_{k+2} \star \cdots \star (a_{n-1} \star a_n) \dots))$, where $n \geq 1$, $a_i = e$, $i = 1, \dots, n$, $0 \leq k \leq n$, and this correspondence is one to one. Thus, for $X = \{e\}$ we have a bijection $W(\{e\}) \longleftrightarrow T$. It is easy to see that, for each n, the number of trees in T_n (and respectively the number of words of the length n) is equal to n.

Chapter 7

CENTRAL SERIES FOR GROUPS WITH ACTION AND LEIBNIZ ALGEBRAS

The well-known construction of E. Witt defines the functor from the category of groups to the category of Lie algebras [90], [83]. The aim of this chapter is to define a category and to give an analogue of Witt's construction for its objects, which will lead us to the category of Leibniz algebras. This problem was stated by J.-L. Loday [62]; later an analogous question for the possibly defined partial Leibniz algebras was proposed to me, which was inspired by the work of Baues and Conduché [10]. Since the main interest was in the absolute case, the author decided to begin with this one. The results obtained in this chapter give the solution to the first problem of J.-L. Loday formulated in the Introduction (see [62, 64]).

7.1. Groups with Action on Itself

Let G be a group that acts on itself from the right side; i.e., we have a map $\varepsilon : G \times G \longrightarrow G$ with

$$\begin{aligned} \varepsilon(g, g' + g'') &= \varepsilon(\varepsilon(g, g'), g''), \\ \varepsilon(g, 0) &= g, \\ \varepsilon(g' + g'', g) &= \varepsilon(g', g) + \varepsilon(g'', g), \end{aligned} \tag{7.1.1}$$

for $g, g', g'' \in G$. Denote $\varepsilon(g, h) = g^h$, for $g, h \in G$. We denote the group operation additively; nevertheless the group is not commutative in general. If (G', ε') is another group with action, then a homomorphism $(G, \varepsilon) \longrightarrow (G', \varepsilon')$ is a group homomorphism $\varphi : G \longrightarrow G'$ for which the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \varepsilon\ } & G \\ {\scriptstyle (\varphi,\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G' \times G' & \xrightarrow[\ \varepsilon'\ ]{} & G' \end{array}$$

commutes. In other words, we have

$$\varphi(g^h) = \varphi(g)^{\varphi(h)}, \qquad g, h \in G.$$

If we consider an action as a group homomorphism $G \xrightarrow{\nu} \operatorname{Aut} G$, then a homomorphism between two groups with action means the commutativity of the diagram

$$\begin{array}{ccc}
G & \xrightarrow{\quad \nu \quad} & \operatorname{Aut} G \subset \operatorname{Hom}(G,G) \\
\Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \operatorname{Hom}(G,\varphi)} \\
 & & \operatorname{Hom}(G,G') \\
 & & \Big\uparrow{\scriptstyle \operatorname{Hom}(\varphi,G')} \\
G' & \xrightarrow{\quad \nu' \quad} & \operatorname{Aut} G' \subset \operatorname{Hom}(G',G')
\end{array}$$

so that $\varphi \cdot (\nu(h)) = \nu'(\varphi(h)) \cdot \varphi$, $h \in G$. Note that the action defined above is a split derived action within the category of groups $\mathbb{G}\mathrm{r}^\bullet$ in the sense of Chap. 3.

Recall [55] that an Ω-group is a group with a system of n-ary algebraic operations Ω $(n \geq 1)$ that satisfies the condition

$$00 \cdots 0\omega = 0, \tag{7.1.2}$$

where 0 is the identity element of G, and 0 on the left side occurs n times if ω is an n-ary operation. In special cases Ω-groups give groups, rings, and groups with action on itself. In the latter case Ω consists of one binary operation, an action, and the condition (7.1.2) is satisfied. We shall denote the category of groups with action on itself by $\mathbb{G}\mathrm{r}^\bullet$. Let $\mathbb{A}\mathrm{b}^\bullet$ denote the category of abelian groups with action on itself; here we mean the action within $\mathbb{G}\mathrm{r}$. We have the functors

$$\mathbb{A}\mathrm{b}^\bullet \underset{A}{\overset{E}{\rightleftarrows}} \mathbb{G}\mathrm{r}^\bullet \begin{array}{c} \xrightarrow{Q_1} \\ \xleftarrow{T} \\ \xrightarrow{Q_2} \\ \xleftarrow{C} \end{array} \mathbb{G}\mathrm{r}$$

where $Q_1(G)$, for $G \in \mathbb{G}\mathrm{r}^\bullet$, is the greatest quotient group of G that makes the action trivial; $Q_2(G)$ is a quotient of G by the equivalence relation generated by the relation $g^h \sim -h + g + h$, $g, h \in G$; A is the abelianization functor; thus $A(G) = G/(G,G)$, where (G,G) is the ideal of G generated by the commutator normal subgroup of G (for the definition of an ideal see Sec. 2). $A(G)$ has the induced operation of action on itself. Each group can be considered as a group with the trivial action or with the action by conjugation; they give functors T and C, respectively. Every object of $\mathbb{A}\mathrm{b}^\bullet$ can be considered as an object of $\mathbb{G}\mathrm{r}^\bullet$; this functor is denoted by E. It is easy to see that the functors Q_1, Q_2, and A are left adjoints to the functors T, C and E, respectively. Let $G \in \mathbb{G}\mathrm{r}^\bullet$. Define the operation of square brackets $[\ ,\ ] : G \times G \longrightarrow G$ on G by

$$[g,h] = -g + g^h, \qquad g,h \in G.$$

Proposition 7.1.1. *For the operation* $[\ ,\ ]$ *we have the following identities*:

(i) $[g, h_1 + h_2] = [g, h_1] + [g + [g, h_1], h_2]$;

(ii) $[g + g', h] = -g' + [g,h] + g' + [g', h]$;

(iii) $[g, 0] = [0, g] = 0$.

Proof. These identities follow directly from (7.1.1). □

Corollary 7.1.2. *For* $g, h \in G$

$$[g^h, -h] = -[g,h]; \quad [-g, h] = g - [g,h] - g.$$

Denote by $\mathbb{G}\mathrm{r}^{[\,]}$ the category of groups with an additional bracket operation $[\ ,\ ]$ satisfying conditions (i)–(iii) of Proposition 7.1.1; morphisms of $\mathbb{G}\mathrm{r}^{[\,]}$ are group homomorphisms preserving the bracket operation. We shall denote the objects of $\mathbb{G}\mathrm{r}^{[\,]}$ by $G^{[\,]}$.

Conversely, if $G^{[\,]} \in \mathbb{G}\mathrm{r}^{[\,]}$, we can define an action of $G^{[\,]}$ on itself due to the bracket operation by

$$g^h = g + [g,h], \qquad g,h \in G^{[\,]}.$$

It is easy to prove that these two procedures are converse to each other and actually we have an isomorphism of categories

$$\mathbb{G}\mathrm{r}^{\bullet} \approx \mathbb{G}\mathrm{r}^{[\,]}.$$

7.2. Ideals and Commutators in $\mathbb{G}\mathrm{r}^{\bullet}$

Let $G \in \mathbb{G}\mathrm{r}^{\bullet}$.

Definition 7.2.1. A nonempty subset A of G is called an ideal of G if it satisfies the following conditions:

1. A is a normal subgroup of G as a group;
2. $a^g \in A$, for $a \in A$, $g \in G$;
3. $-g + g^a \in A$, for $a \in A$ and $g \in G$.

Definition 7.2.2 (Kurosh [55])**.** A nonempty subset A of an Ω-group G is called an ideal if

(a) A is an additive normal subgroup of G;

(b) For any n-any operation ω from Ω, any element $a \in A$, and elements $x_1, x_2, \dots, x_n \in G$,

$$-(x_1 \cdots x_n\omega) + x_1 \cdots x_{i-1}(a + x_i)x_{i+1} \cdots x_n\omega \in A$$

for $i = 1, 2, \dots, n$.

This definition in the case of groups is the definition of a normal subgroup of a group, and in the case of rings is the definition of a two-sided ideal of a ring.

Proposition 7.2.3. *For a group $G \in \mathbb{G}\mathrm{r}^{\bullet}$ considered as an Ω-group, where Ω consists of one binary operation of action, Definitions* 2.1 *and* 2.2 *are equivalent.*

Proof. Condition (b) of Definition 2.2 has the form

$$-x_1^{x_2} + (a + x_1)^{x_2} \in A \quad \text{for} \quad i = 1; \tag{7.2.1}$$

$$-x_1^{x_2} + x_1^{a+x_2} \in A \quad \text{for} \quad i = 2. \tag{7.2.2}$$

Taking $x_1 = 0$ in (7.2.1), we obtain $a^{x_2} \in A$, which is condition 2 of Definition 2.1. Taking $x_2 = 0$ in (7.2.2), we have $-x_1 + x_1^a \in A$, which is condition 3 of Definition 2.1.

Conversely, we shall show that conditions 2 and 3 of Definition 2.1 imply conditions (7.2.1) and (7.2.2). From condition 2 we have $a^{x_2} \in A$; also

$$-x_1^{x_2} + (a + x_1)^{x_2} = -x_1^{x_2} + a^{x_2} + a_1^{x_2},$$

and it is an element of A since A is a normal subgroup of G. By condition 3 of Definition 2.1, $-x_1 + x_1^a \in A$. We have $-x_1^{x_2} + x_1^{a+x_2} = (-x_1 + x_1^a)^{x_2}$, and this is an element of A due to condition 2, which ends the proof. □

Thus an ideal of G is a subobject of G in $\mathbb{G}\mathrm{r}^{\bullet}$. It is clear that G itself and the trivial subobject of G are ideals of G. An intersection of any system of ideals of G is an ideal, and therefore we conclude that there exists the ideal generated by a system of elements of G.

Proposition 7.2.4. *Let A be an ideal of G. For $a_1, a_2 \in A$, $g_1, g_2 \in G$ we have*

$$(a_1 + g_1)^{a_2+g_2} \in g_1^{g_2} + A.$$

Proof. Since A is an ideal of G, there exist $a_1', a_2' \in A$, such that $a_1 + g_1 = g_1 + a_1'$, $a_2 + g_2 = g_2 + a_2'$. Therefore

$$(a_1 + g_1)^{a_2+g_2} = (g_1 + a_1')^{g_2+a_2'} = (g_1^{g_2})^{a_2'} + a_1'^{\,g_2+a_2'} =$$
$$= g_1^{g_2} - g_1^{g_2} + (g_1^{g_2})^{a_2'} + a_1'^{\,g_2+a_2'} \in g_1^{g_2} + A;$$

here we apply $-g_1^{g_2} + (g_1^{g_2})^{a_2'} \in A$. □

Let A and B be subobjects of G. Denote by $\{A, B\}$ the subobject of G generated by A and B, and let $A + B$ denote the subset of G

$$A + B = \{a + b \,|\, a \in A,\ b \in B\}.$$

Proposition 7.2.5. *If A is an ideal of G and B is a subobject of G, then*

$$\{A, B\} = A + B.$$

Proof. It is obvious that $A + B \subset \{A, B\}$. Since A is an ideal, it follows that $A + B$ is a subgroup of G. By Proposition 7.2.4, $(a_1 + b_1)^{a_2+b_2} \in b_1^{b_2} + A$. Since B is a subobject, $b_1^{b_2} \in B$, and since A is an ideal, $b_1^{b_2} + A = A + b_1^{b_2} \in A + B$, which ends the proof. □

For Ω-groups see Propositions 7.2.4 and 7.2.5 in [55].

Proposition 7.2.6. *If A and B are ideals of G, then $A + B$ is also an ideal.*

Proof. For $g \in G$, $a \in A$ and $b \in B$ we have

$$g + (a + b) = (a' + g) + b = a' + b' + g \in A + B + g,$$

for certain $a' \in A$ and $b' \in B$. Thus $g + (A + B) \subset (A + B) + g$. In the same way we show that $(A + B) + g \subset g + (A + B)$ and thus $g + (A + B) = (A + B) + g$. It is obvious that $(a + b)^g \in A + B$. Now we have to show that $-g + g^{a+b} \in A + B$. We have

$$-g + g^{a+b} = -g + g^a - g^a + (g^a)^b \in A + B$$

since $-g + g^a \in A$, $-g^a + (g^a)^b \in B$. □

It is easy to verify that the ideal generated by a system of ideals of G coincides with the additive subgroup of G generated by these ideals. For Ω-groups see [55].

Definition 7.2.1′. Let $G^{[\,]} \in \mathbb{Gr}^{[\,]}$ and A be a nonempty subset of $G^{[\,]}$. A is called an ideal of $G^{[\,]}$ if

1′. A is a normal subgroup of $G^{[\,]}$ as of an additive group;

2′. $[a, g] \in A$, for $a \in A$, $g \in G^{[\,]}$;

3′. $[g, a] \in A$, for $a \in A$, $g \in G^{[\,]}$.

It is easy to see that the isomorphism of categories $\mathbb{Gr}^{\bullet} \approx \mathbb{Gr}^{[\,]}$ carries ideals to ideals.

Proposition 7.2.7. *If A is an ideal of G, then the quotient group G/A with the induced action on itself is an object of $\mathbb{Gr}^{\bullet}$.*

Proof. Straightforward verification. □

In what follows, for $G \in \mathbb{Gr}^{\bullet}$ and $g, g' \in G$, $[g, g']$ will indicate the element $-g + g^{g'}$ of G and (g, g') the commutator $-g - g' + g + g'$. Let A and B be subobjects of G.

Definition 7.2.8. A commutator $[A,B]$ of G generated by A and B is the ideal of $\{A,B\}$ generated by the elements

$$\big\{[a,b],[b,a],(a,b) \mid a\in A,\ b\in B\big\}.$$

Definition 7.2.9 ([55])**.** Let G be an Ω-group, A,B be Ω-subgroups of G, and $\{A,B\}_\Omega$ be the Ω-subgroup of G generated by A and B. The commutator $[A,B]_\Omega$ is the ideal of $\{A,B\}_\Omega$ generated by elements of the form

$$(a,b)=-a-b+a+b,\quad a\in A,\quad b\in B,$$

and

$$[a_1,\dots,a_n;b_1,\dots,b_n;\omega]=-a_1a_2\cdots a_n\omega-b_1b_2\cdots b_n\omega+(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\omega, \quad (7.2.3)$$

where ω is an n-any operation from Ω, $a_1,\dots,a_n\in A$ and $b_1,\dots,b_n\in B$.

If G is a group with the trivial action on itself or with the action by conjugation, then $[A,B]$ in Definition 2.8 is the normal subgroup of G generated in $\{A,B\}$ by commutators (a,b), $a\in A$, $b\in B$, i.e., the usual commutator for the case of groups. The same is true for Definition 2.9; if an Ω-group is a group without multioperations, then the commutator $[A,B]_\Omega$ is the usual commutator (A,B) of a group [55].

Proposition 7.2.10. *In the case of groups with action on itself, Definitions* 2.8 *and* 2.9 *are equivalent.*

Proof. For groups with action, (7.2.3) has the form

$$-a^{a_2}-b_1^{b_2}+(a_1+b_1)^{a_2+b_2}. \quad (7.2.4)$$

Take $a_1=a$, $a_2=b_1=0$, $b_2=b$; then $-a+a^b\in[A,B]_\Omega$. Take in (7.2.4) $a_1=b_2=0$, $a_2=a$, $b_1=b$; then we obtain

$$-b+b^a\in[A,B]_\Omega.$$

Thus we have shown that $[A,B]\subset[A,B]_\Omega$. Conversely, for $x=-a_1^{a_2}-b_1^{b_2}+(a_1+b_1)^{a_2+b_2}\in[A,B]_\Omega$ we have $x=-a_1^{a_2}-b_1^{b_2}+(a_1^{a_2})^{b_2}+(b_1^{a_2})^{b_2}\in\{A,B\}$. Let $\overline{\{A,B\}}=\{A,B\}/[A,B]$ and let $\overline{g}$ be the class of the element $g\in\{A,B\}$ in $\overline{\{A,B\}}$. We have $\overline{a^b}=\overline{a}$, $\overline{b^a}=\overline{b}$ in $\overline{\{A,B\}}$. Thus

$$\overline{x}=\overline{-a_1^{a_2}-b_1^{b_2}+(a_1^{a_2})^{b_2}+(b_1^{a_2})^{b_2}}=\overline{a_1^{a_2}}-\overline{b_1^{b_2}}+\overline{a_1^{a_2}}+\overline{b_1^{a_2}}^{\,b_2}=$$
$$=-\overline{a_1^{a_2}}-\overline{b_1^{b_2}}+\overline{a_1^{a_2}}+\overline{b_1^{b_2}}=\overline{-a_1^{a_2}-b_1^{b_2}+a_1^{a_2}+b_1^{b_2}}=0,$$

which means that $x\in[A,B]$. □

Below we formulate without proofs two statements for Ω-groups from [55], which in the case of groups with action give the corresponding results.

Proposition 7.2.11. *For any Ω-subgroups A and B in G we have*

$$[A,B]_\Omega=[B,A]_\Omega.$$

Proposition 7.2.12. *An Ω-subgroup A is an ideal of G if and only if*

$$[A,G]_\Omega\subseteq A.$$

Corollary 7.2.13. *Any Ω-subgroup A of an Ω-group G that contains the commutator $[G,G]_\Omega$ is an ideal of G.*

Proof. It follows from the inclusions $[A,G]_\Omega\subset[G,G]_\Omega\subset A$. □

7.3. Central Series in $\mathbb{G}\mathrm{r}^\bullet$ and the Main Result

Let $G \in \mathbb{G}\mathrm{r}^\bullet$.

Definition 7.3.1. The (lower) central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots$$

of the object G is defined inductively by

$$G_n = [G_1, G_{n-1}] + [G_2, G_{n-2}] + \cdots + [G_{n-1}, G_1].$$

By definition, we have $[G_n, G_m] \subset G_{n+m}$.

Proposition 7.3.2. *For each $n \geq 1$, G_{n+1} is an ideal of G_n.*

Proof. We have $G_2 = [G_1, G_1]$, which is an ideal of G_1, by definition. $G_3 = [G_1, G_2] + [G_2, G_1]$. By Proposition 7.2.11, $[G_1, G_2] = [G_2, G_1]$. We have

$$[G_1, G_2] \subset [G_1, G_1] = G_2 \subset \{G_1, G_2\}$$

and $[G_1, G_2]$ is an ideal of $\{G_1, G_2\}$; from this it follows that $[G_1, G_2]$ is an ideal of G_2 and therefore, by Proposition 7.2.6, G_3 is an ideal of G_2. We have

$$G_{n+1} = [G_1, G_n] + [G_2, G_{n-1}] + \ldots [G_{n-1}, G_2] + [G_n, G_1].$$

For $1 \leq k \leq n$, $[G_k, G_{n-k+1}]$ is an ideal of $\{G_k, G_{n-k+1}\}$; $G_n \subseteq G_k$, from which it follows that $G_n \subseteq \{G_k, G_{n-k+1}\}$. At the same time

$$[G_k, G_{n-k+1}] \subset [G_k, G_{n-k}] \subset G_n.$$

Therefore $[G_k, G_{n-k+1}]$ is an ideal of G_n for each $1 \leq k \leq n$. Thus each summand of G_{n+1} is an ideal of G_n. By Propositions 7.2.6 and 7.2.11 we conclude that G_{n+1} is an ideal of G_n. $\square$

Since $(G_i, G_i) \subset G_{2i} \subset G_{i+1}$, each G_i/G_{i+1} has an abelian group structure. Let

$$LL_G = G_1/G_2 \oplus G_2/G_3 \oplus \cdots \oplus G_n/G_{n+1} \oplus \cdots, \tag{7.3.1}$$

where $\oplus$ denotes the direct sum of abelian groups.

Let k be a commutative ring with the unit, and A a k-module. For the definitions of Lie and Leibniz algebras we refer the reader to Sec. 5.1. Let k be a commutative ring with the unit and let $\mathbb{L}\mathrm{ie}$ be the category of Lie algebras over k. Morphisms in $\mathbb{L}\mathrm{ie}$ are k-module homomorphisms φ with

$$\varphi(x, y) = (\varphi(x), \varphi(y)).$$

Leibniz algebras considered in Chap. 5 are in fact right Leibniz algebras over a k. The dual notion of a left Leibniz algebra is made out of the dual relation

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]],$$

for $x, y, z \in A$.

A morphism of Leibniz algebras is a k-module homomorphism $\varphi : A \longrightarrow A'$ with $\varphi[x, y] = [\varphi(x), \varphi(y)]$.

In this paper we deal with right Leibniz algebras. Denote this category by $\mathbb{L}\mathrm{eibniz}$.

Definition 7.3.3. A Lie–Leibniz algebra is a k-module A together with two k-module homomorphisms

$$(\ ,\ ), [\ ,\ ] : A \otimes_k A \longrightarrow A$$

called round and square brackets, respectively, such that $(x, x) = 0$ for $x \in A$ and both Jacobi and Leibniz identities hold.

A morphism of Lie–Leibniz algebras is a k-module homomorphism $\varphi : A \longrightarrow A'$ with

$$\varphi(x, y) = (\varphi(x), \varphi(y)),$$
$$\varphi[x, y] = [\varphi(x), \varphi(y)].$$

We denote the corresponding category by $\mathbb{LL}$.

Condition 1. For each $x, y, z \in G$, $G \in \mathbb{G}\mathrm{r}^\bullet$

$$x - x^{(z^x)} + x^{y+z^x} - x + x^z - x^{z+y^z} = 0.$$

It is straightforward to verify that if G satisfies Condition 1; then the group $G^{[\,]}$, which corresponds to G (i.e., $[\ ,\ ]$ is defined by $[g, h] = -g + g^h$, $g, h \in G$), satisfies the following condition.

Condition 1′.

$$[x^y, [y, z]] = [[x, y], z^x] + [-[x, z], y^z], \qquad x, y, z \in G^{[\,]}.$$

Let G be a group. Consider G as a group with the (right) action by conjugation, i.e. $g^{g'} = -g' + g + g'$. Then G satisfies Condition 1, and in this case Condition 1′ is equivalent to the Witt–Hall identity for groups. Each group with the trivial action on itself (i.e., $g^{g'} = g$, $g, g' \in G$) also satisfies Condition 1. For an arbitrary set X, let $\mathcal{F}_X$ be a free group with action on itself generated by X (see Sec. 8.2 for the construction). The quotient $\mathcal{F}_X/_\sim$ of $\mathcal{F}_X$ by the equivalence relation generated by the relation corresponding to Condition 1 is obviously a group that satisfies Condition 1. See also an example at the end of the proof of Theorem 7.3.4.

Denote by $\mathbb{G}\mathrm{r}^{\mathrm{c}}$ a category of groups with action on itself satisfying Condition 1. In an analogous way we define the category $\mathbb{A}\mathrm{b}^{\mathrm{c}}$. It is easy to see that the functors E, A, T, C, Q_1, Q_2, defined in Sec. 1, give the functors between categories $\mathbb{A}\mathrm{b}^{\mathrm{c}}$, $\mathbb{G}\mathrm{r}^{\mathrm{c}}$, and $\mathbb{G}\mathrm{r}$. We shall denote below these functors by the same letters. $\mathcal{F}_X/_\sim$ is a free object in $\mathbb{G}\mathrm{r}^{\mathrm{c}}$ and consequently the action in it is neither the trivial one nor the conjugation.

Let $G \in \mathbb{G}\mathrm{r}^c$. Denote $\overline{G}_m = G_m/G_{m+1}$, then $LL_G = \sum_{m \geq 1} \overline{G}_m$.

Consider the maps $(\ ,\ )_{mn}, [\ ,\ ]_{mn} : G_m \times G_n \longrightarrow G_{m+n}$ defined by round and square brackets in G, respectively:

$$x, y \longmapsto (x, y),$$
$$x, y \longmapsto [x, y].$$

By the definition of G_i, it is clear that if $x \in G_m$, $y \in G_n$, then $(x, y), [x, y] \in G_{m+n}$. For $x \in G_m$, denote by $\overline{x}$ the corresponding class in $\overline{G}_m$.

Theorem 7.3.4. *Let G be a group with action on itself satisfying Condition* 1. *Then we have:*

(a) $\overline{x^y} = \overline{x}$, $\overline{-y + x + y} = \overline{x}$, *for each* $x \in G_m$, $y \in G_n$;

(b) *The maps* $(\ ,\ )_{mn}$ *and* $[\ ,\ ]_{mn} : G_m \times G_n \longrightarrow G_{m+n}$ *induce bilinear maps* $\alpha_{mn}, \beta_{mn} : \overline{G}_m \times \overline{G}_n \longrightarrow \overline{G}_{m+n}$;

(c) *The maps* α_{mn}, β_{mn}, $m, n \geq 1$ *define bilinear maps* $(\ ,\ ), [\ ,\ ] : LL_G \times LL_G \longrightarrow LL_G$, *which give a Lie–Leibniz structure on* LL_G.

Proof. (a) Let $x \in G_m$, $y \in G_n$, $m, n \geq 1$. Then $[x, y] = -x + x^y \in G_{m+n} \subset G_m$, and since $x \in G_m$ we obtain that $x^y \in G_m$. In $\overline{G}_m$ we have $\overline{[x, y]} = -\overline{x} + \overline{x^y}$, but since $[x, y] \subset G_{m+n} \subset G_{m+1}$, we have $\overline{[x, y]} = 0$ in $\overline{G}_m$, and thus in $\overline{G}_m$ we have $\overline{x} = \overline{x^y}$. In the same way we show for the action with conjugation that $\overline{-y + x + y} = \overline{x}$ (see also [83]).

(b) We shall check this condition for a square bracket; for a round bracket the proof is similar [83]. First we shall show that the map $\beta_{mn} : \overline{G}_m \times \overline{G}_n \longrightarrow \overline{G}_{m+n}$ is defined correctly. Let $\overline{x} \in \overline{G}_m$, $\overline{y} \in \overline{G}_n$, where $x \in G_m$, $y \in G_n$. By definition, $\beta_{mn}(\overline{x}, \overline{y}) = [\overline{x}, \overline{y}] = \overline{[x, y]}$, where $[x, y] \in G_{n+m}$. Let $\overline{x} = \overline{x'}$ for $x' \in G_m$, thus $x - x' \in G_{m+1}$. For simplicity, suppose that $x - x' \in [G_{i+1}, G_{m-i}] \subset G_{m+1}$ (a more general case is treated similarly). Then $x = [a, b] + x'$, where $a \in G_{i+1}$, $b \in G_{m-i}$. From this we have in $\overline{G}_{m+n}$:

$$\overline{[x, y]} = \overline{[[a, b] + x', y]} = \overline{-x' + [[a, b], y] + x' + [x', y]} = \overline{-x' + [[a, b], y] + x'} + \overline{[x', y]}. \tag{7.3.2}$$

$[[a, b], y] \in G_{m+n+1} \subset G_{m+n}$. Applying condition (a), we obtain

$$\overline{-x' + [[a, b], y] + x'} = \overline{[[a, b], y]} = 0 \quad \text{in} \quad \overline{G}_{m+n}.$$

Thus from (7.3.2) we have $\overline{[x, y]} = \overline{[x', y]}$. If $x - x' = (a, b) \in [G_{i+1}, G_{m-i}] \subset G_{m+1}$, then by the same argument we have

$$\overline{[x, y]} = \overline{[x' + (a, b), y]} = \overline{-x' + [(a, b), y] + x'} + \overline{[x', y]} = \overline{[(a, b), y]} + \overline{[x', y]} = \overline{[x', y]},$$

since $\overline{[(a, b), y]} = 0$ in $\overline{G}_{m+n}$. The correctness of β_{mn} for the second argument is proved in an analogous way.

Now we shall show that the maps β_{mn} are bilinear. Let $\overline{x}_1, \overline{x}_2 \in \overline{G}_m$ and $\overline{y} \in \overline{G}_n$. We have in $\overline{G}_{m+n}$

$$[\overline{x}_1 + \overline{x}_2, \overline{y}] = \overline{[x_1 + x_2, y]} = \overline{-x_2 + [x_1, y] + x_2} + \overline{[x_2, y]} = \overline{[x_1, y]} + \overline{[x_2, y]};$$

here we again apply condition (a). Let $\overline{x} \in \overline{G}_m$ and $\overline{y}_1, \overline{y}_2 \in \overline{G}_n$. We have in $\overline{G}_{m+n}$

$$[\overline{x}, \overline{y}_1 + \overline{y}_2] = \overline{[x, y_1 + y_2]} = \overline{[x, y_1] + [x^{y_1}, y_2]} =$$
$$= \overline{[x, y_1]} + \overline{[x^{y_1}, y_2]} = [\overline{x}, \overline{y}_1] + [\overline{x^{y_1}}, \overline{y}_2] = [\overline{x}, \overline{y}_1] + [\overline{x}, \overline{y}_2],$$

since, by condition (a) $\overline{x^{y_1}} = \overline{x}$. This proves that maps β_{mn} are bilinear.

(c) The maps α_{mn}, β_{mn} can be continued linearly in a natural way up to the bilinear maps $(\ ,\ ), [\ ,\ ] : LL_G \times LL_G \longrightarrow LL_G$. The proof of the fact that $(\ ,\ )$ satisfies condition (5.1.5) and $(l, l) = 0$ for any $l \in LL_G$ is similar to the proof of the corresponding statement in Witt's theorem (see [83, Proposition 2.3], [90]). It remains to show that the square bracket operation $[\ ,\ ]$ satisfies the Leibniz identity.

The object G satisfies Condition 1; therefore we have Condition 1$'$ for the square bracket in G. Since the square bracket operation in LL_G is linear for both arguments, we can limit ourself to the case where $\overline{x} \in G_m$, $\overline{y} \in \overline{G}_n$, $\overline{z} \in \overline{G}_t$. Applying conditions (a) and (b) of the theorem, we have

$$[\overline{x}, [\overline{y}, \overline{z}]] = [\overline{x^y}, [\overline{y}, \overline{z}]] = \overline{[x^y, [y, z]]};$$
$$[[\overline{x}, \overline{y}], \overline{z}] = [[\overline{x}, \overline{y}], \overline{z^x}] = \overline{[[x, y], z^x]};$$
$$-[[\overline{x}, \overline{z}], \overline{y}] = [-[\overline{x}, \overline{z}], \overline{y^z}] = \overline{[-[x, z], y^z]}.$$

By Condition 1$'$ we obtain

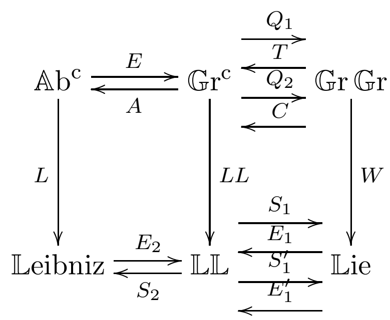
$$[\overline{x}, [\overline{y}, \overline{z}]] = [[\overline{x}, \overline{y}], \overline{z}] - [[\overline{x}, \overline{z}], \overline{y}] \quad \text{in} \quad \overline{G_{m+n+t}},$$

which completes the proof of the theorem. □

The following example is due to the referee.

Example. Let G be the abelian group of integers $\mathbb{Z}^\bullet$, which acts on itself in the following way: $x^y = (-1)^y x$. We have $[x, y] = 0$ for y even, $[x, y] = -2x$ for y odd, and $G_n = 2^{n-1}\mathbb{Z}^\bullet$. It is easy to see that $\mathbb{Z}^\bullet \in \mathbb{G}\mathrm{r}^{\mathrm{c}}$ and $LL_{\mathbb{Z}^\bullet}$ is a free Leibniz algebra generated by a single element over a two element field (see also [65]).

It is easy to see that by Theorem 7.3.4 we have actually constructed the functor $LL : \mathbb{G}\mathrm{r}^{\mathrm{c}} \longrightarrow \mathbb{LL}$. In an analogous way one can construct the functor $L : \mathbb{A}\mathrm{b}^{\mathrm{c}} \longrightarrow \mathbb{L}\mathrm{eibniz}$. For $A \in \mathbb{LL}$ let $S_1(A)$ denote the greatest quotient algebra of A that makes the square bracket in A trivial. Then $S_1(A) \in \mathbb{L}\mathrm{ie}$ and we have the functor $S_1 : \mathbb{LL} \longrightarrow \mathbb{L}\mathrm{ie}$. Similarly, we construct the functor $S_2 : \mathbb{LL} \longrightarrow \mathbb{L}\mathrm{eibniz}$. Let ${S_1}'(A)$ denote the greatest quotient algebra of A that makes the square bracket in A equal to the round bracket; thus ${S_1}'(A) \in \mathbb{L}\mathrm{ie}$, and ${S_1}'$ is a functor. S_1, ${S_1}'$, and S_2 are left adjoints to the embedding functors E_1, ${E_1}'$, and E_2 respectively, where E_1 considers a Lie algebra as a Lie–Leibniz algebra with a trivial bracket operation, ${E_1}'$ considers a Lie algebra as a Lie–Leibniz algebra with square bracket equal to round bracket, and E_2 considers a Leibniz algebra as a Lie–Leibniz algebra with trivial round bracket operation. Denote by $W : \mathbb{G}\mathrm{r} \longrightarrow \mathbb{L}\mathrm{ie}$ the functor defined by Witt's theorem [90], [83]. Thus we have the following functors between the well defined categories:



where $LL\,T = E_1\,W$, $E_2\,L = LL\,E$, $LL\,C = {E_1}'\,W$. A more detailed account of this diagram will be given in the next chapter, where free objects in $\mathbb{G}\mathrm{r}^{\bullet}$ and free Leibniz algebras are studied.

Chapter 8

WITT'S THEOREM FOR GROUPS WITH ACTION AND FREE LEIBNIZ ALGEBRAS

In this chapter we give the solution to the second problem of J.-L. Loday formulated in the Introduction (see [62], [64]). We introduce two more conditions (Condition 2 and Condition 3) between round and square brackets for the objects of $\mathbb{G}\mathrm{r}^c$ defined in Chap. 7, and according to these conditions define the full subcategories $\overline{\mathbb{G}\mathrm{r}}$ and $\overline{\mathbb{LL}}$ of $\mathbb{G}\mathrm{r}^c$ and $\mathbb{LL}$, respectively. We prove that the functor LL defined in Chap. 7 takes free objects from $\overline{\mathbb{G}\mathrm{r}}$ to free objects in $\overline{\mathbb{LL}}$. The composite $S_2\,LL\big|_{\overline{\mathbb{G}\mathrm{r}}} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \mathbb{L}\mathrm{eibniz}$ is the functor we were looking for, which takes free self-acting groups from $\overline{\mathbb{G}\mathrm{r}}$ to free Leibniz algebras, where S_2 is the functor which makes the round bracket operation trivial in Lie–Leibniz algebras. This result is an analogue of E. Witt's theorem [90], [83] for groups with action on itself and Leibniz algebras. The properties of a commutator in $\overline{\mathbb{G}\mathrm{r}}$ and the construction of free objects in this category and in $\mathbb{L}\mathrm{eibniz}$ and other related questions are considered.

8.1. Some Properties of Commutators for Groups with Action

For the definitions of the categories $\mathbb{G}\mathrm{r}^{\bullet}$, $\mathbb{G}\mathrm{r}^c$ we refer the reader to Chap. 7. Recall that we have defined square bracket operation in $\mathbb{G}\mathrm{r}^{\bullet}$ by $[g,h] = -g + g^h$, for $g, h \in G$, $G \in \mathbb{G}\mathrm{r}^{\bullet}$.

We have (see Proposition 7.1.1)

$$\begin{aligned} &[g, h_1 + h_2] = [g, h_1] + [g^{h_1}, h_2] = [g, h_2] + [g, h_1]^{h_2}; \\ &[g + g', h] = [g, h]^{\underline{g'}} + [g', h]; \qquad [g, 0] = [0, g] = 0, \end{aligned} \tag{8.1.1}$$

where $x^{\underline{g}} = -g + x + g, \;\; x, g \in G$.

From (8.1.1) it follows that

$$[g, -h] = -[g^{-h}, h] = -[g, h]^{-h}; \qquad [-g, h] = -[g, h]^{\underline{-g}}. \tag{8.1.2}$$

For the round bracket we have $(g, h) = -(h, g)$ and the identities analogous to (8.1.1) and (8.1.2):

$$\begin{aligned} (g, h_1 + h_2) &= (g, h_1) + (g^{\underline{h_1}}, h_2) = (g, h_2) + (g, h_1)^{\underline{h_2}}; \\ (g + g', h) &= (g, h)^{\underline{g'}} + (g', h); \\ (g, 0) &= (0, g) = 0, \end{aligned} \tag{8.1.3}$$

$$(g, -h) = -(g, h)^{\underline{-h}} \Longleftrightarrow (-g, h) = -(g, h)^{\underline{-g}}. \tag{8.1.4}$$

These identities are well known for groups (see, e.g., [83]) and are special cases of (8.1.1) and (8.1.2).

For the case of groups, it is proved that if A and B are normal subgroups of G, then the commutator (A, B) is also a normal subgroup of G. Below we will show that the analogous statement is true for a certain type of groups with action on itself.

Consider the following conditions:

Condition 2. $\big[x^y, (y, z)\big] = \big[(x, y), z^x\big] + \big[-(x, z), y^z\big]$.

Condition 3. $\big(x^y, [y, z]\big) = \big([x, y], z^x\big) + \big(-[x, z], y^z\big)$.

In Sec. 8.3 we will see that the objects of $\mathbb{G}\mathrm{r}^c$ do not generally satisfy these conditions. Note that for groups with trivial action on itself, or with the action by conjugation, Conditions $1'$, 2, and 3 are always satisfied (see Sec. 7.3 for Condition $1'$). The same is true for the example $\mathbb{Z}^\bullet$ from Chap. 7. For any set X, consider a free object F_X on the set X in the category $\mathbb{G}\mathrm{r}^\bullet$ (see Sec. 8.2 for the construction of free objects in this category). Let $F_X/\sim$ be the quotient object, where $\sim$ is the minimal congruence relation generated by the relations expressed in Conditions $1'$, 2 and 3. Then $F_X/\sim$ is an object of $\mathbb{G}\mathrm{r}^c$ that satisfies the above two conditions.

Denote by $\overline{\mathbb{G}\mathrm{r}}$ the full subcategory of $\mathbb{G}\mathrm{r}^\bullet$ of those objects that satisfy Conditions $1'$, 2, and 3. Thus $\overline{\mathbb{G}\mathrm{r}}$ is the full subcategory of $\mathbb{G}\mathrm{r}^c$.

Since groups with action are Ω-groups, $[A, B]$ is an ideal of G if and only if $[[A, B], G] \subseteq [A, B]$ [55] (see Proposition 7.2.12).

Now we are going to prove statements concerning some properties of elements of $[A, B]$, $\{A, B\}$ and G, where A and B are ideals of G. These statements will readily imply that $[A, B]$ is an ideal of G if A and B are ideals of G and $G \in \overline{\mathbb{G}\mathrm{r}}$. Note that in this case $\{A, B\} = A + B$, and this object is also an ideal of G (Proposition 7.2.5).

Below for $g, h \in G$, $g^{\underline{h}} = -h + g + h$.

Lemma 8.1.1. *Let $a, b, g \in \mathbb{G}\mathrm{r}^\bullet$. Then we have*

(i) $\big(a^{\underline{g}}\big)^b = \big(a^b\big)^{\underline{(g^b)}}$;

(ii) $\big(a^b\big)^{\underline{g}} = \big(a^{\underline{g^{(-b)}}}\big)^b$.

The proof is an easy computation of both sides.

Lemma 8.1.2. *Let A and B be ideals of $G \in \overline{\mathbb{G}\mathrm{r}}$. Then for any $a \in A$, $b \in B$, $g \in G$ the elements*

$$[a, b]^{\underline{g}}, [b, a]^{\underline{g}}, (a, b)^{\underline{g}}, [a, b]^g, [b, a]^g,$$
$$(a, b)^g, \big[g, [a, b]\big], \big[g, [b, a]\big], \big[g, (a, b)\big]$$

belong to $[A, B]$.

Proof. We have

$$\begin{aligned}
[a,b]^{\underline{g}} &= -g + [a,b] + g = -g - a + a^b + g = -g - a + g + (a^b)^{\underline{g}} \\
&= -g - a + g + \big(a^{\underline{g^{(-b)}}}\big)^b = -g - a + g + a^{\underline{g^{(-b)}}} - a^{\underline{g^{(-b)}}} \\
&\quad + \big(a^{\underline{g^{(-b)}}}\big)^b = -g - a + g - g^{(-b)} + a + g^{(-b)} + \big[a^{\underline{g^{(-b)}}}, b\big] \\
&= -g - a + g - g^{(-b)} + a + g^{(-b)} - g + g + \big[a^{\underline{g^{(-b)}}}, b\big] \\
&= (a, b')^{\underline{g}} + \big[a^{\underline{g^{(-b)}}}, b\big],
\end{aligned}$$

where $b' = g - g^{(-b)} \in B$, since B is an ideal, which proves that $[a,b]^{\underline{g}} \in [A,B]$.

We have $[b,a]^{\underline{g}} \in [A,B]$, since $[b,a]^{\underline{g}} \in [B,A]$ by the above-given proof and the equality $[B,A] = [A,B]$ (see Chap. 7). For the round bracket we have

$$(a,b)^{\underline{g}} \in [A,B], \quad \text{since} \quad (a,b)^{\underline{g}} = (a^{\underline{g}}, b^{\underline{g}}).$$

For the next element we have

$$[a,b]^g = -a^g + a^{b+g} = -a^g + a^{g+b'} = [a^g, b'] \in [A,B],$$

where $b' = -g + b + g \in B$; here we apply the fact that B is an ideal of G.

From the previous result and from $[B,A] = [A,B]$ it follows that $[b,a]^g \in [A,B]$.

It is easy to see that

$$(a,b)^g = (a^g, b^g) \in [A,B].$$

For the element $[g,[a,b]]$ we apply Condition $1'$:

$$\big[g,[a,b]\big] = \big[(g^{-a})^a, [a,b]\big] = \big[[g^{-a},a], b^{(g^{-a})}\big] + \big[-[g^{-a},b], a^b\big].$$

This element is from $[A,B]$, since A and B are ideals of G and $[A,B] = [B,A]$.

From the previous result it follows that $[g,[b,a]] \in [A,B]$. In the same way applying Condition 2, we prove that $[g,[a,b]] \in [A,B]$. ☐

Remark. We do not need to check that elements of the type (g,t) belong to $[A,B]$, where t is a generator of $[A,B]$, since

$$(g,t) \in [A,B] \Leftrightarrow (t,g) \in [A,B] \Leftrightarrow t^{\underline{g}} \in [A,B].$$

The latter inclusion has been considered in Lemma 8.1.2.

Lemma 8.1.3. *Let A, B be ideals of G, $G \in \mathbb{G}\mathrm{r}^c$. For $g \in G$, $t, t_i \in [A,B]$, $i = 1,2$*

(a) *If $[g,t_i] \in [A,B]$, $i = 1,2$, then $[g, t_1 + t_2] \in [A,B]$.*

(b) *If $[t_i,g] \in [A,B]$, $i = 1,2$, then $[t_1 + t_2, g] \in [A,B]$.*

(c) *If $[g,t] \in [A,B]$, then $[g,-t] \in [A,B]$.*

The proof follows from (8.1.1) and (8.1.2).

Lemma 8.1.4. *Let A and B be ideals of G, $G \in \overline{\mathbb{G}\mathrm{r}}$. If for $t \in [A,B]$ and any $g \in G$ we have $t^g, t^{\underline{g}}, [g,t] \in [A,B]$, then for any $g_1 \in \{A,B\}$ the elements*

$$\begin{matrix}
(t^{g_1})^g, & (t^{\underline{g_1}})^g, & [g_1,t]^g, \\
(t^{g_1})^{\underline{g}}, & (t^{\underline{g_1}})^{\underline{g}}, & [g_1,t]^{\underline{g}}, \\
[g, t^{g_1}], & [g, t^{\underline{g_1}}], & \big[g,[g_1,t]\big]
\end{matrix}$$

belong to $[A,B]$.

Proof. It is obvious that $(t^{g_1})^g$, $(t^{\underline{g_1}})^{\underline{g}} \in [A,B]$. By Lemma 8.1.1, for the elements $(t^{g_1})^{\underline{g}}$, $(t^{\underline{g_1}})^g$ we have $(t^{g_1})^{\underline{g}} = (t^{\underline{g^{(-g_1)}}})^{g_1} \in [A,B]$. Since $\{A,B\} = A+B$ is an ideal, $(t^{\underline{g_1}})^g = (t^g)^{\underline{(g_1^g)}} \in [A,B]$, and therefore $g_1^g \in \{A,B\}$.

For the element $[g_1,t]^g$ we have

$$[g_1,t]^g = -g_1^g + g_1^{t+g} = -g_1^g + g_1^{g+t'} = [g_1^g, t'] \in [A,B],$$

where $t' = -g+t+g \in [A,B]$ and $g_1^g \in \{A,B\}$.

For the element $[g_1,t]^{\underline{g}}$ we will show that $([g_1,t],g) \in [A,B]$, from which it follows that $[g_1,t]^{\underline{g}} \in [A,B]$. Applying Condition 3, we obtain

$$\big([g_1,t],(g^{-g_1})^{g_1}\big) = \big(g_1^t,[t,g^{-g_1}]\big) - \big(-[g_1,g^{-g_1}],t^{g^{(-g_1)}}\big) \in [A,B].$$

For the element $[g,t^{g_1}]$ we show that $[g,[t,g_1]] \in [A,B]$, from which, by (8.1.1), it follows that $[g,t^{g_1}] + [g,-t]^{g_1} \in [A,B]$. Since $[g,t] \in [A,B]$, we have $[g,-t] \in [A,B] \Rightarrow [g,-t]^{t^{g_1}} \in [A,B]$, which implies that $[g,t^{g_1}] \in [A,B]$.

By Condition $1'$ we have

$$\begin{aligned}\big[g,[t,g_1]\big] &= \big[[g^{-t},t],g_1^{(g^{-t})}\big] + \big[-[g^{-t},g_1],t^{g_1}\big]\\ &\in \big[[A,B,\{A,B\}\big] + \big[\{A,B\},[A,B]\big] \subset [A,B].\end{aligned}$$

For $[g,t^{\underline{g_1}}] \in [A,B]$ we show that $[g,(t,g_1)] \in [A,B]$, which can be done analogously to the previous proof by applying Condition 2.

For the element $[g,[g_1,t]]$ we have

$$\begin{aligned}\big[g,[g_1,t]\big] &= \big[(g^{-g_1})^{g_1},[g_1,t]\big] = \big[[g^{-g_1},g_1],t^{(g^{-g_1})}\big] + \big[-[g^{-g_1},t],g_1\big]\\ &\in \big[\{A,B\},[A,B]\big] + \big[[A,B],\{A,B\}\big] \subset [A,B].\end{aligned}$$

□

Proposition 8.1.5. *Let A and B be ideals of $G \in \overline{\mathbb{G}\mathrm{r}}$. Then the commutator $[A,B]$ is also an ideal of G.*

Proof. By Lemmas 8.1.1–8.1.4 we have proved that the generators of $[A,B]$ (as an ideal of $\{A,B\}$) satisfy the conditions t^g, $t^{\underline{g}}$, $[g,t] \in [A,B]$ for any $g \in G$, where t is any generator of $[A,B]$ (Lemma 8.1.2), and from Lemmas 8.1.3, 8.1.4 it follows that if the generators satisfy these conditions, then any element of $[A,B]$ satisfies the same conditions, which is a necessary and sufficient condition for $[A,B]$ to be an ideal of G, which proves the proposition. □

Remark. From the above proved lemmas we obtain $[[A,B],C] \subset [A,B]$, which is a necessary and sufficient condition for $[A,B]$ to be an ideal of G [55] (see Chap. 7), and this is another similar way to prove Proposition 8.1.5 by applying the same lemmas.

If A,B,C are normal subgroups of a group G, we have

$$\big(A,(B,C)\big) \subset \big(B,(C,A)\big) + \big(C,(A,B)\big), \tag{8.1.5}$$

where (A,B) denotes the commutator subgroup of G (see, e.g., [83]).

For groups with action on itself, the analogous inclusion for square brackets does not hold in general for the ideals A,B,C of G, when $G \in \mathbb{G}\mathrm{r}^\bullet$, nor in the case when G satisfies the Condition $1'$ (i.e., $G \in \mathbb{G}\mathrm{r}^c$).

Proposition 8.1.6. *Let A,B,C be ideals of G, $G \in \overline{\mathbb{G}\mathrm{r}}$. Then we have*

$$\big[A,[B,C]\big] \subset \big[[A,B],C\big] + \big[[A,C],B\big].$$

For the case of groups, this result gives (8.1.5). We have formulated the right side of the inclusion in this form, since it is more convenient for the proof using Conditions $1'$, 2, 3. We need several lemmas. For simplicity, denote

$$D_{A,B,C} = \big[[A,B],C\big] + \big[[A,C],B\big].$$

By Proposition 8.1.5, $[A,[B,C]]$ and D are ideals of G; therefore it is sufficient to prove that the generators of $[A,[B,C]]$ (as an ideal of $\{A,[B,C]\}$) belong to D. By the definition of a commutator, $[A,[B,C]]$ is an ideal of $\{A,[B,C]\}$ generated by the elements

$$\big\{[a,t],[t,a],(a,t) \,|\, a \in A,\ t \in [B,C]\big\}.$$

The commutator $[B,C]$ itself is an ideal of $\{B,C\}$ generated by the elements

$$\big\{[b,c],[c,b],(b,c) \,|\, b \in B,\ c \in C\big\},$$

and we have $\{B,C\} = B + C$, since B and C are ideals of G.

Lemma 8.1.7. *Let A, B and C be ideals of G, $G \in \overline{\mathbb{G}\mathrm{r}}$. For $a \in A$, $b \in B$, $c \in C$ the elements*

$$\big[a,[b,c]\big],\ \big[a,[c,b]\big],\ \big[a,(b,c)\big],\ \big[[b,c],a\big],$$
$$\big[[c,b],a\big],\ \big[(b,c),a\big],\ \big(a,[b,c]\big),\ \big(a,[c,b]\big),\ \big(a,(b,c)\big)$$

belong to D_{ABC}.

Proof. For the first element we apply Condition $1'$. We have

$$\big[a,[b,c]\big] = \big[(a^{-b})^b,[b,c]\big] = \big[(a^{-b},b),c^{a(-b)}\big] + \big[-[(a,c),b^c\big] \in D_{ABC}.$$

For the next element we apply the first result and we have $[a,[c,b]] \in D_{ACB} = D_{ABC}$.

In the same way, applying Conditions 2, 3 and also the corresponding Witt–Hall identity for commutators in groups, we prove that all elements given in the lemma belong to D. ☐

Lemma 8.1.8. *Let A, B, and C be ideals of G, $G \in \overline{\mathbb{G}\mathrm{r}}$, and $t_i \in [B,C]$, $i = 1,2$.*

If $(a,t_i) \in D_{ABC}$, $i = 1,2$ for any $a \in A$, then

$$(a,t_1+t_2) \in D_{ABC}.$$

If $[a,t_i] \in D_{ABC}$, $i = 1,2$ for any $a \in A$, then

$$[a,t_1+t_2] \in D_{ABC}.$$

If $[t_i,a] \in D_{ABC}$, $i = 1,2$ for any $a \in A$, then

$$[t_1+t_2,a] \in D_{ABC}.$$

The proof follows from (8.1.1) and (8.1.3) and the fact that D is an ideal of G.

Lemma 8.1.9. *For any ideal I of G, $G \in \mathbb{G}\mathrm{r}^\bullet$ and elements $g,h \in G$,*

If $[g,h] \in I$, then $[-g,h]$, $[g,-h] \in I$.

If $(g,h) \in I$, then $(-g,h)$, $(g,-h) \in I$.

The proof follows from (8.1.2) and (8.1.4) and the fact that I is an ideal of G.

Lemma 8.1.10. *Let A, B, and C be ideals of G, $G \in \overline{\mathbb{G}\mathrm{r}}$. For any $t \in [B,C]$, any $a \in A$, and any $x \in \{B,C\}$ we have:*

(a) *$[a,t] \in D_{ABC}$, then $[a,t^x] \in D_{ABC}$.*

(b) *$[a,t] \in D_{ABC}$, then $[a,t^{\overline{x}}] \in D_{ABC}$.*

(c) $[a,[x,t]] \in D_{ABC}$.

(a′) *If* $(a,t) \in D_{ABC}$, *then* $(a,t^x) \in D_{ABC}$.

(b′) *If* $(a,t) \in D_{ABC}$, *then* $(a,t^{\underline{x}}) \in D_{ABC}$.

(c′) $(a,[x,t]) \in D_{ABC}$.

(a″) *If* $[t,a] \in D_{ABC}$, *then* $[t^x,a] \in D_{ABC}$.

(b″) *If* $[t,a] \in D_{ABC}$, *then* $[t^{\underline{x}},a] \in D_{ABC}$.

(c″) $[[x,t],a] \in D_{ABC}$.

Proof. We will show that $[a,[t,x]] \in D_{ABC}$, from which it follows that $[a,t^x] \in D_{ABC}$. Since B and C are ideals of G, $\{B,C\} = B + C$, any element $x \in \{B,C\}$ has the form $x = b + c$, $b \in B$, $c \in C$. We have

$$\big[a,[t,b+c]\big] = \big[a,[t,b] + [t^b,c]\big] = \big[a,[t,b]\big] + \big[a^{[t,b]},[t^b,c]\big].$$

By Proposition 8.1.5, $[B,C]$ is an ideal of G. By Lemma 8.1.7 applied to $A,[B,C],B$ and A, $[B,C]$, C we obtain

$$\big[a,[t,b+c]\big] \subset D_{A,[B,C],B} + D_{A,[B,C],C} \subset D_{A,C,B} + D_{A,B,C} = D_{A,B,C},$$

since $[B,C] \subset C$, $[B,C] \subset B$ (since B and C are ideals of G) and $D_{ACB} = D_{ABC}$. We have

$$\big[a,[t,x]\big] = [a,-t+t^x] = [a,t^x] + [a,-t]^{(t^x)}.$$

Since $[a,t] \in D$, by Lemma 8.1.9 $[a,-t] \in D$, and since D is an ideal of G, $[a,-t]^{(t^x)} \in G$. This proves that $[a,t^x] \in D_{ABC}$.

(b) is proved in an analogous way; we prove first that $[a,(t,x)] \in D_{ABC}$ for any $a \in A$, $t \in [B,C]$, $x \in \{B,C\}$, from which it follows that $[a,t^{\underline{x}}] \in D_{ABC}$.

(c) Since $x = b + c$, for $b \in B$, $c \in C$, we have

$$\big[a,[x,t]\big] = \big[a,[b+c,t]\big] = \big[a,[b,t]^{\underline{c}} + [c,t]\big] = \big[a,[b,t]^{\underline{c}}\big] + \big[a^{[b,t]^{\underline{c}}},[c,t]\big].$$

In the same way as in (a), applying Lemma 8.1.7 we can prove that $[a,[b,t]] \in D_{AB[B,C]} \subset D_{ABC}$ and $[a^{[b,t]^{\underline{c}}},[c,t]] \subset D_{AC[B,C]} \subset D_{ACB} \subset D_{ABC}$. By (b) we have $[a,[b,t]^{\underline{c}}] \subset D_{ABC}$, since $[b,t] \in [B,[B,C]] \subset [B,C]$ and $c \in \{B,C\}$.

(a′), (b′), (c′) are proved in a similar way.

For (a″) we first show that $[[t,x],a] \in D_{ABC}$. We have

$$\big[[t,x],a\big] = \big[[t,b+c],a\big] = \big[[t,b] + [t^b,c],a\big] = \big[[t,b],a\big]^{\underline{[t^b,c]}} + \big[[t^b,c],a\big].$$

Applying Lemma 8.1.7, we show that $[[t,b],a] \in D_{AB[B,C]} \subset D_{ABC}$ and since D_{ABC} is an ideal of G, we have $[[t,b],a]^{\underline{[t^b,c]}} \in D_{ABC}$.

Next, we show by Lemma 8.1.7 applied to $t^b \in [B,C]$, $c \in C$, $a \in A$, that the element $[[t^b,c],a]$ from $[A,[[B,C],C]]$ is included in $D_{A[B,C]C}$ and hence in D_{ABC}, since B is an ideal of G and $[B,C] \subset B$.

Applying Lemma 8.1.9, from $[[t,x],a] \in D_{ABC}$ it follows that $[t^x,a] \in D_{ABC}$.

(b″) We begin by proving that $[(t,x),a] \in D_{ABC}$. We have

$$\big[(t,b+c),a\big] = \big[(t,c) + (t,b)^{\underline{c}},a\big] = \big[(t,c),a\big]^{\underline{(t,b)^{\underline{c}}}} + \big[(t,b)^{\underline{c}},a\big].$$

Again by Lemma 8.1.7 $[(t,c),a] \in D_{A[B,C]C} \subset D_{ABC}$, from which $[(t,c),a]^{\underline{(t,b)^c}} \in D_{ABC}$.

For the second summand we have

$$\big[(t,b)^{\underline{c}},a\big] = \big[(t^c,b^c),a\big] \in D_{A[B,C]B} \subset D_{ACB} = D_{ABC};$$

hence $[(t,x),a] \in D_{ABC}$.

We have

$$\big[(t,x),a\big] = \big[-t+t^{\underline{x}},a\big] = \big[-t,a\big]^{\underline{t^{x}}} + \big[t^{\underline{x}},a\big].$$

By Lemma 8.1.9, $[-t,a] \in D_{ABC}$, and therefore $[-t,a]^{\underline{t^{x}}} \in D_{ABC}$, from which $[t^{\underline{x}},a] \in D_{ABC}$.

(c″) We have $\big[[x,t],a\big] = \big[[c+b,t],a\big] = \big[[c,t]^{\underline{b}} + [b,t],a\big] = \big[[c,t]^{\underline{b}},a\big]^{\underline{[b,t]}} + \big[[b,t],a\big]$.

By Lemma 8.1.7,

$$\big[[b,t],a\big] \subset D_{A[B,C]B} \subset D_{ABC}.$$

For the first summand we have $[c,t] \in \big[B,[B,C]\big] \subset [B,C]$; $\big[[c,t],a\big] \in D_{A[B,C]C} \subset D_{ABC}$ by Lemma 8.1.7. Thus for $t' = [c,t]$ we have $[t',a] \in D_{ABC}$. From (b″) we obtain

$$\big[(t')^{\underline{b}},a\big] \in D_{ABC} \quad \text{since} \quad b \in \{B,C\},$$

and therefore

$$\big[[c,t]^{\underline{b}},a\big]^{\underline{[b,t]}} \in D_{ABC},$$

since D_{ABC} is an ideal of G. This ends the proof of the lemma. □

The proof of Proposition 8.1.6 follows from Lemmas 8.1.7–8.1.10.

Lemma 8.1.11. *If $G \in \overline{\mathbb{G}\mathrm{r}}$, then for*

$$G_n = [G_1,G_{n-1}] + [G_2,G_{n-2}] + \cdots + G_{n-1},G_1]$$

we have

$$G_n = [G_{n-1},G], \tag{8.1.6}$$

for $n > 1$, where $G_1 = G$.

Proof. For $n = 2,3$ (8.1.6) is trivial. For $n = 4$ we have

$$G_4 = [G_1,G_3] + [G_2,G_2] + [G_3,G_1].$$

Thus $[G_3,G_1] \subset G_4$, and for $G_4 \subset [G_3,G_1]$ we will show that $[G_2,G_2] \subset [G_3,G_1]$. We have

$$[G_2,G_2] = \big[[G_1,G_1],G_2\big] \subset \big[G_1,[G_1,G_2]\big] + \big[G_1,[G_1,G_2]\big] \subset [G_3,G_1],$$

since $[G_1,G_2] \subset G_3$.

Assume that (8.1.6) is true for any G_l, where $l < n$. For $l = n$ we have $[G_{n-1},G_1] \subseteq G_n$. We have to show that

$$[G_k,G_{n-k}] \subseteq [G_{n-1},G] \quad \text{for} \quad 1 \le k < n. \tag{8.1.7}$$

For $k = 1$, $[G_1,G_{n-1}] = [G_{n-1},G]$.

For $k = 2$, by Proposition 8.1.6,

$$[G_2,G_{n-2}] = \big[[G_1,G_1],G_{n-2}\big] \subset \big[G_1,[G_1,G_{n-2}]\big] + \big[G_1,[G_1,G_{n-2}]\big] = [G_{n-1},G],$$

since $[G_1,G_{n-2}] = G_{n-1}$ by our assumption.

Suppose that (8.1.7) is true for $1 \le k \le t-1$, where $t < n$. We will show (8.1.7) for $k = t$.

By our assumption, $G_t = [G_{t-1},G]$; therefore

$$\begin{aligned}[G_t,G_{n-t}] &= \big[[G,G_{t-1}],G_{n-t}\big] \subset \big[G,[G_{t-1},G_{n-t}\big] + \big[G_{t-1},[G,G_{n-t}\big] \\ &\subset [G,G_{n-1}] + [G_{t-1},G_{n-t+1}] \subset [G,G_{n-1}] + [G_{n-1},G] = [G_{n-1},G];\end{aligned}$$

here we have used the fact that $[G_{t-1},G_{n-t}] \subset G_{n-1}$, $[G,G_{n-t}] \subset G_{n-t+1}$ and that, by our assumption, $[G_{t-1},G_{n-t+1}] \subset [G_{n-1},G]$, which proves the lemma. □

From this lemma the construction of the functor $\mathbb{G}\mathrm{r}^c \longrightarrow \mathbb{LL}$ becomes simpler for the objects of $\overline{\mathbb{G}\mathrm{r}}$. Namely, if $G \in \overline{\mathbb{G}\mathrm{r}}$, then

$$LL(G) = \sum_{n=1}^{\infty} G_n \,/\, [G_n, G]. \tag{8.1.8}$$

Let G be a free object in $\overline{\mathbb{G}\mathrm{r}}$ (see Sec. 8.2 for the construction) and $G_n = [G_{n-1}, G]$, $n > 1$. Let E be the set of all defining identities between the brackets (both round and square) in G_n, $n \geq 1$, and $\overline{E}$ the set of all defining identities that satisfy the elements of the groups $\overline{G}_n = G_n/[G_n, G]$, $n \geq 1$. Under "defining identities" we mean that any identity in G follows from the identities from E.

Remark. We could define G_n from the beginning by (8.1.6), but we would need Propositions 8.1.5 and 8.1.6 for proving $[G_n, G_m] \subset G_{n+m}$, which we have applied in proving Theorem 7.3.6 of Chap. 7.

If G is a free object in $\overline{\mathbb{G}\mathrm{r}}$, then we have Conditions $1'$, 2, 3 for the elements of G, but there can be more identities between the round and round and square brackets. In the case of $\mathbb{A}\mathrm{b}^c$ we have another picture, the only identity we have in $\mathbb{A}\mathrm{b}^c$ is Condition $1'$ (and of course its consequences).

Let G be a free object of $\mathbb{A}\mathrm{b}^c$ and $g_1, \dots, g_k \in G$. Let $P(g_1, \dots, g_k)$ be any expression of the elements $g_i, i = 1, \dots, k$ and bracket operations in G.

We say that P is a pure n-bracket if after decomposing each g_i in terms of brackets it contains only n-brackets. Here we have in mind that $\mathbb{A}\mathrm{b}^\bullet \cong \mathbb{A}\mathrm{b}^{[]}$ and the corresponding isomorphism for $\mathbb{A}\mathrm{b}^c$. For example, for the basis elements x_1, x_2, x_3 of G, $[x_1, [x_2, x_3]]$ is a pure 3-bracket. If g is a pure m-bracket and h is a pure k-bracket, then $[g, h]$ is a pure $m + k$-bracket.

It may happen that according to Condition $1'$, a linear combination of pure n-brackets is an element of G_{n+1}.

Lemma 8.1.12. *Let G be a free object in $\mathbb{A}\mathrm{b}^c$. If $P(g_1, \dots, g_t) \in G_n$ is a linear combination of pure n-brackets in G and $P(g_1, \dots, g_t) \in G_{n+1}$, then $P(\overline{g_1}, \dots, \overline{g_t}) = 0$ in $\overline{G}_n = G_n/G_{n+1}$ is either the Leibniz identity or its consequence.*

Proof. There exists an expression $Q() \in G_{n+1}$ with $n+1$ brackets such that $P() - Q() = 0$. Since G is free, $P() - Q() = 0$ is either equivalent to Condition $1'$ or to its consequence. Now the proof is a direct computation. Take x, y, z as pure k, l, m-brackets, respectively, in Condition $1'$, with $k + l + m = n$. Then from (8.1.1), (8.1.3) and the fact that $g^h = g + [g, h]$, for any $g, h \in G$ we obtain that the pure n-bracket combination part of Condition $1'$ has the form $[x, [y, z]] - [[x, y], z] + [[x, z], y]$. Note that in $\mathbb{A}\mathrm{b}^c$ we have $[-g, h] = -[g, h]$. The same result we have in the case $P() - Q() = 0$ is equivalent to a consequence of Condition $1'$, which ends the proof. □

Proposition 8.1.13. *Let G be a free object in $\mathbb{A}\mathrm{b}^c$. Then the elements of the object $L(G)$ ($L : \mathbb{A}\mathrm{b}^c \longrightarrow$ $\mathbb{L}$eibniz) satisfy only the Leibniz algebra identities for square brackets i.e., the square bracket operation is bilinear and in $\overline{G}_n = G_n/G_{n+1}$, $n \geq 1$we have the Leibniz identity*

$$\big[\overline{x}, [\overline{y}, \overline{z}]\big] = \big[[\overline{x}, \overline{y}], \overline{z}\big] - \big[[\overline{x}, \overline{z}], \overline{y}\big],$$

where $x, y, z \in G$ and $\overline{x} \in \overline{G}_m$, $\overline{y} \in \overline{G}_l$, $\overline{z} \in \overline{G}_t$ denote the corresponding elements with $m + l + t = n$.

Proof. Suppose G is free in $\mathbb{A}\mathrm{b}^c$ and we have in $\overline{G}_n$ the identity or relation $P(\overline{x}_{ji}) = \sum_{j=1}^{l} P_j(\overline{x}_{j1}, \dots, \overline{x}_{jt}) = 0$, where each P_j denotes a bracket element in P, and Σ denotes the sum of these elements in $\overline{G}_n$, $\overline{x}_{ij} \in \overline{G}_{kji}$, $k_{j1} + \cdots + k_{jt} = n, j = 1, \dots, l$. We suppose that each $\overline{x}_{ji} \neq 0$ and P contains at most n brackets. For each inverse image x'_{ji} in G_{kji}, $j = 1, \dots, l$, $i = 1, \dots, t$ (i.e., $x'_{ji} \mapsto \overline{x}_{ji}$, by the natural homomorphism $G_{kji} \longrightarrow \overline{G}_{kji}$) we have $P(x'_{ji}) = \sum_{j=1}^{l} P_J(x'_{j1}, \dots, x'_{jt}) \in G_{n+1}$. Since each $\overline{x}_{ji} \neq 0$, we have $x_{ji} \notin G_{kji+1}$; thus each x_{ji} contains kji brackets as a summand. Hence each $\overline{x}_{ji}$ has an inverse image $\widetilde{x}_{ji} \in G_{kji}$, $\widetilde{x}_{ji} \mapsto \overline{x}_{ji}$, and $\widetilde{x}_{ji}$ is a pure kji-bracket. We have $P(\widetilde{x}_{ji}) = \sum_{j=1}^{l} P_j(\widetilde{x}_{j1}, \ldots, \widetilde{x}_{jt}) \in G_{n+1}$, and each $P_j(\widetilde{x}_{j1}, \ldots, \widetilde{x}_{jt})$ is a pure n-bracket. $P(\overline{\widetilde{x}}_{ji}) = P(\overline{x}_{ji})$ and, by Lemma 8.1.12, $P(\overline{x}_{ji}) = 0$ is either the Leibniz identity or its consequence. □

Remark. In $\mathbb{A}\mathrm{b}^{\bullet}$, Condition 1 has the form

$$-x^{(z^x)} + x^{y+z^x} + x^z - x^{z+y^z} = 0,$$

which is, of course, equivalent to Condition 1′.

Direct computations show that in $\mathbb{A}\mathrm{b}^c$ we have the identities

$$\begin{aligned} [-g, h] &= -[g, h], \\ [g, -h] &= [-g, h]^{-h}, \\ [g, h]^x &= [g^x, h], \quad x, g, h \in G \in \mathbb{A}\mathrm{b}^c. \end{aligned}$$

The first two identities can be obtained from the identities in $\mathbb{G}\mathrm{r}^c$

$$\begin{aligned} \underline{[-g, h]^{g}} &= -[g, h], \\ \underline{[g, -h]^{-g}} &= [-g, h]^{-h}, \quad g, h \in G' \in \mathbb{G}\mathrm{r}^c, \end{aligned}$$

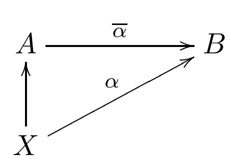
applying the functor $A : \mathbb{G}\mathrm{r}^c \longrightarrow \mathbb{A}\mathrm{b}^c$. It is easy to see that these identities follow from (8.1.2), and all the above identities do not give new identities for $LL(G')$, or $L(G)$.

8.2. Free Objects in $\mathbb{G}\mathrm{r}^{\bullet}$, $\overline{\mathbb{G}\mathrm{r}}$, $\mathbb{A}\mathrm{b}^c$ and $\mathbb{L}$eibniz

In this section we recall the definition of free objects in algebraic categories. We give the construction of free objects in the categories of groups with action and in Leibniz algebras.

Let **A** be any algebraic category.

Definition 8.2.1. Let A be an object in **A**. A is a free object on the set X if there is an injection $X \longrightarrow A$ and for any object $B \in$ **A** and a map $\alpha : X \longrightarrow B$, there exists a unique homomorphism $\overline{\alpha} : A \longrightarrow B$ in **A** such that the diagram



is commutative.

We will deal with free objects in the following algebraic categories: $\mathbb{A}\mathrm{b}$, $\mathbb{A}\mathrm{b}^c$, $\mathbb{G}\mathrm{r}$, $\mathbb{G}\mathrm{r}^{\bullet}$, $\mathbb{G}\mathrm{r}^c$, $\overline{\mathbb{G}\mathrm{r}}$, $\mathbb{L}$eibniz, $\mathbb{L}$ie, $\mathbb{LL}$ and $\overline{\mathbb{LL}}$; the last noted category will be defined in Sec. 8.3.

Let X be a set, and M_X be the free magma generated by X. Recall (see, e.g., [10] or [83]) that a magma is a set M with a (generally nonassociative) binary operation

$$M \times M \longrightarrow M.$$

We write the elements of M_X in the "vertical way"; so the elements of M_X have the form

$$\left(\left(x^{\left(\left(x_{11}^{\,x_{12}}\right)^{x_{13}}\right)^{\cdot^{\cdot^{\cdot^{x_{1i_1}}}}}}\right)^{\cdot^{\cdot^{\cdot^{\left(\left(x_{t-1,1}^{\,x_{t-1,2}}\right)^{x_{t-1,3}}\right)^{\cdot^{\cdot^{\cdot^{x_{t-1,i_{t-1}}}}}}}}}}\right)^{\left(\left(x_{t1}^{\,x_{t2}}\right)^{x_{t3}}\right)^{\cdot^{\cdot^{\cdot^{x_{ti_t}}}}}} \tag{8.2.1}$$

where $x,\ x_{js} \in X$, $j = 1, 2, \dots, t$, $s = 1, 2, \dots i_j$.

We denote this kind of elements by $x^{\square}$ to indicate that the element (8.2.1) is represented by the element $x \in X$.

Let $\mathcal{F}(M_X)$ be a free group generated by M_X. The operation in $\mathcal{F}(M_X)$ is denoted by "+," so the elements of $\mathcal{F}(M_X)$ have the form

$$\pm x_1^{\square} \pm x_2^{\square} \pm \cdots \pm x_n^{\square},$$

where $x_i^{\square}$ is an element of type (8.2.1) for each $i = 1, \dots, n$. The empty word (neutral element) of $\mathcal{F}(M_X)$ is denoted by 0.

Define in $\mathcal{F}(M_X)$ the action of elements by

$$(x_1^{\square} + \cdots + x_n^{\square})^{y_1^{\square} + \cdots + y_m^{\square}} = \left(\left((x_1^{\square})^{(y_1^{\square})}\right)^{\cdot^{\cdot^{\cdot^{(y_{m-1}^{\square})}}}}\right)^{(y_m^{\square})} + \cdots + \left(\left((x_n^{\square})^{(y_1^{\square})}\right)^{\cdot^{\cdot^{\cdot^{(y_{m-1}^{\square})}}}}\right)^{(y_m^{\square})},$$

$$(x_1^{\square} + \cdots + x_n^{\square})^0 = x_1^{\square} + \cdots + x_n^{\square}, \quad 0^{(x_1^{\square} + \cdots + x_n^{\square})} = 0.$$

Now it is easy to see that the following statement holds.

Proposition 8.2.2. *The object $\mathcal{F}(M_X)$ is a group with action on itself and it is a free object in $\mathbb{G}\mathrm{r}^{\bullet}$ on the set X.*

Actually we have defined the functor $\mathcal{F} : \mathbf{Set} \to \mathbb{G}\mathrm{r}^{\bullet}$, which is left adjoint to the forgetful functor U.

Let $\sim$ be a minimal congruence relation on $\mathcal{F}(M_X)$ generated by the relation defined by Condition 1. Then we obtain

Proposition 8.2.3. *The object $\mathcal{F}(M_X)/\sim$ is a free object in $\mathbb{G}\mathrm{r}^c$ on the set X.*

In the same way we construct free objects in $\overline{\mathbb{G}\mathrm{r}}$.

On the other hand, in diagram 7.3.3 of Sec. 7.3 the functor A is left adjoint to the full embedding functor E, and therefore we obtain

Proposition 8.2.4. *$A(\mathcal{F}(M_X)/\sim)$ is a free object in $\mathbb{A}\mathrm{b}^c$ on the set X.*

Here we give the construction of free Leibniz algebras. Let k be a commutative ring with the unit and X be any set. Denote by $W(X)$ the set that contains X and all those formal combinations of square brackets and elements of X that do not contain words of the form $[a,[b,c]]$, where a,b,c are elements of X or combinations of elements of X and brackets. Let $F(W(X))$ be the free k-module generated by the set $W(X)$. Consider the map $\eta : W(X)\times W(X) \longrightarrow F(W(X))$ defined by $\eta(w_1,w_2) = [w_1,w_2]$ if $[w_1,w_2] \in W(X)$; for $[w_1,w_2] \notin W(X)$ we decompose the word $[w_1,w_2]$ according to the Leibniz identity and express it as a sum of the words from $W(X)$ in $F(W(X))$. We define $\eta(w_1,w_2)$ as this final sum. Note that any two different decompositions give the same element of $F(W(X))$. We define the bracket operation on $F(W(X))$ as the k-linear extension of the map η to $F(W(X))$. It is easy to see that the obtained object is a free Leibniz algebra on the set X (cf. [65]).

Remark. Since the functors Q_1, Q_2, S_1, ${S_1}'$, A, and S_2 in the diagram (7.3.3) of Sec. 7.3 are left adjoints to the embedding functors, these functors take free objects to free objects. These new obtained objects are free on the same sets as original taken objects. Moreover, we have $Q_1(G) \approx Q_2(G)$ and $S_1(L) \approx {S_1}'(L)$, where G and L are free in $\mathbb{G}\mathrm{r}^c$ and $\mathbb{LL}$ respectively. Here, e.g., for the case of the functor Q_i we have in mind that $Q_1\mathcal{F}$ and $Q_2\mathcal{F}$ are left adjoints to one and the same functor $UT = UC = U_G$; thus $Q_1\mathcal{F} \approx Q_2\mathcal{F} \approx \mathcal{F}_G$, where $U_G : \mathbb{G}\mathrm{r} \to$ **Set** is an underlying functor and $\mathcal{F}_G$ is its left adjoint, which corresponds to any set ofthe free group generated by this set. Thus $Q_1(G) \approx Q_2(G)$ for free object G. We apply the analogous argument for the other functors above.

8.3. Identities in $\mathbb{G}\mathrm{r}^c$ and the Main Results

In this section all algebras (Lie, Leibniz, Lie–Leibniz) are considered over the ring of integers $\mathbb{Z}$. We investigate the question of the existence of identities between round and square brackets in $\mathbb{G}\mathrm{r}^\bullet$. If E is the set of the identities for the category $\overline{\mathbb{G}\mathrm{r}}$, we define the full subcategory $\overline{\mathbb{LL}}$ of $\mathbb{LL}$ (Lie–Leibniz algebras) of those objects satisfying identities $\overline{\mathrm{E}}$, where $\overline{\mathrm{E}}$ denotes the set of all identities inherited in $\mathbb{LL}$ from E. We prove that if G is the free object in $\overline{\mathbb{G}\mathrm{r}}$ generated by the set X, then every element of $\overline{G}_n = G_n/G_{n+1}$ is represented as a combination of elements of the form

$$\Big[\Big(\cdots\big[\big(\big[\big(\overline{y}_k,\cdots\big[\big(\overline{y}_3,\big[\big(\big[\big(\overline{x},\overline{y}_1\big)\big],\overline{y}_2\big)\big]\ \big)\big]\cdots\big)\big],\overline{y}_m\big)\big]\cdots,\overline{y}_{n-1}\Big)\Big],$$

where two brackets mean that we have either a round or a square bracket for $x, y_1, \dots, y_{n-1} \in X$, and this representation is unique up to identities from $\overline{E}$. By this result we easily prove that the functor LL takes free objects from $\overline{\mathbb{G}\mathrm{r}}$ to free objects in $\overline{\mathbb{LL}}$, and $L(G)$ (resp. $LA(G')$) is a free Leibniz algebra if G (resp. G') is a free object in $\mathbb{A}\mathrm{b}^c$ (resp. in $\overline{\mathbb{G}\mathrm{r}}$).

The category $\overline{\mathbb{G}\mathrm{r}}$ is defined in Sec. 8.1 as the full subcategory of those objects of $\mathbb{G}\mathrm{r}^\bullet$ that satisfy Conditions 1′, 2, 3. We look for possible identities in $\overline{\mathbb{G}\mathrm{r}}$ between the round and square brackets. We have well-known Witt–Hall identities for round brackets in $\mathbb{G}\mathrm{r}$. By Witt's theorem [83], [90] the functor $W : \mathbb{G}\mathrm{r} \longrightarrow \mathbb{L}\mathrm{ie}$ in diagram (7.3.3) takes free objects to free objects. Taking into account the same kind of argument as we have at the end of Sec. 8.1 for the case of groups with action and Lie–Leibniz algebras, we conclude that in $\mathbb{G}\mathrm{r}$ we do not have such identities for the round brackets that "inherit" new identities for Lie algebras. Thus if new identities exist in $\mathbb{G}\mathrm{r}$, they give the same Jacobi identity, the identity $(x,x) = 0$ and the bilinear property for the operation (,) in the corresponding Lie algebra. Below we consider in $\overline{\mathbb{G}\mathrm{r}}$ those "variations" of the well-known identities in $\mathbb{G}\mathrm{r}$ which by applying the usual functors (see diagram (7.3.3) of Chap. 7)

$$\mathbb{A}\mathrm{b}^c \xleftarrow{\ A\ } \mathbb{G}\mathrm{r}^c \xrightarrow{\ Q_2\ } \mathbb{G}\mathrm{r}$$

give the known identities in $\mathbb{A}\mathrm{b}^c$ and $\mathbb{G}\mathrm{r}$. As above, for $x, y \in G$, $G \in \mathbb{G}\mathrm{r}^\bullet$ we denote $x^{\underline{y}} = -y + x + y$. Consider the following expressions:

$$
\begin{aligned}
&a_1 = \big[x^y, (y,z)\big]; && b_1 = \big[(x,y), z^x\big]; && c_1 = \big[-(x,z), y^z\big];\\
&a_2 = \big(x^y, [y,z]\big); && b_2 = \big([x,y], z^x\big); && c_2 = \big(-[x,z], y^z\big);\\
&a_3 = -\big[(y,z), x^y\big]; && b_3 = -\big[z^x, (x,y)\big]; && c_3 = -\big[y^z, -(x,z)\big];\\
&a_4 = \big(x^y, -[z,y]\big); && b_4 = \big(-[y,x], z^x\big); && c_4 = \big([z,x], y^z\big);\\
&a_5 = \big[x^{\underline{y}}, (y,z)\big]; && b_5 = \big[(x,y), z^{\underline{x}}\big]; && c_5 = \big[-(x,z), y^{\underline{z}}\big];\\
&a_6 = \big(x^{\underline{y}}, [y,z]\big); && b_6 = \big([x,y], z^{\underline{x}}\big); && c_6 = \big(-[x,z], y^{\underline{z}}\big);\\
&a_7 = -\big[(y,z), x^{\underline{y}}\big]; && b_7 = -\big[z^{\underline{x}}, (x,y)\big]; && c_7 = -\big[y^{\underline{z}}, -(x,z)\big];\\
&a_8 = \big(x^{\underline{y}}, -[z,y]\big); && b_8 = \big(-[y,x], z^{\underline{x}}\big); && c_8 = \big([z,x], y^{\underline{z}}\big).
\end{aligned}
$$

Consider all kinds of identities

$$a_i = b_j + c_k, \qquad i,j,k = \overline{1,8}. \tag{8.3.1}$$

Applying the functor A or Q_2 to (8.3.1), we obtain that the resulting equalities are true in $\mathbb{A}\mathrm{b}^c$ and $\mathbb{G}\mathrm{r}$ (i.e., when $(\quad) = 0$ or $[\quad] = (\quad)$).

Direct computations give

$$
\begin{aligned}
a_1 &= -x^y + x^{-z+y+z};\\
a_2 &= -x^y - y^z + y + x^y - y + y^z;\\
a_3 &= -z^{(x^y)} - y^{(x^y)} + z^{(x^y)} + y^{(x^y)} - y - z + y + z;\\
a_4 &= -x^y - z + z^y + x^y - z^y + z;\\
a_5 &= -y - x + y - y^{-y-z+y+z} + x^{-y-z+y+z} + y^{-y-z+y+z};\\
a_6 &= -y - x + y - y^z + x + y^z;\\
a_7 &= -z^{-y+x+y} - y^{-y+x+y} + z^{-y+x+y} + y^{-y+x+y} - y - z + y + z;\\
a_8 &= -y - x + y - z + z^y - y + x + y - z^y + z;\\
b_1 &= -y - x + y + x - x^{(z^x)} - y^{(z^x)} + x^{(z^x)} + y^{(z^x)};\\
b_2 &= -x^y + x - z^x - x + x^y + z^x;\\
b_3 &= -z^{-y+x+y} + z^x;\\
b_4 &= -y + y^x - z^x - y^x + y + z^x;\\
b_5 &= -y - x + y + x - x^{-x+z+x} - y^{-x+z+x} + x^{-x+z+x} + y^{-x+z+x};\\
b_6 &= -x^y - z + x^y - x + z + x;\\
b_7 &= -x^{-x-y+x+y} - z^{-x-y+x+y} + x^{-x-y+x+y} - x + z + x;\\
b_8 &= -y + y^x - x - z + x - y^x + y - x + z + x;\\
c_1 &= -x - z + x + y - z^{(y^z)} - x^{(y^z)} + z^{(y^z)} + x^{(y^z)};\\
c_2 &= --x + x^z - y^z - x^z + x + y^z;\\
c_3 &= -y^{-x+z+x} + y^z;\\
c_4 &= -z^x + z - y^z - z + z^x + y^z;\\
c_5 &= -x - z + x + z - z^{-z+y+z} - x^{-z+y+z} + z^{-z+y+z} + x^{-z+y+z};
\end{aligned}
$$

$$c_6 = -x + x^z - z - y + z - x^z + x - z + y + z;$$
$$c_7 = -z^{-z-x+z+x} - y^{-z-x+z+x} + z^{-z-x+z+x} - z + y + z;$$
$$c_8 = -z^x - y + z^x - z + y + z.$$

The checking shows that none of identities (8.3.1) holds for free objects in $\mathbb{G}\mathrm{r}^\bullet$. The same is true for the category $\mathbb{G}\mathrm{r}^c$, since Condition 1 represents any element x by a combination of elements with base element x, and therefore Condition 1 does not help any of identities (8.3.1) to hold in $\mathbb{G}\mathrm{r}^c$. Nevertheless we cannot claim that we do not have identities between round and square brackets in $\mathbb{G}\mathrm{r}^\bullet$ or in $\mathbb{G}\mathrm{r}^c$. The same situation is observed for $\overline{\mathbb{G}\mathrm{r}}$; by definition, here we have two identities from (8.3.1); these are Condition 2 and Condition 3 (for $i = j = k = 1$ and $i = j = k = 2$). Note also that we may have identities in $\mathbb{G}\mathrm{r}$ that do not give new identities for $W(F)$ (where F is a free group and $W(F)$ is the corresponding Lie algebra), but "variations" (with square brackets) of these identities in $\mathbb{G}\mathrm{r}^c$ (or in $\overline{\mathbb{G}\mathrm{r}}$) may give new identities in $LL(G)$, for a free object $G \in \mathbb{G}\mathrm{r}^c$, since, e.g., in $W(G)$ we have $(\overline{x}, \overline{x}) = 0$, but in $LL(G)$, $[\overline{x}, \overline{x}] \neq 0$, $x \in G$.

Let G be a free object in $\overline{\mathbb{G}\mathrm{r}}$. Let E be the set of all defining identities between both kinds of brackets in $\overline{\mathbb{G}\mathrm{r}}$, and let $\overline{\mathrm{E}}$ be the set of corresponding identities that satisfies $LL(G)$ and thus the identities inherited from E.

Denote by $\overline{\mathbb{LL}}$ the full subcategory of $\mathbb{LL}$ consisting of those objects of $\mathbb{LL}$ that satisfy the conditions from $\overline{\mathrm{E}}$. Of course, among the identities in $\overline{\mathrm{E}}$ we have the bilinear properties of [,] and (,), the identity $(x, x) = 0$, the Jacobi identity

$$\big((x, y), z\big) + \big((y, z), x\big) + \big((z, x), y\big) = 0,$$

the Leibniz identity

$$\big[x, [y, z]\big] = \big[[x, y], z\big] - \big[[x, z], y\big], \tag{8.3.2}$$

and also the identities

$$\big[x, (y, z)\big] = \big[(x, y), z\big] - \big[(x, z), y\big],$$
$$\big(x, [y, z]\big) = \big([x, y], z\big) - \big([x, z], y\big)$$

which correspond to the known identities for round and square brackets in $\mathbb{G}\mathrm{r}$ and $\mathbb{G}\mathrm{r}^\bullet$, respectively, Conditions $1'$, 2, and 3 in $\mathbb{G}\mathrm{r}^\bullet$.

For the case of $\mathbb{A}\mathrm{b}^c$, for a free object $G \in \mathbb{A}\mathrm{b}^c$, E contains the usual identities (8.1.1) and only one additional identity, Condition $1'$ (see Chap. 7, Sec. 7.3); by virtue of Proposition 8.1.13 the set of all defining identities $\overline{E}$ (which satisfy the elements of $L(G)$) consists of identity (8.3.2), bilinear properties of the square bracket operation, and of the identity $[x, 0] = [0, x] = 0$. See also the remark after the proof of Proposition 8.1.13.

Proposition 8.3.1. *Let* $G \in \overline{\mathbb{G}\mathrm{r}}$, $G_n = [G_{n-1}, G]$ *for* $n > 1$, *where* $G_1 = G$, *and* $\overline{G}_n = G_n/G_{n+1}$. *If* G *is the free object in* $\overline{\mathbb{G}\mathrm{r}}$ *generated by the set* X, *then* $\overline{G}_1$ *is the free abelian group generated by the same set* X, *and every element of* $\overline{G}_n$, $n > 1$ *has a representation as a combination of elements of the form*

$$\Big[\Big(\cdots\big[\big(\big[\big(\overline{y}_k, \cdots\big[\big(\overline{y}_3, \big[\big(\big[\big(\overline{x}, \overline{y}_1\big)\big], \overline{y}_2\big)\big]\big)\big]\cdots\big)\big], \overline{y}_m\big)\big]\cdots, \overline{y}_{n-1}\Big)\Big] \tag{8.3.3}$$

($n - 1$ *round or square brackets*), *where* $x, y_1, \ldots, y_{n-1} \in X$, *and this representation is unique up to identities from* $\overline{\mathrm{E}}$.

Proof. It is obvious that $\overline{G}_1 = G_1/G_2$ is the free abelian group on the set X. We have $G_2 = [G_1, G_1]$, and, by definition, G_2 is an ideal of G generated by elements of the form $\underline{[(g, h)]}$ (here we mean elements of both forms $[g, h]$ and (g, h)), $g, h \in G$. Since G is a free object in $\overline{\mathbb{G}\mathrm{r}}$, we have

$$g = x_1^{\square} + \cdots + x_n^{\square}, \qquad h = y_1^{\square} + \cdots + y_k^{\square},$$

where $x_i, y_i \in X$, $i = \overline{1,n}$, $j = \overline{1,k}$. Then by (8.1.2) and (8.1.3) we obtain that $[(g,h)]$ has the form

$$[(g,h)] = \sum_{i,j} \left[\left(x_i^{\square}, y_j^{\square}\right)\right]^{\square}; \tag{8.3.4}$$

here for $a \in G$, $a^{\square}$ means that the action operations represented by $\square$ include also actions by conjugation. Now we have to show that if $t, t_1, t_2 \in G_2$ and have form (8.3.4), then t^g, $t^{\overline{g}}$, $[g,t]$, $t_1 + t_2$ have the same form for $g \in G$. It is obvious that t^g, $t^{\overline{g}}$, and $t_1 + t_2$ have form (8.3.4). For $[g,t]$ we have the representation

$$[g,t] = \sum_{l,i,j} \left[\left(z_l^{\square}, \left[\left(x_i^{\square}, y_j^{\square}\right)\right]^{\square}\right)\right]^{\square}.$$

If we open one bracket (square or round, as in the representation) in each summand

$$\begin{aligned} \left[x_i^{\square}, y_j^{\square}\right] &= -x_i^{\square} + x_i^{(y_j^{\square})}, \\ \left(x_i^{\square}, y_i^{\square}\right) &= -x_i^{\square} + x_i^{\overline{(y_j^{\square})}}, \end{aligned} \tag{8.3.5}$$

and then apply (8.1.2) and (8.1.3), we will see that $[(g,t)]$ has the representation of the form (8.3.4). We have $\overline{[(g,t)]} = 0$ in $\overline{G}_2$, since $[(g,t)] \in G_3$, and this is also obvious from (8.3.5) and the fact that $\overline{x_i} = \overline{x_i^{(y_j^0)}}$ in $\overline{G}_2$ for $x_i \in G_2$. In the same way we prove that the elements of $G_3 = [G_2, G]$ have representations of the form

$$\sum \left[\left(z_2^{\square}, \left[\overset{\frown}{\left(x_i^{\square}, y_i^{\square}\right)}\right]^{\square}\right)\right]^{\square},$$

where for $a, b \in G$, $[\overset{\frown}{(a,b)}]$ denotes elements either of the form $[(a,b)]$ or of the form $[(b,a)]$.

Suppose that the elements of G_{n-1} can be represented as $\mathbb{Z}$-combinations of the elements of the form

$$\left[\left(\cdots\left[\left(\left[\left(y_k^{\square}, \cdots \left[\left(y_3^{\square}, \left[\left(\left[\left(x^{\square}, y_1^{\square}\right)\right]^{\square}, y_2^{\square}\right)\right]^{\square}\right)\right]^{\square} \cdots\right)\right]^{\square}, y_m^{\square}\right)\right]^{\square} \cdots, y_{n-2}^{\square}\right)\right]^{\square}.$$

Then we obtain the corresponding result for G_n. These representations are unique up to identities from E. From this it follows that the elements of $\overline{G}_n$ are combinations with coefficients from $\mathbb{Z}$ of elements of the form (8.3.3). Since $\overline{\mathrm{E}}$ is the set of all identities in $L(G) = \sum_{n=1}^{\infty} \overline{G}_n$, these representations of elements of $\overline{G}_n$ are unique up to identities from $\overline{\mathrm{E}}$. $\square$

From Proposition 8.3.1 follows the main result.

Theorem 8.3.2. *Let G be a free object in $\overline{\mathbb{G}\mathrm{r}}$ on the set X. Then the Lie–Leibniz algebra $LL(G)$ is a free object in the category $\overline{\mathbb{LL}}$ on the set X.*

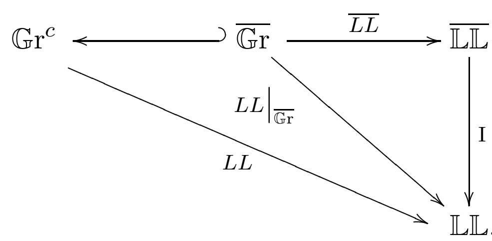
In the same way, applying Proposition 8.1.13 we obtain

Theorem 8.3.3. *Let G be a free object in $\mathbb{A}\mathrm{b}^c$ generated by the set X. Then $L(G)$ is a free Leibniz algebra on the set X.*

Corollary 8.3.4. *Any free Leibniz algebra can be obtained up to an isomorphism by the functor L; i.e., for any free Leibniz algebra A there is an object $G \in \mathbb{A}\mathrm{b}^c$ such that $L(G) \approx A$.*

Proof. Let A be the free Leibniz algebra on the set X. Take the free object G in $\mathbb{A}\mathrm{b}^c$ on the set X. Now, by Theorem 8.3.3, $L(G)$ is the free Leibniz algebra generated by the set X, and therefore $L(G) \approx A$. $\square$

Consider the restriction $LL\big|_{\overline{\mathbb{G}\mathrm{r}}}$. It is obvious that $LL\big|_{\overline{\mathbb{G}\mathrm{r}}}$ factors through $\overline{\mathbb{LL}}$. Thus we have the commutative diagram



Corollary 8.3.5. *Any free Leibniz algebra A can be considered as an object of $\overline{\mathbb{LL}}$, i.e., $E_2(A) \in \overline{\mathbb{LL}}$.*

Proof. It follows from Corollary 8.3.4 and the fact that $\mathbb{A}\mathrm{b}^c \hookrightarrow \overline{\mathbb{G}\mathrm{r}}$ and $E_2 \cdot L = LL\big|_{\mathbb{A}\mathrm{b}^c} = \overline{LL}\big|_{\mathbb{A}\mathrm{b}^c}$. □

Corollary 8.3.6. *There is a full embedding functor $\overline{E_2}$: $\mathbb{L}$eibniz $\longrightarrow \overline{\mathbb{LL}}$ such that $I\overline{E}_2 = E_2$; the functor $\overline{S}_2 = S_2 I$ is a left adjoint to $\overline{E}_2$.*

Proof. Let A be any Leibniz algebra; choose a free Leibniz algebra F_A on the basis A and an epimorphism $F_A \longrightarrow A$. We have $E_2(F_A) \in \overline{\mathbb{LL}}$ by Corollary 8.3.5 and $E_2(A) \in \mathbb{LL}$; from this it follows that the elements of A also satisfy identities from $\overline{E}$; thus $E_2(A) \in \overline{\mathbb{LL}}$, which means that there is a full embedding functor $\overline{E_2}$: $\mathbb{L}$eibniz$\longrightarrow \overline{\mathbb{LL}}$ with $I\overline{E}_2 = E_2$. It is easy to see that $\overline{S}_2$ is a left adjoint to $\overline{E}_2$. □

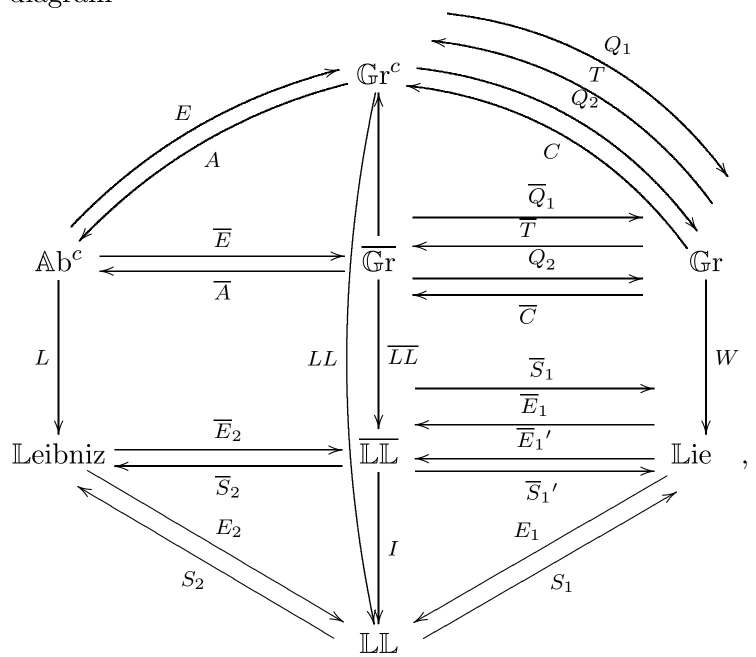
Applying Witt's theorem stating that the functor W takes free objects from $\mathbb{G}\mathrm{r}$ to free objects in $\mathbb{L}$ie, we obtain the following results.

Corollary 8.3.7. *Any free Lie algebra can be obtained up to isomorphism by the functor W.*

Corollary 8.3.8. *Any free Lie algebra A can be considered as an object of $\overline{\mathbb{LL}}$ either with trivial square bracket operation or with the square bracket equal to the round bracket, i.e., $E_1(A) \in \overline{\mathbb{LL}}$, ${E_1}'(A) \in \overline{\mathbb{LL}}$.*

Corollary 8.3.9. *There are full embedding functors $\overline{E}_1, \overline{E'}_1$: $\mathbb{L}$ie $\longrightarrow \overline{\mathbb{LL}}$ such that $I\overline{E}_1 = E_1$ and $I\overline{E'}_1 = {E_1}'$; the functor $\overline{S}_1 = S_1 I$ is a left adjoint to $\overline{E}_1$ and the functor $\overline{S'}_1 = {S_1}' I$ is a left adjoint to $\overline{E'}_1$.*

Thus we have the diagram



(8.3.6)

where $\overline{A} = A|_{\overline{\mathbb{G}\mathrm{r}}}$, $\overline{E}$ is the obvious full embedding, (i.e., it is clear that E factors through $\overline{\mathbb{G}\mathrm{r}}$), $\overline{Q}_i = Q_i|_{\overline{\mathbb{G}\mathrm{r}}}$, $i = 1, 2$. Since Conditions 2 and 3 are satisfied for groups with trivial action or action by conjugation, it follows that T and C factor through $\overline{\mathbb{G}\mathrm{r}}$; this gives the functors $\overline{T}$ and $\overline{C}$. $\overline{Q}_1$ and $\overline{Q}_2$ are left adjoints to $\overline{T}$ and $\overline{C}$ respectively.

Corollary 8.3.10. *For free objects in* $\overline{\mathbb{G}\mathrm{r}}$ *the left and right directional diagrams in* (8.3.6) *commute up to an isomorphism, i.e., if G is a free object in* $\overline{\mathbb{G}\mathrm{r}}$*, then*

$$\begin{aligned} L\,\overline{A}(G) &\approx \overline{S}_2\,\overline{LL}(G) = S_2\,LL|_{\overline{\mathbb{G}\mathrm{r}}}(G), \\ W\,\overline{Q}_1 &\approx \overline{S}_1\,\overline{LL}(G) = S_1\,LL|_{\overline{\mathbb{G}\mathrm{r}}}(G), \\ W\,\overline{Q}_2 &\approx \overline{S'}_1\ ,\overline{LL}(G) = S_1{}'\,LL|_{\overline{\mathbb{G}\mathrm{r}}}(G). \end{aligned}$$

Proof. It is sufficient to mention that the functors $\bar{A}$, $\bar{S}_2$, $\bar{Q}_1$, $\bar{Q}_2$, $\bar{S}_1$, $\bar{S}_1'$ take free objects to free objects and the new obtained objects are free on the same sets as the original ones. This fact can be shown directly or by the analogous argument that we have used in the remark at the end of the Sec. 8.2. □

It may be useful to formulate the result concerning free Leibniz algebras in the following form.

Corollary 8.3.11. *The composites of functors* $\overline{S}_2\,\overline{LL}$, $L\,\overline{A} : \overline{\mathbb{G}\mathrm{r}} \longrightarrow \mathbb{L}\mathrm{eibniz}$ *take free objects from* $\overline{\mathbb{G}\mathrm{r}}$ *to free Leibniz algebras, and for any free Leibniz algebra A there is a free object $G \in \overline{\mathbb{G}\mathrm{r}}$ with* $\overline{S}_2\,\overline{LL}(G) \approx A \approx L\,\overline{A}(G)$.

Let $V : \overline{\mathbb{LL}} \longrightarrow \mathbb{L}\mathrm{eibniz}$ be the obvious forgetful functor. The following commutative diagram, which is obtained from the above stated results, was suggested by the referee of the paper [38]:

$$\begin{array}{ccc} \mathbb{G}\mathrm{r} & \xrightarrow{\ \overline{C}\ } & \overline{\mathbb{G}\mathrm{r}} \\ {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle V\circ\overline{LL}} \\ \mathbb{L}\mathrm{ie} & \longrightarrow & \mathbb{L}\mathrm{eibniz}, \end{array}$$

where $\mathbb{L}\mathrm{ie} \longrightarrow \mathbb{L}\mathrm{eibniz}$ is an obvious inclusion functor.

These results together with Theorem 7.3.6 of Chap. 7 give E. Witt's well-known construction for groups with action on itself and prove an analogue of Witt's theorem for this special kind of groups and Leibniz algebras, which give a solution to the problems of J.-L. Loday stated in [62, 64].

Chapter 9

HOMOTOPICAL AND CATEGORICAL PROPERTIES OF CHAIN FUNCTORS

The material presented in this chapter emphasizes more the algebraic aspects of the category of chain functors $\mathfrak{C}h$ in comparison to [6], where we tried to explain the close analogy with the case of topological spaces or simplicial sets. The objective of [6] is the verification of the closed model properties of Quillen, CM2) – CM5). Property CH1), the existence of finite limits and colimits, is not fulfilled in $\mathfrak{C}h$. Although not every mapping $f \in \mathfrak{C}h(\boldsymbol{A}_*, \boldsymbol{B}_*)$ in the category of chain functors admits a kernel or a cokernel, we prove that all cofibrations have a cokernel, all regular fibrations have a kernel, and the pushout of a cofibration along a cofibration exists in $\mathfrak{C}h$, resp. the dual statement for fibrations. These results are applied in [8]. In Sec. 9.7 we include some basic material about chain functors.

9.1. The Closed Model Properties of $\mathfrak{C}h$

Recall that a mapping $p \in \mathfrak{C}h(\boldsymbol{E}_*, \boldsymbol{B}_*)$ in the category of chain functors is called regular whenever it commutes with κ, φ, and the chain homotopies $\varphi\kappa \simeq 1$, $j_{\#}\varphi \simeq l$ (see Sec. 9.7 for the definition of a chain functor). In what follows we will deal with regular injective mappings of chain functors $A_* \rightarrowtail B$. We will often call this kind of injections "inclusions" and denote it by $A_* \hookrightarrow B_*$.

We briefly record the closed model structure for $\mathfrak{C}h$ from [6]:

(1) The weak equivalences are the homotopy equivalences.

(2) A cofibration $q : \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$ is an inclusion satisfying the homotopy extension property ([6, Definition 4.1]).

(3) A fibration $p : \boldsymbol{E}_* \longrightarrow \boldsymbol{B}_*$ is a mapping having the lifting property with respect to all trivial cofibrations, i.e., to all cofibrations that are also weak equivalences.

In $\mathfrak{C}h$ we have a cylinder object $\boldsymbol{K}_* \times I$ and a cocylinder object $\boldsymbol{K}_*^I, \boldsymbol{K}_* \in \mathfrak{C}h$ ([6, § 1]).

In particular:

Lemma 9.1.1.

(1) *The inclusion* $i_0 : \boldsymbol{K}_* \longrightarrow \boldsymbol{K}_* \times I$ *is a trivial cofibration.*

(2) *The projection* $p_0 : \boldsymbol{K}_*^I \longrightarrow \boldsymbol{K}_*$ *is a trivial fibration.*

(3) *All objects in* $\mathfrak{C}h$ *are fibrant and cofibrant.*

Proof. Assertions (1) and (2) follow from [6, example on p. 114 and § 1, Lemma 3.5]. The inclusion $\{0\} \subset \boldsymbol{A}_*$ will be a cofibration because A_n splits on each level $n \in \mathbb{Z}$, $A_n = A_n \oplus \{0\}$ (cf. [6, Lemma 4.4]). The projection $p : \boldsymbol{A}_* \longrightarrow \{0\}$ has the lifting property with respect to any trivial cofibration $q : \boldsymbol{B}_* \overset{\subset}{\longrightarrow} \boldsymbol{C}_*$: According to [6, Lemma 4.8], there exists a $s : \boldsymbol{C}_* \longrightarrow \boldsymbol{B}_*$ satisfying $sq = 1$. So the commutative square

$$\begin{array}{ccc} \boldsymbol{A}_* & \xrightarrow{p} & \{0\} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle 0} \\ \boldsymbol{B}_* & \xrightarrow[q]{} & \boldsymbol{C}_* \end{array}$$

has a lifting $\overline{F}: \boldsymbol{C}_* \longrightarrow \boldsymbol{A}_* \overline{F} = fs$, satisfying $\overline{F}q = f$, $\overline{F}p = 0$. □

Lemma 9.1.2. *Let* $q: \boldsymbol{A}_* \hookrightarrow \boldsymbol{B}_*$ *be an inclusion in* $\mathfrak{Ch}$; *then the following properties of* q *are equivalent*:

(1) *Let* $j \in \mathfrak{Ch}(\boldsymbol{B}_* \cup \boldsymbol{A}_* \times I, \boldsymbol{B}_* \times I)$ *be the inclusion*; *then there exists*

$$r \in \mathfrak{Ch}(\boldsymbol{B}_* \times I, \boldsymbol{B}_* \cup \boldsymbol{A}_* \times I)$$

satisfying $rj = 1$.

(2) *Let*

$$\begin{array}{ccc} {\boldsymbol{L}_*}^I & \xrightarrow{p_0} & \boldsymbol{L}_* \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle G} \\ \boldsymbol{A}_* & \xrightarrow[q]{} & \boldsymbol{B}_* \end{array} \tag{9.1.1}$$

be commutative, $\boldsymbol{L}_* \in \mathfrak{Ch}$; *then there exists a lifting* $\overline{G}: \boldsymbol{B}_* \longrightarrow {\boldsymbol{L}_*}^I$, *rendering* (9.1.1) *commutative.*

(3) *For every* $B_n(X,U)$ $(B'_n(X,U))$ *there exists a natural isomorphism as an abelian group* (*not necessarily as a chain complex!*), $B_n(X,U) \approx A_n(X,U) \oplus C_n(X,U)$, *resp. for* $B'_n(X,U)$.

(4) q *is a cofibration.*

Proof. Follows from [6, Lemmas 4.2, 4.3, 4.4]. □

Lemma 9.1.3. *The following properties of a mapping* $p \in \mathfrak{Ch}(\boldsymbol{E}_*, \boldsymbol{B}_*)$ *are equivalent*

(1) *Suppose*

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ \boldsymbol{K}_* & \xrightarrow[i_0]{} & \boldsymbol{K}_* \times I \end{array} \qquad \boldsymbol{K}_* \in \mathfrak{Ch} \tag{9.1.2}$$

is commutative; *then there exists a diagonal* $\overline{F} \in \mathfrak{Ch}(\boldsymbol{K}_* \times I, \boldsymbol{E}_*)$ *rendering* (9.1.2) *commutative.*

(2) p *is a fibration.*

Proof. Property (1) was the definition of a (Hurewicz-) fibration in [6, Definition 3.1]. The assertion follows from [6, Theorem 5.1]. □

Theorem 9.5.2 in Sec. 9.5 is a dual statement to Lemma 9.1.2, extending Lemma 9.1.3 considerably for regular fibrations.

Proposition 9.1.4.

(1) *Let* $q: \boldsymbol{A}_* \subset \boldsymbol{B}_*$ *be a* (*trivial*) *cofibration, then*

$$q \times I: \boldsymbol{A}_* \times I \subset \boldsymbol{B}_* \times I$$

is a (*trivial*) *cofibration.*

(2) *Let* $p : \boldsymbol{E}_* \longrightarrow \boldsymbol{B}_*$ *be a* (*trivial*) *fibration, then*

$$p^I : {\boldsymbol{E}_*}^I \longrightarrow {\boldsymbol{B}_*}^I$$

is a (*trivial*) *fibration.*

Proof. (1) Since q is a cofibration, there exists a levelwise isomorphism $B_n(X,U) \approx A_n(X,U) \oplus C_n(X,U)$ as described in Lemma 9.1.2 3). This induces a splitting

$$(B_* \times I)_n(X,U) \approx (A_* \times I)_n(X,U) \oplus \widetilde{C}_n(X,U)$$

for suitable, obviously existing $\widetilde{C}_n$, with the same properties. Hence $q \times I$ is a cofibration. If q is a weak equivalence, there exist $s : \boldsymbol{B}_* \longrightarrow \boldsymbol{A}_*$, homotopies $H : qs \simeq 1,\ G : sq \simeq 1$. Forming $s \times I,\ H \times I, G \times I$ yields a homotopy inverse to $q \times I$.

(2) Let $q : \boldsymbol{A}_* \subset \boldsymbol{C}_*$ be a trivial cofibration,

$$\begin{array}{ccc} {\boldsymbol{E}_*}^I & \xrightarrow{p^I} & {\boldsymbol{B}_*}^I \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ \boldsymbol{A}_* & \xrightarrow[q]{} & \boldsymbol{C}_* \end{array} \tag{9.1.3}$$

be commutative; then the adjoint diagram

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ {\scriptstyle \widetilde{f}}\uparrow & & \uparrow{\scriptstyle \widetilde{F}} \\ \boldsymbol{A}_* \times I & \xrightarrow[q \times I]{} & \boldsymbol{C}_* \times I \end{array} \tag{9.1.4}$$

is also commutative, $q \times I$ according to (1) is a trivial cofibration, hence there exists a diagonal $\overline{\widetilde{F}} : \boldsymbol{C}_* \times I \longrightarrow \boldsymbol{E}_*$, rendering (9.1.4) commutative. So the adjoint $\overline{F} : \boldsymbol{C}_* \longrightarrow {\boldsymbol{E}_*}^I$ is a diagonal for (9.1.3). The remaining part of 2) follows, e.g., by replacing trivial cofibrations in the previous proof by arbitrary ones. □

We include an algebraic proof of the following assertion.

Lemma 9.1.5. *Let* $\rho : \boldsymbol{A}_* \times I \longrightarrow$ *cone* $\boldsymbol{A}_*$ *be the projection*; *then there exists a* $\sigma :$ *cone* $\boldsymbol{A}_* \longrightarrow \boldsymbol{A}_* \times I$ *satisfying*

$$\rho\sigma = 1, \tag{9.1.5}$$

$$i_0\, r + \sigma\rho = 1 \tag{9.1.6}$$

with $\boldsymbol{A}_* \xrightarrow{i_0} \boldsymbol{A}_* \times I \xrightarrow{r} \boldsymbol{A}_*\ ,\ r({a_n}', a_{n-1}, a_n) = {a_n}' + a_n.$

Proof. Recall that *cone* A_* is defined in dimension n as

$$\big\{(a_{n-1}, a_n) \mid\ a_m \in A_m\big\}, \quad (\boldsymbol{A}_* \times I)_n = \big\{({a_n}', a_{n-1}, a_n)\big\};$$

then we set

$$\sigma(a_{n-1}, a_n) = (-a_n, a_{n-1}, a_n)$$

and realize that $\sigma \in \mathfrak{Ch}(\textit{cone}\ \boldsymbol{A}_*, \boldsymbol{A}_* \times I)$ satisfies (9.1.5) and (9.1.6). □

From this, we deduce immediately

Corollary 9.1.6. $p_0 : {\boldsymbol{K}_*}^I \longrightarrow \boldsymbol{K}_*$ *is a trivial fibration.*

Proof. p_0 is by construction a weak equivalence. There is a section $s : \boldsymbol{K}_* \longrightarrow \boldsymbol{K}_*{}^I$, $p_0\, s = 1$. Let

$$\begin{array}{ccc} \boldsymbol{K}_*{}^I & \xrightarrow{p_0} & \boldsymbol{K}_* \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ \boldsymbol{A}_* & \xrightarrow[i_0]{} & \boldsymbol{A}_* \times I \end{array} \tag{9.1.7}$$

be a commutative square; then

$$\overline{F} = fr + sF\sigma\rho$$

is a diagonal, rendering (9.1.7) commutative. Applying Lemma 9.1.3 p is a fibration. ☐

There exists in $\mathfrak{C}h$ a canonical suspension $\Sigma\boldsymbol{A}_*$ (resp. loop object $\Omega\boldsymbol{A}_*$); $\Sigma\boldsymbol{A}_*$ is the cokernel of the mapping

$$\boldsymbol{A}_* \oplus \boldsymbol{A}_* \xrightarrow{i_0 \oplus i_1} \boldsymbol{A}_* \times I \longrightarrow \Sigma\boldsymbol{A}_* \ ,$$

while $\Omega\boldsymbol{A}_*$ is the kernel of the projection

$$\Omega\boldsymbol{A}_* \longrightarrow \boldsymbol{A}_*{}^I \xrightarrow{p_0 \oplus p_1} \boldsymbol{A}_* \oplus \boldsymbol{A}_* \ .$$

As in all categories with $\mathbb{Z}$-graded objects, we have another kind of suspension (resp. loop construction)

$$(\overline{\Sigma}\boldsymbol{A}_*)_n = A_{n-1},$$
$$(\overline{\Omega}\boldsymbol{A}_*)_n = A_{n+1}.$$

Lemma 9.1.7. *Suspension and loop functor are in the homotopy category $\mathfrak{C}h_h$ inverses of each other.*

Proof. This is simply the content of [6, Lemma 8.1, Theorem 8.2]. ☐

According to [6] the category of chain functors with the above given classes of weak equivalences, cofibrations and fibrations satisfies Quillen's axioms CM2) – CM5), also by Corollary 6.2 (resp. Corollary 7.2), the decompositions in the decomposition axiom CM5) for $\mathfrak{C}h$ are canonical.

Furthermore $\mathfrak{C}h$ belongs to the class of categories where the whole model structure, i.e., the class of fibrations, cofibrations, and weak equivalences, is entirely determined by the concept of a homotopy $H : \boldsymbol{K}_* \times I \longrightarrow \boldsymbol{L}_*$ (resp. by its adjoint $G : \boldsymbol{K}_* \longrightarrow \boldsymbol{L}_*{}^I$), i.e., by the cylinder construction and its dual. This follows from the fact that weak equivalences are homotopy equivalences and from Lemma 9.1.2(3) (resp. Lemma 9.1.3).

9.2. The Chain Functor Property of a Special Pushout

Since the category $\mathfrak{C}h$ does not have arbitrary (co-)limits, we are obliged to investigate separately in each case whether a kernel or cokernel exists. In the present section we prove that for a regular injection $q : \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$ the pushout $\boldsymbol{B}_* \cup \mathit{cone}\ \boldsymbol{A}_* = \boldsymbol{P}_*$ carries the structure of a chain functor. Since $\boldsymbol{P}_*$ is also a cokernel, this is a special case of the existence of a cokernel (Theorem 9.3.1 in the next section). This and some other results will be deduced from Theorem 9.2.1. Therefore we present a detailed proof of Theorem 9.2.1. The axioms for a chain functor CH1) – CH7) are recorded in Sec. 9.7.

Let $q : \boldsymbol{A}_* \hookrightarrow \boldsymbol{B}_*$ be an inclusion in $\mathfrak{C}h$; then we form the pushout

$$\boldsymbol{P}_* = \boldsymbol{B}_* \cup_{\boldsymbol{A}_*} \mathit{cone}\ \boldsymbol{A}_*$$

Theorem 9.2.1. *$\boldsymbol{P}_*$ is a chain functor.*

Proof. We verify properties CH1) – CH7) and begin by defining

$$P'_* = B'_* \cup cone\ A'_*,$$

with natural inclusions $l : \boldsymbol{P}'_* \subset \boldsymbol{P}_*,\ i' : P_*(U) \subset P'_*(X,U)$. Since by assumption $\kappa,\ \varphi$ and the related homotopies $\varphi\kappa \simeq 1,\ j_\#\varphi \simeq 1$ in $\boldsymbol{A}_*$ are the restrictions of the associated items in $\boldsymbol{B}_*$, we obtain all this for $\boldsymbol{P}_*$. We have $\kappa i = i'$. All inclusions $k : (X,U) \subset (Y,B)$ induce monomorphisms for $\boldsymbol{A}_*$ and $\boldsymbol{B}_*$, hence also for $\boldsymbol{P}_*$, and $P_*(X,X)$ is clearly acyclic.

This confirms CH1) and CH2).

Ad CH3): Any cycle $\widetilde{z} \in P_*(X,U)$ is of the form $c - \widehat{z}, \widehat{z} \in cone\ \boldsymbol{A}_*, c \in B_*(X,U)$, with $dc = d\widehat{z} = z \in \boldsymbol{A}_* \hookrightarrow cone\ \boldsymbol{A}_*$ (interpreting the injection $\boldsymbol{A}_* \longrightarrow cone\ \boldsymbol{A}_*$, as always, as an inclusion). We apply Lemma 9.7.5 and detect

(1) a $z' \in A'_*(X,U)$, $a_1 \in A_*(U,U)$ such that $lz' + q_\# a_1 \sim z$, $q : (U,U) \subset (X,U)$ in $A_*(X,U)$;

(2) a $c' \in B'_*(X,U)$ such that $lc' + q_\# a_2 = c + dw$, $a_2 \in B_*(U,U)$, $w \in B_*(X,U)$, all $\widehat{(\cdots)} \in cone\ \boldsymbol{A}_*$.

We have

$$c - \widehat{z} = c - (l\widehat{z}' + q_\#\widehat{a}_2) + dv = l(c' - \widehat{z}') + q_\#(a_2 - \widehat{a}_2) + d(v - w).$$

This confirms CH3) for $\boldsymbol{P}_*$.

Ad CH4): $\operatorname{Ker}\psi \subset \operatorname{Ker}\overline{\partial}$:

Suppose $lz' + q_\# a = dw$ in $\boldsymbol{P}_*$, $z' = z_B + \widehat{z}'$, $a = a_B + \widehat{a}$, $w = w_B + \widehat{w}$, $(\cdots)_B \in \boldsymbol{B}_*$, $\widehat{(\cdots)} \in cone\ \boldsymbol{A}_*$; then $dz' = e_B + \widehat{e} \in P_*(U)$, $e_B \in B_*(U)$, $\widehat{e} \in cone\ A_*(U)$. We have

$$lz'_B + q_\# a_B - dw_B = -(l\widehat{z}' + q_\#\widehat{a}) + d\widehat{w} = z^1_A,$$

and deduce that $z^1_A \in A_*(X,U)$. Since $dz^1_A \in A_*(X,U) \cap (cone\ A_*(U))$, we conclude $dz^1_A \in A_*(U)$. Hence we obtain an $\overline{a} \in A_*(U,U)$ such that $z_A = z^1_A + q_\#\overline{a}$ is a cycle in $A_*(X,U)$, which according to CH3) for A_*, implies that

$$z_A = lz'_A + q_\# a^1_A + dw_A,$$
$$lz'_B + q_\#(a_B + \overline{a}) - dw_B = z_A$$

and

$$l(z'_B - z'_A) + q_\#(a_B - \overline{a} - a^1_A) = dv_B.$$

According to CH4) for B_* we obtain

$$d(z'_B - z'_A) = i'\ du_B, \quad u_B \in B_*(U)$$

and

$$dz'_A = i'\ u_A = i'\ d\widehat{u}_A, \quad u_A \in A_*(U), \quad \widehat{u}_A \in cone\ A_*(U).$$

As a result

$$dz'_B = i'\ d(u_B - \widehat{u}_A)$$

and

$$dz' = dz'_B + d\widehat{z}' = i'\ d(u_B - \widehat{u}_A + \widehat{x}),$$
$$i' d\widehat{x} = d\widehat{z}'.$$

This confirms the first part of CH4) for $\boldsymbol{P}_*$.

$$\operatorname{Ker}\ j_* \subset \operatorname{Ker}\ p_*\ \kappa_* :$$

Suppose $j_\# z = dw$, $z = z_B + \widehat{z}$, $w = w_B + \widehat{w}$. We are required to find a $u \in P_*(U)$, $x' \in P_*(X,U)$ such that $\kappa z = i'\ u + dx'$. This will be accomplished by changing z several times in its homology class in P_*.

1) We want to change z such that dz_B is not only contained in $A_*(X)$ but in $\operatorname{Im} i_\#$.

To this end we observe that $dz_B \in A_*(X)$, hence $j_\# z_B \in A_*(X,U)$, and conclude $t = dw_B - j_\# z_B \in A_*(X,U)$ and $dw_B \in A_*(X,U)$. So $dt = j_\# dz_B$ and according to CH4) for A_* we obtain

$$\kappa dz_B = i'\, u_A + dx'_A, \quad u_A \in A_*(U), \quad x'_A \in A'(X,U).$$

Application of φ yields

$$dz_B = i_\# u_A + dy_A, \quad y_A \in A_*(X).$$

As a result there exists an $\widehat{x}$ such that

$$z_B - y_A + \widehat{x} = \overline{z}_B \sim z \quad \text{in} \quad P_*,$$

but now with $d\overline{z}_B \in \operatorname{Im} i_\#$.

So we can from now on assume without loss of generality that $dz_B \in \operatorname{Im} i_\#$ and that $\widehat{z}$ is the cone over $d\, z_B$ in *cone* $\boldsymbol{A}_*$.

2) We calculate

$$j_\# z - dw_B = j_\# z_B - dw_B + j_\# \widehat{z} = d\widehat{w} = j_\# \widehat{z} + e,$$

$e \in A_*(X,U)$, $de \in \operatorname{Im} i_\#$. Property CH3) for A_* allows us to assume that $e \in A'_*(X,U)$ hence we can apply φ and obtain a $\overline{e} \in A_*(X)$, such that $j_\#(z_B - \overline{e}) - dw_{1B} = 0$, with suitable $w_{1B} \in B_*(X,U)$. So $z_B - \overline{e} \in B_*(X)$ is a cycle that is in $\boldsymbol{P}_*$ (not in $\boldsymbol{B}_*$) homologous to z: $z - (z_B - \overline{e}) = \widehat{z} + \overline{e}$ is a cycle in *cone* $\boldsymbol{A}_*$, hence bounding in $\boldsymbol{P}_*$.

Property CH4) for B_* yields a $u_B \in B_*(U)$ and a $x'_B \in B'_*(X,U)$ such that $\kappa(z_B - \overline{e}) = i'\, u_B + dx'_B$. This confirms the second part of CH4) for $\boldsymbol{P}_*$.

Ad CH5): Is obvious.

Ad CH6): Suppose $p : (X,U) \longrightarrow (Y,V)$, as an excision map, is an isomorphism in the homology of $\boldsymbol{B}_*$ and $\boldsymbol{A}_*$. Let $\overline{e} \in B_*(X,U)$ be a cycle such that $p_\# \overline{e} = dw$, $w \in B_*(Y,V)$. Then we find a $\overline{w} \in B_*(X,U)$, such that $\overline{e} = d\overline{w}$, $p_\# \overline{w} = w + dx$.

If on the other hand, $e \in B_*(Y,V)$ is a cycle, then there exists a cycle $\overline{e} \in B_*(X,U)$, such that $p_\# \overline{e} = e + dx$. The same is true for A_*

Let $z = z_B + \widehat{z} \in P_*(X,U)$ be a cycle; then $dz_B = -d\widehat{z} \in A_*(X,U)$. Assume now that $p_\# z$ is bounding, hence that $p_\# z = dw_B + d\widehat{w}$. There exists a $t \in A_*(Y,V)$ such that $p_\# z_B - t$ is a bounding cycle. Hence there exists a $\overline{t} \in A_*(X,U)$ such that $z_B - \overline{t}$ is a bounding cycle. This is equivalent to $z \sim 0$. So p_* for P_* is monic.

Let $z = z_B + \widehat{z}$ be a cycle in $P_*(Y,V)$; then we find a $\overline{z}_B \in B_*(X,U)$, such that $p_\#\, d\overline{z}_B \sim dz_B$, i.e., an $s \in A_*(X,U)$ with $ds = p_\# d\overline{z}_B - dz_B$ and a $x \in B_*(X,U)$, such that $dx = p_\# \overline{z}_B - z_B - s$.

This confirms that p_* for $\boldsymbol{P}_*$ is epic.

Ad CH7): Is obvious by construction of the *cone* $\boldsymbol{A}_*$ and $\boldsymbol{P}_*$.

This completes the proof of Theorem 9.2.1. □

There are of course other colimits that exist trivially in $\mathfrak{C}h$:

Lemma 9.2.2. *Any family of chain functors* $\{\boldsymbol{A}^\iota_*,\ \iota \in J\}$ *has a direct sum* $\bigoplus_{\iota \in J} \boldsymbol{A}^\iota_*$ *in* $\mathfrak{C}h$. *This is simply the direct sum of the chain complexes involved, and the other ingredients of a chain functor are taken for each summand.*

9.3. Cokernels of Cofibrations

In this section we deduce the existence of a cokernel $\boldsymbol{B}_*/\boldsymbol{A}_*$ for a cofibration $q : \boldsymbol{A}_* \hookrightarrow \boldsymbol{B}_*$.

Theorem 9.3.1. *Let* $q : \boldsymbol{A}_* \hookrightarrow \boldsymbol{B}_*$ *be a cofibration in* $\mathfrak{Ch}$*; then the chain complex functor* $\boldsymbol{B}_*/\boldsymbol{A}_*$ *can be equipped with the structure of a chain functor, so that* $r : \boldsymbol{B}_* \longrightarrow \boldsymbol{B}_*/\boldsymbol{A}_*$ *is a cokernel.*

Proof. We apply Theorem 9.2.1 ensuring that $\boldsymbol{P}_* = \boldsymbol{B}_* \cup cone\ \boldsymbol{A}_*$ is a chain functor and observe that $s : \boldsymbol{P}_* \longrightarrow \boldsymbol{B}_*/\boldsymbol{A}_*$ induces an isomorphism of homology groups. Since q is a cofibration, there exists, due to Lemma 9.1.2, a splitting $B_* = \overline{A}_n \oplus A_n$ for each n, which is not a splitting of chain complexes. Setting $d\overline{a} = \overline{a}_1 + a_1$, $a_1 \in A_{n-1}$, $\overline{a}_1 \in \overline{A}_n$, $d[\overline{a}] = [\overline{a}_1]$, we endow $\boldsymbol{B}_*/\boldsymbol{A}_*$ with the structure of a free chain functor functorially. Assume that

$$(\boldsymbol{B}_*/\boldsymbol{A}_*)' = \{[b'] \mid b' \in B'_*\}.$$

Since φ, κ and the chain homotopies $\varphi\kappa \simeq 1$, $j_\#\ \varphi \simeq l$ are preserved by q, we obtain induced mappings $\varphi : (B_*/A_*)'_*(X,U) \longrightarrow (B_*/A_*)_*(X)$, $\kappa : (B_*/A_*)_*(X) \longrightarrow (B_*/A_*)'_*(X,U)$, chain homotopies $\varphi\kappa \simeq 1$, $j_\#\varphi \simeq l : (\boldsymbol{B}_*/\boldsymbol{A}_*)' \subset (\boldsymbol{B}_*/\boldsymbol{A}_*)$ as well as a natural $i' : (B_*/A_*)_*(U) \longrightarrow (B_*/A_*)'_*(X,U)$ satisfying $\kappa\, i = i'$. This confirms property CH1).

Ad CH2): An inclusion $k : (X,U) \subset (Y,V)$ induces a monomorphism for $\boldsymbol{B}_*$ and $\boldsymbol{A}_*$, hence for $\overline{A}_*$. Suppose

$$k_\#\overline{a}_1 = k_\#\overline{a}_2 + a, \quad a \in A_*;$$

then $a = k_\#(\overline{a}_1 - \overline{a}_2) = 0$, hence $\overline{a}_1 = \overline{a}_2$.

Suppose $[c] \in (B_*/A_*)_*(X,X)$ is a cycle, hence $dc \in A_*$; then there exists an $a \in A_*$ such that $da = dc$. So $z = c - a$ is a cycle in $(B_*/A_*)_*(X,X)$, hence bounding, confirming that $[c] \sim 0$ in $(B_*/A_*)_*(X,X)$.

Ad CH3): Let $[z] \in (B_*/A_*)_*(X,U)$ be a cycle; then $dz = z_A \in A_{n-1}$, hence by the acyclicity of $cone\ A_*$ we obtain a $\widetilde{x}_A \in cone\ A_*$ with $d\widetilde{x}_A = z_A$. So $z - \widetilde{x}_A$ is a cycle in $\boldsymbol{P}_*$. Condition CH3) for $\boldsymbol{P}_*$ provides us with chains

$$z - \widetilde{x}_A \sim l\ \widetilde{z}' + q_\#\ \widetilde{b} = l\ z' + q_\#\ b + \widehat{c},$$

$\widetilde{(\cdots)} \in P_*$, $\widehat{c} \in cone\ A_*$. So we conclude

$$[z] \sim l\ [z'] + q_\#\ [b],$$

confirming CH3) for $(\boldsymbol{B}_*/\boldsymbol{A}_*)$.

Ad CH4): Suppose $l\ z' + q_\#\ b = dw + a$, $a \in A_*$. Since $da = 0$, there exists $\widetilde{x}_a \in cone\ A_*$ with $d\widetilde{x}_a = a$, hence $l\ z' + q_\#\ b = d(w + \widetilde{x}_a)$. Due to CH4) for $\boldsymbol{P}_*$, we obtain a $\widetilde{u} \in P_*(U)$, such that $dz' = d\widetilde{u} = d(u + \widehat{c})$, $\widehat{c} \in cone\ A_*$. This confirms

$$\text{Ker}\ \psi \subset \text{Ker}\ \overline{\partial}$$

for $(\boldsymbol{B}_*/\boldsymbol{A}_*)$.

Suppose $j_\#\ [c] = d[w]$, $j : X \subset (X,U)$, $da \in A_*$. Then there exists a $\widehat{a} \in cone\ A_*$, such that $d\widehat{a} = dc$. On the other hand $j_\#\ c = dw + a_1$, hence $j_\#\ \widetilde{z} = j_\#\ c - j_\#\ \widehat{a} = dw + a_1 - j_\#\ \widehat{a}$, hence $j_\#\ \widetilde{z} = dw + a$, $\widetilde{z}$ a cycle in P_*. Since a is a cycle, there exists a $\widehat{a}_2 \in cone\ A_*$, such that $d\widehat{a}_2 = a$. Application of CH4) to P_* furnishes

$$\kappa\widetilde{z} = i'\ \widetilde{u} + d\ \widetilde{y}',$$

$\widetilde{y}' \in P'_*(X,U)$, $\widetilde{u} \in P_*(U)$, $\widetilde{u} = u + \widehat{v}$, $\widetilde{y}' = y' + \widehat{y}$. As a result

$$\kappa(c) = i'\ u + dy' + e, \quad e \in cone\ A_*$$

confirming

$$\text{Ker } j_* \subset \text{Ker } p_* \, \kappa_*$$

for $(\boldsymbol{B}_*/\boldsymbol{A}_*)$.

Ad CH5): Is obvious.

Ad CH6): Holds for $\boldsymbol{P}_*$, hence for $(\boldsymbol{B}_*/\boldsymbol{A}_*)$ because s_* is an isomorphism of homology groups.

CH7): Follows, as already mentioned, from the natural splitting of $\boldsymbol{B}_*$ on each level n. □

The following assertion confirms the existence of a certain pushout. It will become crucial for the verification of the Quillen axiom SM7 for the model structure under discussion [8].

Theorem 9.3.2. *Let $q_1 : \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$, $q_2 : \boldsymbol{A}_* \longrightarrow \boldsymbol{C}_*$ be cofibrations in $\mathfrak{Ch}$; then $\boldsymbol{B}_* \cup_{\boldsymbol{A}_*} \boldsymbol{C}_* = \boldsymbol{D}_*$ is a chain functor.*

Proof. $\boldsymbol{K}_* = \{(a, -a) \mid \ a \in A_*\} \subset \boldsymbol{B}_* \oplus \boldsymbol{C}_*$ can immediately be equipped with the structure of a chain functor. We assert:

(1) *The inclusion $\alpha : \boldsymbol{K}_* \subset \boldsymbol{A}_* \oplus \boldsymbol{A}_*$ is a cofibration.*

Proof. We must display a natural retraction of $K_n \subset A_n \oplus A_n$ on each level $n \in \mathbb{Z}$, which respects A'_*: Each $(a_1, a_2) \in A_n \oplus A_n$ can be written as $(a_1, a_2) = (a_1, -a_1) + (0, a_2 + a_1)$. This defines the splitting, having all required properties.

The inclusion $q_1 \oplus q_2 : \boldsymbol{A}_* \oplus \boldsymbol{A}_* \subset \boldsymbol{B}_* \oplus \boldsymbol{C}_*$ is a cofibration, because q_1, q_2 are. Therefore:

(2) *The inclusion $i : \boldsymbol{K}_* \subset \boldsymbol{B}_* \oplus \boldsymbol{C}_*$ is a cofibration.*

Thus we can apply Theorem 9.3.1 to the result that $(\boldsymbol{B}_* \oplus \boldsymbol{C}_*)/\boldsymbol{K}_* = \text{Coker } i$ is a chain functor (hence contained in $\mathfrak{Ch}$). However, $(\boldsymbol{B}_* \oplus \boldsymbol{C}_*)/\boldsymbol{K}_* = \boldsymbol{B}_* \cup_{\boldsymbol{A}_*} \boldsymbol{C}_*$. □

Corollary 9.3.3. *Let q_1, q_2 be as in Theorem 9.3.2; then $(\boldsymbol{B}_* \oplus \boldsymbol{C}_*) \cup_{\boldsymbol{K}_*}$ cone $\boldsymbol{K}_*$ is an object of $\mathfrak{Ch}$.*

Proof. Follows from (2) and Theorem 9.2.1. □

We call a pair of functors C_*, C'_* into the category of chain complexes with inclusion $l : C'_* \subset C_*$ having all ingredients of a chain functor (without knowing that they fulfill CH1) – CH7) of Sec. 9.7) a *chain complex functor*.

Corollary 9.3.4. *Let $\boldsymbol{A}_*$, $\boldsymbol{B}_*$, $\boldsymbol{A}_* \oplus \boldsymbol{B}_*$ be three chain complex functors. If two of them are chain functors, so is the third.*

Proof. Follows from Lemma 9.2.2 and Theorem 9.3.1. □

9.4. Existence of a Particular Pullback

We will call a mapping $p \in \mathfrak{Ch}(\boldsymbol{E}_*, \boldsymbol{B}_*)$ a regular fibration if it is a regular mapping (see Sec. 9.1) and a fibration.

Let $\boldsymbol{W}_*$ be the pullback in the following diagram.

$$\begin{array}{ccc} \boldsymbol{W}_* & \xrightarrow{\mu} & {\boldsymbol{B}_*}^I \\ \zeta \downarrow & & \downarrow p_0 \\ \boldsymbol{E}_* & \xrightarrow[p]{} & \boldsymbol{B}_* \end{array} \quad . \tag{9.4.1}$$

We have $W_* = \{(e; p(e), b_1, x)\} = \{(e, \omega) \mid \ \omega(0) = p(e)\}$, of course with $e \in E_n$, $(b, b_1, x) \in B_n \oplus B_n \oplus B_{n+1}$, where ω is considered as a path with endpoints $\omega(0) = b$, $\omega(1) = b_1$ (cf. [6, § 1 (7)] concerning the description of path objects).

Theorem 9.4.1. *Let $p \in \mathfrak{Ch}$ be a regular mapping of chain functors; then $\boldsymbol{W}_*$ carries the structure of a chain functor.*

Proof. Since p commutes with κ, φ and the relevant chain homotopies, we have all these items also for $\boldsymbol{W}_*$. Moreover, $\boldsymbol{W}'_*$, l, and i' are obvious. All inclusions induce monomorphisms, and $W_*(X,X)$ is acyclic: Let $(e;p(e),b_1,x) \in W_*(X,X)$ be a cycle; then $e = d\overline{e}$, $b_1 = d\overline{b}_1$, $dx = (-1)^{n+1}(p(e)-b_1)$, hence

$$z = x + (-1)^n(p(\overline{e}) - \overline{b}_1)$$

is a cycle in $B_*(X,X)$, hence bounding. So there exists a $\overline{x}$ with $d\overline{x} = z$, implying

$$d(\overline{e};p(\overline{e}),\overline{b}_1,\overline{x}) = (e;p(e),b_1,x).$$

This confirms CH1) and CH2).

Let $z = (e;p(e),b_1,x) \in W_*$ be a cycle; then $de = 0$, $db_1 = 0$, $dx = (-1)^n(p(e)-b_1)$. We have $e \sim l\ e' + q_\#\ a_E$, $b_1 \sim l\ b'_1 + q_\#\ a_1$, $dw + x = l\ x' + q_\#\ a$, the last according to Lemma 9.7.5. So we obtain

$$z \sim l(e';p(e'),b'_1,x') + q_\#\ (a_E;p(a_E),a_1,a) + d(0;0,0,w).$$

This confirms CH3).

We come to the two parts of CH4):

$$\text{Ker } \psi \subset \text{Ker } \overline{\partial}$$

Suppose

$$d(\overline{e};p(\overline{e}),\overline{b}_1,\overline{x}) = l(e';p(e'),b'_1,x') + q_\#\ (a_E;p(a_E),a_B,a)$$

then $d\overline{e} = l\ e' + q_\#\ a_E$, $d\overline{b}_1 = l\ b'_1 + q_\#\ a_B$,

$$d\overline{x} - (-1)^n(\overline{b}_1 - p(\overline{e})) = l\ x' + q_\#\ a = x + dw.$$

Therefore, $de' = de_A$, $e_A \in E_*(U)$, $db'_1 = db_U$, $b_U \in B_*(U)$. According to Lemma 9.7.5, $l\ \overline{e}' + q_\#\ \overline{a} = \overline{e} + dw_e$, $d\overline{b}'_1 + q_\#\ \overline{a}_B = \overline{b}_1 + dw_1$,

$$d(\overline{x} - w) = x + (-1)^n(p(\overline{e}) - \overline{b}_1) = l(x' + (-1)^n(p(\overline{e}') - \overline{b}'_1)) + q_\#\ b, \quad b \in B_*(U,U),$$

hence there exists a $y \in B_*(U)$ satisfying

$$dy = dx' + (-1)^n(p(e') - b'_1),$$
$$d(e_A;p(e_A),b_u,y) = d(e';p(e'),b'_1,x').$$

This confirms the first part of CH4),

$$\text{Ker } j_* \subset \text{Ker } p_*\ \kappa_*.$$

Suppose

$$j_\#\ (e;p(e),b_1,x) = d(\overline{e};p(\overline{e}),u,w).$$

First we observe that we have an embedding $W_*(\cdot) \subset (E_* \oplus {B_*}^I)(\cdot)$. Let $\alpha = (e;b,b_1,x) \in (E_* \oplus {B_*}^I)(\cdot)$ be any element; then we can write

$$\alpha = (e;p(e),b_1,x) + (0;b-p(e),0,0) = \widehat{\alpha} + \alpha_0, \quad \widehat{\alpha} \in W_*.$$

We know that $(\boldsymbol{E}_* \oplus \boldsymbol{B}^I)$ is a chain functor, hence the derived homology sequence is exact. So, if $j_\#\ z = dw$ in W_* we find elements $\widetilde{u} \in (E_* \oplus {B_*}^I)_*(U)$, $\widetilde{x} \in (E_* \oplus {B_*}^I)_*(X)$ satisfying $z = i\ \widetilde{u} + d\widetilde{x} = i_\#\ u + dx + i_\#\ u_0 + dx_0$. So

$$z = (e_z;p(e_z),b_{1z},x_z), \quad u = (e_u;p(e_u),b_{1u},x_u), \quad dx = (e_x;p(e_x),b_{1x},v_x)$$

Setting $z - i_\#\ u - dx = \rho = i_\#\ u_0 + dx_0 = (0;c,0,0)$ gives

$$e_z - i_\#\ e_u - e_x = 0;$$

but

$$p(e_z + i_\# \, e_u - e_x) = c = 0.$$

Therefore $z = i_\# \, u + dx$. By applying κ we obtain

$$\kappa z = i' \, u + d\kappa x$$

which was the assertion. This confirms CH4).

All remaining properties of a chain functor are left to the reader, because their proofs are either technical or they are immediate. □

We will need another pullback, which is dual to $\boldsymbol{B}_* \cup \mathit{cone}\ \boldsymbol{A}_*$. To this end define $\boldsymbol{A}_*{}^{(I,0)} \subset \boldsymbol{A}_*{}^I$ consisting of those paths with fixed endpoint 0, i.e., $\boldsymbol{A}_*{}^{(I,0)} = \{\omega \mid \omega(0) = 0\} = \{(0,c,x)\} = \mathrm{Ker}\ p_0$, $(0,c,x) \in A_n \oplus A_n \oplus A_{n+1}$ using the terminology of [6, § 1]. We have a mapping $\pi : \boldsymbol{A}_*{}^{(I,0)} \longrightarrow \boldsymbol{A}_*$, by taking the endpoint of a path, i.e., $\pi(0,c,x) = c$. This is immediately seen to be a mapping of chain functors. The fact that $\boldsymbol{A}_*{}^{(I,0)}$ carries the structure of a chain functor is left to the reader. All structural maps are inherited from $\boldsymbol{A}_*{}^I$. We obtain again a pullback diagram

$$\begin{array}{ccc} \boldsymbol{W}_* & \xrightarrow{\mu} & \boldsymbol{B}_*{}^{(I,0)} \\ \zeta \big\downarrow & & \big\downarrow \pi \\ \boldsymbol{E}_* & \xrightarrow[p]{} & \boldsymbol{B}_* \end{array} \quad . \tag{9.4.2}$$

and realize that

$$W_* = \big\{(e;0,p(e),x)\big\} = \Big\{(e;\omega) \mid \omega(0) = 0,\ \omega(1) = p(e)\Big\}.$$

We claim:

Theorem 9.4.2. *Let $p \in \mathfrak{Ch}$ be a regular mapping of chain functors; the pullback $\boldsymbol{W}_*$ in* (9.4.2) *carries the structure of a chain functor.*

Proof. The proof consists in a precise repetition of the proof of Theorem 9.4.1 in this new situation, where an element has now the form $(e;0,p(e),x)$. That makes the steps even simpler than in the proof of Theorem 9.4.1. □

We need more informations about $\boldsymbol{B}_*{}^{(I,0)}$:

Lemma 9.4.3. $\boldsymbol{B}_*{}^{(I,0)}$ *is acyclic.*

Proof. There are natural chain homotopies $0 \simeq 1 : \boldsymbol{B}_*{}^{(I,0)} \longrightarrow \boldsymbol{B}_*{}^{(I,0)}$ that are standard for chain complexes, carrying over immediately to the structure of a chain functor, i.e., they commute with l and i'. □

There is another description of $\boldsymbol{W}_*$ in Theorem 9.4.2. We know already that $\boldsymbol{W}_*$ is a chain functor and that $\boldsymbol{B}_*{}^{(I,0)}$ is a chain functor; we do not know yet that Ker p for a regular fibration p is a chain functor, although we are able to define

$$(\mathrm{Ker}\ p)' = \mathrm{Ker}\ p \cap E'_*, \quad l : (\mathrm{Ker}\ p)'(X,U) \subset (\mathrm{Ker}\ p)(X,U), \quad i' : (\mathrm{Ker}\ p)(U) \subset (\mathrm{Ker}\ p)'(X,U)$$

as well as κ, φ, and the associated chain homotopies, which are inherited from E_*.

Therefore we formulate the following lemma for *chain complex functors*:

Lemma 9.4.4. *Let $\boldsymbol{W}_*$ be the pullback in the diagram* (9.4.2)*, where p is a regular fibration. There is an isomorphism of chain complex functors*

$$\boldsymbol{W}_* \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Ker}\ p \oplus \boldsymbol{B}_*{}^{(I,0)}. \tag{9.4.3}$$

These mappings α and β commute with l and i' (i.e., they are candidates for mappings of chain functors).

Proof. Instead of a purely categorical proof, using the general properties of chain complex functors, we present a direct proof that describes the relevant mappings explicitly. We have a commutative diagram

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{\;p\;} & \boldsymbol{B}_* \\ {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle \pi} \\ {\boldsymbol{B}_*}^0 & \xrightarrow[\;q\;]{} & {\boldsymbol{B}_*}^{(I,0)} \end{array} , \tag{9.4.4}$$

where $\boldsymbol{B}_*^0 = 0$ denotes simply the zero path in $\boldsymbol{B}_*$, and q is a trivial cofibration (observe that, according to Lemma 9.4.3, ${\boldsymbol{B}_*}^{(I,0)}$ is contractible). So there exists a lifting $\overline{\pi} : {\boldsymbol{B}_*}^{(I,0)} \longrightarrow \boldsymbol{E}_*$, rendering (9.4.4) commutative. We set $\overline{\pi}(\omega) = e_\omega$, $p(e_\omega) = \pi(\omega)$ and observe that $de_\omega = e_{d\omega}$, $e_{\omega_1+\omega_2} = e_{\omega_1} + e_{\omega_2}$, and that for $f : (X, U) \longrightarrow (Y, B)$, $f_\#\, e_\omega = e_{f_\#\omega}$.

Now we are able to define

$$\alpha(e, \omega) = (e - e_\omega, \omega), \quad \beta(x, \omega) = (x + e_\omega, \omega).$$

It is easy to verify that α, β display the compatibility properties mentioned in the assertion and that $\beta\,\alpha = 1$, $\alpha\,\beta = 1$. This completes the proof of the lemma. □

9.5. Kernels of Regular Fibrations

The kernel of any regular fibration can be endowed with the structure of a chain functor. Hence:

Theorem 9.5.1. *Every regular fibration p has a kernel*

$$\text{Ker } p \longrightarrow \boldsymbol{E}_* \xrightarrow{\;p\;} \boldsymbol{B}_* .$$

Proof. We have already equipped Ker p with the structural ingredients of a chain functor (cf. Lemma 9.4.4). The assertion now follows from Corollary 9.3.4, Theorem 9.4.2, and Lemma 9.4.4. □

The following theorem is the perfect dual to Lemma 9.1.2 for regular fibrations.

Theorem 9.5.2. *The following properties of a regular mapping $p : \boldsymbol{E}_* \longrightarrow \boldsymbol{B}_*$ are equivalent:*

1) *p is a regular fibration.*
2) *Each commutative square*

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{\;p\;} & \boldsymbol{B}_* \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ \boldsymbol{K}_* & \xrightarrow[\;i_0\;]{} & \boldsymbol{K}_* \times I \end{array} , \qquad \boldsymbol{K}_* \in \mathfrak{Ch} \tag{9.5.1}$$

admits a diagonal $\overline{F} : \boldsymbol{K}_* \times I \longrightarrow \boldsymbol{E}_*$ rendering the square commutative.

3) *Let $\boldsymbol{W}_*$ be the pullback of Theorem 9.4.1, $j : {\boldsymbol{E}_*}^I \longrightarrow \boldsymbol{W}_*$, $j(\nu) = (\nu(0), p^I(\nu))$; then there exists a $s : \boldsymbol{W}_* \longrightarrow {\boldsymbol{E}_*}^I$, with $j\ s = 1$.*
4) *p induces a levelwise isomorphism $E_n \approx B_n \oplus \overline{B}_n$, which is natural and commutes with l and i' (i.e., $E'_n \approx B'_n \oplus \overline{B}'_n$); thus p is surjective and has a natural levelwise section which commutes with l and i'.*

Proof. The equivalence of 1) and 2) (even for non-regular p) is the subject of Lemma 9.1.3.

1) $\Longrightarrow$ 3) : Observe firstly that j (as originating from a pullback diagram for $\boldsymbol{W}_*$) is completely determined by $p_0 = \zeta\ j,\ p_0 : \boldsymbol{E}_*^I \longrightarrow \boldsymbol{E}_*,\ \mu\ j = p^I$. Now consider the commutative diagram

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{\ p\ } & \boldsymbol{B}_* \\ {\scriptstyle \zeta}\uparrow & & \uparrow{\scriptstyle \widetilde{\mu}} \\ \boldsymbol{W}_* & \xrightarrow[i_0]{} & \boldsymbol{W}_* \times I \end{array} \tag{9.5.2}$$

which has a lifting $\widetilde{s} : \boldsymbol{W}_* \times I \longrightarrow \boldsymbol{E}_*$, hence the adjoint $s : \boldsymbol{W}_* \longrightarrow \boldsymbol{E}_*{}^I$. Since $p\widetilde{s} = \widetilde{\mu}$ we deduce $p^I\ s = \mu$. Since $\widetilde{s}\, i_0 = p_0\ s$ we deduce $p_0\ s = \zeta$, $p_0 : \boldsymbol{E}_*^I \longrightarrow \boldsymbol{E}_*$. So $\zeta js = \zeta$, $\mu js = \mu$, implying $js = 1$.

3) $\Longrightarrow$ 1) : Suppose that

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{\ p\ } & \boldsymbol{B}_* \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ \boldsymbol{C}_* & \xrightarrow[i_0]{} & \boldsymbol{C}_* \times I \end{array} \quad , \qquad \boldsymbol{C}_* \in \mathfrak{C}h \tag{9.5.3}$$

is commutative. We want to deduce the existence of a diagonal $\overline{F} : \boldsymbol{C}_* \times I \longrightarrow \boldsymbol{E}_*$ from the existence of an $s : \boldsymbol{W}_* \longrightarrow \boldsymbol{E}_*{}^I$ with $js = 1$. Let $\widetilde{F} : \boldsymbol{C}_* \longrightarrow \boldsymbol{B}_*^I$ be a mapping of chain functors adjoint to F. The pair $(f, \widetilde{F})$ defines a unique mapping $h : \boldsymbol{C}_* \longrightarrow \boldsymbol{W}_*$ with $\mu h = \widetilde{F}$, $\zeta h = f$. Let $\overline{F} = \overline{sh} : \boldsymbol{C}_* \times I \longrightarrow \boldsymbol{E}_*$ be the adjoint of $sh : \boldsymbol{C}_* \longrightarrow \boldsymbol{E}_*^I$. We must show that $p\overline{F} = F$, $\overline{F} i_0 = f$. From the naturality of the adjunction isomorphism

$$\mathrm{Hom}(\boldsymbol{C}_* \times I, \boldsymbol{E}_*) \approx \mathrm{Hom}(\boldsymbol{C}_* . \boldsymbol{E}_*^I),$$

it follows that $p\ \overline{sh} = F$ is equivalent to $p^I\ sh = \widetilde{F}$, which holds because $p^I\ s = \mu$ and $\mu\ h = \widetilde{F}$. On the other hand, the adjunction isomorphism yields $\overline{sh}\ i_0 = p_0^E\ sh$, implying

$$\overline{F}\ i_0 = \overline{sh}\ i_0 = p_0^E\ sh = \zeta jsh = \zeta h = f.$$

3) $\Longrightarrow$ 4) : Recall that

$$W_n = \{(e; p(e), b, y)\}$$

and that there is a projection $\mu : W_n \longrightarrow B^I{}_n$ so that $\mu j = p^I$. We want to construct a $t_n : B_n \longrightarrow E_n$, such that $p_n\ t_n = 1$. So we define $\alpha_n : B_n \longrightarrow W_n$ by

$$\alpha_n(b_n) = (0; 0, b_n, 0),$$

which is functorial, compatible with l, but *not with boundaries.* Set

$$t_n(b_n) = p_1\ s_n\ \alpha_n(b_n)$$

It is now immediate to verify that $p_n\ t_n = 1$ and that t_n has the same compatibility properties like α_n.

4) $\Longrightarrow$ 3) : Suppose we have a $t_n : B_n \longrightarrow E_n$ as before, then we construct $s : \boldsymbol{W}_* \longrightarrow \boldsymbol{E}_*{}^I$ with $js = 1$ by setting

$$s_n(e_n; p(e_n), b_n, b_{n+1}) =$$
$$= (e_n, e_n - t_n\ p_n(e_n) + (-1)^{n+1}\big[d(t_{n+1}(b_{n+1})) - t_n\ d(b_{n+1})\big] + t_n(b_n), t_{n+1}(b_{n+1})).$$

It turns out that 1) $s \in \mathfrak{C}h(\boldsymbol{W}_*, \boldsymbol{E}_*{}^I)$ and 2) $js = 1_{\boldsymbol{W}_*}$. The calculations asserting this fact are straightforward. $\square$

Corollary 9.5.3.

1) *Let* $p : \boldsymbol{E}_* \longrightarrow \boldsymbol{B}_*$ *be a regular mapping, admitting a section* $s : \boldsymbol{B}_* \longrightarrow \boldsymbol{E}_*$, $ps = 1$; *then* p *is a fibration.*

2) *Let* $q : \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$ *be a regular mapping and* $r : \boldsymbol{B}_* \longrightarrow \boldsymbol{A}_*$ *be such that* $rq = 1$; *then* q *is a cofibration.*

Proof. Follows from 9.5.2. 4) resp. 9.1.2. 3). □

We have, concerning j in Theorem 9.5.2. 3),

Corollary 9.5.4. *If* p *is a regular fibration, then the mapping* j *is a regular fibration.*

Proof. Follows because if p is a fibration, j has a section. The regularity of j is immediate. □

Theorem 9.3.2 has of course a dual:

Theorem 9.5.5. *Let* $\pi_i : \boldsymbol{E}_{i*} \longrightarrow \boldsymbol{B}_*$, $i = 1, 2$ *be regular fibrations, then there exists in* $\mathfrak{Ch}$ *a pullback*

$$\begin{array}{ccc} \boldsymbol{P}_* & \longrightarrow & \boldsymbol{E}_{1*} \\ \downarrow & & \downarrow \pi_1 \\ \boldsymbol{E}_{2*} & \xrightarrow[\pi_2]{} & \boldsymbol{B}_* \end{array} . \tag{9.5.4}$$

Proof. The proof is completely dual to that of Theorem 9.3.2: One has to realize that $\boldsymbol{P}_*$ is the kernel of a regular fibration

$$\boldsymbol{P}_* \longrightarrow \boldsymbol{E}_{1*} \oplus \boldsymbol{E}_{2*} \xrightarrow{\pi_1 \oplus \pi_2} \boldsymbol{B}_* \oplus \boldsymbol{B}_* \xrightarrow{r} \boldsymbol{B}_* ,$$

with $r(b_1, b_2) = b_1 - b_2$. □

9.6. Exact Sequences in $\mathfrak{Ch}$

Definition 9.6.1. We call a sequence

$$\boldsymbol{A}_* \xrightarrow{\alpha} \boldsymbol{B}_* \xrightarrow{\beta} \boldsymbol{C}_* \tag{9.6.1}$$

in $\mathfrak{Ch}$ *exact* whenever the following two sequences of chain complexes:

$$A_* \xrightarrow{\alpha} B_* \xrightarrow{\beta} C_* , \tag{9.6.2}$$

$$A'_* \xrightarrow{\alpha'} B'_* \xrightarrow{\beta'} C'_* \tag{9.6.3}$$

are exact.

As a result of the characterizations of cofibrations and regular fibrations, we have:

Lemma 9.6.2. *Suppose that*

$$0 \longrightarrow \boldsymbol{A}_* \xrightarrow{\alpha} \boldsymbol{B}_* \xrightarrow{\beta} \boldsymbol{C}_* \longrightarrow 0 \tag{9.6.4}$$

is exact (i.e., exact in the sense of Definition 9.6.1 *at* $\boldsymbol{A}_*$, $\boldsymbol{B}_*$ *and* $\boldsymbol{C}_*$*); then:*

1) α *is a cofibration, if and only if* β *is a regular fibration.*

2) $\boldsymbol{C}_*$ *is the cokernel of* α *and* $\boldsymbol{A}_*$ *is the kernel of* β.

Proof. 1) is a consequence of the characterization of (co-)fibrations in Lemma 9.1.2 3) (resp. Theorem 9.5.2 4). Assertion 2) is straightforward. □

Definition 9.6.3. The exact sequence (9.6.4) *splits* whenever there exists an isomorphism $f : \boldsymbol{B}_* \xrightarrow{\approx} \boldsymbol{A}_* \oplus \boldsymbol{C}_*$, such that $f\,\alpha = i_{\boldsymbol{A}_*} : \boldsymbol{A}_* \longrightarrow \boldsymbol{A}_* \oplus \boldsymbol{C}_*$, $f = i_{\boldsymbol{C}_*}\,\beta$, where $i_{\boldsymbol{A}_*}$, $i_{\boldsymbol{C}_*} : \boldsymbol{C}_* \longrightarrow \boldsymbol{A}_* \oplus \boldsymbol{C}_*$ are coproduct injections.

Proposition 9.6.4. *The following properties of the exact sequence* (9.6.4) *are equivalent:*

1) *There exists a retraction* $\lambda : \boldsymbol{B}_* \longrightarrow \boldsymbol{A}_*$ *satisfying* $\lambda\alpha = 1$.
2) *There exists a section* $s : \boldsymbol{C}_* \longrightarrow \boldsymbol{B}_*$ *satisfying* $\beta s = 1$.
3) *The sequence* (9.6.4) *is splitting.*

Proof. The proof follows entirely the pattern of the classical proof in homological algebra (cf. [86]). □

Proposition 9.6.5. *Let* $q : \boldsymbol{A}_* \longrightarrow \boldsymbol{B}_*$ *be a cofibration*; *then there exists an isomorphism of chain functors*

$$\mathit{cone}\ \boldsymbol{A}_* \oplus \operatorname{Coker}\ q \approx \boldsymbol{B}_* \cup_{\boldsymbol{A}_*} \mathit{cone}\ \boldsymbol{A}_*.$$

Proof. We consider the exact sequence in $\mathfrak{C}h$

$$0 \longrightarrow \mathit{cone}\ \boldsymbol{A}_* \xrightarrow{i} \boldsymbol{B}_* \cup_{\boldsymbol{A}_*} \mathit{cone}\ \boldsymbol{A}_* \xrightarrow{p} \boldsymbol{B}_*/\boldsymbol{A}_* \longrightarrow 0\ , \tag{9.6.5}$$

with cofibration i (cf. Lemma 9.6.2 2)) which splits:

The commutative square with trivial fibration on the top

$$\begin{array}{ccc} \mathit{cone}\ \boldsymbol{A}_* & \longrightarrow & 0 \\ {\scriptstyle 1}\uparrow & & \uparrow \\ \mathit{cone}\ \boldsymbol{A}_* & \xrightarrow[i]{} & \boldsymbol{B}_* \cup_{\boldsymbol{A}_*} \mathit{cone}\ \boldsymbol{A}_* \end{array} \tag{9.6.6}$$

has a diagonal $\varrho : \boldsymbol{B}_* \cup_{\boldsymbol{A}_*} \mathit{cone}\ \boldsymbol{A}_* \longrightarrow \mathit{cone}\ \boldsymbol{A}_*$. The result follows from Definition 9.6.3 and Proposition 9.6.4 3). □

9.7. Chain Functors and Associated Homology Theories

In this appendix we present for the convenience of the reader some material about the definition and the motivation of chain functors without proofs. Concerning details as well as further references, we refer to [4].

It would be advantageous to define a homology theory $h_*(\ )$ as the derived homology of a functor

$$C_* :\ \mathfrak{K} \longrightarrow \boldsymbol{ch};$$

$\mathfrak{K} =$ the category on which h_* is defined. For us this will be always either a subcategory of the category of all pairs of topological spaces, or of pairs of spectra or of pairs of CW spaces, of CW spectra, or their simplicial counterparts. $\boldsymbol{ch}$ denotes the category of chain complexes (i.e., $C_* = \{C_n,\ d_n,\ n \in \mathbb{Z},\ d^2 = 0\} \in \boldsymbol{ch}$).

Let $(X, A) \in \mathfrak{K}$ be a pair; then one would like to have an exact sequence (writing $C_*(X)$ instead of $C_*(X, \varnothing)$)

$$0 \longrightarrow C_*(A) \xrightarrow{i_\#} C_*(X) \xrightarrow{j_\#} C_*(X, A) \longrightarrow 0 \tag{9.7.1}$$

such that the associated boundary $\overline{\partial} : H_n(C_*(X, A)) \longrightarrow H_{n-1}(C_*(A))$ induces the boundary $\partial : h_n(X, A) \longrightarrow h_{n-1}(A)$ of the homology theory $h_*(\ )$.

In accordance with [5] we call a homology with this property *flat.* Due to a result of R. O. Burdick, P. E. Conner, and E. E. Floyd (see [4] or [3] for further reference) this implies, for $\mathfrak{K} =$ category of CW pairs, that $h_*(\ )$ is a sum of ordinary homology theories, i.e., of those satisfying a dimension axiom, although not necessarily in dimension 0.

We call a functor C_* equipped with a short exact sequence (9.7.1), which determines the boundary operator, a *chain theory* for h_*. The non-existence of such a chain theory gives rise to the theory of chain functors.

A chain functor $\boldsymbol{C}_* = \{C_*, C'_*, l, i', \kappa, \varphi\}$ is a pair of functors $C_*,\ C'_* : \mathfrak{K} \longrightarrow \boldsymbol{ch}$, natural inclusions $i' : C_*(A) \subset C'_*(X,A)$, $l : C'_*(X,A) \subset C_*(X,A)$, and non-natural chain mappings

$$\varphi : C'_*(X,A) \longrightarrow C_*(X),$$
$$\kappa : C_*(X) \longrightarrow C'_*(X,A),$$

satisfying conditions **CH1)** – **CH7)** below:

CH1) *There exist (of course in general non-natural) chain homotopies* $\varphi\kappa \simeq 1$, $j_\#\ \varphi \simeq l \quad (j : X \subset (X,A))$, *as well as an identity*

$$\kappa\ i_\# = i', \quad i :\ A \subset X.$$

CH2) *All inclusions* $k :\ (X,A) \subset (Y,B)$ *are assumed to induce monomorphisms on* C_*. *All* $C_*(X,X)$ *are acyclic.*

It should be observed that the chain complexes $C_*(X,A)$ appearing in (9.7.1) are not identical with the chain complexes $C_*(X,A)$ appearing in a chain functor. The latter have the property that for all pairs (X,A) one has inclusions $C_*(X) = C_*(X,\varnothing) \subset C_*(X,A) \subset C_*(X,X)$. These groups cannot be members of a short exact sequence (9.7.1).

Needless to say, we have that C'_*, as well as ϕ, κ are *not* determined by the functor $C_*(\cdots,\cdots)$ but are additional ingredients of the structure of a chain functor.

Instead of the exact sequence (9.7.1), which we have for *chain theories* in the case of a *chain functors*, we are dealing with the sequence

$$0 \longrightarrow C_*(A) \xrightarrow{i'} C'_*(X,A) \xrightarrow{p} C'_*(X,A)/im\ i' \longrightarrow 0 \tag{9.7.2}$$

and there exists a homomorphism

$$\begin{aligned} \psi :\ H_*(C'_*(X,A)/\operatorname{Im} i') &\longrightarrow H_*(C_*(X,A)) \\ [z'] &\longmapsto [l(z') + q_\#(\overline{a})] \end{aligned} \tag{9.7.3}$$

where $z' \in C'_*(X,A)$, $dz' \in im\ i'$, $q : (A,A) \subset (X,A)$, $\overline{a} \in C_*(A,A)$, $d\overline{a} = -dz'$. By this assignment, ψ is readily defined.

CH3) *It is assumed that* ψ *is epic.*

Since $C_*(A,A)$ is acyclic and $dz' \in im\ i'$, there exists an $\overline{a}$ with $q_\#(\overline{a}) = -dl(z')$, and $[l(z') + q_\#(\overline{a})]$ turns out to become independent of the choice of $\overline{a}$.

This assumption implies that each cycle $z \in C_*(X,A)$ is homologous to a cycle of the form $l(z') + q_\#(\overline{a})$, with z' being a *relative* cycle, the analogue of a classical relative cycle $z \in C_*(X)$ with $dz \in im\ i_\#$, whenever (9.7.1) holds, i.e., whenever we are dealing with a chain theory.

Suppose $\overline{\partial} : H_n(C'_*(X,A)/im\ i') \longrightarrow H_{n-1}(C_*(A))$ is the boundary induced by the exact sequence (9.7.2).

CH4) *We assume*

$$\text{Ker}\ \ \psi \subset \text{Ker}\ \ \overline{\partial}, \tag{9.7.4}$$

Moreover,

$$\text{Ker } j_* \subset \text{Ker } p_* \, \kappa_*, \tag{9.7.5}$$

with κ_* denoting the homomorphism induced by κ for the homology groups; *j_* and p_* have an analogous meaning.*

CH5) *Homotopies $H : (X, A) \times I \longrightarrow (Y, B)$ induce chain homotopies $D(H) : C_*(X, A) \longrightarrow C_{*+1}(Y, B)$ naturally and are compatible with i' and l.*

The derived (or associated) homology of a chain functor

$$h_*(X, A) = H_*(C_*(X, A)),$$

resp. for the induced mappings, is endowed with a boundary operator $\partial : H_n(C_*(X, A)) \longrightarrow H_{n-1}(C_*(A))$, determined by $\overline{\partial}$:

Given $\zeta \in H_n(C_*(X, A))$, we choose a lift z', which exists by **CH3)**, and a representative $l(z') + q_\#(\overline{a}) \in \zeta$, and set

$$\partial \, \zeta = \overline{\partial}[z'] = [i'^{-1} \, d \, z'].$$

This turns out to be independent of the choices involved.

This $h_*(\ )$ satisfies all properties of a homology theory eventually with the exception of an excision. Let us assume that in $\mathfrak{K}$ there are some mappings $p : (X, A) \longrightarrow (X', A')$ serving as *excision maps* (of some kind, e.g., $p : (X, A) \longrightarrow (X/A, \star)$). Then it is convenient to add:

CH6) *Let p be an excision map*; *then $p_* = H_*(C_*(p))$ is required to be an isomorphism.*

This $H_*(C_*(\ )) = h_*(\ )$ turns out to be a homology theory. Moreover, under very general conditions on $\mathfrak{K}$, every homology theory $h_*(\ )$ is isomorphic to the derived homology of some chain functor (see [4] for further references).

Let $\lambda : C_* \longrightarrow L_*$, $\lambda' : C'_* \longrightarrow L'_*$ be natural transformations, where $\boldsymbol{C}_*$, $\boldsymbol{L}_*$ are chain functors, compatible with i', l and the natural homotopies of **CH5)**; then we call $\lambda : \boldsymbol{C_*} \longrightarrow \boldsymbol{L_*}$ a *mapping* or *a transformation of chain functors.* Such a transformation induces obviously a transformation $\lambda_* : H_*(\boldsymbol{C_*}) \longrightarrow H_*(\boldsymbol{L_*})$ of the derived homology. This furnishes a category $\mathfrak{Ch}$ of chain functors. A *weak equivalence* in $\mathfrak{Ch}$ is a $\lambda : \boldsymbol{C_*} \longrightarrow \boldsymbol{L_*}$ which has a homotopy inverse.

Furthermore, we can introduce the homotopy category $\mathfrak{Ch}_h$ with chain homotopy classes of transformations of chain functors as morphisms.

In order to establish all this, it becomes necessary sometimes to assume that a chain functor $\boldsymbol{C_*}$ satisfies:

CH7) *All chain complexes $C_*(X, A)$ are free (i.e., all $C_n(X, A)$ are free abelian groups) with natural basis $\boldsymbol{b}$.*

However, this is not a severe restriction as the following lemma ensures:

Lemma 9.7.1 ([6, Lemma 9.1])**.** *To any chain functor $\boldsymbol{C_*}$ (satisfying **CH1)- CH6)**) there exists a canonically defined chain functor $\boldsymbol{L_*}$ and a transformation of chain functors $\lambda : \boldsymbol{L_*} \longrightarrow \boldsymbol{C_*}$ compatible with φ and κ, inducing an isomorphism of homology, such that:*

L1) *All $L_*(X, A)$ have a natural basis $\boldsymbol{b}$ in all dimensions;*

L2) *$b \in \boldsymbol{b} \Longrightarrow db \in \boldsymbol{b}$; $b \in \boldsymbol{b} \Longrightarrow i'(b) \in \boldsymbol{b}$, $l(b) \in \boldsymbol{b}$, whenever this is defined and makes sense;*

L3) *For every homology class $\zeta \in H_*(C_*(X, A))$ there exists a basic (with respect to the basis in* L1)) *$z \in (\lambda_*)^{-1}\zeta$.*

Lemma 9.7.2. *Suppose $\{C_*, C'_*.i', l, \varphi, \kappa\}$ satisfies all properties of a chain functor eventually without* **CH3), CH4), CH6)**. *Assume that there exists a chain functor $\boldsymbol{L}_* \in \mathfrak{C}h$, $q : \boldsymbol{L}_* \subset C_*$ such that q preserves all structure and induces an isomorphism of homology*; *then C_* is a chain functor.*

Proof. Follows immediately by checking the properties of a chain functor. □

A chain functor $\boldsymbol{K}_*$ is called *flat* whenever φ, κ and the chain homotopies $\varphi\kappa \simeq 1$, $j_\# \varphi \simeq l$ are natural. In the beginning we introduced the concept of a flat homology theory.

Theorem 9.7.3 ([3, Theorem 3.3])**.** *The following conditions for a homology theory are equivalent:*

1) h_* *is flat;*
2) *there exists a flat chain functor associated with* h_*.

Corollary 9.7.4 ([3, Corollary 3.4])**.** *For a homology theory defined on the category of CW spaces, conditions* 1), 2) *are equivalent to* 3)*, and* h_* *is the direct sum of ordinary homology theories.*

Lemma 9.7.5. *Let* $\mathbf{C}_*$ *be any chain functor,* $d\,c = z$, *in* $C_*\,(X,U)$; *then there exist:* $z', c' \in C'_*(X,U)$, $a_i \in C_*(U,U)$, $i = 1,2$, $a_3 \in C_*(U)$ *such that*

$$\begin{cases} l\,z' + q_\#\,a_1 \sim z, & dz' \in \operatorname{Im} i' \\ l\,c' + q_\#\,a_2 = c + dw, & w \in C_*\,(X,U) \\ z' + i'\,a_3 = dc'. \end{cases} \tag{9.7.6}$$

This is Lemma 1.1 of [5] with $k = 1$. □

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