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TO THE PROBLEM OF CONSTRUCTING A POLYNOMIAL WITH SMALL DEVIATIONS ON TWO SEGMENTS

I. BUKHNIKASHVILI

Abstract. In the present paper we consider the problem of constructing (approximately) a normalized Chebyshev polynomial on two segments on one side of the origin.

The exact effective solution of the problem is well-known only in the case of equal in length segments. Approximate solutions suggested in the paper are found under a supplementary restriction to the length of a smaller segment.

The tables are given which show the advantage in achieving small modulus-maxima of polynomials against the Chebyshev ones constructed on the given segments after they are equalized in length.

რეზიუმე. წინამდებარე ნაშრომში განხილულია მიახლოებითი სახით ჩებიშევის ნორმირებული პოლინომის აგების ამოცანა ორ მონაკვეთზე, რომლებიც განლაგებული არიან კოორდინატთა სათავის ერთ მხარეზე. აღნიშნული ამოცანის ზუსტი ეფექტური ამოხსნა ცნობილია მხოლოდ თანაბარი სიგრძის მონაკვეთებისთვის. ნაშრომში მოტანილი მიახლოებითი ამონახსნები აგებულია დამატებითი პირობების დადებით მცირე მონაკვეთის სიგრძეზე.

ნაშრომში მოცემულია ცხრილები, რომლის მიხედვით დასტურდება აგებული პოლინომების უპირატესობა მცირე მოდულ-მაქსიმუმების მიღწევის თვალსაზრისით, შედარებით იმ პოლინომებთან, რომლებიც აიგება მოცემულ მონაკვეთებზე სიგრძეში გატოლების შემდეგ.

The goal of the present paper is to construct a polynomial with small deviations from zero on two unequal in length segments on one side of the origin. Unlike the generally accepted method aimed to make segments equal in length and then to construct a polynomial of the least deviation from zero (or briefly, PLDZ) on two equal segments (see [1]), we suggest the scheme in which the use is made of the third additional segment to achieve desired symmetry. In this case, under certain supplementary conditions, by means of transformations we reduce these three segments to one and then construct on the latter the PLDZ. It turns out that if the smaller segment

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is sufficiently small (see inequality (5)), then the modulus-maximum on the preassigned segments of the polynomial will be less than that of the PLDZ on two equal in length segments.

The second approximate scheme of constructing a polynomial shows weakening of the condition on the sufficient smallness of the smaller segment, however at the expense of additional calculations.

The tables presented in the paper display that the polynomials, suggested by the author (in achieving a small module-maxima) are more advantageous than those suggested in [1]. The above-said gives us all grounds to prefer in some separate cases the first over the second ones in solving such computational problems as, for example, finding of eigen-numbers of a matrix by the method proposed by Gavurin ([2]), or solving a system of linear algebraic equations by the method due to Richardson ([3], p. 253).

1. Statement of the Problem

On the real numerical axis we consider the segments

$$[m_1,\, M_1] \tag{1}$$

$$[m_2,\, M_2]. \tag{2}$$

Assume that the inequalities

$$0 < m_1 < M_1 < m_2 < M_2 \tag{3}$$

are fulfilled.

We try to construct on the segments (1) and (2) the n-th degree polynomial $P_n(x)$ normalized by the condition

$$P_n(0) = 1 \tag{4}$$

such that it competes successfully in achieving small modulus-maxima with the PLDZ, constructed on the segments (1) and (2) by Lebedev's scheme ([1]) after they are equalized in length and written in the form convenient for calculations and allowing one to find zeros and modulus-maxima of the polynomial. To be more precise, the method of construction suggested by us and the necessary for that conditions depend considerably on the fact which of the segments (larger, or smaller) is closer to the point of normalization $x = 0$; for example, if the smaller segment is closer (see inequality (3)), then it is necessary that the condition

$$M_1 < \frac{M_2 + m_2}{2} - \sqrt{(M_2 - m_1)(m_2 - m_1)} \tag{5}$$

is fulfilled (see remark to formula (18)).

Since the value in the right-hand side of inequality (5) is comparatively small, a number of possible practical cases for application of the method is limited. Therefore along with the method described above, we suggest another way of constructing (see Remark to Table 1) under which the restrictive condition (5) can to a certain extent be weakened.

To solve the above-stated problem, we use special scheme of constructing which allows one to apply additional restrictions to the length of the smaller segment and hence to avoid general methods of constructing the PLDZ on two segments. These methods elaborated by N. Achiezer ([4]) require introducing into consideration special functions given in a complicated parametric form which makes it difficult to find zeros and modulus-maxima. For the same reason we gave up the methods of construction described by A. Bogatyrev in [5].

2. Description and Comparison of Numerical Schemes Resolving the Above-Stated Problem

Depending on the location of the smallest of the segments (1) and (2) from the normalization point $x = 0$ (see condition (4)), we distinguish two cases.

Case 1. *The condition*

$$M_2 - m_2 > M_1 - m_1$$

is fulfilled, or to be more precise, the smaller segment is closer to the point of normalization than the larger one (*see inequality* (3)).

Following the general scheme of constructing suggested by V. Lebedev ([1]) for n segments, it is necessary in a particular case of two segments that the segments (1) ans (2) be equal in length taking the segment

$$[m_1,\, M_2 - m_2 + m_1] \tag{6}$$

and then, under the condition

$$2m_2 - M_2 - m_1 > 0 \tag{7}$$

for the segments (6) and (2) to be nonconfluent, to construct on them the PLDZ of even n. Towards this end, we substitute

$$y = x - \frac{M_2 + m_1}{2}, \quad z = y^2 \tag{8}$$

and transform the segments (6) and (2) into the segment

$$\Big[\frac{(2m_2 - M_2 - m_1)^2}{4},\, \frac{(M_2 - m_1)^2}{4}\Big]$$

on which we construct the $\frac{n}{2}$-th degree polynomial of smallest deviations from zero with respect to the variable z and write it (see [3], p. 253) as follows:

$$\cos\frac{n}{2}\arccos\frac{8z - (M_2 - m_1)^2 - (2m_2 - M_2 - m_1)^2}{(M_2 - m_1)^2 - (2m_2 - M_2 - m_1)^2}. \tag{9}$$

The normalizing value (see the condition (4)) of the polynomial (9) by which the latter is divided looks as

$$\cos\frac{n}{2}\arccos\frac{8z_0-(M_2-m_1)^2-(2m_2-M_2-m_1)^2}{(M_2-m_1)^2-(2m_2-M_2-m_1)^2}, \tag{10}$$

where z_0 is the abscissa of the point $x=0$ obtained after transformations (8). Representing in (10) the expression under the sign arccos in the form of the ratio of sums and differences of the same values, after calculations (see [3], p. 253), we obtain the following formula:

$$\cos\frac{n}{2}\arccos\frac{4z_0-(2m_2-M_2-m_1)^2+4z_0-(M_2-m_1)^2}{4z_0-(2m_2-M_2-m_1)^2-[4z_0-(M_2-m_1)^2]}= \\ =\frac{1}{2}\Big(\mathcal{P}^{\frac{n}{2}}+\frac{1}{\mathcal{P}^{\frac{n}{2}}}\Big), \tag{11}$$

where

$$\mathcal{P}=\frac{\sqrt{4z_0-(2m_2-M_2-m_1)^2}+\sqrt{4z_0-(M_2-m_1)^2}}{\sqrt{4z_0-(2m_2-M_2-m_1)^2}-\sqrt{4z_0-(M_2-m_1)^2}}.$$

Taking into account formulas (9), (10) and (11) and substitutions (8), the unknown PLDZ on the segments (6) and (2), satisfying the condition (4) (see the text after formula (9)), can be written in the form

$$T_n(x)=\frac{2}{\mathcal{P}^{\frac{n}{2}}+\frac{1}{\mathcal{P}^{\frac{n}{2}}}}\times \\ \times\cos\frac{n}{2}\arccos\frac{8(x-\frac{M_2+m_1}{2})^2-(M_2-m_1)^2-(2m_2-M_2-m_1)^2}{(M_2-m_1)^2-(2m_2-M_2-m_1)^2}. \tag{12}$$

It is clear that the modulus-maximum of the polynomial (12) on the segments (6) and (2) is calculated by the formula

$$\max|T_n(x)|=\frac{2}{\mathcal{P}^{\frac{n}{2}}+\frac{1}{\mathcal{P}^{\frac{n}{2}}}}. \tag{13}$$

Now, instead of the segment (6) which is obtained as a result of lengthening of the segment (1) up to the whole length of the segment (2), we take the segment

$$[m_1,\overline{M_1}] \tag{14}$$

and choose abscissas of its right end $\overline{M}_1$ such that after transformations

$$y=x-\frac{M_2+m_2}{2},\quad z=y^2 \tag{15}$$

the segments (14) and (2) transformed into the segments

$$\Big[\frac{(M_2+m_2-2\overline{M}_1)^2}{4},\ \frac{(M_2+m_2-2\overline{m}_1)^2}{4}\Big]\cup\Big[0,\ \frac{(M_2-m_2)^2}{4}\Big], \tag{16}$$

are equal in length. The latter condition results in the quadratic equation with respect to $\overline{M}_1$ from which we choose the least root,

$$\overline{M}_1 = \frac{M_2 + m_2}{2} - \sqrt{(M_2 - m_1)(m_2 - m_1)}, \tag{17}$$

and write the segment (14) as follows:

$$\Big[m_1,\ \frac{M_2 + m_2}{2} - \sqrt{(M_2 - m_1)(m_2 - m_1)}\Big]. \tag{18}$$

It is clear that the above scheme of constructing is possible if and only if the segment (1) is smaller than the segment (18). Moreover, after transformations (15) in reverse order and getting back to the variable x, the left of the segments (16) is divided into two (symmetric with respect to the point $x = \frac{1}{2}(M_2 + m_2)$) segments: the segment (17) and the segment

$$\Big[\frac{M_2 + m_2}{2} + \sqrt{(M_2 - m_1)(m_2 - m_1)},\quad M_2 + m_2 - m_1\Big], \tag{19}$$

while the right segment transfers into the segment (2), coming open on both sides of the point of symmetry.

Having obtained the equal in length segments (16), thanks to our choice of $\overline{M}_1$ by formula (17), by means of substitutions

$$\xi = z - \frac{(M_2 + m_2 - 2m_1)^2}{8}, \quad \eta = \xi^2 \tag{20}$$

we can transform the above-mentioned segments into the segment

$$\Big[\frac{[4(M_2 - m_1)(m_2 - m_1) - (M_2 - m_2)^2]^2}{64},\ \frac{(M_2 + m_2 - 2m_1)^4}{64}\Big]. \tag{21}$$

Further, to complete the construction, we can use general methods developed by V. Lebedev ([1]).

To simplify our writing of the subsequent calculations, we introduce brief designation and write the expression for the segment (21) in the form

$$[A,\ B]. \tag{22}$$

Taking as the initial on the segment (22) the first degree Chebyshev polynomial

$$T_1(\eta) = \frac{2\eta - B - A}{B - A} \tag{22}$$

with modulus-maxima equal to unity, we construct the corresponding Chebyshev polynomial of degree $\frac{n}{4}$, assuming n to be divisible by 4 (see, for e.g., [6]) and using the superposition of the functions,

$$T_{\frac{n}{4}}(T_1(\eta)) = \cos\frac{n}{4}\arccos\frac{2\eta - B - A}{B - A}. \tag{23}$$

Taking now into account the transformations (15) and (20), we can write (23) as the n-th degree polynomial of the variable x:

$$\cos\frac{n}{4}\arccos\frac{2\left[\frac{(2x-M_2-m_2)^2}{4}-\sqrt{B}\right]^2-B-A}{B-A}. \qquad (24)$$

The normalizing value (see the condition (4)) of the polynomial (24) by which this latter should be divided is

$$\cos\frac{n}{4}\arccos\frac{2\eta_0-B-A}{B-A}, \qquad (25)$$

where η_0 is the abscissa of the point into which the point $x=0$ transfers after the transformations (15) and (20). Next, performing calculations, analogous to those we have used for obtaining formula (11), we can write

$$\cos\frac{n}{4}\arccos\frac{\eta_0-A+\eta_0-B}{\eta_0-A-(\eta_0-B)}=\frac{1}{2}\Big(\mathcal{P}_1^{\frac{n}{4}}+\frac{1}{\mathcal{P}_1^{\frac{n}{4}}}\Big), \qquad (26)$$

where

$$\mathcal{P}_1=\frac{\sqrt{\eta_0-A}+\sqrt{\eta_0-B}}{\sqrt{\eta_0-A}-\sqrt{\eta_0-B}}.$$

Taking into account formulas (24), (25) and (26), the unknown PLDZ on the segments (2), (18) and (19) can be written as follows:

$$P_n(x)=\frac{2}{\mathcal{P}_1^{\frac{n}{4}}+\frac{1}{\mathcal{P}_1^{\frac{n}{4}}}}\cos\frac{n}{4}\arccos\frac{2\left[\frac{(2x-M_2-m_2)^2}{4}-\sqrt{B}\right]^2-B-A}{B-A}. \qquad (27)$$

It is clear that the modulus-maximum of the polynomial (27) on the segments (2), (18) and (19) is calculated by the formula

$$\max|P_n(x)|=\frac{2}{\mathcal{P}_1^{\frac{n}{4}}+\frac{1}{\mathcal{P}_1^{\frac{n}{4}}}}. \qquad (28)$$

Equating the expression (24) to zero, we obtain the relations:

$$\arccos\frac{2\left[\frac{(2x-M_2-m_2)^2}{4}-\sqrt{B}\right]^2-B-A}{B-A}=\frac{2(2k-1)\pi}{n}, \quad k=1,\dots,\frac{n}{4},$$

from which to find zeros of the polynomial (27) we get the formulas:

$$x_{kij}= \qquad (29)$$

$$=\frac{M_2+m_2}{2}+(-1)^i\sqrt{\sqrt{B}+(-1)^j\frac{1}{\sqrt{2}}\sqrt{B+A+(B-A)\cos\frac{2(2k-1)\pi}{n}}},$$

$$k=1,\dots,\frac{n}{4}, \quad i=1,2, \quad j=1,2.$$

It is not difficult to see that all zeros obtained by formula (29) are real.

Below we present the table. The first column reproduces the values of the modulus-maxima (13) of the polynomial (12); the second column provides us with the values of the modulus-maxima (28) of the polynomial (27); the third and the fourth columns of Table 1 represent abscissas of the ends of the segments (18) and (2), respectively; the fifth column shows order of degrees of the polynomials (12) and (27), which is the same for the both polynomials.

Table 1

$\max\lvert T_n(x)\rvert$	$\max\lvert P_n(x)\rvert$	$[m_1, \overline{M}_1]$	$[m_2, M_2]$	n
10^{-5}	$5\cdot10^{-16}$	1; 2	300,350	80
$7\cdot10^{-6}$	$7\cdot10^{-14}$	1; 2,4	200,250	80
$5\cdot10^{-6}$	$4\cdot10^{-13}$	1; 2,6	170,220	80
$4\cdot10^{-6}$	10^{-12}	1; 2,8	150,200	80
$4\cdot10^{-6}$	$3\cdot10^{-12}$	1; 2,9	140,190	80
$2\cdot10^{-6}$	$5\cdot10^{-11}$	1; 3,5	100,150	80
10^{-6}	10^{-10}	1; 3,7	90,140	80
$7\cdot10^{-7}$	$6\cdot10^{-10}$	1; 4,4	70,120	80
$4\cdot10^{-7}$	10^{-9}	1; 4,8	60,110	80
$2\cdot10^{-7}$	$3\cdot10^{-9}$	1; 5,2	51,101	80

As is seen from the table (see the third and the fourth columns), the segments (18) are small compared to the segments (2), hence a number of practical cases for application of polynomials (27) are limited. Taking this fact into account, below we present the scheme of constructing a polynomial for which, unlike the polynomial (27), we do not introduce the third additional segment, and distribute all zeros on the segments (1) and (2). This allows one to achieve improvement in two directions: to obtain small modulus-maxima and to elongate the segment, but at the expense of additional calculations.

To construct the above-mentioned polynomial, we substitute

$$y = x - \frac{M_2 + m_2}{2}, \quad z = \frac{2y}{M_2 - m_2} \tag{30}$$

and transform the segments (1) and (2), respectively, into the segments

$$[z_1,\, z_2] \cup [-1,\, 1], \tag{31}$$

where

$$z_1 = -\frac{M_2 + m_2 - 2m_1}{M_2 - m_2}. \tag{32}$$

It is not necessary to write out the right end abscissa z_2 of the first of the segments (31), because in the sequel the construction of this segment will be "accomplished" accordingly to the right (see the condition (54)).

We write the initial basis polynomial ([42]) in the form

$$P_3(z) = \delta(1 + \alpha z + \beta z^2 + \gamma z^3) \tag{33}$$

and require that the conditions

$$P_3(-1) = P_3(1) = 0, \tag{34}$$

$$P_3(z_0) = P_3(z_1) \tag{35}$$

are fulfilled, and the normalizing condition

$$P_3(z_{\scriptscriptstyle N}) = 1. \tag{36}$$

In equality (35) we assume that

$$P_3'(z_0) = 0 \tag{37}$$

and

$$|z_0| < 1. \tag{38}$$

It is clear that all zeros of the polynomial (33) having hitherto two real zeros (see equality (34)), are real ones.

It should also be noted that the point $z_{\scriptscriptstyle N}$, appearing in equality (36) and obtained from the point $x = 0$ by transformations (30), is equal to

$$z_{\scriptscriptstyle N} = -\frac{M_2 + m_2}{M_2 - m_2}. \tag{39}$$

On the basis of the conditions (34), we can write the relations $P_3(-1) - P_3(1) = 0$ and $P_3(-1)+P_3(1) = 0$ which result, respectively, in the equalities $\alpha = -\gamma$ and $\beta = -1$. The polynomial (33) can now be rewritten as follows:

$$P_3(z) = \delta(1 - \gamma z - z^2 + \gamma z^3) = \delta(z^2 - 1)(\gamma z - 1), \tag{40}$$

and equality (37) in the form

$$3\gamma z_0^2 - 2z_0 - \gamma = 0. \tag{41}$$

From equality (41) we obtain the formula

$$\gamma = \frac{2z_0}{3z_0^2 - 1}. \tag{42}$$

Considering equality (41) with respect to z_0, resolving it and writing the radical with the sign $-$ (see inequality (38)), we find that

$$z_0 = \frac{1 - \sqrt{1 + 3\gamma^2}}{3\gamma}. \tag{43}$$

In addition, we assume that $\gamma < 0$, because otherwise, the abscissa of the extremum point of the polynomial (40) with the largest in absolute value abscissa will be positive, but this is impossible by virtue of our choice of the new coordinate system (see substitutions (30)).

Taking the above-said into account, from formula (43) we obtain the inequalities

$$0 < z_0 < \frac{1}{\sqrt{3}}. \tag{44}$$

Bearing formula (42) in mind, the condition (35) results in the following cubic equation with respect to z_0,

$$z_0^3 + z_1 z_0^2 - (2z_1^2 - 1)z_0 - z_1 = 0, \tag{45}$$

where z_1 is taken by formula (32).

It can be easily verified that one of the roots of equation (45) is separated by inequalities (44), but we can find it by, for example, the method of bisection of the segment (see [7], p. 118). Next, by formula (42) we can find γ and then, using equality (36), we can write for δ the following formula:

$$\delta = \frac{1}{(z_{_N}^2 - 1)(\gamma z_{_N} - 1)},$$

where $z_{_N}$ is taken by formula (39).

Thus, taking the above-said and formula (40) into account, we can write

$$P_3(z) = \frac{(z^2-1)(\gamma z - 1)}{(z_{_N}^2 - 1)(\gamma z_{_N} - 1)} \tag{46}$$

and regard that

$$\max |P_3(z)| = P_3(z_1) = \frac{(z_1^2-1)(\gamma z_1 - 1)}{(z_{_N}^2 - 1)(\gamma z_{_N} - 1)}. \tag{47}$$

To pass from the polynomial (46) to the corresponding n-th degree polynomial, we have to take the relation

$$\frac{P_3(z)}{P_3(z_1)} = \theta \tag{48}$$

in which $|\theta| < 1$ (see formulas (46) and (47)) and, assuming n as a number divisible by 3, we construct by means of superposition the polynomial of the form

$$P_n(z) = \frac{\cos\frac{n}{3}\arccos\frac{P_3(z)}{P_3(z_1)}}{\cos\frac{n}{3}\arccos\frac{1}{P_3(z_1)}} \tag{49}$$

with $\frac{n}{3}$ Chebyshev zeros with respect to the variable θ (see formula (48)) which are defined from the equation $\cos\frac{n}{3}\arccos\theta = 0$ by the formulas

$$\theta_k = \cos\frac{3(2k-1)\pi}{n}, \quad k = 1, \dots, \frac{n}{3}$$

and with n zeros with respect to the variable zx (see formula (48)) which are defined from $\frac{n}{3}$ cubic equations of the type

$$\frac{P_3(z)}{P_3(z_1)} = \cos\frac{3(2k-1)\pi}{n}, \quad k = 1, \dots, \frac{n}{3}. \tag{50}$$

Clearly, the polynomial (49) satisfies the normalization condition (36).

It can be easily verified that all n roots of equations (50) are real because in the left-hand side of each of the equations there appear the polynomial with simple zeros and the modulus-maxima equal to unity, while in the right-hand side of the equation there appears a free term which is less than, or equal to unity in an absolute value.

For the sake of convenience, in the denominator of formula (49) we introduce the notation $\omega = \frac{1}{P_3(z_1)}$, and reasoning analogously as when deducing formulas (11) and (26), we can write the following relations:

$$\cos\frac{n}{3}\arccos\omega = \cos\frac{n}{3}\arccos\frac{(\omega+1)+(\omega-1)}{(\omega+1)-(\omega-1)} = \frac{1}{2}\Big(\mathcal{P}_2^{\frac{n}{3}} + \frac{1}{\mathcal{P}_2^{\frac{n}{3}}}\Big), \tag{51}$$

where

$$\mathcal{P}_2 = \frac{\sqrt{\omega+1}+\sqrt{\omega-1}}{\sqrt{\omega+1}-\sqrt{\omega-1}}.$$

Taking into account the relations (51), we can rewrite the polynomial (49) in the form

$$P_n(z) = \frac{2}{\mathcal{P}_2^{\frac{n}{3}} + \frac{1}{\mathcal{P}_2^{\frac{n}{3}}}} \cos\frac{n}{3}\arccos\frac{P_3(z)}{P_3(z_1)}, \tag{52}$$

and calculate its modulus-maximum by the formula

$$\max|P_n(z)| = \frac{2}{\mathcal{P}_2^{\frac{n}{3}} + \frac{1}{\mathcal{P}_2^{\frac{n}{3}}}}.$$

Applying now transformations (30) to the polynomial (46) in reverse order, we can get both the third degree polynomial $\overline{P}_3(x)$ of the variable x and the corresponding n-th degree polynomial

$$\overline{P}_n(x). \tag{53}$$

Now all formulas, starting from (47) to (52) inclusive, can be written by means of the variable x. But we do not consider it necessary. We only notice that the polynomial $\overline{P}_3(x)$ has zeros at the points $x = m_2$, $x = M_2$ and two equal modulus-maxima at the points $x = m_1$ and $x = x_0$ ($\overline{P}'_3(x_0) = 0$, $m_2 < x_0 < M_2$). The third point at which we achieve the same modulus-maximum can be obtained after finding the point $x = \widehat{M}_1$ at which $\overline{P}_3(m_1) = -\overline{P}_3(\widetilde{M}_1)$ and introducing into consideration the segment $[\widetilde{m}_1, \widehat{M}_1]$; in addition, the condition

$$M_1 < \widetilde{M}_1 \tag{54}$$

should be fulfilled.

Here we present the table, analogous to Table 1.

Table 2

$\max\|T_n(x)\|$	$\max\|\overline{P}_n(x)\|$	$[m_1, \widetilde{M}_1]$	$[m_2, M_2]$	n
$3 \cdot 10^{-6}$	10^{-15}	[1; 3,7]	[450,500]	90
$3 \cdot 10^{-8}$	$9 \cdot 10^{-20}$	[1; 4]	[400,450]	120
$2 \cdot 10^{-6}$	$3 \cdot 10^{-13}$	[1; 4,9]	[300,350]	90
10^{-6}	$3 \cdot 10^{-11}$	[1; 6,7]	[200,250]	90
$8 \cdot 10^{-7}$	$4 \cdot 10^{-10}$	[1; 8,6]	[150,200]	90
$5 \cdot 10^{-9}$	$5 \cdot 10^{-13}$	[1; 9,1]	[140,190]	120
$5 \cdot 10^{-7}$	$2 \cdot 10^{-9}$	[1; 10,4]	[120,170]	90
$3 \cdot 10^{-7}$	$7 \cdot 10^{-9}$	[1; 12,3]	[100,150]	90
$2 \cdot 10^{-7}$	10^{-8}	[1; 13,7]	[90,140]	90
$6 \cdot 10^{-8}$	$9 \cdot 10^{-8}$	[1; 25,2]	[60,110]	90

Case 2. *The condition*

$$M_2 - m_2 < M_1 - m_1 \tag{55}$$

is fulfilled.

Calculations performed show that if we construct the polynomial under the condition (55) similarly to (27), but making the segment (2) longer than (1) and using the substitutions, we do not obtain in most practical cases any noticeable improvement, but the polynomials of type (53) obtained by superposition from the corresponding third degree polynomial turned out to be more competitive.

Here we present the scheme of constructing the above-mentioned polynomial of type (53). First, by the substitutions

$$y = x - \frac{M_1 + m_1}{2}, \quad z = \frac{2y}{M_1 - m_1} \tag{56}$$

we transform the segments (1) and (2) into the segments

$$[-1,\, 1] \cup [z_1,\, z_2],$$

where

$$z_1 = \frac{2m_2 - M_1 - m_1}{M_1 - m_1}, \tag{57}$$

but we do not write out the abscissa z_2, because the latter will be chosen later on (see the condition (68)).

The initial basis polynomial for the unknown one is sought just as in the previous case, i.e. in the form of the polynomial (33), but under the requirement that the conditions

$$\widetilde{P}_3(-1) = \widetilde{P}_3(1) = \widetilde{P}_3(z_1) \tag{58}$$

and

$$\widetilde{P}_3(1) = -\widetilde{P}_3(z_0), \tag{59}$$

are fulfilled; the relation (37) and inequality (38) remain valid. The normalization condition (36) remains likewise valid, however for

$$z_N = -\frac{M_1 + m_1}{M_1 - m_1}. \tag{60}$$

Remark 1. In the last of equalities (58) we have taken $\widetilde{P}_3(z_1)$ instead of $-\widetilde{P}_3(z_2)$, and hence gave the consideration of the well-known B. Samokish problem ([8]) up, whose resolution (even in our case of the third degree polynomial) is connected with cumbersome calculations, however allows one to obtain comparatively better results in achieving small modulus-maxima.

Remark 2. The third degree polynomial (33) in the case under consideration has real zeros. This automatically follows from the scheme of construction, with alternation of positive and negative modulus-maxima: $\widetilde{P}_3(-1) = -\widetilde{P}_3(z_0) = \widetilde{P}_3(1)$ (see equalities (58) and (59) and inequality (38)), which in its turn indicates the existence of two real zeros.

From equalities (58) we can get the relations $\alpha = -\gamma$ and $\beta = -z_1\gamma$. Then the polynomial (33) and equality (37) can, respectively, be written as follows:

$$\widetilde{P}_3(z) = \delta(1 - \gamma z - z_1\gamma z^2 + \gamma z^3) \tag{61}$$

and

$$3z_0^2 - 2z_1z_0 - 1 = 0. \tag{62}$$

Regarding equality (62) as the equation with respect to z_0, resolving it and taking the radical with the sign $-$ (see inequality (38)), we obtain for z_0 the formula

$$z_0 = \frac{z_1 - \sqrt{z_1^2 + 3}}{3},$$

in which z_1 is taken by formula (57).

From equation (59) we obtain the formula which allows one to find γ:

$$\gamma = \frac{2}{z_1 + z_0 + z_1z_0^2 - z_0^3}. \tag{63}$$

Finally, from the normalization condition (36) we find δ by means of the formula

$$\delta = \frac{1}{1 - \gamma z_{N} - z_1 \gamma z_{N}^2 + \gamma z_{N}^3}, \tag{64}$$

where z_{N} is taken by formula (60).

Taking into account formula (64), we can write the polynomial (61) in the form

$$\widetilde{P}_3(z) = \frac{1 - \gamma z - z_1 \gamma z^2 + \gamma z^3}{1 - \gamma z_{N} - z_1 \gamma z_{N}^2 + \gamma z_{N}^3}, \tag{65}$$

where γ is taken by formula (63).

For the modulus-maximum of the polynomial (65) we write the formula

$$\max |\widetilde{P}_3(z)| = \widetilde{P}_3(1) = \frac{1 - z_1 \gamma}{1 - \gamma z_{N} - z_1 \gamma z_{N}^2 + \gamma z_{N}^3}. \tag{66}$$

Further reasoning and calculations, starting from formulas (48) to (52), inclusive, are the same as above. Therefore we restrict ourselves and deduce the resultant formula for the unknown polynomial:

$$\widetilde{P}_n(z) = \frac{2}{\mathcal{P}_3{}^{\frac{n}{3}} + \frac{1}{\mathcal{P}_3^{\frac{n}{3}}}} \cos \frac{n}{3} \arccos \frac{\widetilde{P}_3(z)}{\widetilde{P}_3(z_1)}. \tag{67}$$

When calculating $\mathcal{P}_3$, it should be mentioned that unlike the previous case, there takes place the equality $\omega = \dfrac{1}{\widetilde{P}_3(1)}$ (see formulas (51), (52) and (66)).

In addition to the above-said, the initial basis polynomial $\widetilde{P}_3(x)$ for the unknown one, obtained from the polynomial (65) by the passage from the variable z to x and substitutions (56), has four equal modulus-maxima at the points $x = m_1$, $x = M_1$, $x = m_2$, $x = x_0$ ($\widehat{P}_3{}'(x_0) = 0$, $m_1 < x_0 < M_1$); the fifth point at which we achieve the same modulus-maximum is obtained only if the point $x = \widehat{M}_2$, at which $\widehat{P}_3(m_2) = -\widehat{P}_3(M_2)$, is found, the segments $[m_2, \widetilde{M}_2]$ are introduced into consideration and the condition

$$M_2 < \widetilde{M}_2 \tag{68}$$

is fulfilled.

Further (see Table 3), by $\widehat{P}_n(x)$ we denote the unknown polynomial which is obtained from (67) by returning to the initial variable x and substitutions (56).

The table below is analogous to Tables 1 and 2.

Table 3

$\max \lvert T_n(x)\rvert$	$\max \lvert \widehat{P}_n(x)\rvert$	$[m_1, M_1]$	$[m_2, M_2]$	n
$3 \cdot 10^{-13}$	$9 \cdot 10^{-17}$	[1; 60,3]	[397,5;400]	210
10^{-7}	$9 \cdot 10^{-10}$	[1; 60,3]	[397,5;400]	120
-10^{-13}	$6 \cdot 10^{-17}$	[1; 60,6]	[296,7;300]	210
$6 \cdot 10^{-8}$	$7 \cdot 10^{-10}$	[1; 60,6]	[296,7;300]	120
$3 \cdot 10^{-8}$	$5 \cdot 10^{-10}$	[1; 61,2]	[215,2;220]	120
$2 \cdot 10^{-8}$	$4 \cdot 10^{-10}$	[1; 61,5]	[194,5;200]	120
$7 \cdot 10^{-9}$	$3 \cdot 10^{-10}$	[1; 62,1]	[163,2;170]	120
10^{-13}	$3 \cdot 10^{-15}$	[1; 62,8]	[141,8;150]	180
$3 \cdot 10^{-9}$	$3 \cdot 10^{-10}$	[1; 62,8]	[141,8;150]	120
$2 \cdot 10^{-11}$	$9 \cdot 10^{-13}$	[1; 62,8]	[141,8;150]	150

Remark 3. It should be noted that the values of the polynomial $T_n(x)$ in the first column of the table are taken under the most possible favorable conditions allowing one to achieve small modulus-maxima when the right end of the smallest equalizable segment is at the point $x = m_2$, and equalization occurs to the left of the above-mentioned point by the value $M_1 - m_1$.

Remark 4. In all the above tables, the condition (7) is fulfilled for the segments (1) and (2) to be equal in length.

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Author's address:

A. Razmadze Mathematical Institute,
Georgian Academy of Sciences,
1, M. Aleksidze St, 0193 Tbilisi
Georgia