Journal of Pure and Applied Algebra 167 (2002) 195–202

JOURNAL OF
PURE AND
APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Kan extensions of internal functors. Nonconnected case

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Received 22 November 1999; received in revised form 8 February 2001

Communicated by J. Adámek

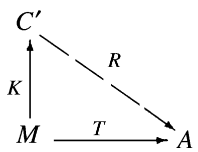
Abstract

An investigation of internal Kan extensions started in Datuashvili (Georgian Math. J. 6 (2) (1999) 127–148) is continued. The necessary and sufficient conditions for its existence are given, which generalizes the result obtained in Datuashvili for the case when the domain internal categories in the Kan extension diagram are connected.

MSC: 18D35; 18B40

1. Introduction

In [1–3] we defined and gave the complete computations of cohomologies of internal categories (equivalently crossed modules) in categories of groups with operations, described cohomologically trivial internal categories (crossed modules) and studied related questions. Here we consider the problem of the necessary and sufficient conditions for the existence of internal Kan extensions in the category of groups. In [4] according to the equivalence of categories $\mathrm{Cat}(\mathbb{G}\mathrm{r}) \simeq X\,\mathrm{Mod}(\mathbb{G}\mathrm{r})$ [7] we considered internal categories as crossed modules and treated the same question in the case when in the diagram of the Kan extension



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0022-4049/02/$ - see front matter
PII: S0022-4049(01)00035-4

two domain internal categories C and M are connected. In the present paper we follow the same crossed module approach, consider a more general nonconnected case and under certain assumptions obtained the necessary and sufficient conditions for the existence of internal Kan extensions.

Since every internal category in the category of groups is a groupoid, these kind of questions can be treated by means of the category theory (groupoid) methods. Note that in our case the groupoid approach did not give the desirable result. Due to Mac Lane-Whitehead's well-known classification of connected cell complexes according to their 3-type [6] we can also consider a topological approach to this problem. It will be the subject of the forthcoming paper.

In Section 2 we present definitions and results on internal functors and internal Kan extensions from [2,4]. In Section 3 we prove the main theorem and also consider a case of abelian groups as a special case of internal categories.

2. Definitions and summary of relevant previous results

Let $\mathrm{Cat}(\mathbb{G}\mathrm{r})$ denote the category of internal categories in the category of groups $\mathbb{G}\mathrm{r}$, and $X\,\mathrm{Mod}(\mathbb{G}\mathrm{r})$ the category of crossed modules in $\mathbb{G}\mathrm{r}$. By the equivalence of categories $\mathrm{Cat}(\mathbb{G}\mathrm{r}) \simeq X\,\mathrm{Mod}(\mathbb{G}\mathrm{r})$ [7], the internal nature of categories and the problem itself enables us to consider them as crossed modules.

Thus if A, C are internal categories in $\mathbb{G}\mathrm{r}$, we denote them as crossed modules $A : A_1 \xrightarrow{d^A} A_0$ and $C : C_1 \xrightarrow{d^C} C_0$. Recall [8] that by definition a crossed module is a pair of groups (A_0, A_1), where A_0 acts on A_1, and there is a group homomorphism $d^A : A_1 \longrightarrow A_0$ which satisfies the conditions

$$d^A(a_0 \cdot a_1) = a_0 + d^A(a_1) - a_0,$$

$$d^A(a_1) \cdot a_1' = a_1 + a_1' - a_1,$$

$a_0 \in A_0,\ a_1, a_1' \in A_1$.

Let $R : C \longrightarrow A$ be an internal functor. As shown in [2], R is a pair of group homomorphisms $R = (R_0, R_1),\ R_0 : C_0 \longrightarrow A_0,\ R_1 : C_1 \longrightarrow A_1$ satisfying the conditions

$$\begin{aligned} &R_1(r \cdot c) = R_0(r) \cdot R_1(c), \quad r \in C_0,\ c \in C_1, \\ &R_0 d^C = d^A R_1. \end{aligned} \tag{2.1}$$

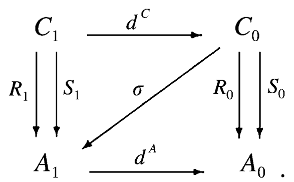
In Whitehead's terminology R_1 is an operator homomorphism associated with R_0 [8]. Let $S = (S_0, S_1) : C \longrightarrow A$ be another internal functor and $\sigma : S \longrightarrow R$ a morphism between them. Then σ is a crossed homomorphism $\sigma : C_0 \longrightarrow A_1$ associated with S_0 (for terminology see again [8]):

$$\sigma(r + r') = \sigma(r) + S_0(r) \cdot \sigma(r'),$$

which satisfies conditions

$$d^A\sigma = R_0 - S_0,$$
$$\sigma d^C = R_1 - S_1.$$

For the proof see [2]. The picture is



Since every internal category in $\mathbb{G}$r is a groupoid, any morphism $\sigma : S \longrightarrow R$ between internal functors is an isomorphism (see also [2]).

Note that if $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ then the action of A_0 on A_1 is trivial, so that σ is a group homomorphism, satisfying the above two conditions.

If α and α' are morphisms of internal functors

$$F \xrightarrow{\alpha} F' \xrightarrow{\alpha'} F'', \quad F, F', F'' : C \longrightarrow C' \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$$

then the composition $\alpha'\alpha$ is a map $\alpha' + \alpha : C_0 \longrightarrow C_1'$, and it satisfies the corresponding conditions [2].

Let $A, C, M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$ and $K = (K_0, K_1) : M \longrightarrow C$ and $T = (T_0, T_1) : M \longrightarrow A$ are internal functors.

Definition 2.1. An internal right Kan extension of $T = (T_0, T_1)$ along $K = (K_0, K_1)$ is a pair $(R = (R_0, R_1), \varepsilon)$, where R is an internal functor $C \longrightarrow A$, $\varepsilon : RK \longrightarrow T$—a morphism of internal functors, such that for each internal functor $S = (S_0, S_1) : C \longrightarrow A$ and a morphism $\alpha : SK \longrightarrow T$, there is a unique morphism of internal functors $\sigma : S \longrightarrow R$ with $\alpha = \varepsilon + \sigma k$.

Here σk is the morphism $SK \longrightarrow RK$, i.e., the composition

$$M_0 \xrightarrow{K_0} C_0 \xrightarrow{\sigma} A_1,$$

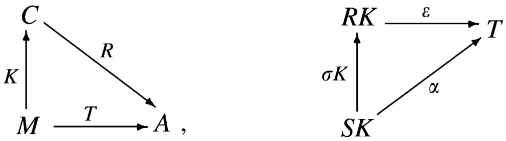
which satisfies the corresponding conditions:

$$d^A \cdot \sigma K_0 = R_0 K_0 - S_0 K_0,$$

$$\sigma K_0 \cdot d^M = R_1 K_1 - S_1 K_1,$$

$$\sigma K_0(m + m') = \sigma K_0(m) + S_0 K_0(m) \cdot \sigma K_0(m'), \quad m, m' \in M_0.$$

The diagram is



(2.2)

For the definition of the Kan extension for ordinary categories, see [5].

Note that as it is for the case of groupoids, internal right and left Kan extensions coincide, so we shall omit the word "right".

Proposition 2.2. *Let* $\varepsilon : RK \longrightarrow T$ *be a morphism of internal functors in the diagram* (2.2). (R, ε) *is an internal Kan extension of* T *along* K *if and only if for each internal functor* $S : C \longrightarrow A$ *the assignment* $\sigma \mapsto \sigma K_0$ $(K = (K_0, K_1))$ *determines a bijection*

$$\mathrm{Hom}(S, R) \longrightarrow \mathrm{Hom}(SK, RK).$$

Proof. It follows from the composition

$$\mathrm{Hom}(S, R) \longrightarrow \mathrm{Hom}(SK, RK) \xrightarrow{\mathrm{Hom}(SK, \varepsilon)} \mathrm{Hom}(SK, T),$$

where $\mathrm{Hom}(SK, \varepsilon)$ is a bijection. □

For $M = (M_0, M_1) \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, $A = (A_0, A_1) \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ we denote by $\mathrm{Hom}_{\mathrm{cat}}(M, A)$ an abelian group of internal functors from M to A, and by $\mathrm{Hom}_{\mathrm{oph}}(M_1, A_1)$ an abelian group of operator homomorphisms $T_1 : M_1 \longrightarrow A_1$ associated with trivial group homomorphism $M_0 \longrightarrow A_0$; i.e., T_1 is a group homomorphism with the condition:

$$T_1(m_0 \cdot m_1) = T_1(m_1), \quad m_0 \in M_0, \ m_1 \in M_1.$$

The same meaning will have the notation $\mathrm{Hom}_{\mathrm{oph}}(M_1, A_0)$.

Proposition 2.3. *Let* $M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, *and* $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$. *Then* $\mathrm{Hom}_{\mathrm{Cat}}(M, A)$ *is a pull-back of the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Cat}}(M, A) & \longrightarrow & \mathrm{Hom}_{\mathrm{oph}}(M_1, A_1) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(M_1,\ d^A)} \\ \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_0) & \xrightarrow{\mathrm{Hom}(d^M,\ A_0)} & \mathrm{Hom}_{\mathrm{oph}}(M_1, A_0). \end{array}$$

The proof follows directly from (2.1).

Denote by $\widetilde{\mathrm{Hom}}_{\mathrm{Cat}}(M, A)$ the set of all isomorphism classes of internal functors from M to A $(A \in \mathrm{Cat}(\mathbb{A}\mathrm{b}))$. Obviously, this set has an abelian group structure. We have homomorphisms

$$\mathrm{Hom}(d^M, A_1) : \ \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1) \longrightarrow \mathrm{Hom}_{\mathrm{oph}}(M_1, A_1),$$
$$\mathrm{Hom}(M_0, d^A) : \ \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1) \longrightarrow \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_0),$$

such that the following compositions are equal:

$$\mathrm{Hom}(M_1, d^A) \circ \mathrm{Hom}(d^M, A_1) = \mathrm{Hom}(d^M, A_0) \circ \mathrm{Hom}(M_0, d^A).$$

From this it follows that we have an induced homomorphism

$$\varphi_M : \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1) \longrightarrow \mathrm{Hom}_{\mathrm{Cat}}(M, A)$$

and it is easy to prove.

Proposition 2.4 (Datuashvili [4]). *For* $M \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, $A \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ *we have*

$$\widetilde{\mathrm{Hom}}_{\mathrm{Cat}}(M, A) = \mathrm{Coker}\, \varphi_M.$$

Proposition 2.5 (Datuashvili [4]). $\mathrm{Ker}\, \varphi_M = \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Coker}\, d^M, \mathrm{Ker}\, d^A)$.

3. The main result and its proof

In this section, we give the necessary and sufficient conditions for the existence of internal Kan extensions. We consider the case when K in diagram (2.2) is an epimorphic internal functor. The case where K is monomorphic is considered in [4], where we proved the theorem for the case when all internal categories A, C, M from (2.2) are in $\mathrm{Cat}(\mathbb{A}\mathrm{b})$ and M and C are connected. Note that these two cases (K is an epi and K is a mono) in general do not provide an answer for arbitrary K, even for $A, C, M \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ and connected M and C. For details see [4].

Let $K = (K_0, K_1)$ be an internal functor $M \longrightarrow C$. Denote by $\mathrm{Ker}\, K$ an internal category $d^{\mathrm{Ker}\, K} : \mathrm{Ker}\, K_1 \longrightarrow \mathrm{Ker}\, K_0$, where $d^{\mathrm{Ker}\, K} = d^M \mid_{\mathrm{Ker}\, K_1}$.

Theorem 3.1. *Let* $C = (C_0, C_1)$, $M = (M_0, M_1) \in \mathrm{Cat}(\mathbb{G}\mathrm{r})$, $A = (A_0, A_1) \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$, $K = (K_0, K_1) : M \longrightarrow C$ *be an epimorphic internal functor such that an injection* $\mathrm{Ker}\, K_0 \longrightarrow M_0$ *has a retraction. There exists a Kan extension of* $T = (T_0, T_1) : M \longrightarrow A$ *along* $K : M \longrightarrow C$ *if and only if* $T \mid_{\mathrm{Ker}\, K} \approx 0$ *and* $\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Coker}\, d^{\mathrm{Ker}\, K}, \mathrm{Ker}\, d^A) = 0$.

Proof. We have an exact sequence in $\mathrm{Cat}(\mathbb{G}\mathrm{r})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}\, K_1 & \xrightarrow{I_1} & M_1 & \xrightarrow{K_1} & C_1 & \longrightarrow & 0 \\ & & \Big\downarrow d^{\mathrm{Ker}\, K} & & \Big\downarrow d^M & & \Big\downarrow d^C & & \\ 0 & \longrightarrow & \mathrm{Ker}\, K_0 & \xrightarrow{I_0} & M_0 & \xrightarrow{K_0} & C_0 & \longrightarrow & 0. \end{array} \tag{3.1}$$

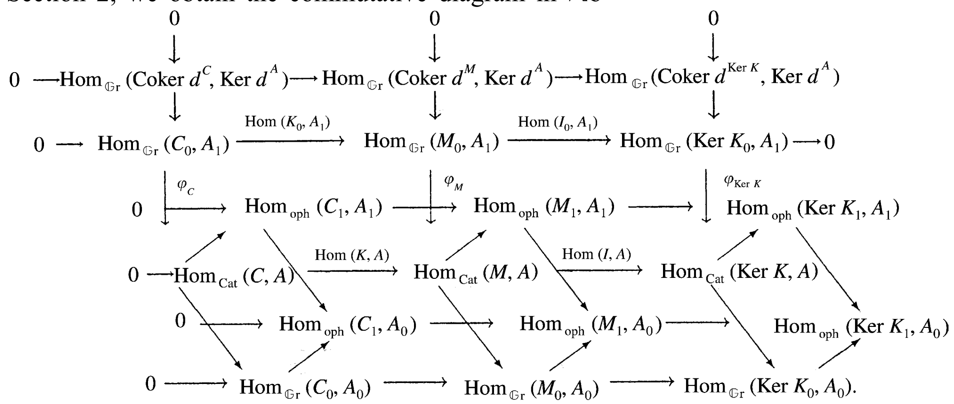
Consider abelian group homomorphisms

$$\varphi_C : \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(C_0, A_1) \longrightarrow \mathrm{Hom}_{\mathrm{Cat}}(C, A),$$
$$\varphi_M : \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1) \longrightarrow \mathrm{Hom}_{\mathrm{Cat}}(M, A),$$
$$\varphi_{\mathrm{Ker}\, K} : \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Ker}\, K_0, A_1) \longrightarrow \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Ker}\, K, A),$$

defined in the Section 2. By Proposition 2.5 of Section 2, we have

$$\begin{aligned}
&\operatorname{Ker}\varphi_C = \operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^C, \operatorname{Ker} d^A),\\
&\operatorname{Ker}\varphi_M = \operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^M, \operatorname{Ker} d^A),\\
&\operatorname{Ker}\varphi_{\operatorname{Ker} K} = \operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A).
\end{aligned}$$

Applying the functor $\operatorname{Hom}_{\mathrm{Cat}}(-,A)$ to diagram (3.1) and Propositions 2.3 and 2.5 of Section 2, we obtain the commutative diagram in $\mathbb{A}\mathrm{b}$



(3.2)

Since $I_0 \colon \operatorname{Ker} K_0 \mapsto M_0$ has a retraction, $\operatorname{Hom}(I_0, A_1)$ is an epimorphism. Each row and column in diagram (3.2) is exact. Applying Snake Lemma to (3.2) and Proposition 2.4 we obtain an exact sequence of abelian groups

$$\begin{aligned}
&\operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^M, \operatorname{Ker} d^A) \xrightarrow{\varphi} \operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) \xrightarrow{\psi} \widetilde{\operatorname{Hom}}_{\mathrm{Cat}}(C,A)\\
&\xrightarrow{\widetilde{\operatorname{Hom}}(K,A)} \widetilde{\operatorname{Hom}}_{\mathrm{Cat}}(M,A) \xrightarrow{\widetilde{\operatorname{Hom}}(I,A)} \widetilde{\operatorname{Hom}}_{\mathrm{Cat}}(\operatorname{Ker} K, A).
\end{aligned} \tag{3.3}$$

Suppose that the conditions of the theorem hold. We shall show that there exists a Kan extension of T along K.

Since $T\,|_{\operatorname{Ker} K} \approx 0$, it means that $\widetilde{\operatorname{Hom}}(I,A)(\operatorname{cl} T) = 0$. From the exactness of (3.3), there exists an internal functor $R \in \operatorname{Hom}_{\mathrm{Cat}}(C,A)$, such that $\widetilde{\operatorname{Hom}}(K,A)(\operatorname{cl} R) = \operatorname{cl} T$; which is equivalent to the condition that there exists an isomorphism $\varepsilon \colon RK \xrightarrow{\approx} T$. Suppose that there is an internal functor $S \colon C \longrightarrow A$ with $\alpha \colon SK \xrightarrow{\approx} T$. This gives an equality $\operatorname{cl} SK = \operatorname{cl} RK = \operatorname{cl} T$. Since $\operatorname{Hom}_{\mathbb{G}\mathrm{r}}(\operatorname{Coker} d^{\operatorname{Ker} K}, \operatorname{Ker} d^A) = 0$, $\widetilde{\operatorname{Hom}}(K,A)$ in (3.3) is a monomorphism. Thus we have an isomorphism $S \approx R$. We have to show that there exists a unique isomorphism $\sigma \colon S \longrightarrow R$, with $\sigma K_0 = -\varepsilon + \alpha$. From (3.2) we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & -\varepsilon+\alpha & \xmapsto{\operatorname{Hom}(I_0,A_1,A)} & x\\
 & & \big\downarrow & & \big\downarrow \varphi_{\operatorname{Ker} K}\\
R-S & \xmapsto{\operatorname{Hom}(K,A)} & RK-SK & \xmapsto{\operatorname{Hom}(I,A)} & 0\ .
\end{array} \tag{3.4}$$

Since $\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Coker}\, d^{\mathrm{Ker}\, K}, \mathrm{Ker}\, d^A) = 0$, $\varphi_{\mathrm{Ker}\, K}$ is a monomorphism. So in (3.4) $x = 0$. From the exactness of the corresponding row in (3.2), we conclude, that there exists a unique $\sigma \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(C_0, A_1)$ such that $\sigma \overset{K_0}{\mapsto} -\varepsilon + \alpha$.

Since the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(C_0, A_1) & \xrightarrow{\mathrm{Hom}(K_0, A_1)} & \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1) \\ \varphi_C \downarrow & & \downarrow \varphi_M \\ \mathrm{Hom}_{\mathrm{Cat}}(C, A) & \xrightarrow{\mathrm{Hom}(K, A)} & \mathrm{Hom}_{\mathrm{Cat}}(M, A) \end{array}$$

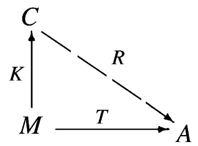
is commutative and $\mathrm{Hom}(K, A)$ is a monomorphism, we obtain $\varphi_C(\sigma) = R - S$, which means that σ is a morphism of internal functors $\sigma : S \longrightarrow R$; it proves that (R, ε) is a Kan extension.

Now suppose, that there exists a Kan extension (R, ε) of T along K. Then from (3.2) we have that $\mathrm{Hom}(I, A)(T) \approx 0$, which means that $T\,|_{\,\mathrm{Ker}\, K} \approx 0$. Since R is unique up to isomorphism with the property that $RK \xrightarrow{\approx} T$, from the exact sequence (3.3) we obtain that $\widetilde{\mathrm{Hom}}(K, A)$ is a monomorphism, so that $\psi = 0$ in (3.3). Now we shall show that $\varphi = 0$ in (3.3), from which will follow that $\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Coker}\, d^{kr\, K}, \mathrm{Ker}\, d^A) = 0$. Let $\beta \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(M_0, A_1)$ and $\varphi_M(\beta) = 0$; therefore we can consider β as a morphism of internal functors $\beta : RK \longrightarrow RK$. By Proposition 2.2 we have a bijection $\mathrm{Hom}(R, R) \leftrightarrow \mathrm{Hom}(RK, RK)$. Thus, there is a morphism $\gamma : R \longrightarrow R$ with $\gamma K_0 = \beta$. In diagram (3.2) we have $\gamma \in \mathrm{Hom}_{\mathbb{G}\mathrm{r}}(C_0, A_1)$ and $\mathrm{Hom}(K_0, A_1)(\gamma) = \beta$. From this it follows that $\varphi = 0$, which completes the proof of the theorem. $\square$

In the case when $A, C, M \in \mathrm{Cat}(\mathbb{A}\mathrm{b})$ the condition an injection $I_0 : \mathrm{Ker}\, K_0 \mapsto M_0$ has a retraction can be replaced by one of the following conditions:
(i) A_1 is injective in $\mathbb{A}\mathrm{b}$;
(ii) C_0 is projective in $\mathbb{A}\mathrm{b}$;
(iii) d^C is a split epimorphism;
(iv) d^A is a split monomorphism.
We do not state here the proof of this theorem, since it is based on analogous arguments as for the connected case given in [4, Theorem 3.4], where we use the groups $\mathrm{Ext}^1_{\mathrm{Cat}(\mathbb{A}\mathrm{b})}(A, B)$ (extensions of internal categories or equivalently of crossed modules in $\mathbb{A}\mathrm{b}$) and their properties.

Note that the condition (iv) means that A is internally equivalent to the discrete internal category $0 \longrightarrow \mathrm{Coker}\, d^A$ (see [2, Proposition 4.4]) and if (iv) holds then the condition $\mathrm{Hom}_{\mathbb{G}\mathrm{r}}(\mathrm{Coker}\, d^{\mathrm{Ker}\, K}, \mathrm{Ker}\, d^A) = 0$ is automatically satisfied. Analogously, it can be proved that condition (iii) means that C is internally equivalent to the connected internal category $\mathrm{Ker}\, d^C \longrightarrow 0$, i.e., to the abelian group $\mathrm{Ker}\, d^C$ considered as an internal category.

Suppose that all categories A, C, M are connected internal categories in $\mathbb{G}\mathrm{r}$ with only one object, i.e., $A_0 = C_0 = M_0 = 0$. These conditions imply that A_1, C_1, M_1 are abelian groups, internal functors are abelian group homomorphisms and an isomorphism between internal functors is an equality. In this case the notion of an internal Kan extension reduces to the notion of a unique extension of a homomorphism in $\mathbb{A}$b



In both cases where K is mono [4] and K is epi our theorems give the corresponding well-known results for abelian groups.

Acknowledgements

I would like to thank G. Janelidze, who suggested me to consider this problem, as well as the questions investigated earlier by me in [1,3]. The work was partially supported by the grant: INTAS-97-31961.

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