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MATHEMATICS

I. Bukhnikashvili

On an Approximate Solution of the Matric Equation with a Matrix Spectrum on Two Segments

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ABSTRACT. **The iterative method of finding an approximate solution for the linear matrix equation with a symmetric matrix A of nonfixed sign is suggested. The method does not provide us with the generally accepted transfer to the matrix A^2, but allows one to apply to the corresponding iterative scheme the single and k-multiple zeros of specially chosen polynomial which is constructed on two spectral segments of the matrix located on the negative and positive numerical semiaxes. The above-mentioned method makes it possible to attain significant acceleration of iterative convergence to the exact solution.**

Key words: modulus-maximum, k-multiple zero, iteration, cycle, polynomial, matrix.

Consider the equation

$$A\varphi = f \tag{1}$$

with the symmetric matrix A whose eigennumbers $\{\lambda_i\}$ are located on the segments

$$[-N, -m] \cup [m, M], \quad M > N > m > 0. \tag{2}$$

To solve equation (1) approximately, according to the general method of V. Lebedev [1] we have to pass from segments (2) to the adjusted in length segments

$$[-M, -m] \cup [m, M] \tag{3}$$

and then, having constructed on segments (3) the normalized Chebyshev second degree polynomial [2]

$$T_2(x) = 1 - \frac{2}{M^2 + m^2} x^2, \tag{4}$$

to apply zeros γ_1 and γ_2 of polynomial (4) to the cyclic iterative sceme [1]

$$\varphi_s = \varphi_{s-1} - \frac{1}{\gamma_s}(A\varphi_{s-1} - f), \quad \varphi_0 = 0 \tag{5}$$

for the conditions $\gamma_{2+s} = \gamma_s$ $(s = 1, 2)$. The scheme described above is equivalent both to the passage from equation (1) to the equation $A^2\varphi = Af$ and to the iteration of the obtained equation. Moreover, if we take into account that

$$\max|T_2(x)| = \frac{M^2 - m^2}{M^2 + m^2}, \tag{6}$$

then after n cycles we obtain for the error φ_{2n} of approximation to the exact solution φ [1] the following estimate:

$$\|\varphi_{2n} - \varphi\| \le \frac{\|f\|}{m} \max|T_2^n(x)| = \frac{\|f\|}{m}\left(\frac{M^2 - m^2}{M^2 + m^2}\right)^n \tag{7}$$

Remark. In formulas (6) and (7) as well as in those below, the modulus-maxima for the polynomial (4) is assumed to be taken on segments (3).

Suppose that in the iterative scheme (5) we take instead of zeros of polynomial (4) those of the polynomial of the kind

$$P_{k+1}(x) = \left(1-\frac{x}{a}\right)\left(1-\frac{x}{M}\right)^k \qquad (8)$$

It can be easily seen that if

$$a = -\frac{m\left[1+\left(\frac{M-m}{M+m}\right)^k\right]}{1-\left(\frac{M-m}{M+m}\right)^k} \qquad (9)$$

and

$$|a| > N, \qquad (10)$$

then the correlations

$$P_{k+1}(-m) = P_{k+1}(m) = \max\left|P_{k-1}(x)\right| = \frac{2\left(1-\frac{m^2}{M^2}\right)^k}{\left(1+\frac{m}{M}\right)^{k+} + \left(1-\frac{m}{M}\right)^k} \qquad (11)$$

are valid.

Note that a number of iterative steps made by the scheme (5) is equal to a number of zeros (with regard for the multiplicity of each of zeros) of the corresponding polynomial raised to the appropriate power. Taking all the above said into consideration and writing the polynomials of type (4) and (8) equal to the power orders, we first consider the polynomial (4) raised in power $k+1$ and then compare it by the value of modulus-maxima with the squared polynomial (8): in both cases we obtain the polynomial of order $2(k+1)$.

Remark. In correlations (11) as weel as in what follows, the modulus-maximum of the polynomial (8) is assumed to be taken on segments (2).

According to the above-said, we can prove the following

Theorem. *For all natural numbers $k>1$, if condition (10) is fulfilled, the inequality*

$$\max P_{k+1}^2(x) < \max\left|T_2^{k+1}(x)\right| \qquad (12)$$

is valid.

Taking into account inequality (7), inequality (12) gives all grounds to prefer zeros of polynomial (8) to those of polynomial (4) in the iterative scheme (5) in case we are required to perform a great number of cycles equal, for instance, to lk. In the latter case instead of inequality (12) we shall have the inequality

$$\max P_{k+1}^{2l}(x) < \max\left|T_2^{k+1}(x)\right|^l$$

with a possible significant decrease of the corresponding modulus-maxima for sufficiently large l. Alongside with the above-said, it should be taken into account that if we apply in scheme (5) zeros of polynomial (8) instead of those of polynomial (4), the cycle length increases from two to $k+1$ (a number of different zeros remains unchanged, equal to two), and therefore when constructing the polynomials (8) it is not advisable to take large k.

The fifth and the sixth columns of the table reproduce the values of the modulus-

Table 1

M	a	k	l	$\max\lvert T_2^{(k+1)l}(x)\rvert$	$\max P_{k+1}^{2l}(x)$
10	−1.3106	10	67	$4\cdot10^{-7}$	$5\cdot10^{-29}$
20	−6.7111	3	800	10^{-7}	$4\cdot10^{-11}$
20	−5.0624	4	605	$3\cdot10^{-7}$	$8\cdot10^{-14}$
20	−4.0797	5	470	$8\cdot10^{-7}$	$6\cdot10^{-16}$
20	−3.43	6	460	10^{-7}	$2\cdot10^{-21}$
20	−2.63	8	350	10^{-7}	$2\cdot10^{-27}$
20	−2.3685	9	310	$2\cdot10^{-7}$	$3\cdot10^{-30}$
20	−2.1624	10	280	$2\cdot10^{-7}$	$5\cdot10^{-33}$
30	−10.0296	3	1800	10^{-7}	$4\cdot10^{-11}$
30	−6.0532	5	1150	$2\cdot10^{-7}$	$3\cdot10^{-17}$
30	−5.0647	6	1000	$2\cdot10^{-7}$	$6\cdot10^{-21}$
30	−4.3616	7	870	$2\cdot10^{-7}$	$5\cdot10^{-24}$
30	−3.4315	9	720	10^{-7}	10^{-31}
40	−5.7713	7	1600	10^{-7}	$6\cdot10^{-25}$

maxima of polynomials $T_2^{(k+1)l}(x)$ and $P_{k+1}^{2l}(x)$ for the values M, k and l which are respectively given in the first, third and fourth columns of the table; the second column shows the values of a (see formula (9)).

Let us introduce into consideration the value

$$\sqrt{\sum_i R_n^2(\lambda_i)} \tag{13}$$

in which the nodes $\{\lambda_i\}$ are distributed uniformly on segments (2). To a considerable extent the value (13) represents the degree of fitness of the polynomial $R_n(\lambda)$ to the Richardson's method, since it is the error norm of the n-th approximation to the exact solution of equation (1), if the correlation

$$f = A\sum_i v_i = \sum_i \lambda_i v_i$$

is fulfilled and all zeros of the polynomial $R_n(\lambda)$ are correlated with the iterative scheme (5).

Table 2 gives the quantities of value (13) for the polynomials $P_{k+1}^{2l}(x)$ and $T_2^{(k+1)l}(x)$ (in the fifth and sixth columns, respectively) for fixed M=40 and m=1, but for different values k, l and N (in the first, second and third columns, respectively); the fourth column

Table 2

k	l	N	h	$\sqrt{\sum_i P_{k+1}^{4l}(\lambda_i)}$	$\sqrt{\sum T_2^{2l(k+1)}(\lambda_i)}$	$2l(k+1)$
6	150	−6	1	$7\cdot10^{-7}$	$7\cdot10^{-3}$	2100
5	540	−8	0.5	$4\cdot10^{-10}$	$2\cdot10^{-4}$	6480
8	150	−5	0.25	$5\cdot10^{-5}$	0.1063	2700
9	150	−4	0.25	$4\cdot10^{-6}$	$8\cdot10^{-2}$	3000
9	170	−4	0.125	$9\cdot10^{-6}$	0.1124	3400
10	260	−4	0.125	$4\cdot10^{-10}$	$2\cdot10^{-2}$	5720
7	320	−5	0.0625	$5\cdot10^{-6}$	$5\cdot10^{-2}$	5120
3	400	−13	0.0625	$6\cdot10^{-2}$	0.2224	3200

of the table gives the quantities of the subinterval value h for uniformly divided segments (2) and, finally, the seventh column reproduces the power order values of polynomials, equal to $2(k+1)$.

Georgian Academy of Sciences
A. Razmadze Mathematical Institute

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