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ON A NONLOCAL BOUNDARY-VALUE PROBLEM FOR TWO-DIMENSIONAL ELLIPTIC EQUATION

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Dedicated to Raytcho Lazarov on the occasion of his 60th birthday.

Abstract — A boundary-value problem with a nonlocal integral condition is considered for a two-dimensional elliptic equation with constant coefficients and a mixed derivative. The existence and uniqueness of a weak solution of this problem are proved in a weighted Sobolev space. A difference scheme is constructed using the Steklov averaging operators. It is proved that the difference scheme converges in discrete $W_2^1(\omega,\rho)$ norm with the rate $O(h^{m-1})$, $m \in (1;3]$, when the solution of the problem belongs to the space $W_2^m(\Omega)$.

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1. Introduction

Boundary-value problems for differential equations with a nonlocal condition occur in many applications. Problems with integral conditions were considered by various authors (see, e.g., [1, 8, 9]). In the present paper, a nonlocal boundary problem with integral restriction is considered in a domain $\Omega = (0,1)^2$ for a second order elliptic equation with constant coefficients.

In Section 2, existence and uniqueness of a weak solution of this problem in the weighted Sobolev space $W_2^1(\Omega,\rho)$, $\rho(x) = x_1^\varepsilon$, $\varepsilon \in (0;1)$ is proved.

In Section 3, the corresponding difference scheme is constructed. Under the assumption that the solution to the original problem belongs to Sobolev spaces, the estimate of convergence rate

$$||y-u||_{W_2^1(\omega,\rho)} \leqslant ch^{m-1}||u||_{W_p^m(\Omega)}, \quad m \in (1;3] \tag{1}$$

is obtained, where ω is a uniform grid in Ω with the step h, $p = 2$ for $\varepsilon \in (0.5;1)$, $p > 1/\varepsilon$ for $\varepsilon \in (0;0.5]$.

2. Solvability of a nonlocal problem

Let $\Omega = \{(x_1,x_2) : 0 < x_k < 1, k = 1,2\}$ be a unit square with a boundary Γ, and let $\Gamma_1 = \{(0,x_2) : 0 < x_2 < 1\}$, $\Gamma_* = \Gamma\backslash\Gamma_1$.

Consider the nonlocal boundary-value problem with constant coefficients

$$Lu = f(x),\ x \in \Omega, \quad u(x) = 0,\ x \in \Gamma_*, \quad l(u) = 0,\ 0 < x_2 < 1, \tag{2}$$

where

$$Lu = -\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + a_0 u, \quad l(u) = \int_0^1 \beta(x_1)u(x)\,dx_1,\ \ \beta(t) = \varepsilon t^{\varepsilon-1},\ \varepsilon \in (0;1)$$

and with the coefficients satisfying the following conditions:

$$\sum_{i,j=1}^{2} a_{ij}t_i t_j \geqslant \nu_1(t_1^2 + t_2^2), \quad \nu_1 > 0, \ \ a_0 \geqslant 0. \tag{3}$$

Let

$$(u, v) = \int_\Omega u(x)v(x)\,dx, \ \ ||u|| = (u, u)^{1/2}.$$

By $L_2(\Omega, \rho)$ we denote the weighted Lebesgue space of all real-valued functions $u(x)$ on Ω with the inner product and the norm

$$(u, v)_{L_2(\Omega,\rho)} = \int_\Omega \rho(x)u(x)v(x)\,dx, \ \ ||u||_{L_2(\Omega,\rho)} = (u, u)^{1/2}_{L_2(\Omega,\rho)}.$$

The weighted Sobolev space $W_2^1(\Omega, \rho)$ is usually defined as a linear set of all functions $u(x) \in L_2(\Omega, \rho)$, whose derivatives $\partial u/\partial x_k,\ k = 1, 2$ (in the generalized sense) belong to $L_2(\Omega, \rho)$. It is a normed linear space if equipped with the norm

$$||u||_{W_2^1(\Omega,\rho)} = \Big(||u||^2_{L_2(\Omega,\rho)} + |u|^2_{W_2^1(\Omega,\rho)}\Big)^{1/2}, \quad |u|^2_{W_2^1(\Omega,\rho)} = \left\|\frac{\partial u}{\partial x_1}\right\|^2_{L_2(\Omega,\rho)} + \left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_2(\Omega,\rho)}.$$

Let us choose weight function $\rho(x)$ in the following way: $\rho(x) = \rho(x_1) = \int_0^{x_1} \beta(t)\,dt = x_1^\varepsilon$.

It is well-known (see, e.g., [4, p.10], [5, Theorem 3.1]) that $W_2^1(\Omega, \rho)$ is a Banach space and $C^\infty(\bar\Omega)$ is dense in $W_2^1(\Omega, \rho)$ and in $L_2(\Omega, \rho)$. As an immediate consequence, we can define the space $W_2^1(\Omega, \rho)$ as the closure of $C^\infty(\bar\Omega)$ with respect to the norm $||\cdot||_{W_2^1(\Omega,\rho)}$, and these both definitions are equivalent.

Define the subspace of the space $W_2^1(\Omega, \rho)$ which can be obtained by closing the set

$$\overset{*}{C^\infty}(\bar\Omega) = \left\{u \in C^\infty(\bar\Omega) : \operatorname{supp} u \cap \Gamma_* = \emptyset, \ \int_0^1 \beta(x_1)\,u(x)\,dx_1 = 0,\ 0 < x_2 < 1\right\}$$

with the norm $||\cdot||_{W_2^1(\Omega,\rho)}$. Denote it by $\overset{*}{W_2^1}(\Omega, \rho)$.

Let the right-hand side $f(x)$ in equation (2) be a linear continuous functional on $\overset{*}{W_2^1}(\Omega, \rho)$ which can be represented as

$$f = f_0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, \quad f_k(x) \in L_2(\Omega, \rho), \quad k = 0, 1, 2. \tag{4}$$

We say that the function $u \in \overset{*}{W}{}_2^1(\Omega,\rho)$ is a *weak solution* of problem (2)–(4), if the relation

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in \overset{*}{W}{}_2^1(\Omega,\rho) \tag{5}$$

holds, where

$$a(u,v) = \int\limits_{\Omega} \left(a_{11} x_1^{\varepsilon} \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + (a_{12}+a_{21}) x_1^{\varepsilon} \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_1} + a_{22}\frac{\partial u}{\partial x_2} G\frac{\partial v}{\partial x_2} + a_0 u\, Gv \right) dx, \tag{6}$$

$$\langle f, v\rangle = \int\limits_{\Omega} f_0\, Gv\, dx - \int\limits_{\Omega} x_1^{\varepsilon} f_1 \frac{\partial v}{\partial x_1}\, dx - \int\limits_{\Omega} f_2\, G\frac{\partial v}{\partial x_2}\, dx, \tag{7}$$

$$Gv(x) = \rho v(x) - \int\limits_0^{x_1} \beta(t)\, v(t,x_2)\, dt. \tag{8}$$

Equality (5) formally is obtained from $(Lu - f, Gv) = 0$ by integration by parts.

To prove the existence of the unique solution of problem (5) (weak solution of problem (2)–(4)) we will apply the Lax-Milgram lemma [2]. First we will prove some auxiliary results.

Lemma 1. *Let $u, v \in L_2(\Omega,\rho)$ and v satisfy the condition $l(v) = 0$. Then*

$$|(u, Gv)| \leqslant \frac{1+\varepsilon}{1-\varepsilon}\, ||u||_{L_2(\Omega,\rho)} ||v||_{L_2(\Omega,\rho)}, \tag{9}$$

$$||v||^2_{L_2(\Omega,\rho)} \leqslant (v, Gv), \tag{10}$$

$$||v||_{L_2(\Omega,\rho^2)} \leqslant ||Gv|| \leqslant (2\varepsilon+1) ||v||_{L_2(\Omega,\rho^2)}. \tag{11}$$

Proof. Due to the density $C^\infty(\bar{\Omega})$ in $L_2(\Omega,\rho)$ it suffices to prove the lemma for an arbitrary functions from the class $C^\infty(\bar{\Omega})$. By virtue of the Cauchy inequality we have

$$|(u, Gv)| \leqslant ||u||_{L_2(\Omega,\rho)} \big(||v||_{L_2(\Omega,\rho)} + \varepsilon\, J_1(v) \big), \tag{12}$$

where

$$J_1^2(v) = \int\limits_{\Omega} x_1^{-\varepsilon} \Big(\int\limits_0^{x_1} t^{\varepsilon-1}\, v(t,x_2)\, dt \Big)^2 dx = -\frac{2}{1-\varepsilon} \int\limits_{\Omega} v(x) \int\limits_0^{x_1} t^{\varepsilon-1}\, v(t,x_2)\, dt\, dx$$
$$\leqslant \frac{2}{1-\varepsilon} ||v||_{L_2(\Omega,\rho)} \cdot J_1(v).$$

Thus, $J_1(v) \leqslant 2(1-\varepsilon)^{-1} ||v||_{L_2(\Omega,\rho)}$ and the estimate (9) follows from (12).

Inequality (10) follows from the easily verifiable identity

$$(v, Gv) = ||v||^2_{L_2(\Omega,\rho)} + \frac{\varepsilon(1-\varepsilon)}{2} J_1^2(v).$$

The first inequality in (11) is sequent of the identity

$$||Gv||^2 = \int\limits_{\Omega} x_1^{2\varepsilon}\, v^2(x)\, dx + (\varepsilon^2+\varepsilon) J_2(v), \quad J_2(v) = \int\limits_{\Omega} \Big(\int\limits_0^{x_1} t^{\varepsilon-1}\, v(t,x_2)\, dt \Big)^2 dx$$

and in order to prove the second inequality of (11), it is enough to observe that

$$J_2(v) = -2\int\limits_{\Omega} x_1^{\varepsilon}\, v(x) \int\limits_0^{x_1} t^{\varepsilon-1}\, v(t,x_2)\,dt\,dx \leqslant 2||v||_{L_2(\Omega,\rho^2)}(J_2(v))^{1/2},$$

i.e., $J_2(v) \leqslant 4||v||^2_{L_2(\Omega,\rho^2)}$. This completes the proof of the lemma. □

Lemma 2. *Let* $u \in \overset{*}{W}{}_2^1(\Omega,\rho)$*. Then*

$$|u|_{W_2^1(\Omega,\rho)} \leqslant ||u||_{W_2^1(\Omega,\rho)} \leqslant c_1|u|_{W_2^1(\Omega,\rho)}, \quad c_1 = (4(1+\varepsilon)^{-2}+1)^{1/2}.$$

Proof. Due to the density $\overset{*}{C}{}^{\infty}(\bar\Omega)$ in $\overset{*}{W}{}_2^1(\Omega,\rho)$, it is sufficient to prove the lemma for an arbitrary $u \in \overset{*}{C}{}^{\infty}(\bar\Omega)$. The first inequality of the lemma is obvious. Integrating by parts, we obtain

$$\int\limits_{\Omega} x_1^{\varepsilon}u^2(x)\,dx = -\int\limits_{\Omega}\left(\varepsilon x_1^{\varepsilon}\,u^2(x) + 2x_1^{\varepsilon+1}\,u(x)\frac{\partial u}{\partial x_1}\right)dx.$$

Therefore,

$$(1+\varepsilon)\int\limits_{\Omega} x_1^{\varepsilon}\,u^2(x)\,dx = -2\int\limits_{\Omega} x_1^{\varepsilon+1}\,u\frac{\partial u}{\partial x_1}\,dx \leqslant 2||u||_{L_2(\Omega,\rho)}\left(\int\limits_{\Omega} x_1^{\varepsilon+2}\left|\frac{\partial u}{\partial x_1}\right|^2 dx\right)^{1/2},$$

that is

$$||u||_{L_2(\Omega,\rho)} \leqslant \frac{2}{1+\varepsilon}\left(\int\limits_{\Omega} x_1^{\varepsilon+2}\left|\frac{\partial u}{\partial x_1}\right|^2 dx\right)^{1/2},$$

which proves the lemma. □

Application of both lemmas 1, 2 and condition (3), (6) gives the continuity

$$|a(u,v)| \leqslant c_2||u||_{W_2^1(\Omega,\rho)}||v||_{W_2^1(\Omega,\rho)}, \ \ c_2>0, \quad \forall u,\, v \in \overset{*}{W}{}_2^1(\Omega,\rho)$$

and W_2^1-ellipticity

$$a(u,u) \geqslant c_3||u||^2_{W_2^1(\Omega,\rho)}, \ \ c_3>0, \quad \forall u \in \overset{*}{W}{}_2^1(\Omega,\rho)$$

of the bilinear form $a(u,v)$.

By appliyng lemmas 1, 2 from (7) we obtain the continuity of linear form $\langle f,v\rangle$:

$$|\langle f,v\rangle| \leqslant c_4||v||_{W_2^1(\Omega,\rho)}, \ \ c_4>0, \quad \forall v \in \overset{*}{W}{}_2^1(\Omega,\rho).$$

Thus, all conditions of the Lax-Milgram lemma are fulfilled. Therefore, the following theorem is true.

Theorem 1. *The problem (2)–(4) has unique weak solution from* $\overset{*}{W}{}_2^1(\Omega,\rho)$.

3. Finite-difference scheme

Consider the following grid domains in Ω: $\bar{\omega}_\alpha = \{x_\alpha = i_\alpha h : i_\alpha = 0, 1, \cdots, n,\ h = 1/n\}$, $\omega_\alpha = \bar{\omega}_\alpha \cap (0,1)$, $\omega_\alpha^+ = \bar{\omega}_\alpha \cap (0;1]$, $\omega_\alpha^- = \bar{\omega}_\alpha \cap [0;1)$, $\alpha = 1,2$, $\omega = \omega_1 \times \omega_2$, $\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2$, $\gamma_* = \Gamma_* \cap \bar{\omega}$. Let us denote $\hbar = h/2$ for $x_1 = 0$, and $\hbar = h$ for $x_1 \neq 0$.

For grid functions and difference ratios, we use the standard notation from [6].

Define the following averaging operators:

$$S_1^- u = \frac{1}{h} \int\limits_{x_1-h}^{x_1} u(t, x_2)\, dt, \qquad S_1^+ u = \frac{1}{h} \int\limits_{x_1}^{x_1+h} u(t, x_2)\, dt, \qquad T_1 u = \frac{1}{2}(T_1^- + T_1^+)u,$$

$$T_1^+ u = \frac{2}{h^2} \int\limits_{x_1}^{x_1+h} (h + x_1 - t) u(t, x_2)\, dt, \quad T_1^- u = \frac{2}{h^2} \int\limits_{x_1-h}^{x_1} (h - x_1 + t) u(t, x_2)\, dt.$$

The operators $S_2^\pm, T_2$ are defined likewise.

We introduce the notation

$$\beta^+ = T_1^+ \beta, \qquad \beta^- = T_1^- \beta, \qquad \beta_k = \frac{1}{2}(\beta^+(kh) + \beta^-(kh)), \quad \beta_0^- = \beta_n^+ = 0,$$

$$\rho^+ = \rho + \frac{h}{2}\beta^+, \quad \rho^- = \rho - \frac{h}{2}\beta^-, \quad \rho_i = \sum_{k=0}^{i} h\beta_k - \frac{h}{2}\beta_i^+, \qquad \bar{\rho} = \frac{1}{2}(\rho^+ + \rho^-).$$

It is not hard to check that

$$\rho_i = \rho(ih), \quad \rho^+ = S_1^+ \rho, \quad \rho^- = S_1^- \rho, \quad \bar{\rho}_0 = \frac{h}{4}\beta_0^+.$$

We will define the difference analogue of the operator G from (8) in the following way:

$$G_h y = \bar{\rho} y - P y, \quad P y(ih, x_2) = \sum_{k=0}^{i} h\beta_k y(kh, x_2) - \frac{h}{2}\beta_i y(ih, x_2). \tag{13}$$

A set of grid-functions given on $\bar{\omega}$ and satisfying the condition

$$y = 0, \quad x \in \gamma_*, \qquad l_h(y) \equiv \sum_{k=0}^{n} \beta_k y(kh, x_2) = 0, \quad x_2 \in \omega_2 \tag{14}$$

will be denoted by H. On the set H let us introduce the inner product and the norm

$$(y, v)_{\widetilde{\omega}} = \sum_{\widetilde{\omega}} h^2 y v, \quad ||y||_{\widetilde{\omega}} = (y, y)_{\widetilde{\omega}}^{1/2}, \quad \widetilde{\omega} \subseteq \bar{\omega}.$$

Let, moreover,

$$(y, v)_0 = \sum_{\omega_1^- \times \omega_2} \hbar h y v, \quad ||y||_0 = (y, y)_0^{1/2}, \quad ||y||_\rho^2 = \sum_{\omega_1^- \times \omega_2} \hbar h \bar{\rho} y^2, \quad ||y]|_\rho^2 = \sum_{\omega_1^- \times \omega_2^+} \hbar h \bar{\rho} y^2,$$

$$||y||_1^2 = ||y||_0^2 + ||\nabla y||^2, \quad ||\nabla y||^2 = ||y_{\bar{x}_1}||_{(1)}^2 + ||y_{\bar{x}_2}||_{(2)}^2, \quad ||y_{\bar{x}_1}||_{(1)}^2 = (\rho^- y_{\bar{x}_1}, y_{\bar{x}_1})_{\omega_1^+ \times \omega_2},$$

$$||y_{\bar{x}_2}||_{(2)} = ||y_{\bar{x}_2}]|_\rho, \quad ||y||_*^2 = \sum_{\omega_2} h y^2, \quad ||y]|_*^2 = \sum_{\omega_2^+} h y^2.$$

We approximate problem (2)–(4) by the difference scheme

$$L_h y = -a_{11} y_{\bar{x}_1 x_1} - 2a_{12} y_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} - a_{22} y_{\bar{x}_2 x_2} + a_0 y = \varphi(x), \quad x \in \omega, \quad y \in H, \tag{15}$$

where

$$\varphi = T_1 T_2 f_0 + (S_1^- T_2 f_1)_{x_1} + (T_1 S_2^- f_2)_{x_2}.$$

Lemma 3. *The estimates*

$$(y, G_h y)_\omega \geqslant ||y||_\rho^2, \quad (y, G_h y)_{\omega_1 \times \omega_2^+} \geqslant ||y]|_\rho^2$$

are true for grid functions $y(x)$, satisfying the conditions $l_h(y) = 0,\ y(1, x_2) = 0, \quad x_2 \in \omega_2$.

Proof. It is not difficult to verify that

$$-\sum_{i=1}^{n-1} h y(ih, x_2) P y(ih, x_2) = \frac{1}{2\beta_1} \left(\frac{h}{2} \beta_0^+ y(0, x_2) \right)^2 + J_3, \tag{16}$$

where

$$J_3 = 0, \quad n = 2, \qquad J_3 = \frac{1}{2} \sum_{i=2}^{n-1} \left(\frac{1}{\beta_i} - \frac{1}{\beta_{i-1}} \right) \left(P y(ih, x_2) - \frac{h}{2} \beta_i y(ih, x_2) \right)^2, \quad n > 2.$$

Due to $J_3 \geqslant 0$ because of $(1/\beta_i) - (1/\beta_{i-1}) > 0$, and also $\beta_0^+ > \beta_1$, the validity of Lemma 3 follows from (16). □

Lemma 4. *For any $y \in H$ the inequality*

$$(L_h y, G_h y)_\omega \geqslant c_5 ||y||_1^2, \quad c_5 = \nu/4 \tag{17}$$

holds.

Proof. Using summation by parts, we get

$$\sum_{\omega_1} h v_{x_1} G_h y = -\sum_{\omega_1^+} h \rho^- v y_{\bar{x}_1}, \quad \sum_{\omega_1} h v_{\bar{x}_1} G_h y = -\sum_{\omega_1^-} h \rho^+ v y_{x_1},$$

where v is an arbitrary grid function. Hence

$$-(y_{\bar{x}_1 x_1}, G_h y)_\omega = \frac{1}{2} \sum_{\omega_1^+ \times \omega_2} h^2 \rho^- (y_{\bar{x}_1})^2 + \frac{1}{2} \sum_{\omega_1^- \times \omega_2} h^2 \rho^+ (y_{x_1})^2, \tag{18}$$

$$-(y_{\overset{\circ}{x}_1 \overset{\circ}{x}_2}, G_h y)_\omega = \frac{1}{2} \sum_{\omega_1^- \times \omega_2} h^2 \rho^+ y_{x_1} y_{\overset{\circ}{x}_2} + \frac{1}{2} \sum_{\omega_1^+ \times \omega_2} h^2 \rho^- y_{\bar{x}_1} y_{\overset{\circ}{x}_2}. \tag{19}$$

Besides, applying Lemma 3, we have

$$-(y_{\bar{x}_2 x_2}, G_h y)_\omega \geqslant ||y_{\bar{x}_2}||_{(2)}^2. \tag{20}$$

Let

$$\hat{\rho} = \rho + \frac{h}{2} \beta^+ - \frac{h}{4} \beta_0^+, \quad \check{\rho} = \rho - \frac{h}{2} \beta^- + \frac{h}{4} \beta_0^+.$$

Then $\bar{\rho} = \frac{1}{2}(\hat{\rho} + \check{\rho}), \quad \hat{\rho}_0 = \frac{h}{4}\beta_0^+$, and after some transformations we obtain

$$-(y_{\bar{x}_1 x_1}, G_h y)_\omega = \frac{1}{2} \sum_{\omega_1^+ \times \omega_2} h^2 \check{\rho} (y_{\bar{x}_1})^2 + \frac{1}{2} \sum_{\omega_1^- \times \omega_2} h^2 \hat{\rho} (y_{x_1})^2, \tag{21}$$

$$-(y_{\overset{\circ}{x}_1 \overset{\circ}{x}_2}, G_h y)_\omega = \frac{1}{2} \sum_{\omega_1^+ \times \omega_2} h^2 \check{\rho} y_{\bar{x}_1} y_{\overset{\circ}{x}_2} + \frac{1}{2} \sum_{\omega_1^- \times \omega_2} h^2 \hat{\rho} y_{x_1} y_{\overset{\circ}{x}_2}, \tag{22}$$

$$-(y_{\bar{x}_2 x_2}, G_h y)_\omega \geqslant \frac{1}{2} \sum_{\omega_1^- \times \omega_2^+} h^2 \hat{\rho} (y_{\bar{x}_2})^2 + \frac{1}{2} \sum_{\omega_1 \times \omega_2^-} h^2 \check{\rho} (y_{x_2})^2 \tag{23}$$

from (18), (19), and (20) respectively.

Taking into account (21)–(23), from (15) we have

$$\begin{aligned} 4(L_h y, G_h y)_\omega \geqslant & \sum_{\omega_1^+ \times \omega_2^-} h^2 \check{\rho} F(y_{\bar{x}_1}, y_{x_2}) + \sum_{\omega_1^+ \times \omega_2^+} h^2 \check{\rho} F(y_{\bar{x}_1}, y_{\bar{x}_2}) \\ & + \sum_{\omega_1^- \times \omega_2^-} h^2 \hat{\rho} F(y_{x_1}, y_{x_2}) + \sum_{\omega_1^- \times \omega_2^+} h^2 \hat{\rho} F(y_{x_1}, y_{\bar{x}_2}) + a_0 (y, G_h y)_\omega, \end{aligned} \tag{24}$$

where $F(t_1, t_2) = a_{11} t_1^2 + 2a_{12} t_1 t_2 + a_{22} t_2^2$.

Taking into account

$$\check{\rho} = \frac{1}{h} \int\limits_{x_1 - h}^{x_1} \rho(t)\, dt + \frac{1}{2h} \int\limits_0^h \rho(t)\, dt > 0, \qquad \hat{\rho} = \frac{1}{h} \int\limits_{x_1}^{x_1 + h} \rho(t)\, dt - \frac{1}{2h} \int\limits_0^h \rho(t)\, dt > 0,$$

due to the condition of ellipticity the estimate

$$(L_h y, G_h y)_\omega \geqslant \nu_1 ||\nabla y||^2$$

follows from (24), which together with (see [1])

$$||y||_0^2 \leqslant 4||y_{\bar{x}_1}||^2 \leqslant 4||\nabla y||^2$$

prove Lemma 4. □

Thus, if $\varphi(x) = 0,\ x \in \omega$, then $y(x) = 0,\ x \in \bar{\omega}$ and, consequently, the solution of difference scheme (15) exists and it is unique.

Lemma 5. *If the grid function y defined on $\bar{\omega}$ satisfies the conditions $l_h(y) = 0$, $y(1, x_2) = 0, \quad x_2 \in \omega_2$, then*

$$\left| \sum_{\omega_1} h v G_h y \right| \leqslant c \left(\sum_{\omega_1} h \bar{\rho} v^2 \right)^{1/2} \left(\sum_{\omega_1} h \bar{\rho} y^2 \right)^{1/2},$$

where $v(x)$ is an arbitrary grid function.

Proof. By the definition of the operator G_h, we have

$$\left|\sum_{\omega_1} hvG_hy\right| \leqslant \left(\sum_{\omega_1} h\bar\rho v^2\right)^{1/2}\left[\left(\sum_{\omega_1} h\bar\rho y^2\right)^{1/2} + J_4(y)\right], \tag{25}$$

where

$$J_4^2(y) = \sum_{\omega_1} h(\bar\rho)^{-1}(Py)^2.$$

Let

$$2(\widetilde{P}y)_i = \sum_{k=0}^{i} h\beta_k y(kh,x_2), \quad \sigma_i = \sum_{k=1}^{i}\frac{h}{\bar\rho_k}, \quad \sigma_0 = 0.$$

Then

$$(\widetilde{P}y)_i + (\widetilde{P}y)_{i-1} = (Py)_i, \quad (\widetilde{P}y)_i - (\widetilde{P}y)_{i-1} = \frac{h\beta_i}{2}y(ih,x_2), \quad (\widetilde{P}v)_{n-1} = 0, \quad \sigma_i - \sigma_{i-1} = \frac{h}{\bar\rho_i}$$

and we will have

$$\begin{aligned} J_4^2(y) &\leqslant 2\sum_{i=1}^{n-1}(\sigma_i - \sigma_{i-1})\big((\widetilde{P}y)_i^2 + (\widetilde{P}y)_{i-1}^2\big) = -2\sum_{i=1}^{n-1}(\sigma_i + \sigma_{i-1})\big((\widetilde{P}y)_i^2 - (\widetilde{P}y)_{i-1}^2\big) \\ &= -\sum_{i=1}^{n-1}(\sigma_i + \sigma_{i-1})h\beta_i y(ih,x_2)(Py)_i. \end{aligned} \tag{26}$$

It is possible to show that $(\sigma_i + \sigma_{i-1})\beta_i \leqslant c$. Consequently, the inequality

$$J_4^2(y) \leqslant c\sum_{\omega_1} h|y\,Py| \leqslant c\left(\sum_{\omega_1} h\bar\rho y^2\right)^{1/2} J_4(y), \quad \text{i.e. } J_4(y) \leqslant c\left(\sum_{\omega_1} h\bar\rho y^2\right)^{1/2}$$

follows from (26). This together with (25) completes the proof of Lemma 5. □

To investigate the convergence and accuracy of scheme (15), we consider the error of the method $z = y - u$, where y is a solution to problem (15) and $u = u(x)$ is a solution to problem (2)–(4). Substituting $y = u + z$ into (15) , we obtain the problem

$$L_h z = \psi,\ x \in \omega, \quad z = 0,\ x \in \gamma_*, \quad l_h(z) = \chi(x_2),\ x_2 \in \omega_2, \tag{27}$$

where

$$\begin{gathered} \psi = a_{11}\eta_{11\bar x_1 x_1} + a_{12}\eta_{12 x_1 x_2} + a_{22}\eta_{22\bar x_2 x_2} + a_0\eta_0, \\ \eta_0 = T_1T_2u - u, \quad \eta_{\alpha\alpha} = u - T_{3-\alpha}u, \quad \alpha = 1,2, \\ \eta_{12} = \frac{1}{2}(u + u^{(-1_1)} + u^{(-1_2)} + u^{(-1_1,-1_2)}) - 2S_1^- S_2^- u(x), \quad \chi = l(u) - l_h(u). \end{gathered}$$

If we notice that

$$l_h(u) = \sum_{\omega_1^+}\int_{x_1-h}^{x_1}\beta(t)\left(\frac{x_1 - t}{h}u(x_1 - h, x_2) + \frac{t - x_1 + h}{h}u(x_1,x_2)\right)dt,$$

then we can write the error χ as follows:

$$\chi = \sum_{\omega_1^+} \eta, \quad \eta = \int\limits_{x_1-h}^{x_1} \beta(t) \frac{t-x_1}{h} \int\limits_{x_1-h}^{t} (\xi - x_1 + h) \frac{\partial^2 u(\xi, x_2)}{\partial \xi^2} \, d\xi \, dt$$
$$+ \int\limits_{x_1-h}^{x_1} \beta(t) \frac{t-x_1+h}{h} \int\limits_{t}^{x_1} (\xi - x_1) \frac{\partial^2 u(\xi, x_2)}{\partial \xi^2} \, d\xi \, dt.$$

It is evident that $\chi = 0$ for $u(x) = 1 - x_1$. Consequently, $l_h(1-x_1) = l(1-x_1) = 1/(1+\varepsilon)$ and the substitution

$$z(x) = \widetilde{z}(x) + \frac{1-x_1}{1+\varepsilon} \chi(x_2) \tag{28}$$

turns problem (27) (in which the nonlocal condition is not homogeneous) into the problem with the homogeneous conditions

$$L_h \widetilde{z} = \widetilde{\psi}, \quad x \in \omega, \qquad \widetilde{z} = 0, \quad x \in \gamma_*, \qquad l_h(\widetilde{z}) = 0, \quad x_2 \in \omega_2, \tag{29}$$

where

$$\widetilde{\psi} = \psi + 2a_{12} \left(\frac{1-x_1}{1+\varepsilon} \chi \right)_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} + a_{22} \left(\frac{1-x_1}{1+\varepsilon} \chi \right)_{\bar{x}_2 x_2} - a_0 \frac{1-x_1}{1+\varepsilon} \chi.$$

Applying Lemma 4 to the solution of problem (29) we come to

$$||\widetilde{z}||_1^2 \leqslant c(\widetilde{\psi}, G_h \widetilde{z})_\omega.$$

Using Lemma 5 gives

$$||\widetilde{z}||_1 \leqslant c\big(||\eta_{11\bar{x}_1}||_{\omega_1^+ \times \omega_2} + ||\eta_{12x_2}||_{\omega_1^+ \times \omega_2} + ||\eta_{22\bar{x}_2}||_{\omega_1 \times \omega_2^+} + ||\eta_0||_\omega + ||\chi||_* + ||\chi_{\bar{x}_2}||_*\big). \tag{30}$$

For the error of the method, according to (28), we can write

$$||z||_1 \leqslant ||\widetilde{z}||_1 + c(||\chi||_* + ||\chi_{\bar{x}_2}||_*)$$

which together with (30) gives

$$||z||_1 \leqslant c\big(||\eta_{11\bar{x}_1}||_{\omega_1^+ \times \omega_2} + ||\eta_{12x_2}||_{\omega_1^+ \times \omega_2} + ||\eta_{22\bar{x}_2}||_{\omega_1 \times \omega_2^+} + ||\eta_0||_\omega + ||\chi||_* + ||\chi_{\bar{x}_2}||_*\big). \tag{31}$$

In order to estimate the convergence rate of finite-difference scheme (15), it is enough to estimate the norm of error functionals on the right-hand side of (31). For this we apply the standard technique (see, e.g., [3, 7]).

First, for each summands of $\chi_{\bar{x}_2}$ we write

$$|\eta_{\bar{x}_2}| \leqslant ch^{-1} \int\limits_{x_1-h}^{x_1} \beta(t) \, dt \, h^{m-2/p} |u|_{W_p^m(e)}, \quad pm > 1, \quad m \in (1;3], \quad e = (x_1 - h, x_1) \times (x_2 - h, x_2).$$

Next,

$$|\eta_{\bar{x}_2}| \leqslant c \left(\int\limits_{x_1-h}^{x_1} t^{(\varepsilon-1)p/(p-1)} \, dt \right)^{(p-1)/p} |u|_{W_p^m(e)} h^{m-1-1/p},$$

therefore,

$$|\chi_{\bar{x}_2}| \leqslant ch^{m-1-1/p} \left(\int\limits_0^1 t^{(\varepsilon-1)p/(p-1)}\, dt \right)^{(p-1)/p} |u|_{W_p^m(\bar{e})}, \quad \bar{e} = (0;1) \times (x_2 - h; x_2).$$

Taking into account the inequality

$$\sum_{\omega_2} |u|^2_{W_p^m(\bar{e})} \leqslant ch^{-1+2/p} |u|^2_{W_p^m(\Omega)},$$

we will have

$$||\chi_{\bar{x}_2}||_* \leqslant ch^{m-1} |u|_{W_p^m(\Omega)}.$$

The analogous estimate is obtained for $||\chi||_*$.

With the well-known estimates for η_{11}, η_{12}, η_{22}, η_0 (see [3,7]), (31) yields the convergence theorem.

Theorem 2. *The finite-difference scheme (15) converges and the convergence rate estimate (1) holds.*

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