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ON THE CONVERGENCE RATE ANALYSIS OF ONE DIFFERENCE SCHEME FOR BURGERS' EQUATION

Abstract. We consider an initial boundary value problem for the 1D nonlinear Burgers' equation. A three-level finite difference scheme is studied. Two-level scheme is used to find the values of unknown function on the first level. The obtained algebraic equations are linear with respect to the values of the unknown function for each new level. It is proved that the scheme is convergent at rate $O(\tau^{k-1}+h^{k-1})$ in discrete L_2-norm when an exact solution belongs to the Sobolev space W_2^k, $2<k\leq 3$.

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რეზიუმე. განხილულია ერთგანზომილებიანი არაწრფივი ბურგერსის განტოლებისთვის დასმული საწყის-სასაზღვრო ამოცანა. შესწავლილია სამშრიანი სასრულ-სხვაობიანი სქემა. უცნობი ფუნქციის მნიშვნელობების მოსაძებნად პირველ შრეზე ორშრიანი სქემაა გამოყენებული. მიღებული ალგებრული განტოლებები წრფივია უცნობი ფუნქციის მნიშვნელობების მიმართ ყოველ ახალ შრეზე. დამტკიცებულია, რომ თუ ზუსტი ამონახსნი მიეკუთვნება სობოლევის W_2^k, $2<k\leq 3$, სივრცეს, მაშინ დისკრეტული L_2 ნორმით სქემის კრებადობის სიჩქარეა $O(\tau^{k-1}+h^{k-1})$.

1. Introduction

We will study the finite difference method for a numerical solution of initial boundary value problem for a forced Burgers' equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - \nu\,\frac{\partial^2 u}{\partial x^2} = f, \quad (x,t)\in Q, \tag{1.1}$$

$$u(0,t)=u(1,t)=0, \quad t\in[0,T), \quad u(x,0)=\varphi(x), \quad x\in[0,1], \tag{1.2}$$

where $Q=(0,1)\times(0,T)$, and parameter $\nu = \mathit{const} > 0$ defines the kinematic viscosity.

Assume that a solution of this problem belongs to the fractional-order Sobolev space $W_2^k(Q)$, $k>2$, whose norms and seminorms we denote by $\|\cdot\|_{W_2^k(Q)}$ and $|\cdot|_{W_2^k(Q)}$, respectively.

Certain numerical methods (Galerkin, least squares, collocation, method of lines, finite differences, etc.) are devoted to problems posed for Burgers' equation (see, e.g., [1, 2, 3, 7, 10, 11, 14, 15, 16, 19]). In some cases, the Hopf–Cole transformation [9, 13] is used before approximation in order to reduce Burgers' equation to a linear heat equation.

H. Sun and Z. Z. Sun [19] investigated a three-level difference scheme for the problem (1.1), (1.2) and ascertained a second-order convergence in the maximum-norm under the assumption that the exact solution belongs to $\mathcal{C}^{4,3}(\overline{Q})$.

In the present article, a three-level difference scheme is studied for the problem (1.1), (1.2). All the obtained algebraic equations are linear with respect to the values of an unknown function on the upper level. It is proved that the scheme is convergent at rate $O(\tau^{k-1}+h^{k-1})$ when an exact solution belongs to the Sobolev space $W_2^k(Q)$, $2<k\le 3$. The error estimate is derived by using the certain well-known techniques (see, e.g., [18, 4]) that employ the generalized Bramble–Hilbert Lemma. For the upper layers, the difference equations are the same as in [19] and are obtained by using the well known approximations for derivatives. For the first layer, the difference equations are constructed with the help of approximation of $\partial(u)^2/\partial x$ by the way offered in [5, 6]. In the case of sufficiently smooth solutions, they represent the second order approximations for obtaining additional initial data. At the same time, they represent approximation of the equation (1.1) to within the accuracy $O(\tau+h^2)$.

Despite the last circumstance, the order of convergence by discrete L_2-norm does not decrease and remains still second order on sufficiently smooth solutions. "The study of the local approximation is insufficient for determination of the order of the difference approximation and proper evaluation of the quality of a difference operator" (Samarskii [17, Chapter 2, Section 1.3, Example 1]).

2. A Finite Difference Scheme and Main Results

The finite domain $[0,1]\times[0,T]$ is divided into rectangle grids by the points $(x_i,t_j)=(ih,j\tau)$, $i=0,1,\dots,n$, $j=0,1,2,\dots,J$, where $h=1/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

Let $\overline{\omega}=\{x_i:\ i=0,1,\dots,n\}$, $\omega=\{x_i:\ i=1,2,\dots,n-1\}$, $\omega^+=\{x_i:\ i=1,2,\dots,n\}$.

The value of the mesh function U at the node (x_i,t_j) is denoted by U_i^j, that is, $U(ih,j\tau)=U_i^j$. For the sake of simplicity sometimes we will use notation without subscripts: $U_i^j=U$, $U_i^{j+1}=\widehat{U}$, $U_i^{j-1}=\check{U}$. Moreover, let

$$\overline{U}^{\,0}=\frac{U^1+U^0}{2}, \quad \overline{U}^{\,j}=\frac{U^{j+1}+U^{j-1}}{2}, \quad j=1,2,\dots .$$

We define the difference quotients in x and t directions as follows:

$$(U_i)_{\overline{x}}=\frac{U_i-U_{i-1}}{h}, \quad (U_i)_{\overset{\circ}{x}}=\frac{1}{2h}(U_{i+1}-U_{i-1}), \quad (U_i)_{\overline{x}x}=\frac{U_{i+1}-2U_i+U_{i-1}}{h^2},$$

$$(U^j)_t=\frac{U^{j+1}-U^j}{\tau}, \quad (U^j)_{\overset{\circ}{t}}=\frac{U^{j+1}-U^{j-1}}{2\tau}, \quad (U^j)_{\overline{t}t}=\frac{U^{j+1}-2U^j+U^{j-1}}{\tau^2}.$$

Let H_0 be a set of functions defined on the mesh $\overline{\omega}$ and equal to zero at $x=0$ and $x=1$. On H_0 we define the following inner product and norm:

$$(U,V)=\sum_{x\in\omega}hU(x)V(x), \quad \|U\|=(U,U)^{1/2}.$$

Let, moreover,

$$(U,V]=\sum_{x\in\omega^+}hU(x)V(x),\quad \|U]|=(U,U]^{1/2}.$$

We need the following averaging operators for the functions defined on Q:

$$\widehat{\mathcal{S}}v:=\frac{1}{\tau}\int\limits_{t}^{t+h}v(x,\xi)\,d\xi,\quad \mathring{\mathcal{S}}v:=\frac{1}{2\tau}\int\limits_{t-h}^{t+h}v(x,\xi)\,d\xi,$$

$$\widehat{\mathcal{P}}v:=\frac{1}{h}\int\limits_{x}^{x+h}v(\xi,t)\,d\xi,\quad \mathcal{P}v:=\frac{1}{h^2}\int\limits_{x-h}^{x+h}\big(h-|x-\xi|\big)v(\xi,t)\,d\xi.$$

Note that

$$\mathring{\mathcal{S}}\,\frac{\partial v}{\partial t}=v_{\mathring{t}},\quad \widehat{\mathcal{S}}\,\frac{\partial v}{\partial t}=v_t,\quad \mathcal{P}\,\frac{\partial^2 v}{\partial x^2}=v_{\overline{x}\,x},\quad \mathcal{P}\,\frac{\partial v}{\partial x}=\widehat{\mathcal{P}}v_{\overline{x}}.$$

We approximate the problem (1.1), (1.2) by of the difference scheme:

$$\mathcal{L}U_i^j=F_i^j,\ \ i=1,2,\dots,n-1,\ \ j=0,1,\dots,J-1,\tag{2.1}$$

$$U_0^j=U_n^j=0,\ \ j=0,1,\dots,J,\quad U_i^0=\varphi(x_i),\ \ i=0,1,\dots,n.\tag{2.2}$$

where

$$\begin{gathered}\mathcal{L}U^0:=(U^0)_t+\frac{1}{3}\,\Lambda U^0-\nu(\overline{U}^{\,0})_{\overline{x}\,x},\\ \Lambda U^0:=U^0(\overline{U}^{\,0})_{\mathring{x}}+(U^0\overline{U}^{\,0})_{\mathring{x}},\quad F^0:=\mathcal{P}\overline{f}^{\,0},\\ \mathcal{L}U^j:=(U^j)_{\mathring{t}}+\frac{1}{3}\Lambda U^j-\nu(\overline{U}^{\,j})_{\overline{x}\,x},\ \ j=1,2,\dots,\\ \Lambda U^j=U^j(\overline{U}^{\,j})_{\mathring{x}}+(U^j\overline{U}^{\,j})_{\mathring{x}},\quad F^j:=\mathcal{P}\overline{f}^{\,j}.\end{gathered}$$

Theorem 2.1. *The finite difference scheme* (2.1), (2.2) *is uniquely solvable.*

Proof. Note that

$$(YV_{\mathring{x}}+(YV)_{\mathring{x}},V)=0,\ \text{ if }\ V\in H_0.\tag{2.3}$$

Considering inner products $(\mathcal{L}U^j,\overline{U}^{\,j})$ and $(\mathcal{L}U^0,\overline{U}^{\,0})$, we obtain

$$\frac{1}{4\tau}\left(\|U^{j+1}\|^2-\|U^{j-1}\|^2\right)+\nu\|\overline{U}^{\,j}_{\overline{x}}]|^2=(F^j,\overline{U}^{\,j}),\ \ j=1,2,\dots,\tag{2.4}$$

$$\frac{1}{2\tau}\left(\|U^1\|^2-\|U^0\|^2\right)+\nu\|\overline{U}^{\,0}_{\overline{x}}]|^2=(F^0,\overline{U}^{\,0}).\tag{2.5}$$

Summing up the equalities (2.4) with respect to j from 1 to k, we get

$$\frac{1}{2\tau}\left(\|U^{k+1}\|^2+\|U^k\|^2-\|U^1\|^2-\|U^0\|^2\right)+2\nu\sum_{j=1}^{k}\|\overline{U}^{\,j}_{\overline{x}}]|^2=2\sum_{j=1}^{k}(F^j,\overline{U}^{\,j}).\tag{2.6}$$

Adding the equalities (2.5) and (2.6) gives

$$\frac{1}{2\tau}\left(\|U^{k+1}\|^2+\|U^k\|^2\right)+2\nu\sum_{j=0}^{k}\sigma_j\|\overline{U}^{\,j}_{\overline{x}}]|^2=\frac{1}{\tau}\,\|U^0\|^2+2\sum_{j=0}^{k}\sigma_j(F^j,\overline{U}^{\,j}),\ \ k=1,2,\dots,\tag{2.7}$$

where $\sigma_j=1$ for $j\geq 1$ and $\sigma_0=1/2$.

If we rewrite the equality (2.5) in the form

$$\frac{1}{2\tau}\left(\|U^1\|^2+\|U^0\|^2\right)+\nu\|\overline{U}^{\,0}_{\overline{x}}]|^2=\frac{1}{\tau}\,\|U^0\|^2+(F^0,\overline{U}^{\,0}),\tag{2.8}$$

we will see that the equalities (2.7), (2.8) can be written all in the same key

$$\frac{1}{2}\left(\|U^{j+1}\|^2+\|U^j\|^2\right)+2\nu\tau\sum_{k=0}^{j}\sigma_k\|\overline{U}^{\,k}_{\overline{x}}]|^2=\|\varphi\|^2+2\tau\sum_{k=0}^{j}\sigma_k(F^j,\overline{U}^{\,j}),\ \ j=0,1,2,\dots.\tag{2.9}$$

Since the difference scheme (2.1), (2.2) is linear on each new level with respect to the unknown values, its unique solvability follows directly from (2.9). □

Remark. Let the external source $f(x,t)$ be equal to 0. Then we rewrite (2.9) as

$$E(U^j)+\nu\sum_{k=0}^{j}\sigma_k\tau\|\overline{U}_{\overline{x}}^{k}\|^2=0.5\|\varphi\|^2,\ \ j=0,1,\dots .$$

The left-hand side of this equality is the energy of the system at time $t=t_j$. As we see, the difference scheme is energy conservative and, besides, kinetic energy

$$E(U^j):=\frac{\|U^{j+1}\|^2+\|U^j\|^2}{4}$$

is monotonically decreasing, i.e.,

$$E(U^{j+1})\le E(U^j)\ \text{ for }\ j\ge 0.$$

Theorem 2.2. *Let the exact solution of the initial boundary value problem* (1.1), (1.2) *belong to* $W_2^k(Q)$, $2<k\le 3$. *Then the convergence rate of the finite difference scheme* (2.1), (2.2) *is determined by the estimate*

$$\|U^j-u^j\|\le c(\tau^{k-1}+h^{k-1})\|u\|_{W_2^k(Q)},$$

where $c=c(u)$ *denotes the positive constant, independent of* h *and* τ.

The correctness of Theorem 2.2 follows from the consequence of Lemmas 3.1, 4.2 and 4.4, proved in the next sections.

3. A Priori Estimate of Discretization Error

Let $Z:=U-u$, where u is an exact solution of the problem (1.1), (1.2), and U is a solution of the finite difference scheme (2.1), (2.2). Substituting $U=Z+u$ into (2.1), (2.2), we obtain

$$Z^j_{\overset{\circ}{t}}-\nu\overline{Z}^j_{\overline{x}x}=-\frac{1}{3}\,(\Lambda U^j-\Lambda u^j)+\Psi^j, \tag{3.1}$$

$$Z^0_t-\nu\overline{Z}^0_{\overline{x}x}=-\frac{1}{3}\,(\Lambda U^0-\Lambda u^0)+\Psi^0, \tag{3.2}$$

$$Z^0=0,\quad Z^j_0=Z^j_n=0,\ \ j=0,1,2,\dots , \tag{3.3}$$

where $\Psi^j:=F^j-\mathcal{L}u^j$.

Denote

$$B_j:=\|Z^j\|^2+\|Z^{j-1}\|^2,\ \ j=1,2,\dots .$$

Lemma 3.1. *For a solution of the problem* (3.1)–(3.3)*, the relations*

$$B_1\le\|\tau\Psi^0\|^2, \tag{3.4}$$

$$B_{j+1}\le c_1B_1+c_2\tau\sum_{k=1}^{j}\|\Psi^k\|^2,\ \ j=1,2,\dots , \tag{3.5}$$

are valid, where

$$c_1=\exp\left(\frac{Tc_*^2}{3\nu}\right),\ \ c_2=\frac{c_1}{2\nu}\,,\ \ c_*=\|u\|_{C^1(\overline{Q})}.$$

Proof. Multiplying (3.2) by $\overline{Z}^0$, we obtain

$$(Z^0_t,\overline{Z}^0)+\nu(\overline{Z}^0_{\overline{x}},\overline{Z}^0_{\overline{x}})=-\frac{1}{3}\,(\Lambda U^0-\Lambda u^0,\overline{Z}^0)+(\Psi^0,\overline{Z}^0).$$

Taking into account $U^0=u^0$ we have

$$\Lambda U^0-\Lambda u^0=u^0\overline{Z}^0_{\overset{\circ}{x}}+(u^0\overline{Z}^0)_{\overset{\circ}{x}},$$

therefore due to (2.3)

$$(\Lambda U^0-\Lambda u^0,\overline{Z}^0)=0$$

and we get

$$(Z_t^0, \overline{Z}^0) + \nu(\overline{Z}_{\overline{x}}^0, \overline{Z}_{\overline{x}}^0) = (\Psi^0, \overline{Z}^0).$$

From this, via $Z^0 = 0$, we see that

$$\frac{1}{2\tau}\,\|Z^1\|^2 + \frac{\nu}{4}\,\|Z_{\overline{x}}^1\|^2 = \frac{1}{2}\,(\Psi^0, Z^1),$$

or

$$\|Z^1\|^2 + \frac{\nu\tau}{2}\,\|Z_{\overline{x}}^1\|^2 = (\tau\Psi^0, Z^1),$$

where

$$\|Z^1\|^2 + \frac{\nu\tau}{2}\,\|Z_{\overline{x}}^1\|^2 \le \frac{1}{4}\,\|\tau\Psi^0\|^2 + \|Z^1\|^2$$

and

$$\|Z_{\overline{x}}^1\|^2 \le \frac{\tau}{2\nu}\,\|\Psi^0\|^2,$$

and also

$$\|Z^1\|^2 \le \|\tau\Psi^0\|\,\|Z^1\|$$

and

$$\|Z^1\| \le \|\tau\Psi^0\|.$$

On the basis of the above consideration, we come to the conclusion that (3.4) is true.

Now, let us multiply (3.1) by $\overline{Z}^j$ scalarly:

$$\frac{1}{4\tau}\,\big(\|Z^{j+1}\|^2 - \|Z^{j-1}\|^2\big) + \nu\|\overline{Z}_{\overline{x}}^j]|^2 = -\frac{1}{3}\,(\Lambda U^j - \Lambda u^j, \overline{Z}^j) + (\Psi^j, \overline{Z}^j), \ \ j = 1, 2, \dots . \tag{3.6}$$

Noticing in the right-hand side of (3.6) that

$$\Lambda U^j - \Lambda u^j = (U^j \overline{Z}_{\overset{\circ}{x}}^j + (U^j \overline{Z}^j)_{\overset{\circ}{x}}) + (Z^j \overline{U}_{\overset{\circ}{x}}^j + (Z^j \overline{U}^j)_{\overset{\circ}{x}}),$$

and taking into account (2.3), we obtain

$$(\Lambda U^j - \Lambda u^j, \overline{Z}^j) = (Z^j \overline{u}_{\overset{\circ}{x}}^j + (Z^j \overline{u}^j)_{\overset{\circ}{x}}, \overline{Z}^j) = (Z^j \overline{u}_{\overset{\circ}{x}}^j, \overline{Z}^j) - (Z^j \overline{u}^j, \overline{Z}_{\overset{\circ}{x}}^j) = (Z^j \overline{Z}^j, \overline{u}_{\overset{\circ}{x}}^j) - (Z^j \overline{Z}_{\overset{\circ}{x}}^j, \overline{u}^j).$$

Applying here the Cauchy–Bunyakovsky inequality, the ε-inequality, and finally the Friedrichs' inequality

$$\|V\|^2 \le \frac{1}{8}\,\|V_{\overline{x}}]|^2,$$

we obtain

$$\begin{aligned} \big|(\Lambda U^j - \Lambda u^j, \overline{Z}^j)\big| &\le c_*\big(\|Z^j\|\,\|\overline{Z}^j\| + \|Z^j\|\,\|\overline{Z}_{\overline{x}}^j]|\big) \\ &\le c_*\Big(\frac{\varepsilon}{2}\,\|Z^j\|^2 + \frac{1}{2\varepsilon}\,\|\overline{Z}^j\|^2 + \frac{\varepsilon}{2}\,\|Z^j\|^2 + \frac{1}{2\varepsilon}\,\|\overline{Z}_{\overline{x}}^j]|^2\Big) \le c_*\Big(\varepsilon\|Z^j\|^2 + \frac{9}{16\varepsilon}\,\|\overline{Z}_{\overline{x}}^j]|^2\Big). \end{aligned} \tag{3.7}$$

Now, let us estimate the second term in the right-hand side of (3.6)

$$|(\Psi^j, \overline{Z}^j)| \le \|\Psi^j\|\,\|\overline{Z}^j\| \le \frac{\varepsilon}{2c_*}\|\Psi^j\|^2 + \frac{c_*}{2\varepsilon}\|\overline{Z}^j\|^2 \le \frac{\varepsilon}{2c_*}\,\|\Psi^j\|^2 + \frac{c_*}{16\varepsilon}\,\|\overline{Z}_{\overline{x}}^j]|^2. \tag{3.8}$$

After substituting (3.7) and (3.8) in (3.6), we arrive at

$$\begin{aligned} \frac{1}{4\tau}\,\big(\|Z^{j+1}\|^2 - \|Z^{j-1}\|^2\big) + \nu\|\overline{Z}_{\overline{x}}^j]|^2 &\le c_*\Big(\frac{\varepsilon}{3}\,\|Z^j\|^2 + \frac{3}{16\varepsilon}\,\|\overline{Z}_{\overline{x}}^j]|^2\Big) + \frac{\varepsilon}{2c_*}\,\|\Psi^j\|^2 + \frac{c_*}{16\varepsilon}\,\|\overline{Z}_{\overline{x}}^j]|^2 \\ &\le \frac{\varepsilon c_*}{3}\,\|Z^j\|^2 + \frac{\varepsilon}{2c_*}\,\|\Psi^j\|^2 + \frac{c_*}{4\varepsilon}\,\|\overline{Z}_{\overline{x}}^j]|^2. \end{aligned}$$

Here choose $\varepsilon = \frac{c_*}{4\nu}$. Then we obtain

$$\frac{1}{4\tau}\,\big(\|Z^{j+1}\|^2 - \|Z^{j-1}\|^2\big) \le \frac{1}{8\nu}\,\|\Psi^j\|^2 + \frac{c_*^2}{12\nu}\,\|Z^j\|^2,$$

that is,

$$\|Z^{j+1}\|^2 - \|Z^{j-1}\|^2 \le \frac{\tau}{2\nu}\,\|\Psi^j\|^2 + \frac{c_*^2\tau}{3\nu}\,\|Z^j\|^2, \ \ j = 1, 2, \dots . \tag{3.9}$$

Suppose

$$a := \frac{c_*^2}{3\nu}\,, \quad b := \frac{1}{2\nu}\,.$$

From (3.9) we find

$$B_{j+1} \le (1+a\tau)B_j + b\tau\|\Psi^j\|^2, \;\; j=1,2,\dots,$$

whence

$$B_{j+1} \le (1+a\tau)^j B_1 + b\tau(1+a\tau)^{j-1}\sum_{k=1}^{j}\|\Psi^k\|^2, \;\; j=1,2,\dots. \tag{3.10}$$

Since $j \le T/\tau$, we obtain

$$(1+a\tau)^j \le (1+a\tau)^{T/\tau} \le \exp(Ta),$$

and on the basis of (3.10), the validity of (3.5) follows directly. Thus Lemma 3.1 is proved. □

4. Estimation of the Truncation Error

In order to determine the rate of convergence of the finite difference scheme (2.1), (2.2) with the help of Lemma 3.1, it is sufficient to estimate a truncation error eventuated while replacing a differential equation by a difference scheme, Ψ. Towards this end, we will need the following result.

Lemma 4.1. *Assume that the linear functional $l(u)$ is bounded in $W_2^k(E)$, where $k=\overline{k}+\epsilon$, $\overline{k}$ is an integer, $0<\epsilon\le 1$, and $l(P)=0$ for every polynomial P of degree $\overline{k}$ in two variables. Then, there exists a constant c, independent of u, such that $|l(u)| \le c|u|_{W_2^k(E)}$.*

This lemma is a particular case of the Dupont–Scott approximation theorem [12] and represents a generalization of the Bramble–Hilbert lemma [8] (see, e.g., [18, p. 29]).

Let us introduce the elementary rectangles $e=e(x,t)=\{(x,t):\;|x-x_i|\le h,\;|t-t_j|\le\tau\}$, $e_0=(x_{i-1},x_{i+1})\times(0,\tau)$, $Q_\tau=(0,1)\times(0,\tau)$, $Q_j=(0,1)\times(t_{j-1},t_{j+1})$.

Lemma 4.2. *If a solution u of the problem* (1.1), (1.2) *belongs to the Sobolev space $W_2^k(Q)$, $2<k\le 3$, then for the truncation error $\Psi^j=F^j-\mathcal{L}u^j$ the estimate*

$$\|\Psi^j\|^2 \le c(\tau+h)^{2k-3}\|u\|^2_{W_2^k(Q_j)}, \;\; j\ge 1,$$

is true, where the constant $c>0$ does not depend on the mesh steps.

Proof. Apply operator $\mathcal{P}$ to the equation (1.1):

$$\frac{1}{2}\,\mathcal{P}\Big(\frac{\partial u^{j-1}}{\partial t}+\frac{\partial u^{j+1}}{\partial t}+\Big(u\,\frac{\partial u}{\partial x}\Big)^{j+1}+\Big(u\,\frac{\partial u}{\partial x}\Big)^{j-1}\Big)-\frac{\nu}{2}\,(u^{j+1}+u^{j-1})_{\overline{x}x}=F^j.$$

With the help of this equality, the expression Ψ can be written in the form

$$\Psi=\chi_1+\chi_3+\frac{1}{6}\,\chi_4,$$

where

$$\chi_1=\mathcal{P}\Big(\frac{\partial\overline{u}}{\partial t}\Big)+\overset{\circ}{\mathcal{S}}\Big(\frac{\partial u}{\partial t}\Big),$$

$$\chi_2:=\frac{1}{4}\,\mathcal{P}\Big(\frac{\partial(\widehat{u})^2}{\partial x}+\frac{\partial(\check{u})^2}{\partial x}\Big)-\frac{1}{2}(u)^2_{\overset{\circ}{x}}, \quad \chi_4:=3(u)^2_{\overset{\circ}{x}}-2\Lambda u.$$

We assert that the following inequalities hold for $\alpha=1,2,3$:

$$|\chi_\alpha|\le c(\tau+h)^{k-2}\|u\|_{W_2^k(e)}, \;\; 2<k\le 3. \tag{4.1}$$

First of all, note that χ_1, as a linear functional with respect to $u(x,t)$, vanishes on the polynomials of second degree and is bounded in W_2^k, $k>1$. Consequently, using Lemma 4.1 and the well known techniques from [18], we see that the estimate (4.1) for $\alpha=1$ is true.

Now, let us note that

$$\chi_2=\chi_2(u)=\ell(v):=\frac{1}{2}\,(\widehat{\mathcal{P}}\overset{\circ}{\mathcal{S}}v_{\overline{x}}-v_{\overset{\circ}{x}}), \;\; v:=(u)^2.$$

The linear functional $\ell(v)$ is bounded for $v\in W_2^k$, $k>2$, and vanishes on polynomials of second degree. For this functional the estimate

$$|\ell(v)|\le c(\tau+h)^{k-2}\|v\|_{W_2^k(e)}, \;\; 2<k\le 3, \tag{4.2}$$

is obtained.

Since Sobolev space $W_2^k(Q)$, $k > 1$, is an algebra with respect to a pointwise multiplication, consequently, $\|uu\|_{W_2^k(e)} \le c\|u\|_{W_2^k(e)}$, $c = c(u)$. Therefore, (4.2) proves the validity of (4.1) in the case where $\alpha = 2$.

We will present estimates χ_3 in a more convenient form. We have

$$\begin{aligned}\chi_3 &= 3(u)^2_{\overset{\circ}{x}} - u(\widehat{u} + \check{u})_{\overset{\circ}{x}} - (u(\widehat{u} + \check{u}))_{\overset{\circ}{x}}\\ &= 3(u)^2_{\overset{\circ}{x}} - u(\widehat{u} - 2u + \check{u})_{\overset{\circ}{x}} - \big(u(\widehat{u} - 2u + \check{u})\big)_{\overset{\circ}{x}} - 2uu_{\overset{\circ}{x}} - 2(uu)_{\overset{\circ}{x}}\\ &= (u)^2_{\overset{\circ}{x}} - 2uu_{\overset{\circ}{x}} - \tau^2 uu_{\bar{t}\,t\overset{\circ}{x}} - \tau^2 (uu_{\bar{t}\,t})_{\overset{\circ}{x}},\end{aligned}$$

whence

$$\chi_3 = h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x} - \tau^2 uu_{\bar{t}\,t\overset{\circ}{x}} - \tau^2 (uu_{\bar{t}\,t})_{\overset{\circ}{x}} := \chi_3' + \chi_3'' + \chi_3''', \tag{4.3}$$

since

$$(u)^2_{\overset{\circ}{x}} - 2uu_{\overset{\circ}{x}} = u_{\overset{\circ}{x}}(u_{i+1} + u_{i-1}) - 2uu_{\overset{\circ}{x}} = h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x}.$$

When $u \in W_2^k(Q)$, $2 < k \le 3$, the terms in the right-hand side of (4.3) can be estimated as follows:

$$\begin{gathered}|\chi_3'| \le h^2\|u\|_{C^1(\overline{Q})} |u_{\overline{x}\,x}| \le c(\tau + h)^{k-2}\|u\|_{W_2^2(e)} \le c(\tau + h)^{k-2}\|u\|_{W_2^{k-2}(e)},\\ |\chi_3''| \le \tau^2\|u\|_{C^1(\overline{Q})} |u_{\bar{t}\,t\overset{\circ}{x}}| \le c(\tau + h)^{k-2}\|u\|_{W_2^{k-2}(e)},\\ |\chi_3'''| = \tau^2|u_{i+1}u_{\bar{t}\,t\overset{\circ}{x}} + u_{\overset{\circ}{x}} u_{\bar{t}\,t,i-1}| \le \|u\|_{C^1(\overline{Q})}(|u_{\bar{t}\,t\overset{\circ}{x}}| + |u_{\bar{t}\,t,i-1}|) \le c(\tau + h)^{k-2}\|u\|_{W_2^{k-2}(e)}\end{gathered}$$

and therefore (4.1) is true for $\alpha = 3$ also.

Finally, (4.1) yields

$$\|\chi_\alpha\|^2 = \sum_{x\in\omega} h|\chi_\alpha|^2 \le c(\tau + h)^{2k-3}\|u\|^2_{W_2^k(Q_j)}, \quad \alpha = 1, 2, 3,$$

which completes the proof of Lemma 4.2. □

Lemma 4.3. *For any function $v \in W_2^k(Q)$, $1 < k \le 3$, the inequalities*

$$\|v^0_{\overset{\circ}{x}t}\| \le c(\tau + h)^{k-3}\|v\|_{W_2^k(Q)}, \tag{4.4}$$

$$\|v^0_{\overline{x}\,x}\| \le c(\tau + h)^{k-3}\|v\|_{W_2^k(Q)} \tag{4.5}$$

are true.

Proof. $v^0_{\overset{\circ}{x}t}$ is bounded when $v \in W_2^\lambda(Q)$, $\lambda > 1$, and vanishes on the first degree polynomials. Therefore for $1 < \lambda \le 2$ we have

$$|v^0_{\overset{\circ}{x}t}| \le c(\tau + h)^{\lambda-3}\|v\|_{W_2^\lambda(e_0)},$$

$$\|v^0_{\overset{\circ}{x}t}\|^2 = \sum_{\omega} h|v^0_{\overset{\circ}{x}t}|^2 \le c(\tau + h)^{2\lambda-5}\|v\|^2_{W_2^\lambda(Q_\tau)},$$

which confirms the validity of (4.4) in the case where $1 < k \le 2.5$. Further,

$$|v^0_{\overset{\circ}{x}t}| = \frac{1}{2\tau h}\bigg|\int_0^\tau\int_{x_{i-1}}^{x_{i+1}} \frac{\partial^2 v}{\partial x\partial t}\,dx\,dt\bigg| \le (2\tau h)^{-1/2}\Big\|\frac{\partial^2 v}{\partial x\partial t}\Big\|_{L_2(e_0)},$$

$$\|v^0_{\overset{\circ}{x}t}\| \le c\tau^{-1/2}\Big\|\frac{\partial^2 v}{\partial x\partial t}\Big\|_{L_2(Q_\tau)}. \tag{4.6}$$

In order to obtain the desired estimate, it is sufficient to use the inequality giving estimate of the L_2-norm of the function in the near-border stripe via its W_2^λ-norm in the domain (cf. [18, p. 161])

$$\|v\|_{L_2(Q_\tau)} \le c\tau^{1/2}\|v\|_{W_2^\lambda(Q)}, \quad 0.5 < \lambda \le 1.$$

This relation along with (4.6) confirms the validity of (4.4) for $2.5 < k \le 3$.

When $1 < k \le 2.5$, (4.5) can be proved similarly to the previous case. In the event of $2.5 < k \le 3$, we use the relation

$$|u_{\overline{x}\,x}| \le |\mathcal{P}\widehat{\mathcal{S}}\,\frac{\partial^2 u}{\partial x^2}| + |(u - \widehat{\mathcal{S}})_{\overline{x}\,x}|.$$

Here the first term in the right-hand side is estimated again analogously to the previous case, and for the second term Lemma 4.1 is used. □

Lemma 4.4. *If a solution u of the problem* (1.1), (1.2) *belongs to the Sobolev space* $W_2^k(Q)$, $k > 2$, *then for the truncation error* $\Psi^0 = F^0 - \mathcal{L}u^0$ *the estimate*

$$\|\Psi^0\| \le c(\tau+h)^{k-2}\|u\|^2_{W_2^k(Q)}, \;\; 2 < k \le 3,$$

is true, where the constant $c > 0$ *does not depend on the mesh steps.*

Proof. Apply operator $\mathcal{P}$ to the equation (1.1):

$$F^0 = \frac{1}{2}\,\mathcal{P}(f^0+f^1) = \frac{1}{2}\,\mathcal{P}\Big(\frac{\partial u^0}{\partial t} + \frac{\partial u^1}{\partial t}\Big) + \frac{1}{4}\,\mathcal{P}\bigg(\frac{\partial (u)^2}{\partial x}\Big|_{t=0} + \frac{\partial (u)^2}{\partial x}\Big|_{t=\tau}\bigg) - \nu \overline{u}_{\overline{x}\,x}.$$

Via this equality we rewrite Ψ^0 as

$$\Psi^0 = \zeta_1 - \frac{1}{6}\,\zeta_2 - \frac{1}{2}\,\zeta_3, \;\; t=0,$$

where

$$\begin{aligned}\zeta_1 &:= \mathcal{P}\,\frac{\partial \overline{u}}{\partial t} - u^0_t, \quad \zeta_2 := 2(u\overline{u}_{\overset{\circ}{x}} + (u\overline{u})_{\overset{\circ}{x}}) - \frac{3}{2}\,\big((\widehat{u})^2 + (u)^2\big)_{\overset{\circ}{x}}, \\ \zeta_3 &:= \frac{1}{2}\,\big((\widehat{u})^2 + (u)^2\big)_{\overset{\circ}{x}} - \frac{1}{2}\,\mathcal{P}\bigg(\frac{\partial (u)^2}{\partial x}\Big|_{t=0} + \frac{\partial (u)^2}{\partial x}\Big|_{t=\tau}\bigg).\end{aligned} \tag{4.7}$$

We assert that the inequalities

$$\|\zeta_\alpha\| \le c(\tau+h)^{k-2}\|u\|_{W_2^k(Q)}, \;\; 2 < k \le 3, \tag{4.8}$$

hold for $\alpha = 1,2,3$.

Expression ζ_1 can be estimated similarly to χ_1.

Further, notice that

$$\zeta_3 = \zeta_3(u) = I(v) := \frac{1}{2}\,(\widehat{v}+v)_{\overset{\circ}{x}} - \frac{1}{2}\,\mathcal{P}\Big(\frac{\partial \widehat{v}}{\partial x} + \frac{\partial v}{\partial x}\Big), \;\; v := (u)^2.$$

It is easy to verify that $I(v)$, as a linear functional with respect to v, vanishes on the polynomials of second degree and is bounded when $v \in W_2^k(Q)$, $k > 2$. For that functional we can derive the following estimate

$$\|I(v)\| \le c(\tau+h)^{k-2}\|v\|_{W_2^k(Q)}, \;\; 2 < k \le 3.$$

The latter along with $\|uu\|_{W_2^k(Q)} \le c\|u\|^2_{W_2^k(Q)}$, $k > 1$, states the validity of (4.8) in the case $\alpha = 3$, as well.

Now, let us pass to the estimation of ζ_2. If we take into account that

$$2u\overline{u}_{\overset{\circ}{x}} = 2uu_{\overset{\circ}{x}} + \tau uu_{\overset{\circ}{x}t}, \quad 2uu_{\overset{\circ}{x}} = (u)^2_{\overset{\circ}{x}} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x},$$

(4.7) will give

$$\begin{aligned}\zeta_2 &= \tau uu_{\overset{\circ}{x}t} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x} + \frac{1}{2}\,\big(4u\overline{u} - 3(\widehat{u})^2 - (u)^2\big)_{\overset{\circ}{x}} \\ &= \tau uu_{\overset{\circ}{x}t} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x} - \frac{1}{2}\,\Big(2\big[(\widehat{u})^2 - (u)^2\big] + \big[(\widehat{u})^2 - 2\widehat{u}u + (u)^2\big]\Big)_{\overset{\circ}{x}}\end{aligned}$$

or

$$\zeta_2 = \tau uu_{\overset{\circ}{x}t} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}\,x} - \tau (u)^2_{\overset{\circ}{x}t} - \frac{\tau^2}{2}\,(u_t)^2_{\overset{\circ}{x}} := \zeta_2' + \zeta_2'' + \zeta_2''' + \zeta_2''''. \tag{4.9}$$

In the right-hand side of (4.9), the first and the second terms can be estimated by using Lemma 4.3:

$$\begin{aligned}\|\zeta_2'\| &\le c\tau\|u\|_{C(\overline{Q})}\|u_{\overset{\circ}{x}t}\| \le c(\tau+h)^{k-2}\|u\|_{W_2^k(Q)}, \;\; 2 < k \le 3, \\ \|\zeta_2''\| &\le ch\|u\|_{C(\overline{Q})}\|u_{\overline{x}\,x}\| \le c(\tau+h)^{k-2}\|u\|_{W_2^k(Q)}, \;\; 2 < k \le 3.\end{aligned}$$

The term ζ_2''' can be estimated in a similar way, if we make replacement $(u)^2 := v$ in it.

Change the term ζ_2'''' as follows:

$$\frac{\tau^2}{2}\,(u_t)^2_{\overset{\circ}{x}} = \frac{\tau^2}{2}\,\frac{(u_t)^2_{i+1} - (u_t)^2_{i-1}}{2h} = \frac{\tau^2}{2}\,\frac{(u_{t,i+1} - u_{t,i-1})(u_{t,i+1} + u_{t,i-1})}{2h} = \tau^2 u_{\overset{\circ}{x}t}\,\frac{u_{t,i+1} + u_{t,i-1}}{2}\,,$$

from which again via Lemma 4.3 we get

$$\|\zeta_2''''\| \le c\tau|u_{\overset{\circ}{x}t}| \le c(\tau+h)^{k-2}\|u\|_{W_2^k(Q)}, \quad 2 < k \le 3.$$

Finally, all of these estimates confirm the validity of (4.8) in the case $\alpha = 2$.

The inequalities (4.8) prove Lemma 4.4. □

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