Memoirs on Differential Equations and Mathematical Physics
Volume 38, 2006, 1–131

G. Berikelashvili

CONSTRUCTION AND ANALYSIS OF DIFFERENCE SCHEMES FOR SOME ELLIPTIC PROBLEMS, AND CONSISTENT ESTIMATES OF THE RATE OF CONVERGENCE

Abstract. In the present work we present new results connected with the construction and analysis of difference schemes for:

(a) the second order elliptic equation (the Dirichlet problem, mixed boundary value problem, nonlocal problems);

(b) general systems of elliptic second order equations (the Dirichlet problem);

(c) systems of equations of the statical theory of elasticity (the first mixed, the third boundary-value, the nonlocal problems with integral restriction);

(d) the fourth order elliptic equation (the first boundary-vale problem);

(e) the problem of bending of an orthotropic plate freely supported over the contour;

For the construction of difference schemes the Steklov averaging operators are used. The correctness is investigated by the energy method. The estimate of the rate of convergence is based on the corresponding a priori estimates and on the generalized Bramble-Hilbert lemma. Investigation of the solvability of nonlocal problems for the second order elliptic equation is based on the Lax–Milgram lemma.

2000 Mathematics Subject Classification: 65N06, 65N12, 35J25.

Key words and phrases: Difference schemes, elliptic equation, nonlocal boundary-value problem, weighted spaces, high accuracy, elasticity theory, a priori estimates.

რეზიუმე. ნაშრომში წარმოდგენილია სხვაობიანი სქემების აგებასა და ანალიზთან დაკავშირებული ახალი შედეგები:

– მეორე რიგის ელიფსური ტიპის განტოლებისათვის (დირიხლეს, შერეული სასაზღვრო და არალოკალური ამოცანები);

– მეორე რიგის ელიფსურ განტოლებათა ზოგადი სისტემისათვის (დირიხლეს ამოცანა);

– დრეკადობის სტატიკური თეორიის განტოლებათა სისტემისათვის (პირველი, მესამე და შერეული სასაზღვრო ამოცანები, ინტეგრალურ პირობიანი არალოკალური ამოცანა);

– მეოთხე რიგის ელიფსური განტოლებისათვის (პირველი სასაზღვრო ამოცანა);

– კონტურის გასწვრივ თავისუფლად დაყრდნობილი ორთოტროპული ფირფიტის ღუნვის ამოცანისათვის.

სხვაობიანი სქემები აგებულია სტეკლოვის გასაშუალების ოპერატორების საფუძველზე. კორექტულობა გამოკვლეულია ენერგეტიკული უტოლობების მეთოდით. კრებადობის სიჩქარის შეფასება ემყარება შესაბამის აპრიორულ შეფასებებს და ბრემბლ–ჰილბერტის განზოგადებულ ლემას. მეორე რიგის ელიფსური განტოლებისათვის არალოკალური ამოცანების ამოხსნადობა გამოკვლეულია ლაქს–მილგრამის ლემის გამოყენებით.

Introduction

The method of finite differences is one of the most widespread and universal methods of numerical solution of boundary value problems for differential equations, in particular, those of elliptic type.

In approximate methods as well as in practice the main attention is paid to the issue of accuracy. When a solution of the initial problem is sufficiently smooth, for example, belongs to a class of continuously differentiable functions C^k, we can find in the theory of the finite-difference method a great deal of fundamental investigations devoted to the estimation of accuracy and convergence rate. However, it is worth mentioning that the input data in a number of practical problems are not always smooth, so one have to consider them in other, more general spaces. In this respect, the most suitable ones are the Sobolev classes W_p^k. Development and validation of difference schemes with coefficients and solutions of problems from the Sobolev space became very topical.

During the last twenty years, methods of constructing and analysis of difference schemes with convergence rate consistent with the smoothness of the sought for solution have arisen. In this respect we mention the works due to A. A. Samarskiĭ, R. D. Lazarov, V. L. Makarov, W. Weinelt, S. A. Voĭtsekhovskiĭ, I. P. Gavrilyuk, B. S. Jovanović, P. P. Matus, M. N. Moskal'kov, V. G. Prikazchikov, etc. Later on, such kind of estimates were called consistent ([71]). For elliptic problems they have the form

$$\|y-u\|_{W_2^s(\omega)} \le c|h|^{m-s}\|u\|_{W_2^m(\Omega)}, \quad m > s \ge 0,$$

where u is a solution of the original differential problem, y is an approximate solution, s and m are real numbers, $\|\cdot\|_{W_2^s(\omega)}$ and $\|\cdot\|_{W_2^m(\Omega)}$ are Sobolev norms on a set of functions of discrete and continuous argument.

The aim of the present paper is to construct and analyze different schemes which approximate some classical and nonlocal problems for elliptic equations and systems, as well as to obtain a scale of a priori estimates of the convergence rate, depending on the smoothness of the solution of the initial problem.

Of the obtained results, the following ones are worth mentioning:

(a) A method of averaging the coefficients which preserves ellipticity at the discrete level and is efficient for boundary value problems with rapidly changing or unbounded coefficients.

(b) For the problems with derivatives in the boundary conditions, the convergence rate is, as a rule, reduced by half. The reason of such a reduction is shown, and a way of its elimination is suggested.

(c) A difference analogue of the second principal inequality for the solution of the Dirichlet problem for the second order elliptic equation, without restriction to the mesh step.

(d) In difference schemes for fourth order equations, the values of the approximate solution are defined at outer nods as well. Therefore when investigating the error, there arises the need to extend the exact solution outside of the domain of integration, preserving the smoothness. We suggest a method of estimation of the convergence rate which does not require such an extension.

(e) A new effective method of decomposition of the problem of bending of the orthotropic plate.

(f) Solvability of some nonlocal boundary value problems in weighted Sobolev spaces, validation of the corresponding difference schemes by means of the energy method.

CHAPTER 1

Difference Schemes for Elliptic Equations of Second Order

In this chapter we suggest a method of averaging the coefficients of the differential operator which on the discrete level does not violate the condition of ellipticity. An estimate of the inner product of mesh function traces on the boundary is obtained which allows one to establish a consistent estimate of the convergence rate in the case of the mixed type boundary value problem. Without restriction to the mesh step, the difference analogue of the second principal inequality of the solution of the Dirichlet problem is proved.

In Section 1 we introduce notation and some preliminaries which will be used in the sequel. In Sections 2–4 we study difference schemes for the Dirichlet problem and in Section 5 for the problem with mixed boundary conditions. The energy method allows us to investigate these schemes in Sobolev lattice spaces. The obtained consistent estimates of the convergence rate are based on a generalization of the Bramble–Hilbert lemma.

1. Notation, Auxiliary Results

In this section we introduce notation and indicate the results which we will need in our subsequent discussion. These problems have been considered in detail in [30], [57], [58], [71] and [75].

In what follows, by Ω we denote a rectangular domain in the two-dimensional Euclidean space $\mathbf{R}^2$ with the boundary Γ:

$$\Omega = \{x = (x_1, x_2) : \ \ 0 < x_\alpha < \ell_\alpha, \ \ \alpha = 1, 2\}, \ \ \Gamma = \partial\Omega, \ \ \overline{\Omega} = \Omega \cap \Gamma.$$

For $\ell_1 = \ell_2 = \ell$ we write $\Omega = \Omega_{(\ell)} = (0; \ell)^2$. Let $\Gamma_{\pm\alpha} = \{x \in \Gamma : \ \ x_\alpha = (1 \pm 1)\ell_\alpha/2, \ \ 0 < x_{3-\alpha} < \ell_{3-\alpha}\}$, $\alpha = 1, 2$. Let

$$D_\alpha := \frac{\partial}{\partial x_\alpha}, \ \ D_\alpha^k := \frac{\partial^k}{\partial x_\alpha^k}, \ \ D^{\boldsymbol{s}} := \frac{\partial^{|\boldsymbol{s}|}}{\partial x_1^{s_1} \partial x_2^{s_2}}\,,$$

where $\boldsymbol{s} = (s_1, s_2)$ is the multiindex, $s_1, s_2 \geq 0$ are integers, $|\boldsymbol{s}| = s_1 + s_2$.

As usual, by $W_p^m(\Omega)$ we denote Sobolev–Slobodetski's spaces. For integers $m \geq 0$, the norm in $W_p^m(\Omega)$ is given by the formula

$$\|u\|_{W_p^m(\Omega)} = \Big(\sum_{k=0}^{m} |u|^p_{W_p^k(\Omega)}\Big)^{1/p}.$$

Here

$$|u|_{W_p^k(\Omega)} = \Big(\sum_{|\boldsymbol{s}|=k} \|D^{\boldsymbol{s}} u\|^p_{L_p(\Omega)} \Big)^{1/p}$$

is the higher semi-norm in the space $W_p^m(\Omega)$.

If $m = \overline{m} + \lambda$, $\overline{m}$ is the integer part of m, and $0 < \lambda < 1$, then $\|u\|_{W_p^m(\Omega)} = \big(\|u\|^p_{W_p^{\overline{m}}(\Omega)} + |u|^p_{W_p^m(\Omega)} \big)^{1/p}$, where

$$|u|_{W_p^m(\Omega)} = \bigg(\sum_{|\boldsymbol{s}|=\overline{m}} \int\limits_\Omega \int\limits_\Omega \frac{|D^{\boldsymbol{s}} u(x) - D^{\boldsymbol{s}} u(t)|^p}{|x-t|^{2+\lambda p}}\, dx\, dt \bigg)^{1/p}.$$

For $p = 2$, we denote $\|u\|_{W_2^m(\Omega)} = \|u\|_{m,\Omega}$.

Let S be an interval in $\mathbf{R}$. The Sobolev–Slobodetski's space $W_2^m(S)$ with the real positive index $m = \overline{m} + \lambda$, where $\overline{m}$ is an integer and $0 < \lambda < 1$, is defined as the set of all those functions $u(x) \in W_2^{\overline{m}}(S)$ for which the norm $\|u\|_{m,S} = \big(\|u\|^2_{\overline{m},S} + |u|^2_{m,S} \big)^{1/2}$, where

$$|u|^2_{m,S} = \int\limits_S \int\limits_S \frac{|u^{(\overline{m})}(x) - u^{(\overline{m})}(y)|^2}{|x-y|^{1+2\lambda}}\, dx\, dy$$

is finite.

Theorem 1.1 ([57], p. 332). *$W_2^{1/2}(S)$ is a Banach space.*

By $L_2(\Omega, r)$ we denote the weighted space consisting of all real functions $u(x)$, defined on Ω, with the norm

$$\|u\|_{\Omega,r} = \bigg(\int\limits_\Omega r(x) |u(x)|^2\, dx \bigg)^{1/2},$$

where $r(x)$ is the weight function, i.e. $r(x)$ is a measurable and almost everywhere (a.e.) positive on Ω.

The weighted Sobolev space $W_2^k(\Omega, r)$ is usually defined as the linear space of the given on Ω functions $u(x)$ whose derivatives (in a general sense) $D^{\boldsymbol{s}} u$ of order $|\boldsymbol{s}| \le k$ belong to the space $L_2(\Omega, r)$. This space will become a linear normed space if we introduce the norm

$$\|u\|_{k,\Omega,r} = \Big(\sum_{i=0}^k |u|^2_{i,\Omega,r} \Big)^{1/2},$$

where $|u|^2_{i,\Omega,r} = \sum\limits_{|s|=i} \|D^s u\|^2_{\Omega,r}$, $|u|_{0,\Omega,r} = |u|_{\Omega,r}$.

By $C^\infty(\overline{\Omega})$ we denote the set of real-valued functions $u(x)$ defined on $\overline{\Omega}$ such that the derivatives $D^{\boldsymbol{s}} u$ can be continuously extended to $\overline{\Omega}$ for all multiindices $\boldsymbol{s}$. The following theorem is valid (see, e.g., [56], p. 10; [65], Theorem 3.1).

Theorem 1.2. *If $r \in L_{1,loc}(\Omega)$ and $r^{-1} \in L_{1,loc}(\Omega)$, then $W_2^k(\Omega, r)$, $k = 0, 1, 2, \ldots$, is a Banach space and $C^\infty(\overline{\Omega})$ is dense in it.*

By $c, c_1, c_2, \ldots$, and so on, we denote constants which may be different in different formulas.

Under the belonging of a vector-function (or a matrix) to the space W_2^k we mean that every component of the vector (element of the matrix) belongs to that space.

The imbedding theorems are of great importance in the theory of Sobolev spaces.

Theorem 1.3. *Let Ω be an open domain in $\mathbf{R}^2$ with the Lipschitz-continuous boundary. Then the following imbeddings hold:*

(a) *$W_{p_2}^{m_2}(\Omega) \subset W_{p_1}^{m_1}(\Omega)$ for $0 \le m_1 \le m_2 < \infty$, $1 < p_2 \le p_1 < \infty$ and $2/p_2 - m_2 \le 2/p_1 - m_1$;*

(b) *$W_p^m(\Omega) \subset C^k(\overline{\Omega})$ for $mp > 2$, where k is the least integer larger than or equal to $(m-2)$.*

Let $\boldsymbol{\pi}_k = \left\{ P(x) : \ P(x) = \sum\limits_{|\boldsymbol{s}| \le k} c_{\boldsymbol{s}} x_1^{s_1} x_2^{s_2} \right\}$ denote the set of polynomials of two variables x_1, x_2 of degree $\le k$.

The obtained by us estimates of the convergence rate of the difference solution are based mainly on the following facts.

First of all, we present here the result which follows from the Dupont–Scott approximation theorem ([38]) and is generalization of the Bramble–Hilbert lemma ([33]).

Theorem 1.4 (The Bramble–Hilbert lemma). *Let E be an open convex bounded domain in $\mathbf{R}^2$ with piecewise smooth boundary, and let a linear functional $\ell(u)$ be bounded in $W_p^m(E)$, where $m > 0$, $m = \overline{m} + \lambda$, $\overline{m}$ is an integer and $0 < \lambda \le 1$. If $\ell(u)$ vanishes in $\boldsymbol{\pi}_{\overline{m}}$, then there exists a constant $c > 0$, depending on E but independent of $u(x)$, such that the estimate $|\eta(u)| \le c|u|_{W_p^m(E)}$ holds.*

In the sequel, we will need the inequality that provides an estimate for the L_2-norm of a function in a strip near the boundary in terms of the W_2^m-norm in the domain Ω (see [66], p. 20; [71], p. 26).

Theorem 1.5. *Let the boundary Γ of the domain Ω belong to the class C^1. Then for every function $u(x) \in W_2^m(\Omega)$, the estimate*

$$\|u\|_{L_2(\Omega_\delta)} \le c_1 c(\delta) \|u\|_{W_2^m(\Omega)}$$

is valid, where

$$c_1 = \text{const} > 0, \quad c(\delta) = \begin{cases} \delta^m, & 0 \le m \le 1/2, \\ \delta^{1/2} |\ln \delta|, & m = 1/2, \\ \delta^{1/2}, & 1/2 < m \le 1 \end{cases}$$

A defined on Ω function g is said to be a pointwise multiplicator (or simply, a multiplicator) for the space $W_p^m(\Omega)$, if $gu \in W_2^m(\Omega)$ for all $u \in W_p^m(\Omega)$; the set of all multiplicators for $W_p^m(\Omega)$ is denoted by $M(W_p^m(\Omega))$.

Lemma 1.1. *Let $u,\, v \in W_2^m(\Omega)$, $m > 1$. Then*

$$\|uv\|_{m,\Omega} \le c\|u\|_{m,\Omega}\|v\|_{m,\Omega}, \tag{1.1}$$

where the constant $c > 0$ does not depend on $u(x)$, $v(x)$.

The inequality (1.1) is an obvious consequence of Peetre's lemma ([67]).

Lemma 1.2. *Let $u \in W_2^m(\Omega)$, $v \in W_2^{m+1}(\Omega)$, $0 < m \le 1$. Then*

$$\|uv\|_{m,\Omega} \le c\|u\|_{m,\Omega}\|v\|_{m+1,\Omega}, \tag{1.2}$$

where the constant $c > 0$ does not depend on $u(x)$, $v(x)$.

Proof. Taking into account the imbedding $W_2^2(\Omega) \subset C(\Omega)$, we can see that

$$\|uv\|_{1,\Omega}^2 \le \big(\|u\|_{0,\Omega}^2 + 2\|D_1u\|_{0,\Omega}^2 + 2\|D_2u\|_{0,\Omega}^2\big)\|v\|_{C(\overline{\Omega})}^2 + 2I_1,$$

that is,

$$\|uv\|_{1,\Omega}^2 \le c_1\|u\|_{1,\Omega}^2\|v\|_{2,\Omega}^2 + 2I_1, \tag{1.3}$$

where

$$I_1 = \int\limits_{\Omega} u^2(x)\big(|D_1v|^2 + |D_2v|^2\big)\,dx.$$

Using Hölder's inequality and the imbedding $W_2^1(\Omega) \subset L_4(\Omega)$, we obtain $I_1 \le 2\|u\|_{L_4(\Omega)}^2\big(\|D_1v\|_{L_4(\Omega)}^2 + \|D_2v\|_{L_4(\Omega)}^2\big) \le c_2\|u\|_{1,\Omega}^2\|v\|_{2,\Omega}^2$, which together with (1.3) proves the estimate (1.2) for $m = 1$.

Let now $0 < m < 1$. Then it is easy to show that

$$\begin{aligned}\|uv\|_{m,\Omega}^2 &\le \|u\|_{0,\Omega}^2\|v\|_{C(\overline{\Omega})}^2 + 2I(u,v) + 2I(v,u) \le \\ &\le c_3\|u\|_{m,\Omega}^2\|v\|_{m+1,\Omega}^2 + 2I(u,v),\end{aligned} \tag{1.4}$$

where

$$I(u,v) = \int\limits_{\Omega}\int\limits_{\Omega} \frac{|u(x)|^2|v(x)-v(y)|^2}{|x-y|^{2+2m}}\,dx\,dy.$$

We estimate this integral by using Hölder's inequality

$$\begin{aligned} I(u,v) \le &\Bigg(\int\limits_{\Omega}\int\limits_{\Omega} \frac{|u(x)|^{2/(1-m)}}{|x-y|^{2\varepsilon/(1-m)}}\,dx\,dy\Bigg)^{1-m} \times \\ &\times \Bigg(\int\limits_{\Omega}\int\limits_{\Omega} \frac{|v(x)-v(y)|^{2/m}}{|x-y|^{(2+2m-2\varepsilon)/m}}\,dx\,dy\Bigg)^{m}. \end{aligned}$$

Let the parameter ε be chosen from the interval $\max(0; 1-2m) < \varepsilon < 1-m$. Then

$$I(u,v) \le c_4\|u\|_{L_{2/(1-m)}(\Omega)}^2 |v|_{W_{2/m}^{1-\varepsilon}(\Omega)}^2 \le c\|u\|_{m,\Omega}^2\,\|v\|_{m+1,\Omega}^2 \tag{1.5}$$

since $W_2^m(\Omega) \subset L_{2/(1-m)}(\Omega)$, $W_2^{m+1}(\Omega) \subset W_{2/m}^{1-\varepsilon}(\Omega)$.

Thus the inequalities (1.4) and (1.5) complete the proof of the lemma. □

Lemma 1.3. *Let $u \in W_2^1(\Omega)$ and $a \in W_{2+\varepsilon}^1(\Omega)$, where $\varepsilon > 0$ is an arbitrary number. Then $\|au\|_{1,\Omega} \le c\,\|a\|_{W_{2+\varepsilon}^1(\Omega)}\,\|u\|_{1,\Omega}$, where $c > 0$ does not depend on $u(x)$, $a(x)$.*

Proof. We have

$$\|au\|_{1,\Omega}^2 = \|au\|_{0,\Omega}^2 + \|D_1(au)\|_{0,\Omega}^2 + \|D_2(au)\|_{0,\Omega}^2. \tag{1.6}$$

But $\|D_i(au)\|_{0,\Omega}^2 \le \|D_i a\|_{L_{2p}(\Omega)} + \|u\|_{L_{2q}(\Omega)} + \|a\|_{C(\overline{\Omega})}\|D_i u\|_{0,\Omega}$, $i = 1, 2$, where $p = (\varepsilon+2)/2$, $q = (\varepsilon+2)/\varepsilon$.

By the imbedding theorem, $W_2^1(\Omega) \subset L_{2q}(\Omega)$ for $q < \infty$. Consequently,

$$\|D_1(au)\|_{0,\Omega} \le c\|a\|_{W_{2+\varepsilon}^1(\Omega)}\|u\|_{1,\Omega},$$

which together with (1.6) proves the lemma. □

Lemma 1.4. *If a, $u \in W_2^\lambda(\Omega)$, $0 < \lambda < 1$, then $au \in L_p(\Omega)$, $1 \le p \le 1/(1-\lambda)$. If a, $u \in W_2^1(\Omega)$, then $au \in L_p(\Omega)$, $1 \le p < \infty$.*

Here we introduce the mesh domains $\overline{\omega}_\alpha = \{x_\alpha = i_\alpha h_\alpha : \ i_\alpha = 0, 1, \ldots, N_\alpha, \ h_\alpha = \ell_\alpha/N_\alpha\}$, $\omega_\alpha = \overline{\omega}_\alpha \cap (0, \ell_\alpha)$, $\overline{\omega} = \overline{\omega}_1 \times \overline{\omega}_2$, $\omega = \overline{\omega} \cap \Omega$, $\gamma = \overline{\omega} \setminus \omega$, $\omega_\alpha^+ = \overline{\omega}_\alpha \cap (0, \ell_\alpha]$, $\omega_\alpha^- = \overline{\omega}_\alpha \cap [0, \ell_\alpha)$, $\omega^\pm = \omega_1^\pm \times \omega_2^\pm$, $\omega_{(1)} = \omega_1 \times \overline{\omega}_2$, $\omega_{(2)} = \overline{\omega}_1 \times \omega_2$, $\omega_{(1+)} = \omega_1^+ \times \overline{\omega}_2$, $\omega_{(2+)} = \overline{\omega}_1 \times \omega_2^+$, $\gamma_{\pm\alpha} = \Gamma_{\pm\alpha} \cap \overline{\omega}$, $\omega_{(+\alpha)} = \omega \cup \gamma_{+\alpha}$, $\gamma^- = \{(-h_1, x_2), (\ell_1 + h_1, x_2), (x_1, -h_2), (x_1, \ell_2 + h_2) : \ x_1 \in \omega_1, \ x_2 \in \omega_2\}$, $\hbar_\alpha = h_\alpha, x_\alpha \in \omega_\alpha$; $\hbar_\alpha = h_\alpha/2$, $x_\alpha = 0, \ell_\alpha$, $|h| = (h_1^2 + h_2^2)^{1/2}$, $\alpha = 1, 2$.

A function $y = y(x)$ of the discrete argument is called a mesh function. The value of the mesh function $y(x)$ at the nod (ih_1, jh_2) is denoted by y_{ij}, i.e. $y(ih_1, jh_2) = y_{ij}$. In the cases where the nod number (ij) is not of importance, it will be omitted.

Denote by $H = H(\overline{\omega})$ the set of mesh functions defined on $\overline{\omega}$ and introduce on it the inner product and the norm $(y, v) = \sum_{\overline{\omega}} \hbar_1 \hbar_2 y v$, $\|y\| = (y, y)^{1/2}$, transforming thus H into a finite-dimensional Hilbert space of mesh functions.

Let for the mesh functions

$$(y, v)_{(\alpha)} = \sum_{\omega_{(\alpha)}} h_\alpha \hbar_{3-\alpha} y v, \ \ \|y\|_{(\alpha)} = (y, y)_{(\alpha)}^{1/2}, \ \ \alpha = 1, 2,$$

$$\|v\|_{(\alpha+)}^2 = \sum_{\omega_{(\alpha+)}} h_\alpha \hbar_{3-\alpha} v^2, \ \ (y, v)_{\widetilde{\omega}} = \sum_{\widetilde{\omega}} h_1 h_2 y v, \ \ \|y\|_{\widetilde{\omega}} = (y, y)_{\widetilde{\omega}}^{1/2},$$

$$\|y\|_{L_p(\widetilde{\omega})} = \Big(\sum_{\widetilde{\omega}} h_1 h_2 |y|^p\Big)^{1/p}, \ \ p \ge 1, \ \ \widetilde{\omega} \subseteq \overline{\omega}.$$

For the mesh functions and difference ratios we will use the following notation:

$$y^{(\pm 0.5_1)}(x) = y(x_1 \pm 0.5h_1, x_2),\quad y^{(\pm 0.5_2)}(x) = y(x_1, x_2 \pm 0.5h_2),$$
$$y^{(\pm 1_1)}(x) = I_1^{\pm} y(x) = y(x_1 \pm h_1, x_2),\quad y^{(\pm 1_2)}(x) = I_2^{\pm} y(x) = y(x_1, x_2 \pm h_2),$$
$$y_{x_\alpha} = (y^{(+1_\alpha)} - y)/h_\alpha,\quad y_{\overline{x}_\alpha} = (y - y^{(-1_\alpha)})/h_\alpha,$$
$$y_{\overset{\circ}{x}_\alpha} = (y^{(+1_\alpha)} - y^{(-1_\alpha)})/(2h_\alpha),\quad \alpha = 1,2.$$

The second difference derivative in the direction of x_α is defined by the formula $y_{\overline{x}_\alpha x_\alpha} = (y^{(-1_\alpha)} - 2y + y^{(+1_\alpha)})/h_\alpha^2$, $\alpha = 1,2$.

Here we introduce the Sobolev mesh spaces $W_2^k(\omega)$, $k = 1,2$, in which the norm is defined by the formulas where $\|y\|^2_{W_2^1(\omega)} = |y|^2_{W_2^1(\omega)} + \|y\|^2$, $|y\|^2_{W_2^2(\omega)} = |y|^2_{W_2^2(\omega)} + \|y\|^2_{W_2^1(\omega)}$, where $|y|^2_{W_2^1(\omega)} = \|\nabla y\|^2 = \|y_{\overline{x}_1}\|^2_{(1+)} + \|y_{\overline{x}_2}\|^2_{(2+)}$, $|y|^2_{W_2^2(\omega)} = \|\Delta_h y\|^2 = \|y_{\overline{x}_1 x_1}\|^2_{(1)} + \|y_{\overline{x}_2 x_2}\|^2_{(2)} + 2\|y_{\overline{x}_1 \overline{x}_2}\|^2_{\omega^+}$.

In different problems, depending on the boundary conditions, a type of the mesh inner product as well as of the norm can be defined concretely. Thus, for example, in the case of homogeneous Dirichlet conditions we have

$$(y,v) = \sum_{\omega} h_1 h_2 y v,\quad |y|^2_{W_2^1(\omega)} = \sum_{\omega_1^+ \times \omega_2} h_1 h_2 (y_{\overline{x}_1})^2 + \sum_{\omega_1 \times \omega_2^+} h_1 h_2 (y_{\overline{x}_2})^2,$$
$$|y|^2_{W_2^2(\omega)} = \sum_{\omega} h_1 h_2 \big((y_{\overline{x}_1 x_1})^2 + (y_{\overline{x}_2 x_2})^2\big) + 2\sum_{\omega^+} h_1 h_2 (y_{\overline{x}_1 \overline{x}_2})^2.$$

Let S_α be the averaging Steklov's operator in the direction of x_α ($\alpha = 1,2$):

$$S_1 u(x) = \frac{1}{h_1} \int\limits_{x_1 - h_1/2}^{x_1 + h_1/2} u(\xi_1, x_2)\, d\xi_1,\quad S_2 u(x) = \frac{1}{h_2} \int\limits_{x_2 - h_2/2}^{x_2 + h_2/2} u(x_1, \xi_2)\, d\xi_2.$$

The classical averaging Steklov's operator in $\mathbf{R}^2$ is defined by the equality $S = S_1 S_2$.

We will also need the following averaging operators:

$$S_1^- u(x) = \frac{1}{h_1} \int\limits_{x_1 - h_1}^{x_1} u(\xi_1, x_2)\, d\xi_1,\quad S_1^+ u(x) = \frac{1}{h_1} \int\limits_{x_1}^{x_1 + h_1} u(\xi_1, x_2)\, d\xi_1,$$

$$\check{S}_1 u(x) = \frac{2}{h_1} \int\limits_{x_1 - h_1/2}^{x_1} u(\xi_1, x_2)\, d\xi_1,\quad \widehat{S}_1 u(x) = \frac{2}{h_1} \int\limits_{x_1}^{x_1 + h_1/2} u(\xi_1, x_2)\, d\xi_1,$$

and the operator

$$T_1 u(x) = \frac{1 + \delta_0 + \delta_1}{h_1^2} \int\limits_{x_1 - h_1(1-\delta_0)}^{x_1 + h_1(1-\delta_1)} (h_1 - |x_1 - \xi_1|) u(\xi_1, x_2)\, d\xi_1,$$

introduced in [71] (pp. 58 and 156), where $\delta_0 = \delta(0, x_1)$, $\delta_1 = \delta(\ell_1, x_1)$, $\delta(\cdot\,,\cdot)$ is the Kronecker symbol.

Let

$$T_1^- u(x) = \frac{2}{h_1^2} \int\limits_{x_1-h_1}^{x_1} (h_1 - x_1 + \xi_1) u(\xi_1, x_2)\, d\xi_1,$$

$$T_1^+ u(x) = \frac{2}{h_1^2} \int\limits_{x_1}^{x_1+h_1} (h_1 + x_1 - \xi_1) u(\xi_1, x_2)\, d\xi_1.$$

The operators $S_2^\pm$, $\check{S}_2$, $\widehat{S}_2$, $T_2^\pm$ and T_2 are defined analogously.

It can be easily verified that for the averaging operators T_1 and T_2,

$$T_\alpha = S_\alpha^2 = S_\alpha^+ S_\alpha^- = S_\alpha^- S_\alpha^+, \quad T_\alpha = \begin{cases} T_\alpha^+, & x_\alpha = 0, \\ 0.5(T_\alpha^- + T_\alpha^+), & x_\alpha \in \omega_\alpha, \\ T_\alpha^-, & x_\alpha = \ell_\alpha. \end{cases}$$

We denote

$$\overline{S}_\alpha = \begin{cases} S_\alpha^+, & x_\alpha = 0, \\ 0.5(S_\alpha^+ + S_\alpha^-), & x_\alpha \in \omega_\alpha, \quad \alpha = 1, 2. \\ S_\alpha^-, & x_\alpha = \ell_\alpha, \end{cases}$$

2. The Dirichlet Problem. Convergence in the Norm W_2^1

History of the matter. Results of Section 2 have been published in [7]. The estimate (2.23) has been obtained: in [83] for the constants a_{ij} and the variable $a_0 \in L_\infty(\Omega)$; in [52] and [53] for $a_{ij} \in W_\infty^{m-1}(\Omega)$, $a_{ij} = a_{ji}$, $a_0 \in W_\infty^{m-2}(\Omega)$; in [55] for $a_{ij} \in W_2^{m-1}(\Omega)$, $a_0 \in W_2^{m-2}(\Omega)$.

1^0. Statement of the problem. Difference scheme. We consider the difference approximation of the boundary value problem

$$Lu \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\,\frac{\partial u}{\partial x_j}\Big) + a_0 u = f, \;\; x \in \Omega, \;\; u(x) = 0, \;\; x \in \Gamma. \quad (2.1)$$

The conditions of uniform ellipticity

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \ge \nu(\xi_1^2 + \xi_2^2), \;\; 0 < \nu = \text{const}, \;\; x \in \Omega, \quad (2.2)$$

are assumed to be fulfilled. Let there exist a unique solution $u(x) \in W_2^m(\Omega)$, $2 < m \le 3$, of the problem (2.1) and the conditions

$$\begin{gathered} a_{ij}(x) \in W_2^{m-1}(\Omega), \;\; i,j = 1,2, \;\; 0 \le a_0(x) \in W_2^{m-2}(\Omega), \\ f(x) \in W_2^{m-2}(\Omega), \;\; 2 < m \le 3, \end{gathered} \quad (2.3)$$

be fulfilled.

We approximate the problem (2.1) by the difference scheme

$$Ay=\varphi(x),\ \ \varphi(x)=S_1S_2f,\ \ x\in\omega,\ \ y(x)=0,\ \ x\in\gamma, \tag{2.4}$$

where

$$Ay\equiv-\frac{1}{2}\sum_{i,j=1}^{2}\big((a_{ij}y_{\overline{x}_j})_{x_i}+(a_{ij}y_{x_j})_{\overline{x}_i}\big)+ay,\quad a(x)=S_1S_2a_0(x). \tag{2.5}$$

Let H be the space of mesh functions defined on $\overline{\omega}$ and equal to zero on γ, with the inner product $(y,v)=(y,v)_\omega$ and with the norm $\|y\|=\|y\|_\omega$. The notation $\|\cdot\|_{(\alpha+)}$ takes the form

$$\|v\|_{(1+)}^2=\sum_{\omega_1^+\times\omega_2}h_1h_2v^2,\quad \|v\|_{(2+)}^2=\sum_{\omega_1\times\omega_2^+}h_1h_2v^2.$$

The operator A is positive definite in the space H ([70], p. 262), and hence there exists the unique solution of the problem (2.5).

2^0. A priori estimate of error of the method. In this subsection we investigate the convergence rate of the difference scheme (2.4). For the error $z=y-u$ we obtain the problem

$$Az=\psi,\ \ x\in\omega,\ \ z(x)=0,\ \ x\in\gamma, \tag{2.6}$$

where $\psi=\varphi-Au=S_1S_2f-Au$.

Taking into account the properties of the operators S_1 and S_2:

$$S_1\frac{\partial u}{\partial x_1}=u_{x_1}\Big(x_1-\frac{h_1}{2},x_2\Big),\quad S_2\frac{\partial u}{\partial x_2}=u_{x_2}\Big(x_1,x_2-\frac{h_2}{2}\Big),$$

from the equation (2.1) we find that

$$S_1S_2f=-S_2\Big(a_{11}\frac{\partial u}{\partial x_1}\Big)_{x_1}\Big(x_1-\frac{h_1}{2},x_2\Big)-S_2\Big(a_{12}\frac{\partial u}{\partial x_2}\Big)_{x_1}\Big(x_1-\frac{h_1}{2},x_2\Big)-$$
$$-S_1\Big(a_{21}\frac{\partial u}{\partial x_1}\Big)_{x_2}\Big(x_1,x_2-\frac{h_2}{2}\Big)-S_1\Big(a_{22}\frac{\partial u}{\partial x_2}\Big)_{x_2}\Big(x_1,x_2-\frac{h_2}{2}\Big)+S_1S_2(a_0u).$$

Therefore the expression for the error ψ can be reduced to the form

$$\psi=(\eta_{11}+\eta_{12})_{x_1}+(\eta_{21}+\eta_{22})_{x_2}+\eta, \tag{2.7}$$

where

$$\eta_{11}=\frac{1}{2}\big(a_{11}u_{\overline{x}_1}+(a_{11}u_{x_1})(x_1-h_1,x_2)\big)-S_2\Big(a_{11}\frac{\partial u}{\partial x_1}\Big)\Big(x_1-\frac{h_1}{2},x_2\Big),$$
$$\eta_{12}=\frac{1}{2}\big(a_{12}u_{\overline{x}_2}+(a_{12}u_{x_2})(x_1-h_1,x_2)\big)-S_2\Big(a_{12}\frac{\partial u}{\partial x_2}\Big)\Big(x_1-\frac{h_1}{2},x_2\Big),$$
$$\eta_{21}=\frac{1}{2}\big(a_{21}u_{\overline{x}_1}+(a_{21}u_{x_1})(x_1,x_2-h_2)\big)-S_1\Big(a_{21}\frac{\partial u}{\partial x_1}\Big)\Big(x_1,x_2-\frac{h_2}{2}\Big),$$
$$\eta_{22}=\frac{1}{2}\big(a_{22}u_{\overline{x}_2}+(a_{22}u_{x_2})(x_1,x_2-h_2)\big)-S_1\Big(a_{22}\frac{\partial u}{\partial x_2}\Big)\Big(x_1,x_2-\frac{h_2}{2}\Big),$$
$$\eta=S_1S_2(a_0u)-S_1S_2a_0u.$$

Using the equality (2.7), from (2.6) we find that

$$(Az,z)=(\psi,z)=\big((\eta_{11}+\eta_{12})_{x_1}+(\eta_{22}+\eta_{21})_{x_2}+\eta,z\big)\leq$$
$$\leq\|\eta_{11}+\eta_{12}\|_{(1+)}\|\nabla z\|+\|\eta_{22}+\eta_{21}\|_{(2+)}\|\nabla z\|+\|\eta\|\,\|z\|. \quad (2.8)$$

On the basis of (2.2), we arrive at the inequality

$$\nu\|\nabla z\|^2\leq(Az,z). \quad (2.9)$$

Using the estimate (2.9) and the difference analogue of Friedrichs inequality ([70], p. 309)

$$\|y\|\leq\frac{\ell_0}{4}\|\nabla y\|,\ \ \ell_0=\max(\ell_1;\ell_2), \quad (2.10)$$

from (2.8) we obtain

$$\nu\|\nabla z\|^2\leq\|\nabla z\|\Big(\|\eta_{11}+\eta_{12}\|_{(1+)}+\|\eta_{22}+\eta_{21}\|_{(2+)}+\frac{\ell_0}{4}\|\eta\|\Big),$$

that is,

$$\|z\|_{W_2^1(\omega)}\leq$$
$$\leq\frac{1}{\nu}\Big(\|\eta_{11}\|_{(1+)}+\|\eta_{12}\|_{(1+)}+\|\eta_{21}\|_{(2+)}+\|\eta_{22}\|_{(2+)}+\frac{\ell_0}{4}\|\eta\|\Big). \quad (2.11)$$

3^0. Estimation of the convergence rate. *Estimation of functionals η_{11} and η_{22}.* We rewrite the expression η_{11} in the form

$$\eta_{11}(x)=-\frac{1}{2}\left(a_{11}+a_{11}(x_1-h_1,x_2)\right)\ell_{11}^{(1)}(u)+$$
$$+\frac{\partial u}{\partial x_1}\Big(x_1-\frac{h_1}{2},x_2\Big)\ell_{11}^{(2)}(a_{11})+\ell_{11}^{(3)}\Big(a_{11}\,\frac{\partial u}{\partial x_1}\Big), \quad (2.12)$$

where

$$\ell_{11}^{(1)}(u)=\frac{\partial u}{\partial x_1}\Big(x_1-\frac{h_1}{2},x_2\Big)-u_{\overline{x}_1},$$
$$\ell_{11}^{(2)}(a)=\frac{1}{2}\left(a(x)+a(x_1-h_1,x_2)\right)-a\Big(x_1-\frac{1}{2}\,h_1,x_2\Big),$$
$$\ell_{11}^{(3)}(v)=(E-S_2)v(x_1-0.5h_1,x_2),\quad Eu\equiv u.$$

Let $e=e(x)=\big\{\xi=(\xi_1,\xi_2):\ |\xi_\alpha-x_\alpha|\leq h_\alpha,\ \alpha=1,2\big\}\cap\Omega$. By $\widetilde{u}(t)$ we denote the function obtained from $u(\xi)$ by the substitution of variables $\xi_\alpha=x_\alpha+t_\alpha h_\alpha$, $\alpha=1,2$, which maps the domain $e(x)$ into $\widetilde{e}=\big\{t=(t_1,t_2):\ |t_\alpha|\leq 1,\ \alpha=1,2\big\}$.

Since $u(\xi)=u(x_1+t_1h_1,x_2+t_2h_2)\equiv\widetilde{u}(t)$, $a_{11}(\xi)\equiv\widetilde{a}_{11}(t)$, therefore

$$\frac{\partial\widetilde{u}(t)}{\partial t_\alpha}=\frac{\partial u(\xi)}{\partial t_\alpha}=h_\alpha\,\frac{\partial u(\xi)}{\partial\xi_\alpha},\quad \alpha=1,2.$$

Using the imbedding $W_2^m \subset C^n$ for $m > n+1$, we obtain

$$\begin{aligned}|\ell_{11}^{(1)}(u)| &= \Big|\frac{1}{h_1}\Big(\frac{\partial\widetilde{u}(-0.5;0)}{\partial t_1} - \widetilde{u}(0;0) + \widetilde{u}(-1;0)\Big)\Big| \le \\ &\le \frac{c}{h_1}\,\|\widetilde{u}\|_{C^1(\widetilde{e})} \le \frac{c}{h_1}\,\|\widetilde{u}\|_{W_2^m(\widetilde{e})}, \quad m > 2.\end{aligned}$$

Taking into account the fact that the expression under consideration (as a functional of $\widetilde{u}$) vanishes on $\boldsymbol{\pi}_2$, by the Bramble–Hilbert lemma we have $|\ell_{11}^{(1)}(u)| \le \frac{c}{h_1}\,|\widetilde{u}|_{W_2^m(\widetilde{e})}$, $2 < m \le 3$, or, getting back to our previous variables,

$$|\ell_{11}^{(1)}(u)| \le \frac{c|h|^{m-1}}{(h_1h_2)^{1/2}}\,|u|_{W_2^m(e)}, \;\; x \in \omega, \;\; 2 < m \le 3. \tag{2.13}$$

Analogously,

$$\begin{aligned}|\ell_{11}^{(2)}(a_{11})| &= \big|0.5\,\big(\widetilde{a}_{11}(0;0) + \widetilde{a}_{11}(-1;0)\big) - \widetilde{a}_{11}(-0.5;0)\big| \le \\ &\le c\|\widetilde{a}_{11}\|_{C(\widetilde{e})} \le c\|\widetilde{a}_{11}\|_{W_2^\alpha(\widetilde{e})}, \;\; \alpha > 1.\end{aligned}$$

Since $\ell_{11}^{(2)}$ vanishes on $\boldsymbol{\pi}_1$, by the Bramble–Hilbert lemma we obtain $|\ell_{11}^{(2)}(a_{11})| \le c|\widetilde{a}_{11}|_{W_2^\alpha(\widetilde{e})}$, $1 < \alpha \le 2$, or, passing again to the previous variables,

$$|\ell_{11}^{(2)}(a_{11})| \le \frac{c|h|^\alpha}{(h_1h_2)^{1/2}}\,|a_{11}|_{W_2^\alpha(e)}, \;\; 1 < \alpha \le 2. \tag{2.14}$$

Next,

$$|\ell_{11}^{(3)}(a_{11}D_1u)| \equiv |(E - S_2)v| \le \frac{c|h|^\alpha}{(h_1h_2)^{1/2}}\,|v|_{W_2^\alpha(e)}, \;\; v = a_{11}D_1u, \;\; 1 < \alpha \le 2,$$

that is,

$$|\ell_{11}^{(3)}(a_{11}D_1u)| \le |a_{11}D_1u|_{W_2^\alpha(e)}, \;\; 1 < \alpha \le 2. \tag{2.15}$$

By means of the estimates (2.13)–(2.15), from (2.12) we get

$$\begin{aligned}|\eta_{11}| \le \frac{c|h|^{m-1}}{(h_1h_2)^{1/2}}\Big(&\|a_{11}\|_{C(\Omega)}|u|_{W_2^m(e)} + |a_{11}|_{W_2^{m-1}(e)}\|D_1u\|_{C(\Omega)} + \\ &+ |a_{11}D_1u|_{W_2^{m-1}(e)}\Big), \;\; 2 < m \le 3, \;\; x \in \omega_1^+ \times \omega_2.\end{aligned} \tag{2.16}$$

For the estimate η_{22} we analogously obtain

$$\begin{aligned}|\eta_{22}| \le \frac{c|h|^{m-1}}{(h_1h_2)^{1/2}}\Big(&\|a_{22}\|_{C(\Omega)}|u|_{W_2^m(e)} + |a_{22}|_{W_2^{m-1}(e)}\|D_2u\|_{C(\Omega)} + \\ &+ |a_{22}D_2u|_{W_2^{m-1}(e)}\Big), \;\; 2 < m \le 3, \;\; x \in \omega_1 \times \omega_2^+.\end{aligned} \tag{2.17}$$

Estimation of the functionals η_{12} and η_{21}. We represent the expression η_{12} in the form

$$\eta_{12} = \ell_{12}^{(1)}(a_{12}D_2u) + \frac{h_2}{4}\,\ell_{12}^{(2)}(a_{12})S_1S_2(D_2^2u) + 0.5a_{12}\,\ell_{12}^{(3)}(u) + \\ + \ell_{12}^{(4)}(u) + \ell_{12}^{(5)} + 0.5a_{12}(x_1 - h_1, x_2)\,\ell_{12}^{(6)}(u), \tag{2.18}$$

where

$$\ell_{12}^{(1)}(v) = 0.5\big(v + v(x_1 - h_1, x_2)\big) - S_2v(x_1 - 0.5h_1, x_2),$$
$$\ell_{12}^{(2)}(a) = h_1S_1S_2(D_1a) - a + a(x_1 - h_1, x_2),$$
$$\ell_{12}^{(3)}(u) = u_{\overline{x}_2} - D_2u + \frac{h_2}{2}\,S_1S_2(D_2^2u),$$
$$\ell_{12}^{(4)}(u) = \frac{h_1h_2}{4}\,\Big(S_1S_2(D_1a_{12}D_2^2u) - S_1S_2(D_1a_{12})S_1S_2(D_2^2u)\Big),$$
$$\ell_{12}^{(5)} = -\frac{h_1h_2}{4}\,S_1S_2(D_1a_{12}D_2^2u),$$
$$\ell_{12}^{(6)}(u) = u_{x_2}(x_1 - h_1, x_2) - D_2u(x_1 - h_1, x_2) - \frac{h_2}{2}\,S_1S_2(D_2^2u).$$

To estimate the summands of the equality (2.18), we apply the same method as we have used for the estimation (2.16), $|\ell_{12}^{(1)}(v)| \le c\|\widetilde{v}\|_{C(\widetilde{e})} \le c\|\widetilde{v}\|_{W_2^\alpha(\widetilde{e})}$, $\alpha > 1$.

Since $\ell_{12}^{(1)}(v)$ vanishes on $\boldsymbol{\pi}_1$, using the Bramble–Hilbert lemma, we obtain

$$|\ell_{12}^{(1)}(v)| \le c|\widetilde{v}|_{W_2^\alpha(\widetilde{e})} \le \frac{c|h|^\alpha}{\sqrt{h_1h_2}}\,|v|_{W_2^\alpha(e)},\ \ 1 < \alpha \le 2,$$

that is,

$$\big|\ell_{12}^{(1)}(a_{12}D_2u)\big| \le \frac{c|h|^\alpha}{\sqrt{h_1h_2}}\,\|a_{12}D_2u\|_{W_2^\alpha(e)},\ \ 1 < \alpha \le 2.$$

Next,

$$|\ell_{12}^{(2)}(a_{12})| = \bigg|\int\limits_{-1/2}^{1/2}\int\limits_{-1/2}^{1/2} \frac{\partial\widetilde{a}_{12}}{\partial t_1}\,dt_1\,dt_2 - \widetilde{a}_{12}(0,0) + \widetilde{a}_{12}(-1,0)\bigg| \le \\ \le c\|\widetilde{a}_{12}\|_{W_2^\alpha(\widetilde{e})},\ \ \alpha > 1.$$

Since $\ell_{12}^{(2)}(a_{12})$ vanishes on $\boldsymbol{\pi}_1$, we obtain

$$|\ell_{12}^{(2)}(a_{12})| \le c|\widetilde{a}_{12}|_{W_2^\alpha(\widetilde{e})} \le \frac{c|h|^\alpha}{\sqrt{h_1h_2}}\,|a_{12}|_{W_2^\alpha(e)},\quad 1 < \alpha \le 2.$$

It can be easily seen that

$$|\ell_{12}^{(3)}(u)|\leq\frac{1}{h_2}\left|\widetilde{u}(0,0)-\widetilde{u}(0,1)-\frac{\partial\widetilde{u}}{\partial t_2}\,(0,0)+\frac{1}{2}\int\limits_{-1/2}^{1/2}\int\limits_{-1/2}^{1/2}\frac{\partial^2\widetilde{u}(t)}{\partial t_2^2}\,dt_1\,dt_2\right|\leq$$

$$\leq\frac{c}{h_2}\,\|\widetilde{u}\|_{W_2^m(\widetilde{e})},\quad m>2,$$

and hence

$$|\ell_{12}^{(3)}(u)|\leq\frac{c}{h_2}\,|\widetilde{u}|_{W_2^m(\widetilde{e})}\leq\frac{c|h|^{m-1}}{\sqrt{h_1h_2}}\,|u|_{W_2^m(e)},\quad 2<m\leq 3.$$

Using Cauchy–Buniakowski's inequality, we get

$$|\ell_{12}^{(4)}(u)|\leq c\|D_1a_{12}\|_{L_2(e)}\|D_2^2u\|_{L_2(e)}\leq$$
$$\leq\frac{c}{h_2}\,\|a_{12}\|_{W_2^1(\Omega)}\|D_2^2\widetilde{u}\|_{L_2(\widetilde{e})}\leq\frac{c}{h}\,\|a_{12}\|_{W_2^1(\Omega)}\|\widetilde{u}\|_{W_2^m(\widetilde{e})},\;\; m>2,$$

and since $\ell_{12}^{(4)}(u)$ vanishes on $u\in\boldsymbol{\pi}_2$, therefore

$$|\ell_{12}^{(4)}(u)|\leq\frac{c}{h}\,\|a_{12}\|_{W_2^1(\Omega)}|\widetilde{u}|_{W_2^m(\widetilde{e})}\leq\frac{c|h|^{m-1}}{\sqrt{h_1h_2}}\,\|a_{12}\|_{W_2^1(\Omega)}|u|_{W_2^m(\widetilde{e})},\;\;2<m\leq 3.$$

Applying Hölder's inequality with $p=2/(m-2)$, $q=2/(4-m)$, we obtain

$$|\ell_{12}^{(5)}|\leq\frac{1}{4}\Big(\int\limits_e 1^p\,d\xi\Big)^{1/p}\Big(\int\limits_e\Big|\frac{\partial a_{12}}{\partial\xi_1}\,\frac{\partial^2u}{\partial\xi_2^2}\Big|^q\,d\xi\Big)^{1/q}\leq$$
$$\leq c\,\frac{|h|^{m-1}}{\sqrt{h_1h_2}}\,\Big\|\frac{\partial a_{12}}{\partial x_1}\,\frac{\partial^2u}{\partial x_2^2}\Big\|_{L_{\frac{2}{4-m}}(e)},\;\;2<m\leq 3.$$

Finally, for η_{12} we obtain the estimate

$$|\eta_{12}|\leq\frac{c|h|^{m-1}}{(h_1h_2)^{1/2}}\Big(|a_{12}D_2u|_{W_2^{m-1}(e)}+\|D_1a_{12}D_2^2u\|_{L_{\frac{2}{4-m}}(e)}+$$
$$+\|a_{12}\|_{W_2^{m-1}(\Omega)}|u|_{W_2^m(e)}\Big),\;\;2<m\leq 3,\;\;x\in\omega_1^+\times\omega_2.\quad(2.19)$$

Similarly,

$$|\eta_{21}|\leq\frac{c|h|^{m-1}}{(h_1h_2)^{1/2}}\Big(|a_{21}D_1u|_{W_2^{m-1}(e)}+\|D_2a_{21}D_1^2u\|_{L_{\frac{2}{4-m}}(e)}+$$
$$+\|a_{21}\|_{W_2^{m-1}(\Omega)}|u|_{W_2^m(e)}\Big),\;\;2<m\leq 3,\;\;x\in\omega_1\times\omega_2^+.\quad(2.20)$$

Estimation of the functional $\eta(x)$. Let $R^{(i)}(v)=S_1S_2((\xi_i-x_i)v)$, $R^{(i+2)}(v)=-0.5S_1S_2((\xi_i-x_i)^2v)$, $i=1,2$, $R^{(5)}(v)=-S_1S_2((\xi_1-x_1)(\xi_2-$

$x_2)v)$. To estimate the expression $\eta(x)$, we represent it in the form $\eta = R + Q$, where

$$R = \sum_{i=1}^{2} \Big[R^{(i)}\Big(a_0\,\frac{\partial u}{\partial \xi_i}\Big) + R^{(i+2)}\Big(a_0\,\frac{\partial^2 u}{\partial \xi_i^2}\Big)\Big] + R^{(5)}\Big(a_0\,\frac{\partial^2 u}{\partial \xi_1 \partial \xi_2}\Big),$$
$$Q = S_1 S_2(a_0 u) - S_1 S_2 a_0 u - R.$$

Using the above-mentioned transformation of variables, we obtain

$$|R^{(i)}(v)| = \Big|h_i \int\limits_{\widetilde{e}} t_i \widetilde{v}(t)\,dt\Big| \le ch_i \|\widetilde{v}\|_{L_2(\widetilde{e})}, \quad i = 1, 2,$$

and since $R^{(i)}$, $i = 1, 2$, vanishes for $v \in \boldsymbol{\pi}_0$,

$$|R^{(i)}(v)| \le ch_i |\widetilde{v}|_{W_2^{m-2}(\widetilde{e})} \le \frac{c|h|^{m-1}}{\sqrt{h_1 h_2}}\,|v|_{W_2^{m-2}(e)}, \ \ 2 < m \le 3, \ \ i = 1, 2.$$

Using Hölder's inequality with $p = 2/(m-2)$, $q = 2/(4-m)$, we obtain

$$|R^{(i+2)}(v)| \le \frac{1}{h_1 h_2} \Bigg(\int\limits_{e} |\xi_i - x_i|^{2p}\,d\xi\Bigg)^{1/p} \Bigg(\int\limits_{e} |v(\xi)|^q\,d\xi\Bigg)^{1/q} \le$$
$$\le \frac{c|h|^{m-1}}{\sqrt{h_1 h_2}}\,\|v\|_{L_{\frac{2}{4-m}}(e)}, \ \ 2 < m \le 3, \ \ i = 1, 2.$$

Further,

$$|R^{(5)}(v)| \le \frac{c|h|^{m-1}}{\sqrt{h_1 h_2}}\,\|v\|_{L_{\frac{2}{4-m}}(e)}, \ \ 2 < m \le 3.$$

Finally, for R we obtain the estimate

$$|R| \le \frac{c|h|^{m-1}}{\sqrt{h_1 h_2}} \Big(|a_0 D_1 u|_{W_2^{m-2}(e)} + |a_0 D_2 u|_{W_2^{m-2}(e)} +$$
$$+\|a_0 D_1 D_2 u\|_{L_{\frac{2}{4-m}}(e)} + \|a_0 D_1^2 u\|_{L_{\frac{2}{4-m}}(e)} + \|a_0 D_2^2 u\|_{L_{\frac{2}{4-m}}(e)}\Big), \ \ 2 < m \le 3.$$

We now proceed to obtaining an estimate for Q. First of all, it should be noted that $Q = Q(u)$ vanishes on $u \in \boldsymbol{\pi}_2$. We will have to estimate anew the summands of the functional R so that the norm of the function $u(x)$ would emerge. We have

$$\Big|R^{(i)}\Big(a_0\,\frac{\partial u}{\partial \xi_i}\Big)\Big| = \Big|\frac{1}{2} \int\limits_{\widetilde{e}} t_i \widetilde{a}_0(t)\,\frac{\partial u}{\partial t_i}\,dt\Big| \le \Big|\frac{1}{4} \int\limits_{\widetilde{e}} \widetilde{a}_0(t)\,\frac{\partial \widetilde{u}}{\partial t_i}\,dt\Big| \le$$
$$\le \frac{1}{4} \Big(\int\limits_{\widetilde{e}} |\widetilde{a}_0(t)|^2\Big)^{1/2} \Bigg(\int\limits_{\widetilde{e}} \Big|\frac{\partial \widetilde{u}}{\partial t_i}\Big|^2 dt\Bigg)^{1/2} =$$
$$= \frac{1}{4\sqrt{h_1 h_2}} \Big(\int\limits_{e} |a_0(t)|^2\Big)^{1/2} \Bigg(\int\limits_{\widetilde{e}} \Big|\frac{\partial \widetilde{u}}{\partial t_i}\Big|^2 dt\Bigg)^{1/2} =$$

$$= \frac{1}{4\sqrt{h_1h_2}}\,\|a_0\|_{L_2(e)}\Big\|\frac{\partial\widetilde{u}}{\partial t_i}\Big\|_{L_2(\widetilde{e})} \le \frac{c}{\sqrt{h_1h_2}}\,\|a_0\|_{L_2(\Omega)}\|\widetilde{u}\|_{W_2^1(\widetilde{e})},\ \ i=1,2.$$

Next,

$$\Big|R^{(i+2)}\Big(a_0\,\frac{\partial^2u}{\partial\xi_i^2}\Big)\Big| \le \max_e(\xi_i-x_i)^2\,\frac{1}{h_1h_2}\int\limits_e\Big|a_0\,\frac{\partial^2u}{\partial\xi_i^2}\Big|\,d\xi =$$

$$= \frac{h_1}{4h_2}\int\limits_e\Big|a_0\,\frac{\partial^2u}{\partial\xi_i^2}\Big|\,d\xi \le \frac{h_1}{4h_2}\,\|a_0\|_{L_2(e)}\Big\|\frac{\partial^2u}{\partial\xi_i^2}\Big\|_{L_2(e)} \le$$

$$\le \frac{h_1}{4h_2}\,\|a_0\|_{L_2(\Omega)}\Big\|\frac{\partial^2u}{\partial\xi_i^2}\Big\|_{L_2(e)} \le \frac{c}{\sqrt{h_1h_2}}\,\|a_0\|_{L_2(\Omega)}\Big\|\frac{\partial^2\widetilde{u}}{\partial\xi_i^2}\Big\|_{L_2(\widetilde{e})},\ \ i=1,2.$$

$R^{(5)}(a_0\,\frac{\partial^2u}{\partial\xi_1\partial\xi_2})$ is estimated analogously, and as a result, we obtain

$$|R| \le \frac{c}{\sqrt{h_1h_2}}\,\|a_0\|_{L_2(\Omega)}\|\widetilde{u}\|_{W_2^2(\widetilde{e})}.$$

Now we estimate the remaining summands of the functional Q. We have

$$|S_1S_2(a_0u)| \le \|u\|_{C(e)}\,\frac{1}{h_1h_2}\int\limits_e|a_0(\xi)|\,d\xi \le \frac{1}{\sqrt{h_1h_2}}\,\|\widetilde{u}\|_{C(\widetilde{e})}\|a_0\|_{L_2(\Omega)} \le$$

$$\le \frac{c}{\sqrt{h_1h_2}}\,\|a_0\|_{L_2(\Omega)}\|\widetilde{u}\|_{W_2^2(\widetilde{e})};\quad |S_1S_2a_0\,u| \le \frac{c}{\sqrt{h_1h_2}}\,\|\widetilde{u}\|_{W_2^2(\widetilde{e})}.$$

Finally, we obtain $|Q| \le \frac{c}{\sqrt{h_1h_2}}\,\|a_0\|_{L_2(\Omega)}\|\widetilde{u}\|_{W_2^2(\widetilde{e})}$. Consequently, using the Bramble–Hilbert lemma, we obtain

$$|Q| \le \frac{c|h|^{m-1}}{\sqrt{h_1h_2}}\,\|a_0\|_{L_2(\Omega)}\|\widetilde{u}\|_{W_2^m(\widetilde{e})},\ \ 2<m\le 3.$$

Finally,

$$\begin{aligned}|\eta| \le \frac{c|h|^{m-1}}{\sqrt{h_1h_2}}\,\Big(&|u|_{W_2^m(e)} + |a_0D_1u|_{W_2^{m-2}(e)} +\\ &+|a_0D_2u|_{W_2^{m-2}(e)} + +\|a_0D_1D_2u\|_{L_{\frac{2}{4-m}}(e)} +\\ &+\|a_0D_1^2u\|_{L_{\frac{2}{4-m}}(e)} + \|a_0D_2^2u\|_{L_{\frac{2}{4-m}}(e)}\Big),\quad 2<m\le 3.\end{aligned} \tag{2.21}$$

Estimate several summands in the right-hand side of the inequality (2.11). By (2.16), we have

$$\begin{aligned}&\|\eta_{11}\|^2_{(1+)} = \sum_{x\in\omega_1^+\times\omega_2} h_1h_2|\eta_{11}|^2 \le\\ &\le c|h|^{2m-2}\big(\|u\|^2_{W_2^m(\Omega)} + |a_{11}D_1u|^2_{W_2^{m-1}(\Omega)}\big),\ \ 2<m\le 3.\end{aligned} \tag{2.22}$$

Note that by virtue of Lemma 1.1,

$$|a_{11}D_1u|_{W_2^{m-1}(\Omega)} \le c\|a_{11}\|_{W_2^{m-1}(\Omega)}\|u\|_{W_2^m(\Omega)},\ \ 2<m\le 3.$$

Therefore form (2.22) it follows that $\|\eta_{11}\|_{(1+)} \le c|h|^{m-1}\|u\|_{W_2^m(\Omega)}$, $2 < m \le 3$.

The rest of the summands in the right-hand side of (2.11) are estimated analogously. Note that upon summation of the summands of the type $|v|_{L_{\frac{2}{4-m}}(e)}$, by virtue of the fact (see [48], p. 43, Theorem 22) that $\sum\limits_\alpha b_\alpha^t \le \Big(\sum\limits_\alpha b_\alpha\Big)^t$, $\forall b_\alpha \ge 0$, $t \ge 1$, we write

$$\sum_\omega |v|^2_{L_{\frac{2}{4-m}}(e)} \le \sum_\omega \bigg(\int_e |v|^{\frac{2}{4-m}}\,dx\bigg)^{4-m} \le \bigg(\sum_\omega \int_e |v|^{\frac{2}{4-m}}\,dx\bigg)^{4-m} \le \|v\|^2_{L_{\frac{2}{4-m}}(\Omega)}.$$

Using now Cauchy–Buniakowski's inequality $\|av\|_{L_{\frac{2}{4-m}}(\Omega)} \le \|a\|_{L_{\frac{4}{4-m}}(\Omega)}\|v\|_{L_{\frac{4}{4-m}}(\Omega)}$ and the imbedding $W_2^{m-2} \subset L_{\frac{4}{4-m}}$, we can see that the inequalities

$$\|a_0 D_i D_j u\|_{L_{\frac{2}{4-m}}(\Omega)} \le c\|a_0\|_{W_2^{m-2}(\Omega)}\|u\|_{W_2^m(\Omega)}, \ \ 2 < m \le 3, \ \ i,j = 1,2,$$

$$\big\|D_i a_{i,3-i} D^2_{3-i} u\big\|_{L_{\frac{2}{4-m}}(\Omega)} \le c\|a_{i,3-i}\|_{W_2^{m-1}(\Omega)}\|u\|_{W_2^m(\Omega)}, \ \ 2<m\le 3, \ \ i=1,2,$$

are valid. Consequently, form (2.21) we finally obtain the estimate

$$\|y-u\|_{W_2^1(\omega)} \le c|h|^{m-1}\|u\|_{W_2^m(\Omega)}, \ \ 2 < m \le 3. \tag{2.23}$$

Thus we have proved the following

Theorem 2.1. *Let the solution of the problem* (2.1) $u \in W_2^m(\Omega)$, $2 < m \le 3$, *and let the conditions* (2.2) *and* (2.3) *be fulfilled. Then the convergence of the difference scheme in the norm* $W_2^1(\omega)$ *is characterized by the estimate* (2.23).

3. Difference Schemes with Averaged Coefficients

History of the matter. The results of this section have been published in [12]. The estimate (3.7) has been obtained in [76] for $a_{12} = a_{21} = 0$, $a_{ii} \in W_\infty^1(\Omega)$, $a_0 \in L_\infty(\Omega)$; in [79] for $a_{ii} \in W_\infty^1(\Omega)$, $a_{12} = a_{21} = 0$, $a_0 \in L_\infty(\Omega)$; in [52], [53] for $a_{ij} \in W_\infty^1(\Omega)$, $a_{12} = a_{21}$, $a_0 \in L_\infty(\Omega)$; in [55] for $a_{ij} \in W_2^{1+\delta}(\Omega)$, $\delta > 0$, $a_{12} = a_{21}$, $a_0 \in L_{2+\varepsilon}(\Omega)$, $\varepsilon > 0$.

1^0. Statement of the problem. Difference scheme. In this subsection we consider difference schemes which approximate the problem (2.1), (2.2), where the functions f, a_{ij}, a_0 satisfy the following restrictions:

$$f(x)\in L_2(\Omega), \ \ a_{ij} \in W^1_{2+\varepsilon}(\Omega), \ \ i,j=1,2, \ \ 0 \le a_0(x) \in L_2(\Omega), \ \ \forall \varepsilon > 0. \tag{3.1}$$

As is known [58], there exists a unique generalized solution $u(x) \in W_2^2(\Omega)$ of the problem (2.1), (2.2), (3.1) for which the estimate $\|u\|_{W_2^2(\Omega)} \le c\|f\|_{L_2(\Omega)}$

is valid. We approximate the problem (2.1) by the difference scheme

$$\begin{aligned} &Ay \equiv -\frac{1}{2}\sum_{\alpha,\beta=1,2}^{2}\big[(a^+_{\alpha\beta}y_{x_\beta})_{\overline{x}_\alpha}+(a^-_{\alpha\beta}y_{\overline{x}_\beta})_{x_\alpha}\big]+ay=\varphi,\quad x\in\omega,\\ &y(x)=0,\quad x\in\gamma, \end{aligned} \tag{3.2}$$

where $a^+_{\alpha\beta}(x)=S_1^+S_2^+k_{\alpha\beta}$, $a^-_{\alpha\beta}(x)=a^+_{\alpha\beta}(x_1-h_1,x_2-h_2)$, $a(x)=T_1T_2a_0$, $\varphi(x)=T_1T_2f$.

H is assumed to be the space of mesh functions defined on $\overline{\omega}$ and equal to zero on γ, with the inner product $(y,v)=(y,v)_\omega$ and the norm $\|y\|=\|y\|_\omega$. The notation $\|\cdot\|_{(\alpha+)}$ takes the form $\|v\|^2_{(1+)}=\sum\limits_{\omega_1^+\times\omega_2}h_1h_2v^2$, $\|v\|^2_{(2+)}=\sum\limits_{\omega_1\times\omega_2^+}h_1h_2v^2$. Using the formulas of summation by parts, it is not difficult to show that

$$\begin{aligned} (Ay,y)=\frac{1}{2}\sum_{\omega}h_1h_2\Big(&\sum_{\alpha,\beta=1}^{2}a^+_{\alpha\beta}y_{x_\alpha}y_{x_\beta}+\sum_{\alpha,\beta=1}^{2}a^-_{\alpha\beta}y_{\overline{x}_\alpha}y_{\overline{x}_\beta}\Big)+\\ &+\frac{1}{2}\sum_{\omega_1}h_1h_2\big(a^+_{22}y^2_{x_2}(x_1,0)+a^-_{22}y^2_{\overline{x}_2}(x_1,\ell_2)\big)+\\ &+\frac{1}{2}\sum_{\omega_2}h_1h_2\big(a^+_{11}y^2_{x_1}(0,x_2)+a^-_{11}y^2_{\overline{x}_1}(\ell_1,x_2)\big). \end{aligned}$$

Taking here into account implications from the ellipticity condition (2.2)

$$\begin{aligned} &\sum_{\alpha,\beta=1}^{2}a^{\pm}_{\alpha\beta}(x)\xi_\alpha\xi_\beta\ge\nu(\xi_1^2+\xi_2^2),\quad x\in\omega,\\ &a^+_{11}(0,x_2)\ge\nu,\quad a^-_{11}(\ell_1,x_2)\ge\nu,\quad x_2\in\omega_2,\\ &a^+_{22}(x_1,0)\ge\nu,\quad a^-_{22}(x_1,\ell_2)\ge\nu,\quad x_1\in\omega_1, \end{aligned}$$

we obtain

$$\begin{aligned} (Ay,y)\ge\frac{\nu}{2}\sum_{\omega}h_1h_2\big(y^2_{x_1}+y^2_{x_2}+y^2_{\overline{x}_1}+y^2_{\overline{x}_2}\big)+\\ +\frac{\nu}{2}\sum_{\omega_1}h_1h_2\big(y^2_{x_2}(x_1,0)+y^2_{\overline{x}_2}(x_1,\ell_2)\big)+\frac{\nu}{2}\sum_{\omega_2}h_1h_2\big(y^2_{x_1}(0,x_2)+y^2_{\overline{x}_1}(\ell_1,x_2)\big), \end{aligned}$$

so $(Ay,y)\ge\nu\|\nabla y\|^2$, $\forall\, y\in H$. This estimate together with the difference analogue of the Friedrichs inequality ([70], p. 309)

$$\Big(\frac{8}{\ell_1^2}+\frac{8}{\ell_2^2}\Big)\|y\|^2\le\|\nabla y\|^2 \tag{3.3}$$

yields

$$\|y\|^2_{W_2^1(\omega)}\le\frac{1}{\nu}\Big(1+\frac{\ell_1^2\ell_2^2}{8(\ell_1^2+\ell_2^2)}\Big)(Ay,y),\quad\forall\, y\in H. \tag{3.4}$$

Thus the operator A is positive definite in H, and hence the problem (3.2) is uniquely solvable.

Remark 3.1. For $a_{12} \equiv a_{21}$, the operator A is self-adjoint in H.

2^0. A priori estimate of error. The error $z = y - u$ of the scheme (3.2) is a solution of the problem

$$Az = \psi, \quad x \in \omega, \quad z(x) = 0, \quad x \in \gamma, \tag{3.5}$$

where the approximation error $\psi = T_1 T_2 f - Au$ can be reduced to the form

$$\psi = (\psi_{11} + \psi_{12})_{x_1} + (\psi_{21} + \psi_{22})_{x_2} + \psi_0,$$

$$\psi_0 = T_1 T_2 (a_0 u) - T_1 T_2 a_0 u, \quad \psi_{\alpha\alpha} = S_\alpha^- \overline{S}_\beta a_{\alpha\alpha} u_{\overline{x}_\alpha} - S_\alpha^- T_\beta \Big(a_{\alpha\alpha} \frac{\partial u}{\partial \zeta_\alpha} \Big),$$

$$\psi_{\alpha\beta} = 0.5 \big[S_1^- S_2^- a_{\alpha\beta} u_{\overline{x}_\beta} + S_\alpha^- S_\beta^+ a_{\alpha\beta} u_{\overline{x}_\beta}^{(-1_\alpha)} \big] - S_\alpha^- T_\beta \Big(a_{\alpha\beta} \frac{\partial u}{\partial \zeta_\beta} \Big),$$

$$\beta = 3 - \alpha, \quad \alpha = 1, 2.$$

Using Hölder's inequality and the imbedding of $L_p(\omega)$, $p > 1$, in $W_2^1(\omega)$, we find that

$$|(z, \psi_0)| \le \|\psi_0\|_{L_q(\omega)} \|z\|_{L_p(\omega)} \le \|\psi_0\|_{L_q(\omega)} \|z\|_{W_2^1(\omega)}, \quad \Big(\frac{1}{p}\Big) + \Big(\frac{1}{q}\Big) = 1, \quad q > 1.$$

Consequently, on the basis of (3.4), for the solution of the problem (3.5) we obtain an a priori estimate

$$\|z\|_{W_2^1(\omega)} \le c \Big(\sum_{\alpha,\beta=1}^{2} \|\psi_{\alpha\beta}\|_{(\alpha+)} + \|\psi_0\|_{L_q(\omega)} \Big), \quad \forall\, q > 1. \tag{3.6}$$

3^0. Estimation of the convergence rate. In this subsection we will investigate the convergence rate of the difference scheme (3.2). Towards this end, it suffices to estimate the summands in the right-hand side of the inequality (3.6). The operators $S_\alpha^\pm$, $\overline{S}_\alpha$, T_α are assumed to act with respect to the variable ζ_α, and let $\zeta = (\zeta_1, \zeta_2)$. Let $e_\alpha \equiv e_\alpha(x) = (x_1 - h_1, x_1 + (\alpha - 1)h_1) \times (x_2 - h_2, x_2 + (2-\alpha)h_2)$, $\alpha = 1, 2$, $e \equiv e(x) = (x_1 - h_1, x_1 + h_1) \times (x_2 - h_2, x_2 + h_2)$, $|h|^2 = h_1^2 + h_2^2$. We rewrite the mesh functions $\psi_{\alpha\beta}$, ψ_0 in the form

$$\psi_{\alpha\alpha} = \eta_{\alpha\alpha}^{(1)} + \eta_{\alpha\alpha}^{(2)}, \quad \psi_{\alpha\beta} = \eta_{\alpha\beta}^{(1)} + \eta_{\alpha\beta}^{(2)} + \eta_{\alpha\beta}^{(3)}, \quad \psi_0 = \eta_0^{(1)} + \eta_0^{(2)} + \eta_0^{(3)},$$

where

$$\eta_{\alpha\alpha}^{(1)} = S_\alpha^- \overline{S}_\beta \Big(a_{\alpha\alpha} \frac{\partial u}{\partial \zeta_\alpha} \Big) - S_\alpha^- T_\beta \Big(a_{\alpha\alpha} \frac{\partial u}{\partial \zeta_\alpha} \Big),$$

$$\eta_{\alpha\alpha}^{(2)} = S_\alpha^- \overline{S}_\beta \Big(a_{\alpha\alpha}(\zeta) u_{\overline{x}_\alpha}(x) - a_{\alpha\alpha}(\zeta) \frac{\partial u(\zeta)}{\partial \zeta_\alpha} \Big),$$

$$\eta_{\alpha\beta}^{(1)} = \frac{1}{2} S_1^- S_2^- \Big(a_{\alpha\beta}(\zeta) u_{\overline{x}_\beta}(x) - a_{\alpha\beta}(\zeta) \frac{\partial u(\zeta)}{\partial \zeta_\beta} \Big),$$

$$\eta^{(2)}_{\alpha\beta} = \frac{1}{2}\, S^-_\alpha S^+_\beta \Big(a_{\alpha\beta}(\zeta) u^{(-1_\alpha)}_{x_\beta}(x) - a_{\alpha\beta}(\zeta)\, \frac{\partial u(\zeta)}{\partial \zeta_\beta} \Big),$$

$$\eta^{(3)}_{\alpha\beta} = S^-_\alpha \overline{S}_\beta \Big(a_{\alpha\beta}\, \frac{\partial u}{\partial \zeta_\beta} \Big) - S^-_\alpha T_\beta \Big(a_{\alpha\beta}\, \frac{\partial u}{\partial \zeta_\beta} \Big), \ \ \beta = 3-\alpha, \ \ \alpha = 1,2,$$

$$\eta^{(1)}_0 = T_1 T_2 \Bigg(a_0(\zeta) u(\zeta) - a_0(\zeta) u(x) - - a_0(\zeta)\, \frac{\partial u(\zeta)}{\partial \zeta_1}\, (\zeta_1 - x_1) - a_0(\zeta)\, \frac{\partial u(\zeta)}{\partial \zeta_2}\, (\zeta_2 - x_2) \Bigg),$$

$$\eta^{(2)}_0 = T_1 T_2 \Big(a_0(\zeta)\, \frac{\partial u(\zeta)}{\partial \zeta_1}\, (\zeta_1 - x_1) \Big),$$

$$\eta^{(3)}_0 = T_1 T_2 \Big(a_0(\zeta)\, \frac{\partial u(\zeta)}{\partial \zeta_2}\, (\zeta_2 - x_2) \Big).$$

To estimate each group of summands, we apply the well-known method of investigation [83] which uses the Bramble–Hilbert lemma. As a result, we obtain

$$|\eta^{(1)}_{\alpha\alpha}| \le \frac{c|h|^{m-1}}{(h_1 h_2)^{1/2}}\, |a_{\alpha\alpha} D_\alpha u|_{W_2^{m-1}(e)}, \ \ 1 < m \le 3,$$

$$|\eta^{(2)}_{\alpha\alpha}| \le \frac{c|h|^{m-1}}{(h_1 h_2)^{1/2}}\, \|a_{\alpha\alpha}\|_{L_\infty(\Omega)} |u|_{W_2^m(e)}, \ \ 1 < m \le 2,$$

The functions $\eta^{(1)}_{\alpha\beta}$, $\eta^{(2)}_{\alpha\beta}$ are estimated analogously to $\eta^{(2)}_{\alpha\alpha}$, and the function $\eta^{(3)}_{\alpha\beta}$ is estimated analogously to $\eta^{(1)}_{\alpha\alpha}$.

For $\eta^{(1)}_0$, $\eta^{(2)}_0$ we obtain

$$|\eta^{(\alpha)}_0| \le \frac{1}{h_{3-\alpha}} \int_e \Big| a_0\, \frac{\partial u}{\partial \xi_\alpha} \Big|\, d\xi \le \frac{|h|}{(h_1 h_2)^{1/q}}\, \|a_0 D_\alpha u\|_{L_q(e)},$$

$$q = 3/2, \ \ \alpha = 1,2.$$

For the estimation of $\eta^{(3)}_0$, we note that it vanishes for $u(x) \in \boldsymbol{\pi}_1$. Moreover,

$$|T_1 T_2 a_0 u| \le T_1 T_2 a_0 |\widetilde{u}|_{C(\widetilde{e})} \le (h_1 h_2)^{-1/q} \|a_0\|_{L_q(e)} \|\widetilde{u}\|_{W_2^2(\widetilde{e})},$$

$$|T_1 T_2 (a_0 u)| \le T_1 T_2 a_0 |\widetilde{u}|_{C(\widetilde{e})} \le (h_1 h_2)^{-1/q} \|a_0\|_{L_q(e)} \|\widetilde{u}\|_{W_2^2(\widetilde{e})},$$

$$\Big| T_1 T_2 \Big(a_0(\xi)\, \frac{\partial u}{\partial \xi_\alpha}\, (\xi_\alpha - x_\alpha) \Big) \Big| \le$$

$$\le \frac{1}{h_{3-\alpha}}\, \|a_0\|_{L_q(e)} \Big\| \frac{\partial u}{\partial x_\alpha} \Big\|_{L_p(e)} \le (h_1 h_2)^{-1/q} \|a_0\|_{L_q(e)} \Big\| \frac{\partial \widetilde{u}}{\partial t_1} \Big\|_{L_p(\widetilde{e})},$$

and hence

$$|\eta^{(3)}_0| \le \frac{c}{(h_1 h_2)^{1/q}}\, \|a_0\|_{L_q(e)} \|\widetilde{u}\|_{W_2^2(\widetilde{e})} \le \frac{c}{(h_1 h_2)^{1/q}}\, \|a_0\|_{L_q(e)} |\widetilde{u}|_{W_2^2(\widetilde{e})} \le$$

$$\leq \frac{c|h|}{(h_1h_2)^{1/q}} \|a_0\|_{L_q(e)} |u|_{W_2^2(\Omega)}.$$

The expressions $\widetilde{u}$ and $\widetilde{e}$ are of the same sense as in Section 2.

Relying on the above inequalities, we can see that the following lemma is valid.

Lemma 3.1. *For the functionals $\psi_{\alpha\beta}$, $\alpha, \beta = 1, 2$, and ψ_0 the estimates*

$$|\psi_{\alpha\beta}| \leq \frac{c|h|}{(h_1h_2)^{1/2}} \left(\|a_{\alpha\beta}\|_{L_\infty(\Omega)} |u|_{W_2^2(e_\alpha)} + |a_{\alpha\beta} D_\beta u|_{W_2^1(e_\alpha)}\right), \quad \alpha, \beta = 1, 2,$$

$$|\psi_0| \leq \frac{c|h|}{(h_1h_2)^{1/q}} \Big(\|a_0\|_{L_q(e)} |u|_{W_2^2(\Omega)} + \sum_{\alpha=1}^{2} \|a_0 D_\alpha u\|_{L_q(e)}\Big), \quad q > 1,$$

hold.

Using Lemma 3.1, from (3.6) it follows

Lemma 3.2. *For the solution of the problem* (3.5) *the estimate*

$$\|z\|_{W_2^1(\omega)} \leq c|h|\Bigg(\Big(\sum_{\alpha,\beta=1}^{2} \|a_{\alpha\beta}\|_{L_\infty(\Omega)} + \|a_0\|_{L_q(\Omega)}\Big)|u|_{W_2^2(\Omega)} + \\ + \sum_{\alpha,\beta=1}^{2} |a_{\alpha\beta} D_\beta u|_{W_2^1(\Omega)} + \sum_{\alpha=1}^{2} \|a_0 D_\alpha u\|_{L_q(\Omega)}\Bigg), \quad q > 1,$$

is valid.

In this subsection we have so far assumed that $q > 1$ was an arbitrary number. Now we will choose it from the interval $(1, 2)$. Let $q = 3/2$. We can show that

$$|a_{\alpha\beta} D_\beta u|_{W_2^1(\Omega)} \leq c\|a_{\alpha\beta}\|_{W_{2+\varepsilon}^1(\Omega)} \|u\|_{W_2^2(\Omega)},$$
$$\|a_0 D_\alpha u\|_{L_q(\Omega)} \leq c\|a_0\|_{L_2(\Omega)} \|u\|_{W_2^2(\Omega)}, \quad \forall\, \varepsilon > 0, \quad \alpha, \beta = 1, 2.$$

These inequalities together with Lemma 3.2 prove the following

Theorem 3.1. *Let the assumptions* (2.2) *and* (3.1) *be fulfilled. Then the difference scheme* (3.2) *converges in the mesh norm $W_2^1(\omega)$. Moreover, the estimate*

$$\|y - u\|_{W_2^1(\omega)} \leq c|h|\, \|f\|_{L_2(\Omega)} \tag{3.7}$$

is valid.

The following statement holds.

Theorem 3.2. *Let the condition* (2.2) *be fulfilled,*

$$a_{\alpha\beta} \in W_\infty^1(\Omega), \quad \alpha, \beta = 1, 2, \quad a_0 \in W_{2+\varepsilon}^1(\Omega), \quad \forall\, \varepsilon > 0, \quad f \in L_2(\Omega),$$

and the corresponding coefficients of the scheme (3.2) *be calculated by the formulas*

$$a^{\pm}_{\alpha\beta}(x)=a_{\alpha\beta}\Big(x_1\pm\frac{h_1}{2},x_2\pm\frac{h_2}{2}\Big),\ \ \alpha,\beta=1,2,\ \ a(x)=a_0(x),$$
$$\varphi(x)=T_1T_2f.$$

Then the difference scheme (3.2) *converges in the mesh norm, and the estimate* (3.7) *is valid.*

Proof. If in (3.2) we choose $a=a_0$, then the corresponding summand ψ_0 in the approximation error has the form

$$\psi_0=T_1T_2(a_0u)-a_0u=a_0(x)(T_1T_2u-u)+T_1T_2\big(u(\zeta)(a_0(\zeta)-a_0(x))\big),$$

whence

$$|\psi_0|\le\frac{c|h|}{(h_1h_2)^{1/2}}\,\|a_0\|_{C(\Omega)}|u|_{W_2^2(e)}+\frac{c|h|}{(h_1h_2)^{1/(2+\varepsilon)}}\,\|u\|_{C(\Omega)}|a_0|_{W^1_{2+\varepsilon}(e)},$$
$$\|\psi_0\|\le c|h|\,\|u\|_{W_2^2(\Omega)}\|a_0\|_{W^1_{2+\varepsilon}(\Omega)},$$

which proves a part of the theorem dealing with the coefficient a.

The validity of the theorem regarding to the coefficients $a^{\pm}_{\alpha\beta}$ is proved analogously [76]. □

4. The Dirichlet Problem. Convergence in the Norm W_2^2

History of the matter. The questions of stability of the difference solution of the Dirichlet problem for elliptic equations in the mesh metric W_2^2 were investigated by many authors. An analogue of the estimate (4.9) has been obtained in [64] for the elliptic nondivergent operator containing no lowest terms; in [4], [37] and [39] this estimate is obtained only for sufficiently small h.

In this section, the estimate (4.9) is proved without restriction to the mesh step. A consistent estimate of the convergence rate (4.17) is obtained. The results of the present section have been published in [6] for the equation containing no lowest terms, and also in [13]. Analogous results have been obtained in [52], where relying on [37] a consistent estimate has been obtained in case $a_{ij}\in W_\infty^{m-1}$, $a_0\in W_\infty^{m-2}$, $a_1=a_2=0$.

1^0. Consider in a rectangle Ω the boundary value problem

$$\mathcal{L}u=f,\ \ x\in\Omega,\quad u(x)=0,\ \ x\in\Gamma, \tag{4.1}$$

where

$$\mathcal{L}u=-\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial x_\alpha}\Big(a_{\alpha\beta}(x)\,\frac{\partial u}{\partial x_\beta}\Big)+\sum_{\alpha=1}^{2}a_\alpha(x)\,\frac{\partial u}{\partial x_\alpha}+a_0(x)u.$$

The condition of uniform ellipticity

$$\sum_{\alpha,\beta=1}^{2} a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \nu(\xi_1^2+\xi_2^2), \quad \nu = \text{const} > 0, \quad x \in \Omega, \tag{4.2}$$

is assumed to be fulfilled. Let, moreover,

$$\begin{gathered} a_\alpha \in C(\overline{\Omega}), \quad \alpha=0,1,2, \quad a_{\alpha\beta} \in W_2^{m-1}(\Omega), \quad f \in W_2^{m-2}(\Omega), \quad m \in (3,4], \\ a_{\alpha\alpha} \leq \mu_0, \quad \Big|\frac{\partial a_{\alpha\beta}}{\partial x_\alpha}\Big| \leq \mu_1, \quad \mu_2 \leq a_0 \leq \mu_3, \quad |a_\alpha| \leq \mu_4, \quad \alpha,\beta=1,2, \end{gathered} \tag{4.3}$$

where $\mu_0 > 0$, $\mu_\alpha \geq 0$, $\alpha = 1,2,3,4$, are constants.

The problem (4.1)–(4.3) is assumed to be uniquely solvable in the class $W_2^m(\Omega)$, $3 < m \leq 4$. We approximate the problem (4.1) by the difference scheme [70]

$$\Lambda y = f, \quad x \in \omega, \quad y(x) = 0, \quad x \in \gamma, \tag{4.4}$$

where

$$\Lambda = A + B, \quad By = \sum_{\alpha=1}^{2} a_\alpha y_{\overset{\circ}{x}_\alpha},$$

$$Ay = -\frac{1}{2}\sum_{\alpha,\beta=1}^{2} \big[(a_{\alpha\beta}y_{\overline{x}_\beta})_{x_\alpha} + (a_{\alpha\beta}y_{x_\beta})_{\overline{x}_\alpha}\big] + a_0 y.$$

By H we denote the space of mesh functions defined on $\overline{\omega}$ and equal to γ, with the inner product $(y,v) = (y,v)_\omega$ and the norm $\|y\| = \|y\|_\omega$.

In the space H, the notations $\|\cdot\|_{(\alpha+)}$ and $\|\cdot\|_{(\alpha)}$ take the form

$$\|v\|_{(1+)}^2 = \sum_{\omega_1^+\times\omega_2} h_1h_2v^2, \quad \|v\|_{(2+)}^2 = \sum_{\omega_1\times\omega_2^+} h_1h_2v^2, \quad \|v\|_{(\alpha)} = \|v\|.$$

2⁰. Find sufficient conditions for which the operator Λ is positive definite, and hence the problem (4.4) is uniquely solvable. Let

$$\widetilde{a} = \max_{x\in\overline{\Omega}} \sum_{\alpha=1}^{2} |a_\alpha - \widetilde{a}_\alpha|^2,$$

where the independent of x_α functions $\widetilde{a}_\alpha = \widetilde{a}_\alpha(x_{3-\alpha})$ are chosen in such a way that the value $\widetilde{a}$ is minimal. If it is difficult to find such $\widetilde{a}_\alpha$, we can use anyone with the property (conf. [37], p. 116)

$$\max_{x\in\overline{\Omega}} \sum_{\alpha=1}^{2} |a_\alpha - \widetilde{a}_\alpha|^2 \leq \max_{x\in\overline{\Omega}} \sum_{\alpha=1}^{2} a_\alpha^2.$$

It is not difficult to notice that

$$\Big|\sum_{x\in\omega} h_1h_2a_\alpha y_{\overset{\circ}{x}_\alpha} y\Big| =$$

$$= \Big|\sum_{x\in\omega} h_1h_2(a_\alpha - \widetilde{a}_\alpha)y_{\overset{\circ}{x}_\alpha} y\Big| \leq \frac{\varepsilon c_1}{2}\,\|y_{\overset{\circ}{x}_\alpha}\|^2 + \frac{1}{2\varepsilon c_1}\,\big\|(a_\alpha - \widetilde{a}_\alpha)y\big\|^2, \tag{4.5}$$

where $\varepsilon > 0$ is an arbitrary number, and c_1 is the constant from Friedrich's inequality

$$\|y\| \le c_1\|\nabla y\|, \quad c_1 = (\ell_1\ell_2)^{1/2}/4. \tag{4.6}$$

Taking into account $\|y_{\overset{\circ}{x}_\alpha}\| \le \|y_{\overline{x}_\alpha}\|_{(\alpha+)}$, from (4.5) we obtain

$$|(By,y)| \le (\varepsilon c_1/2)\|\nabla y\|^2 + \frac{\widetilde{a}}{2\varepsilon c_1}\,\|y\|^2,$$

and since on the basis of (4.2), we have $(Ay,y) \ge \nu\|\nabla y\|^2 + \mu_2\|y\|^2$, therefore

$$(\Lambda y,y) \ge (\nu - c_1\varepsilon)\|\nabla y\|^2 + \Big(\mu_2 - \frac{\widetilde{a}}{2\varepsilon c_1}\Big)\|y\|^2 + \frac{c_1\varepsilon}{2}\,\|\nabla y\|^2.$$

Applying the inequality (4.6) to the last summand, we find that

$$(\Lambda y,y) \ge (\nu - c_1\varepsilon)\|\nabla y\|^2 + \Big(\mu_2 - \frac{\widetilde{a}}{2\varepsilon c_1} + \frac{\varepsilon}{2c_1}\Big)\|y\|^2.$$

We choose ε basing on the condition that the coefficient is equal to zero. Then

$$(\Lambda y,y) \ge c_2\|\nabla y\|^2, \quad c_2 = \nu + c_1^2\mu_2 - (c_1^4\mu_2^2 + c_1^2\widetilde{a})^{1/2}. \tag{4.7}$$

If the coefficients a_α do not depend on x_α, then we assume that $\widetilde{a}_\alpha \equiv a_\alpha$, and hence $\widetilde{a} = 0$ and $c_2 = \nu > 0$. In the general case, for the coefficient c_2 to be positive, we assume

$$\nu + c_1^2\mu_2 > (c_1^4\mu_2^2 + c_1^2\widetilde{a})^{1/2}. \tag{4.8}$$

Thus the following lemma is valid.

Lemma 4.1. *Let either the coefficients a_α, $\alpha = 1,2$ be independent of x_α, or the condition* (4.8) *be fulfilled. Then the operator Λ is positive definite in H, and the estimate* (4.7) *is valid.*

3^0. To estimate the convergence rate of the difference scheme (4.4) in the mesh metric W_2^2, we will need

Lemma 4.2. *Under the conditions of Lemma* 4.1 *the estimate*

$$|y|_{W_2^2(\omega)} \le c_3\|\Lambda y\|, \ \ \forall\, y \in H, \tag{4.9}$$

is fulfilled with

$$c_3 = (2\mu_0/\nu^2)\Big(1 + \frac{(c_1\mu_3 + 2^{1/2}\mu_4 + 2^{3/2}\mu_1)c_1}{c_2}\Big).$$

Proof. Using the formula for differencing a product ([70], p. 255), we obtain

$$a_{11}y_{\overline{x}_1x_1} + \frac{(a_{12}+a_{21})(y_{x_1\overline{x}_2} + y_{\overline{x}_1x_2})}{2} + a_{22}y_{\overline{x}_2x_2} = F, \tag{4.10}$$

where

$$F(x) = ay - Ay - \frac{1}{2}\Big(a_{11x_1}y_{x_1} + a_{11\overline{x}_1}y_{\overline{x}_1} + a_{22x_2}y_{x_2} +$$
$$+ a_{22\overline{x}_2}y_{\overline{x}_2} + a_{12x_1}\widehat{y}_{12} + a_{12\overline{x}_1}\check{y}_{12} + a_{21x_2}\widehat{y}_{21} + a_{21\overline{x}_2}\check{y}_{21}\Big), \tag{4.11}$$
$$\widehat{y}_{\alpha\beta} \equiv (y^{(+1_\alpha)})_{\overline{x}_\beta}, \quad \check{y}_{\alpha\beta} \equiv (y^{(-1_\alpha)})_{x_\beta}, \quad \alpha, \beta = 1, 2.$$

Multiplying the equality (4.9) by $y_{\overline{x}_1x_1}/a_{22}$, we have

$$a_{22}^{-1}\Big[a_{11}y^2_{\overline{x}_1x_1} + (a_{12}+a_{21})y_{\overline{x}_1x_1}\,\frac{y_{\overline{x}_1x_2}+y_{x_1\overline{x}_2}}{2} + a_{22}\Big(\frac{y_{\overline{x}_1x_2}+y_{x_1\overline{x}_2}}{2}\Big)^2\Big] =$$
$$= F\,\frac{y_{\overline{x}_1x_1}}{a_{22}} + I(y),$$

where $I(y) = \Big(\frac{y_{\overline{x}_1x_2}+y_{x_1\overline{x}_2}}{2}\Big)^2 - y_{\overline{x}_1x_1}y_{\overline{x}_2x_2}$. Whence by virtue of (4.2), we obtain $\frac{\nu}{\mu_0}\,y^2_{\overline{x}_1x_1} \le \frac{1}{\nu}\,|Fy_{\overline{x}_1x_1}| + I(y)$. Consequently, taking into account the estimate

$$|Fy_{\overline{x}_1x_1}| \le \frac{\mu_0}{2\nu^2}\,F^2 + \frac{\nu^2}{2\mu_0}\,y^2_{\overline{x}_1x_1},$$

we find that

$$\frac{\nu}{2\,\mu_0}\,y^2_{\overline{x}_1x_1} \le \frac{\mu_0}{2\nu^3}\,F^2 + I(y). \tag{4.12}$$

Let us now show that $\sum\limits_{x\in\omega} h_1h_2I(y) \le 0$, $\forall\, y \in H$.

Indeed, this follows from

$$\sum_{x\in\omega} h_1h_2I(y) \le \frac{1}{2}\sum_{\omega} h_1h_2\big(y^2_{\overline{x}_1x_2} + y^2_{x_1\overline{x}_2}\big) - \sum_{\omega} h_1h_2y_{\overline{x}_1x_1}\,y_{\overline{x}_2x_2},$$

with regard for the identities

$$\sum_{\omega} h_1h_2y^2_{\overline{x}_1x_2} = \sum_{\omega} h_1h_2y_{\overline{x}_1x_1}y_{\overline{x}_2x_2} - \sum_{\omega_1} h_1h_2y^2_{\overline{x}_1x_2}(x_1, 0) -$$
$$- \sum_{\omega_2^+} h_1h_2y^2_{\overline{x}_1\overline{x}_2}(\ell_1, x_2),$$
$$\sum_{\omega} h_1h_2y^2_{x_1\overline{x}_2} = \sum_{\omega} h_1h_2y_{\overline{x}_1x_1}y_{\overline{x}_2x_2} - \sum_{\omega_1} h_1h_2y^2_{x_1\overline{x}_2}(x_1, \ell_2) -$$
$$- \sum_{\omega_2^+} h_1h_2y^2_{x_1\overline{x}_2}(0, x_2).$$

Summing (4.12) over the mesh ω, we find that $\|y_{\overline{x}_1x_1}\| \le (\mu_0/\nu^2)\|F\|$. The estimate $\|y_{\overline{x}_2x_2}\| \le (\mu_0/\nu^2)\|F\|$ is obtained analogously. Consequently,

$$\big\|y_{\overline{x}_1x_1} + y_{\overline{x}_2x_2}\big\|^2 \le 2\big(\|y_{\overline{x}_1x_1}\|^2 + \|y_{\overline{x}_2x_2}\|^2\big) \le (4\mu_0^2/\nu^4)\|F\|^2,$$

that is,

$$|y|_{W_2^2(\omega)} \le \frac{2\mu_0}{\nu^2}\,\|F\|. \tag{4.13}$$

Estimate now the right-hand side of the inequality (4.13). It follows from (4.11) that $\|F\| \le \mu_3\|y\| + \|Ay\| + 2\mu_1\big(\|y_{\overline{x}_1}\|_{(1+)} + \|y_{\overline{x}_2}\|_{(2+)}\big)$, so

$$\|F\| \le \mu_3\|y\| + \|\Lambda y\| + \|By\| + 2^{3/2}\mu_1\|\nabla y\|. \tag{4.14}$$

On the basis of the easily verifiable estimate $\|By\| \le 2^{1/2}\mu_4\|\nabla y\|$ and the inequality (4.6), from (4.14) we get

$$\|F\| \le \|\Lambda y\| + \big(c_1\mu_3 + 2^{1/2}\mu_4 + 2^{3/2}\mu_1\big)\|\nabla y\|. \tag{4.15}$$

From (4.6) and (4.7) we obtain the estimate $\|\nabla y\| \le \frac{c_1}{c_2}\|\Lambda y\|$, which together with (4.13) and (4.15) completes the proof of Lemma 4.2. □

Corollary 4.1. *Note that* $\|\cdot\|_{W_2^2(\omega)}$ *and* $|\cdot|_{W_2^2(\omega)}$ *are equivalent. Therefore in the conditions of Lemma* 4.2 *the difference scheme* (4.4) *is stable in the mesh metric* W_2^2.

4^0. We now investigate the convergence rate of the difference scheme (4.4). For the error $z = y - u$ we have the problem

$$\Lambda z = \psi, \quad x \in \omega, \quad z(x) = 0, \quad x \in \gamma, \tag{4.16}$$

where the approximation error $\psi = f - \Lambda u$ is represented by the sum

$$\psi = \sum_{\alpha,\beta=1}^{2} \eta_{\alpha\beta} + \sum_{\alpha=1}^{2} \eta_\alpha, \quad \eta_{\alpha\beta} = \sum_{k=0}^{3} \eta_{\alpha\beta}^{(k)}, \quad \eta_\alpha = a_\alpha\Big(\frac{\partial u}{\partial x_\alpha} - u_{\overset{\circ}{x}_\alpha}\Big),$$

$$\eta_{\alpha\beta}^{(0)} = a_{\alpha\beta}\Big(\frac{u_{\overline{x}_\alpha x_\beta} + u_{x_\alpha \overline{x}_\beta}}{2} - \frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}\Big), \quad \eta_{\alpha\beta}^{(1)} = a_{\alpha\beta\overset{\circ}{x}_\alpha}\Big(\frac{\widehat{u}_{\alpha\beta} + \check{u}_{\alpha\beta}}{2} - \frac{\partial u}{\partial x_\beta}\Big),$$

$$\eta_{\alpha\beta}^{(2)} = \frac{\partial u}{\partial x_\beta}\Big(a_{\alpha\beta\overset{\circ}{x}_\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x_\alpha}\Big), \quad \eta_{\alpha\beta}^{(3)} = \frac{h_\alpha}{4}\, a_{\alpha\beta\overline{x}_\alpha x_\alpha}(\widehat{u}_{\alpha\beta} - \check{u}_{\alpha\beta}).$$

Lemma 4.3. *Let the solution of the problem* (4.1)–(4.3) *belong to the space* $W_2^m(\Omega)$. *Then for the components of the approximation error the estimate*

$$\sum_{\alpha,\beta=1,2}^{2} \|\eta_{\alpha\beta}\| + \sum_{\alpha=1}^{2} \|\eta_\alpha\| \le c|h|^{m-2}\|u\|_{W_2^m(\Omega)}, \quad m \in (3,4],$$

is valid.

Proof. Here, to estimate the values $\eta_{\alpha\beta}^{(k)}$, η_α we use the well-known method based on the Bramble–Hilbert lemma. As a result we obtain

$$|\eta_{\alpha\beta}^{(k)}| \le c|h|^{m-2}(h_1h_2)^{-1/2}\|a_{\alpha\beta}\|_{C^k(\overline{\Omega})}\|u\|_{W_2^{m-k}(e)}, \quad k = 0,1,$$

$$|\eta_{\alpha\beta}^{(k+1)}| \le c|h|^{m-2}(h_1h_2)^{-1/2}\|u\|_{C^k(\overline{\Omega})}\|a_{\alpha\beta}\|_{W_2^{m-k}(e)}, \quad k = 1,2,$$

$$|\eta_\alpha| \le c|h|^{m-2}(h_1h_2)^{-1/2}\|a_\alpha\|_{C(\overline{\Omega})}\|u\|_{W_2^{m-1}(e)}, \quad \alpha,\beta = 1,2, \quad m \in (3,4],$$

where $e \equiv e(x) = (x_1 - h_1, x_1 + h_1) \times (x_2 - h_2, x_2 + h_2)$. Consequently,

$$|\eta_{\alpha\beta}| \leq c|h|^{m-2}(h_1h_2)^{-1/2}\big(\|u\|_{W_2^m(e)} + \|a_{\alpha\beta}\|_{W_2^{m-1}(e)}\|u\|_{W_2^m(\Omega)}\big),$$
$$|\eta_\alpha| \leq c|h|^{m-2}(h_1h_2)^{-1/2}\|u\|_{W_2^m(e)}, \quad m \in (3,4], \quad \alpha, \beta = 1, 2,$$

which proves our lemma. □

On the basis of Lemmas 4.2 and 4.3, from (4.16) we have the following

Theorem 4.1. *Let the solution of the problem* (4.1)–(4.3) *belong to the space* $W_2^m(\Omega)$, $m \in (3,4]$, *and let the condition* (4.8) *be fulfilled. Then the convergence rate of the difference scheme* (4.4) *is characterized by the estimate*

$$\|y - u\|_{W_2^2(\omega)} \leq c|h|^{m-2}\|u\|_{W_2^m(\Omega)}, \quad m \in (3,4], \tag{4.17}$$

where the constant $c > 0$ *does not depend on* h *and* $u(x)$.

5. The Mixed Boundary Value problem

History of the matter. The results of this section have been published in [24]. Analogous results have been obtained in [71] for the Poisson equation, in [10] for the elliptic equation in the case of constant coefficients, and in [35] in case $s = 1$, $m \in (2,3]$.

1^0. Suppose that in Ω we seek for a solution of equation (2.1) satisfying the boundary conditions

$$\begin{gathered} Lu \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\right) + a_0(x)u = f, \quad x \in \Omega, \\ a_{11}\,\frac{\partial u}{\partial x_1} + a_{12}\,\frac{\partial u}{\partial x_2} = \sigma u - g, \quad x \in \Gamma_{-1}, \quad u = 0, \quad x \in \Gamma_0, \quad \Gamma_0 = \Gamma \backslash \Gamma_{-1}. \end{gathered} \tag{5.1}$$

Let the conditions

$$\sum_{i,j=1}^{2} a_{ij}(x)\chi_i\chi_j \geq \nu(\chi_1^2 + \chi_2^2), \quad \nu = \text{const} > 0, \quad a_0(x), \sigma(x_2) \geq 0, \quad x \in \Omega, \tag{5.2}$$

be fulfilled, and the problem (5.1) with the right-hand side $f \in W_2^m(\Omega)$, $g \in W_2^{m-3/2}(\Gamma_{-1})$ is assumed to be uniquely solvable in $W_2^m(\Omega)$, $1 < m \leq 3$. Consequently, the coefficients of the problem must belong to the following classes of multiplicators: $a_{ij} \in M(W_2^{m-1}(\Omega))$, $a_0 \in M(W_2^m(\Omega) \to W_2^{m-2}(\Omega))$, $\sigma \in M(W_2^{m-1/2}(\Gamma_{-1}) \to W_2^{m-3/2}(\Gamma_{-1}))$ for which we have the following sufficient conditions: $a_{ij} \in W_2^{m-1}(\Omega)$ for $2 < m \leq 3$, $a_{ij} \in W_{2+\varepsilon}^1(\Omega)$ for $m = 2$, $a_{ij} \in W_{2/(m-1)}^{m-1+\varepsilon}(\Omega)$ for $1 < m < 2$, $\varepsilon > 0$, $i,j = 1,2$, $a_0 \in W_2^{m-2}(\Omega)$ for $2 \leq m \leq 3$, $a_0 \in L_{2/(3-m)}(\Omega)$ for $1 < m < 2$, $\sigma \in W_2^{m-3/2}(\Gamma_{-1})$ for $3/2 < m \leq 3$, $\sigma \in L_{1/(2-m)}(\Gamma_{-1})$ for $1 < m \leq 3/2$.

2^0. Let $\gamma_0=\gamma\setminus\gamma_{-1}$. We approximate the problem (5.1) by the difference scheme

$$Ay=\varphi,\quad x\in\omega\cup\gamma_{-1},\quad y(x)=0,\quad x\in\gamma_0, \tag{5.3}$$

where

$$A=\sum_{i,j=1}^{2}A_{ij}+a\,E+\delta(\gamma_1)\widetilde{\sigma}E,\quad Ey\equiv y,$$

$$A_{11}=-\begin{cases}\dfrac12\,(a_{11}y_{x_1})_{\overline{x}_1}+\dfrac12(a_{11}y_{\overline{x}_1})_{x_1}, & x\in\omega,\\ \dfrac{2}{h_1}\,\dfrac{a_{11}(0,x_2)+a_{11}(h_1,x_2)}{2}\,y_{x_1}(0,x_2)\,, & x\in\gamma_{-1},\end{cases}$$

$$A_{12}=-\begin{cases}\dfrac12\,(a_{12}y_{x_2})_{\overline{x}_1}+\dfrac12\,(a_{12}y_{\overline{x}_2})_{x_1}, & x\in\omega,\\ \dfrac{a_{12}(0,x_2)}{h_1}\,y_{x_2}(0,x_2)+\dfrac{a_{12}(h_1,x_2)}{h_1}\,y_{\overline{x}_2}(h_1,x_2), & x\in\gamma_{-1},\end{cases}$$

$$A_{21}=-\begin{cases}\dfrac12\,(a_{21}y_{x_1})_{\overline{x}_2}+\dfrac12\,(a_{21}y_{\overline{x}_1})_{x_2}, & x\in\omega,\\ (a_{21}y_{x_1})_{\overline{x}_2}, & x\in\gamma_{-1},\end{cases}$$

$$A_{22}=-\begin{cases}\dfrac12\,(a_{22}y_{x_2})_{\overline{x}_2}+\dfrac12\,(a_{22}y_{\overline{x}_2})_{x_2}, & x\in\omega,\\ (a_{22}y_{x_2})_{\overline{x}_2}, & x\in\gamma_{-1},\end{cases}\qquad a=T_1T_2a,\quad \widetilde{\sigma}=T_2\sigma,$$

$$\varphi(x)=\begin{cases}T_1T_2f, & x\in\omega,\\ T_1^+T_2f+\dfrac{2}{h_1}\,T_2g, & x\in\gamma_{-1},\end{cases}\qquad \delta(\gamma_{-1})=\begin{cases}\dfrac{2}{h_1}\,, & x\in\gamma_{-1},\\ 0, & x\notin\gamma_{-1}.\end{cases}$$

By H we denote the space of mesh functions defined on $\overline{\omega}$ and equal to γ_0, with the inner product and the norm $(y,v)=\sum\limits_{\omega_1^-\times\omega_2}\hbar_1h_2y(x)v(x)$, $\|y\|=(y,y)^{1/2}$. In the space H, the notation $\|\cdot\|_{(\alpha+)}$ takes the form $\|v\|_{(1+)}^2=\sum\limits_{\omega_1^+\times\omega_2}h_1h_2v^2$, $\|v\|_{(2+)}^2=\sum\limits_{\omega_1^-\times\omega_2^+}\hbar_1h_2v^2$. Denote also

$$(y,v)_{\gamma_{-1}}=\sum_{\gamma_{-1}}h_2y(x)v(x),\quad \|y\|_{L_p(\gamma_{-1})}=\Big(\sum_{\gamma_{-1}}h_2|y(x)|^p\Big)^{1/p}.$$

Lemma 5.1. *The operator A is positive definite in the space H and the estimate*

$$\|y\|^2_{W_2^1(\omega)}\le c_1(Ay,y),\quad c_1=(1+\ell_1\ell_2/8)/\nu,\quad \forall\,y\in H, \tag{5.4}$$

is valid.

Proof. Using the formula of summation by parts, it is not difficult to verify that

$$(A_{ij}y,y)=\frac12\sum_{\omega^-}h_1h_2a_{ij}y_{x_i}y_{x_j}+\frac12\sum_{\omega^+}h_1h_2a_{ij}y_{\overline{x}_i}y_{\overline{x}_j},\quad i,j=1,2.$$

Therefore taking into account the conditions (5.2), we obtain $(Ay, y) \geq \nu\|\nabla y\|^2$, $\forall\, y \in H$, which together with the difference analogue of Friedrichs inequality

$$\|y\| \leq \sqrt{\frac{\ell_1 \ell_2}{8}}\,\|\nabla y\| \tag{5.5}$$

results in (5.4). □

By Lemma 5.1, a solution of the problem (5.3) exists and is unique.

Remark 5.1. For $a_{12}(x) = a_{21}(x)$, the operator A is self-adjoint in the space H.

3^0. The error of the solution of the difference scheme (5.3), $z = y - u$, is a solution of the problem

$$Az = \psi(x), \quad x \in \omega \cup \gamma_{-1}, \quad z \in H, \tag{5.6}$$

where $\psi = \varphi - Au$ is the approximation error.

Let

$$\eta_{\alpha\beta} = 0.5\big(a_{\alpha\beta}\, u_{\overline{x}_\beta} + (a_{\alpha\beta}\, u_{x_\beta})^{(-1_\alpha)}\big) - S_\alpha^- T_{3-\alpha}\Big(a_{\alpha\beta}\,\frac{\partial u}{\partial x_\beta}\Big), \quad \alpha, \beta = 1, 2,$$

$$\overline{\eta}_{2\beta} = \frac{h_1}{2}\left(a_{2\beta}\, u_{x_\beta} - T_1^+ S_2^+\Big(a_{2\beta}\,\frac{\partial u}{\partial x_\beta}\Big)\right), \;\; \beta = 1, 2, \;\; \eta = T_1 T_2(a_0 u) - T_1 T_2 a_0 u,$$

$$\overline{\eta} = \frac{h_1}{2}\left(T_1^+ T_2(a_0 u) - T_1^+ T_2 a_0 u\right), \quad \eta_\sigma = T_2(\sigma u) - T_2 \sigma u.$$

It is not difficult to verify that

$$\begin{aligned}
T_1 T_2\,\frac{\partial}{\partial x_\alpha}\left(a_{\alpha\beta}\,\frac{\partial u}{\partial x_\beta}\right) &= -\,A_{\alpha\beta} u - \eta_{\alpha\beta x_\alpha}, \quad x \in \omega,\\
T_1^+ T_2\,\frac{\partial}{\partial x_2}\left(a_{2\beta}\,\frac{\partial u}{\partial x_\beta}\right) &= -\,A_{2\beta} u - \frac{2}{h_1}\,\overline{\eta}_{2\beta\overline{x}_2}, \quad x \in \gamma_{-1},\\
T_1^+ T_2\,\frac{\partial}{\partial x_1}\left(a_{1\beta}\,\frac{\partial u}{\partial x_\beta}\right) &= -\,A_{1\beta} u - \frac{2}{h_1}\,(\eta_{1\beta})^{(+1_1)} -\\
&\quad - \frac{2}{h_1}\,T_2\Big(a_{1\beta}\,\frac{\partial u}{\partial x_\beta}\Big), \quad x \in \gamma_{-1}.
\end{aligned}$$

Therefore the approximation error can be reduced to the form

$$\psi = \begin{cases} \eta_{11 x_1} + \eta_{12 x_1} + \eta_{21 x_2} + \eta_{22 x_2} + \eta, & x \in \omega,\\ \dfrac{2}{h_1}\left((\eta_{11} + \eta_{12})^{(+1_1)} + \overline{\eta}_{21\overline{x}_2} + \overline{\eta}_{22\overline{x}_2} + \eta_\sigma + \overline{\eta}\right), & x \in \gamma_{-1}. \end{cases}$$

Lemma 5.2. *For every mesh function $\eta(x)$ given on the mesh ω^-, and for every $z \in H$ the inequality $|(\eta_{\overline{x}_2}, z)_{\gamma_{-1}}| \leq \|\nabla z\|\left(\|\eta_{\overline{x}_1}\|_{\omega\cup\gamma_{-2}} + \|\eta_{\overline{x}_2}\|_{\omega\cup\gamma_{-1}}\right)$ is valid.*

Proof. Taking into account that $z(\ell_1,x_2)=0$, we write $(\eta_{\overline{x}_2},z)_{\gamma_{-1}}=-\sum\limits_{\omega_1^+}h_1\sum\limits_{\omega_2}h_2(\eta_{\overline{x}_2}z)_{\overline{x}_1}$. Using here the formula for differencing a product, we find that $(\eta_{\overline{x}_2},z)_{\gamma_{-1}}=-\sum\limits_{\omega}h_1h_2\eta_{\overline{x}_1\overline{x}_2}z-\sum\limits_{\omega_1^-\times\omega_2}h_1h_2\eta_{\overline{x}_2}z_{x_1}$, so using the formula of summation by parts we obtain $(\eta_{\overline{x}_2},z)_{\gamma_{-1}}=-\sum\limits_{\omega_1\times\omega_2^-}h_1h_2\eta_{\overline{x}_1}z_{x_2}-\sum\limits_{\omega_1^-\times\omega_2}h_1h_2\eta_{\overline{x}_2}z_{x_1}$. This implies that

$$|(\eta_{\overline{x}_2},z)_{\gamma_{-1}}|\le\Big(\sum_{\omega_1\times\omega_2^-}h_1h_2\eta_{\overline{x}_1}^2\Big)^{1/2}\Big(\sum_{\omega_1\times\omega_2^-}h_1h_2z_{x_2}^2\Big)^{1/2}+$$
$$+\Big(\sum_{\omega_1^-\times\omega_2}h_1h_2\eta_{\overline{x}_2}^2\Big)^{1/2}\Big(\sum_{\omega_1^-\times\omega_2}h_1h_2z_{x_1}^2\Big)^{1/2}.$$

Taking into account that

$$\sum_{\omega_1\times\omega_2^-}h_1h_2z_{x_2}^2\le\sum_{\omega_1^-\times\omega_2^+}h_1h_2z_{\overline{x}_2}^2\le\|\nabla z\|^2,\quad \sum_{\omega_1^-\times\omega_2}h_1h_2z_{x_1}^2=\sum_{\omega_1^+\times\omega_2}h_1h_2z_{\overline{x}_1}^2\le\|\nabla z\|^2,$$

we obtain the inequality of the lemma. □

Now we multiply scalarly both parts of (5.6) by z and apply the estimate (5.4). Next, for the summands with $\eta_{\alpha\beta x_\alpha}$ we use both the summation by parts and the Cauchy inequality, for the summands with $\overline{\eta}_{2\beta\overline{x}_2}$ we apply Lemma 5.2 and the Cauchy inequality, and for the summands with η, $\overline{\eta}$, η_σ we apply Hölder's inequality and imbeddings $W_2^1(\omega)\subset L_q(\omega)$, $W_2^1(\omega)\subset L_q(\gamma_{-1})$, $\forall\, q>1$. As a result, we arrive at the following statement.

Lemma 5.3. *For the solution z of the problem* (5.6) *the estimate $\|z\|_{W_2^1(\omega)}\le cJ(\eta)$ is valid, where*

$$J(\eta)=\sum_{\alpha,\beta=1}^{2}\|\eta_{\alpha\beta}\|_{\omega\cup\gamma_{+\alpha}}+\|\eta\|_{L_p(\omega)}+$$
$$+\|\overline{\eta}\|_{L_p(\gamma_{-1})}+\|\eta_\sigma\|_{L_p(\gamma_{-1})}+\sum_{\alpha,\beta=1}^{2}\|\overline{\eta}_{2\beta\overline{x}_{3-\alpha}}\|_{\omega\cup\gamma_{-\alpha}},\quad p>1. \tag{5.7}$$

4^0. To obtain an estimate for the convergence rate, it is sufficient to estimate the error functionals appearing in (5.7).

For η and $\eta_{\alpha\beta}$, the estimates

$$\|\eta\|_{L_p(\omega)}\le c|h|^{m-1}\|u\|_{W_2^m(\Omega)},\quad 1<m\le 3, \tag{5.8}$$
$$\|\eta_{\alpha\beta}\|_{\omega\cup\gamma_{+\alpha}}\le c|h|^{m-1}\|u\|_{W_2^m(\Omega)},\quad 1<m\le 3, \tag{5.9}$$

where $1<p<2/(3-m)$ for $1<m\le 2$, and $p=2$ for $m>3$, are valid.

To obtain the estimate (5.8) for $2 < m \le 3$, we write

$$\eta = \frac{1}{2} T_{\xi_1} T_{\xi_2} T_{\zeta_1} T_{\zeta_2} \big((a_0(\xi) - a_0(\zeta))(u(\xi) - u(\zeta)) \big) + T_1 T_2 a_0 (T_1 T_2 u - u),$$

where the subscripts ξ_α and ζ_α indicate the variable of integration in the operators T_α.

This easily results in the inequality $|\eta| \le c|h| \big(\|u\|_{C^1(\overline{\Omega})} |a|_{W_2^1(e)} + \|a\|_{C(\overline{\Omega})} |u|_{W_2^2(e)} \big)$, which proves (5.8) for $2 < m \le 3$.

In case $1 < m \le 2$, we choose $t = 2p/(2-(3-m)p)$ and by the Bramble–Hilbert lemma we find that $|\eta| \le c|h|^{m-1} (h_1 h_2)^{-1/p} \|a_0\|_{L_{2/(3-m)}(e)} |u|_{W_t^{m-1}(e)}$, where $e = e(x) = \big\{ \xi = (\xi_1, \xi_2) : \; |\xi_j - x_j| \le h_j, \; j = 1, 2 \big\}$.

Summation along with Hölder's inequality provides us with $\|\eta\|_{L_p(\omega)} \le c|h|^{m-1} \|a_0\|_{L_{2/(3-m)}(\Omega)} |u|_{W_2^m(\Omega)}$, and since $W_2^m \subset W_t^{m-1}$, we arrive at (5.8). The inequality (5.9) is proved analogously.

Estimation of the functional $\overline{\eta}$. Let us show that

$$\|\overline{\eta}\|_{L_p(\gamma_{-1})} \le c|h|^{m-1} \|u\|_{W_2^m(\Omega)}, \quad 1 < m \le 3, \tag{5.10}$$

where $1 < p < 2/(3-m)$ for $1 < m \le 2$, and $p = 2$ for $m > 2$.

Let

$$e_h = \Big\{ \xi = (\xi_1, \xi_2) : \; 0 \le \xi_1 \le h_1, \; |\xi_2 - x_2| \le h_2 \Big\},$$

$$\Omega_h = \Big\{ x = (x_1, x_2) : \; 0 < x_1 < \frac{h_1}{2}, \; 0 < x_2 < \ell_2 \Big\}.$$

For $1 < m \le 2$, just as in the foregoing case for η, we have $|\overline{\eta}| \le c|h|^m (h_1 h_2)^{-1/p} \|a_0\|_{L_{2/(3-m)}(e_h)} |u|_{W_t^{m-1}(e_h)}$, from which it follows that $\|\overline{\eta}\|_{L_p(\gamma_{-1})} \le c|h|^{m-1/p} \|a_0\|_{L_{2/(3-m)}(\Omega_h)} |u|_{W_t^{m-1}(\Omega_h)}$, i.e. (5.1) is valid for the case $1 < m \le 2$.

Using the imbedding $W_2^m(\Omega) \subset C^1(\overline{\Omega})$, $m > 2$, we obtain $|\overline{\eta}| \le c|h| \, \|a_0\|_{L_2(e_h)} \|u\|_{W_2^m(\Omega)}$, $m > 2$. Consequently, $\|\overline{\eta}\|_{L_2(\gamma_{-1})} \le c|h|^{3/2} \times \|a_0\|_{L_2(\Omega_h)} \|u\|_{W_2^m(\Omega)}$, $m > 2$, which means that the estimate (5.10) is valid for $2 < m \le 5/2$. For $5/2 < m \le 3$ we apply Theorem 1.4: $\|a_0\|_{L_2(\Omega_h)} \le c|h|^{1/2} \|a_0\|_{W_2^{m-2}(\Omega)}$ and arrive again at (5.10).

Estimation of the functional η_σ. Let us show that

$$\|\eta_\sigma\|_{L_p(\gamma_{-1})} \le c|h|^{m-1} \|u\|_{W_2^m(\Omega)}, \quad 1 < m \le 3, \tag{5.11}$$

where $2/(3-m) < p < 1/(2-m)$ for $1 < m \le 3/2$, and $p = 2$ for $m > 3/2$.

η_σ, being a linear functional with respect to $u(x)$, is bounded for $u \in W_t^{m-1}(\Omega)$, $1 < m < 2$ and vanishes on $\boldsymbol{\pi}_0$. Hence

$$|\eta_\sigma| \le c|h|^{m-1} h_2^{-1/p} \|\sigma\|_{L_{1/(2-m)}(e_\gamma)} |u|_{W_t^{m-1}(e_\gamma)},$$

where $t = p/(1-(2-m)p)$, $\;2/(3-m) < p < 1/(2-m)$, $e_\gamma = e_\gamma(x_2) = \big\{ \xi_2 : \; |\xi_2 - x_2| \le h_2 \big\}$.

Summation together with Hölder's inequality yields $\|\eta_\sigma\|_{L_p(\gamma_{-1})} \leq c|h|^{m-1}\|\sigma\|_{L_{1/(2-m)}(\Gamma_1)}|u|_{W_t^{m-1}(\Gamma_1)}$, and since $W_2^m(\Omega) \subset W_2^{m-1/2}(\Gamma_1) \subset W_t^{m-1}(\Gamma_1)$ for $1 < m < 2$, and $W_2^{m-3/2}(\Gamma_1) \subset L_{1/(2-m)}(\Gamma_1)$ for $3/2 < m < 2$, we obtain (5.11).

Represent now η_σ in the form of the sum $\eta_\sigma = T_2\sigma\,(T_2u-u)+\big(T_2(\sigma u)-T_2\sigma\,T_2u\big) = \eta'_\sigma+\eta''_\sigma$. The linear with respect to $u(x)$ functional η'_σ vanishes on $\boldsymbol{\pi}_1$ and is bounded in W_2^m, $m>1$. Consequently,

$$|\eta'_\sigma| \leq c|h|^{1/2}\|\sigma\|_{L_2(e_\gamma)}|u|_{W_2^{3/2}(e_\gamma)} \leq c|h|^{1/2}\|\sigma\|_{L_2(e_\gamma)}\|u\|_{W_2^2(\Omega)}.$$

Further, for η''_σ we have

$$|\eta''_\sigma| \leq c|h|^{-1}\int\limits_{e_\gamma}\int\limits_{e_\gamma}\frac{|\sigma(\zeta)-\sigma(\tau)|}{|\zeta-\tau|}\,d\zeta\,d\tau\int\limits_{e_\gamma}\Big|\frac{\partial u(0,\tau)}{\partial\tau}\Big|\,d\tau,$$

so, using the Cauchy inequality, we obtain

$$|\eta''_\sigma| \leq c|h|^{1/2}\|\sigma\|_{W_2^{1/2}(e_\gamma)}\|u\|_{W_2^1(e_\gamma)} \leq c|h|^{1/2}\|\sigma\|_{W_2^{1/2}(e_\gamma)}\|u\|_{W_2^2(\Omega)}.$$

The above estimates result in (5.11) for $m=2$.

For $2<m\leq 3$, the estimate (5.11) follows from

$$|\eta'_\sigma|\leq c|h|^{m-3/2}\|\sigma\|_{L_\infty(\Gamma_1)}|u|_{W_2^{m-1}(e_\gamma)},\quad |\eta''_\sigma|\leq c|h|^{m-3/2}\|\sigma\|_{W_2^{m-2}(e_\gamma)}\|u\|_{W_\infty^1(\Omega)}$$

with regard for the imbeddings $W_2^m(\Omega)\subset W_2^{m-1}(\Gamma_{-1})$, $W_2^m(\Omega)\subset W_\infty^1(\Omega)$, $W_2^{m-3/2}(\Gamma_{-1})\subset L_\infty(\Gamma_{-1})$.

Estimation of the functional $\overline{\eta}_{2\beta\overline{x}_{3-\alpha}}$. Let us prove that

$$\|\overline{\eta}_{2\beta\overline{x}_{3-\alpha}}\|_{\omega\cup\gamma_{-\alpha}} \leq c|h|^{m-1}\|u\|_{W_2^m(\Omega)},\quad 1<m\leq 3. \tag{5.12}$$

For $1<m\leq 2$ we write

$$\overline{\eta}_{2\beta\overline{x}_{3-\alpha}} = \eta'+\eta''+\eta''', \tag{5.13}$$

where

$$\eta' = \frac{h_1}{2}\,a_{2\beta\overline{x}_{3-\alpha}}u_{x_\beta},\quad \eta'' = \frac{h_1}{2}\,(a_{2\beta})^{(-1_{3-\alpha})}u_{x_\beta\overline{x}_{3-\alpha}},$$

$$\eta''' = -\frac{h_1}{2}\,T_1^+S_2^+\Big(a_{2\beta}\,\frac{\partial u}{\partial x_\beta}\Big)_{\overline{x}_{3-\alpha}}.$$

It is not difficult to notice that

$$\begin{aligned}\|\eta'''\|_{\omega\cup\gamma_{-\alpha}} &\leq c|h|^{m-1}\Big|a_{2\beta}\,\frac{\partial u}{\partial x_\beta}\Big|_{W_2^{m-1}(\Omega)},\\ \|\eta''\|_{\omega\cup\gamma_{-\alpha}} &\leq c|h|^{m-1}\,\|a_{2\beta}\|_{C(\overline{\Omega})}|u|_{W_2^m(\Omega)}.\end{aligned} \tag{5.14}$$

Let $e_i = e_i(x) = \big\{\xi=(\xi_1,\xi_2):\ x_i\leq\xi_i\leq x_i+h_i,\ |\xi_{3-i}-x_{3-i}|\leq h_{3-i}\big\}$, $i=1,2$. Using the Bramble–Hilbert lemma, we obtain

$$|\eta'| \leq c|h|^{r+t-1}(h_1h_2)^{-1/2}|u|_{W_q^t(e_\alpha)}|a_{2\beta}|_{W_{2q/(q-2)}^r(e_\alpha)},$$

and hence $\|\eta'\|_{\omega\cup\gamma_{-\alpha}} \le c|h|^{r+t-1}|u|_{W_q^t(\Omega)}|a_{2\beta}|_{W^r_{2q/(q-2)}(\Omega)}$, where $2/q < t \le m + 2/q - 1$, $(q-2)/q < r \le (q-2)/q + \varepsilon$ for $1 < m < 2$, and $(q-2)/q < r \le (q-2)/q + \varepsilon/(\varepsilon+2)$ for $m = 2$, $q > 2$, $\varepsilon > 0$.

Since in the case under consideration $W_2^m \subset W_q^t$, $W^{m-1+\varepsilon}_{2/(m-1)} \subset W^r_{2q/(q-2)}$, $W^1_{2+\varepsilon} \subset W^r_{2q/(q-2)}$, choosing $r + t = m$ we have

$$\begin{aligned} &\|\eta'\|_{\omega\cup\gamma_{-\alpha}} \le c|h|^{m-1}\|u\|_{W_2^m(\Omega)}\|a_{2\beta}\|_{W^{m-1+\varepsilon}_{2/(m-1)}(\Omega)}, \quad 1 < m < 2, \\ &\|\eta'\|_{\omega\cup\gamma_{-\alpha}} \le c|h|\|u\|_{W_2^2(\Omega)}\|a_{2\beta}\|_{W^1_{2+\varepsilon}(\Omega)}. \end{aligned} \tag{5.15}$$

From (5.13)–(5.15) it follows that the estimate (5.12) is valid for $1 < m \le 2$.

For $2 < m \le 3$ we write $\overline{\eta}_{2\beta\overline{x}_{3-\alpha}} = \ell' + \ell'' + \ell'''$, where

$$\ell' = \frac{h_1}{2}\, a_{2\beta\overline{x}_{3-\alpha}}\Big(u_{x_\beta} - \frac{\partial u}{\partial x_\beta}\Big), \quad \ell'' = \frac{h_1}{2}\,(a_{2\beta})^{(-1_{3-\alpha})}\Big(u_{x_\beta} - \frac{\partial u}{\partial x_\beta}\Big)_{\overline{x}_{3-\alpha}},$$

$$\ell''' = \frac{h_1}{2}\,\Big(a_{2\beta}\,\frac{\partial u}{\partial x_\beta} - T_1^+ S_2^+\Big(a_{2\beta}\,\frac{\partial u}{\partial x_\beta}\Big)\Big)_{\overline{x}_{3-\alpha}}.$$

For ℓ', ℓ'', ℓ''' we obtain the analogous inequalities (5.14) and (5.15), and hence the proof of (5.13) is complete. The inequalities (5.8)–(5.12) together with Lemma 5.3 prove the following

Theorem 5.1. *The difference scheme* (5.3) *converges, and the a priori estimate*

$$\|y-u\|_{W_2^1(\omega)} \le c|h|^{m-1}\|u\|_{W_2^m(\Omega)}, \quad 1 < m \le 3, \tag{5.16}$$

is valid.

CHAPTER 2

Difference Schemes for Elliptic Systems and Equations of Higher Order

This chapter deals with the difference schemes for approximate solution of: the Dirichlet problem for elliptic systems of general type (Section 6); mixed type boundary value problem for a system of the statical theory of elasticity (Section 7); the third boundary value problem of elasticity (Sections 8, 9); the first boundary value problem for the fourth order elliptic equation (Section 10). For the construction of difference schemes we use the Steklov averaging operators. The correctness of difference schemes in discrete Sobolev spaces is established by means of the energy method. Consistent estimates of the convergence rate are obtained.

A new approach of obtaining an a priori error estimate of the difference scheme is suggested which does not require for the solution of the fourth order equation to be continued outside the domain of integration.

6. The Dirichlet Problem for Systems

History of the matter. The results of the present section have been published in [7] and [12] (1987).

In [83], for the Lamé system with constant coefficients a difference scheme is constructed and a consistent estimate of the convergence rate is established for $s = 0$, $m = 1, 2$ and $s = 1$, $m = 2$; in [78] (1987) and [77] (1989), difference schemes are constructed for the system of equilibrium of an inhomogeneous anisotropic elastic rigidly fixed solid body. When the elastic coefficients belong to the spaces W^1_∞ and W^∞_2, consistent estimates with $s = 1$, $m = 2$ and $s = 1$, $2 < m \leq 3$ are respectively found.

1^0. We consider a difference scheme with approximates the following problem:

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(K_{ij}\,\frac{\partial \boldsymbol{u}}{\partial x_j}\Big) + K_0\boldsymbol{u} = \boldsymbol{f},\ \ x \in \Omega, \quad \boldsymbol{u}(x) = 0,\ \ x \in \Gamma. \qquad (6.1)$$

Here $\boldsymbol{u} = \big(u_1(x), u_2(x), \dots, u_n(x)\big)$, $\boldsymbol{f} = \big(f_1(x), f_2(x), \dots, f_n(x)\big)$, K_{ij} and K_0 are matrices of the n-th order with the elements $K^{\alpha\beta}_{ij}(x)$, $K^{\alpha\beta}_0(x)$ ($\alpha = 1, 2, \dots, n$; $\beta = 1, 2, \dots, n$).

The conditions of strong ellipticity

$$\frac{1}{2}\sum_{i,j=1}^{2}(K_{ij}+K_{ji}^{T})\boldsymbol{t}_j\cdot\boldsymbol{t}_i\geq\nu\sum_{i=1}^{2}\boldsymbol{t}_i\cdot\boldsymbol{t}_i,\ \ \forall\, x\in\Omega, \tag{6.2}$$

are assumed to be fulfilled; here $\boldsymbol{t}_i=(t_{i1},t_{i2},\dots,t_{in})$, $i=1,2$, are arbitrary real vectors, and $\nu>0$ is a constant number. In what follows, the symbol $\boldsymbol{t}_1\cdot\boldsymbol{t}_2$ denotes the inner product of n-dimensional vectors, i.e. $\sum\limits_{i=1}^{n}t_{1i}t_{2i}$.

Let $\|\cdot\|_*$ be any norm of the function. Then by the norm $\|\boldsymbol{v}\|_*$ of the vector-function $\boldsymbol{v}$ we mean the value $\|\boldsymbol{v}\|_*=\big(\sum\limits_{\alpha=1}^{n}\|v_\alpha\|_*^2\big)^{1/2}$, and by the norm $\|K\|_*$ of the variable matrix $K(x)$ we mean the value $\|K\|_*=\big(\sum\limits_{\alpha,\beta=1}^{n}\|K^{\alpha\beta}\|_*^2\big)$.

On the mesh $\overline{\omega}$ we consider mesh vector functions, for example, $\boldsymbol{y}=(y_1,y_2,\dots,y_n)$, where y_i, $i=1,2,\dots,n$ are mesh functions defined on $\overline{\omega}$.

Let there exist a unique solution $\boldsymbol{u}\in W_2^m(\Omega)$, $2<m\leq 3$ of the problem (6.1), and the conditions

$$K_{ij}(x)\in W_2^{m-1}(\Omega),\ \ i,j=1,2,\quad 0\leq K_0(x)\in W_2^{m-2}(\Omega),\\ \boldsymbol{f}(x)\in W_2^{m-2}(\Omega),\ \ 2<m\leq 3, \tag{6.3}$$

be fulfilled. We approximate the problem (6.1) by the difference scheme

$$A\boldsymbol{y}=\boldsymbol{\varphi},\ \ \boldsymbol{\varphi}=S_1S_2\boldsymbol{f},\ \ x\in\omega,\quad \boldsymbol{y}(x)=0,\ \ x\in\gamma, \tag{6.4}$$

where $A\boldsymbol{y}=-\frac{1}{2}\sum\limits_{i,j=1}^{2}\big((K_{ij}\boldsymbol{y}_{\overline{x}_j})_{x_i}+(K_{ij}\boldsymbol{y}_{x_j})_{\overline{x}_i}\big)+\overline{K}\boldsymbol{y}$, $\overline{K}=S_1S_2K_0(x)$.

Let H be the space of mesh vector-functions defined on $\overline{\omega}$ and equal to zero on γ, with the inner product and the norm $(\boldsymbol{y},\boldsymbol{v})=\sum\limits_{x\in\omega}h_1h_2\boldsymbol{y}(x)\cdot\boldsymbol{v}(x)$, $\|\boldsymbol{y}\|=(\boldsymbol{y},\boldsymbol{y})^{1/2}$, $\boldsymbol{y}$, $\boldsymbol{v}\in H$.

Define also the norms $\|\boldsymbol{y}\|_{(1+)}^2=\sum\limits_{\omega_1^+\times\omega_2}h_1h_2\boldsymbol{y}\cdot\boldsymbol{y}$, $\|\boldsymbol{y}\|_{(2+)}^2=\sum\limits_{\omega_1\times\omega_2^+}h_1h_2\boldsymbol{y}\cdot\boldsymbol{y}$, $\|\boldsymbol{y}\|_{W_2^1(\omega)}^2\equiv\|\nabla\boldsymbol{y}\|^2=\|\boldsymbol{y}_{\overline{x}_1}\|_{(1+)}^2+\|\boldsymbol{y}_{\overline{x}_2}\|_{(2+)}^2$.

The operator A is positive definite in the space H ([37], Ch. IV, § 3) since there exists the unique solution of the problem (6.4).

2^0. Now we investigate the convergence rate of the difference scheme (6.4). For the error $\boldsymbol{z}=\boldsymbol{y}-\boldsymbol{u}$ we obtain the problem

$$A\boldsymbol{z}=\boldsymbol{\psi},\ \ x\in\omega,\ \ \boldsymbol{z}\in H. \tag{6.5}$$

The approximation error $\boldsymbol{\psi}=\boldsymbol{\varphi}-A\boldsymbol{u}$ can be reduced to the form

$$\boldsymbol{\psi}=(\boldsymbol{\eta}_{11}+\boldsymbol{\eta}_{12})_{x_1}+(\boldsymbol{\eta}_{21}+\boldsymbol{\eta}_{22})_{x_2}+\boldsymbol{\eta}, \tag{6.6}$$

where

$$\boldsymbol{\eta}_{11}=\frac{1}{2}\big(K_{11}\boldsymbol{u}_{\overline{x}_1}+(K_{11}\boldsymbol{u}_{x_1})(x_1-h_1,x_2)\big)-S_2(K_{11}D_1\boldsymbol{u})\Big(x_1-h_12,x_2\Big),$$

$$\boldsymbol{\eta}_{12} = \frac{1}{2}\left(K_{12}\boldsymbol{u}_{\overline{x}_2} + (K_{12}\boldsymbol{u}_{x_2})(x_1-h_1,x_2)\right) - S_2(K_{12}D_2\boldsymbol{u})\Big(x_1-\frac{h_1}{2},x_2\Big),$$
$$\boldsymbol{\eta}_{21} = \frac{1}{2}\left(K_{21}\boldsymbol{u}_{\overline{x}_1} + (K_{21}\boldsymbol{u}_{x_1})(x_1,x_2-h_2)\right) - S_1\Big(K_{21}D_1\boldsymbol{u}\Big)\Big(x_1,x_2-\frac{h_2}{2}\Big),$$
$$\boldsymbol{\eta}_{22} = \frac{1}{2}\left(K_{22}\boldsymbol{u}_{\overline{x}_2} + (K_{22}\boldsymbol{u}_{x_2})(x_1,x_2-h_2)\right) - S_1(K_{22}D_2\boldsymbol{u})\Big(x_1,x_2-\frac{h_2}{2}\Big),$$
$$\boldsymbol{\eta} = S_1S_2(K_0\boldsymbol{u}) - S_1S_2K_0\,\boldsymbol{u}.$$

Using the equality (6.6), from the equation (6.5) we obtain

$$(A\boldsymbol{z},\boldsymbol{z}) \le \|\boldsymbol{\eta}_{11}+\boldsymbol{\eta}_{12}\|_{(1+)}\|\nabla\boldsymbol{z}\| + \|\boldsymbol{\eta}_{22}+\boldsymbol{\eta}_{21}\|_{(2+)}\|\nabla\boldsymbol{z}\| + \|\boldsymbol{\eta}\|\,\|\boldsymbol{z}\|. \quad (6.7)$$

By virtue of (6.2) we arrive at the inequality

$$\nu\|\nabla\boldsymbol{z}\|^2 \le (A\boldsymbol{z},\boldsymbol{z}). \quad (6.8)$$

Using the estimate (6.8) and the difference analogue of Friedrichs inequality ([70], p. 309)

$$\|\boldsymbol{z}\| \le \frac{\ell_0}{4}\,\|\nabla\boldsymbol{z}\|, \quad \ell_0 = \max(\ell_1;\ell_2), \quad (6.9)$$

from (6.7) we find that

$$\|\boldsymbol{z}\|_{W_2^1(\omega)} \le \frac{1}{\nu}\Big(\sum_{i,j=1}^{2}\|\boldsymbol{\eta}_{ij}\|_{(i+)} + \frac{\ell_0}{4}\,\|\boldsymbol{\eta}\|\Big). \quad (6.10)$$

3^0. We rewrite the expression $\boldsymbol{\eta}_{11}$ as follows:

$$\boldsymbol{\eta}_{11}(x) = -\frac{1}{2}\left(K_{11}+K_{11}(x_1-h_1,x_2)\right)\ell_{11}^{(1)}(\boldsymbol{u}) + \\ + D_1\boldsymbol{u}\Big(x_1-\frac{h_1}{2},x_2\Big)\ell_{11}^{(2)}(K_{11}) + \ell_{11}^{(3)}(K_{11}D_1\boldsymbol{u}), \quad (6.11)$$

where $\ell_{11}^{(1)}$, $\ell_{11}^{(2)}$, $\ell_{11}^{(3)}$ have the same meaning as in Section 2. We will need the estimates of the type

$$\|K_{ij}\boldsymbol{v}\|_{(i+)} \le \|K_{ij}\|_{C(\overline{\Omega})}\|\boldsymbol{v}\|_{(i)}, \quad i,j=1,2, \quad (6.12)$$
$$\|K_{ij}\boldsymbol{v}\|_{(i+)} \le \|K_{ij}\|_{(i)}\,\|\boldsymbol{v}\|_{C(\overline{\Omega})}, \quad i,j=1,2. \quad (6.13)$$

These estimates hold for any continuous vector-function $\boldsymbol{v}(x)$.

Indeed,

$$\|K_{ij}\boldsymbol{v}\|_{(1+)}^2 = \sum_{\alpha=1}^{n}\sum_{\omega_1^+\times\omega_2} h_1h_2\Big(\sum_{\beta=1}^{n}K_{ij}^{\alpha\beta}v_\beta\Big)^2 \le$$
$$\le \sum_{\alpha=1}^{n}\sum_{\omega_1^+\times\omega_2} h_1h_2\Big(\sum_{\beta=1}^{n}\max_{\overline{\Omega}}|K_{ij}^{\alpha\beta}|\,|v_\beta|\Big)^2 \le$$
$$\le \sum_{\alpha=1}^{n}\sum_{\omega_1^+\times\omega_2} h_1h_2\sum_{\beta=1}^{n}\big(\max_{\overline{\Omega}}|K_{ij}^{\alpha\beta}|\big)^2\sum_{\beta=1}^{n}|v_\beta|^2 = \|K_{ij}\|_{C(\overline{\Omega})}^2\|\boldsymbol{v}\|_{(1+)}^2,$$

and hence the estimate (6.12) is valid for $i = 1$. The estimates (6.12) and (6.13) are proved similarly. Applying the estimates (6.12) and (6.13) $(i = j = 1)$, from (6.11) we find that

$$\|\boldsymbol{\eta}_{11}\|_{(1+)} \le \|K_{11}\|_{C(\overline{\Omega})} \|\ell_{11}^{(1)}(\boldsymbol{u})\|_{(1+)} + \\ + \|\ell_{11}^{(2)}(K_{11})\|_{(1+)} \|D_1 \boldsymbol{u}\|_{C(\overline{\Omega})} + \big\|\ell_{11}^{(3)}(K_{11} D_1 \boldsymbol{u})\big\|_{(1+)}. \qquad (6.14)$$

Using the imbedding $W_2^m \subset C^n$ for $m > n+1$, on the basis of the Bramble–Hilbert lemma we obtain

$$\big|\ell_{11}^{(1)}(u_\alpha)\big| \le \frac{c|h|^{m-1}}{\sqrt{h_1 h_2}}\, |u_\alpha|_{W_2^m(e)}, \quad x \in \omega, \quad 2 < m \le 3, \quad \alpha = 1, 2, \ldots, n, \qquad (6.15)$$

$$\big|\ell_{11}^{(2)}(K_{11}^{\alpha\beta})\big| \le \\ \le \frac{c|h|^{m-1}}{\sqrt{h_1 h_2}}\, |K_{11}^{\alpha\beta}|_{W_2^{m-1}(e)}, \quad x \in \omega, \quad 2 < m \le 3, \quad \alpha, \beta = 1, 2, \ldots, n, \qquad (6.16)$$

$$\big|\ell_{11}^{(3)}((K_{11} D_1 \boldsymbol{u})_\alpha)\big| \le \\ \le \frac{c|h|^{m-1}}{\sqrt{h_1 h_2}}\, \big|(K_{11} D_1 \boldsymbol{u})_\alpha\big|_{W_2^{m-1}(e)}, \quad x \in \omega, \quad 2 < m \le 3, \quad \alpha = 1, 2, \ldots, n, \qquad (6.17)$$

where $e = e(x) = \big\{\xi = (\xi_1, \xi_2) : \; x_1 - h_1 \le \xi_1 \le x_1, \; |x_2 - \xi_2| \le h_2/2\big\}$.

By means of the estimates (6.15)–(6.17), from (6.14) we find the estimate

$$\|\boldsymbol{\eta}_{11}\|_{(1+)} \le c|h|^{m-1}\big(\|K_{11}\|_{C(\overline{\Omega})}\, |\boldsymbol{u}|_{W_2^m(\Omega)} + \\ + |K_{11}|_{W_2^{m-1}(\Omega)}\, \|\boldsymbol{u}\|_{W_2^m(\Omega)} + |K_{11} D_1 \boldsymbol{u}|_{W_2^{m-1}(\Omega)}\big), \quad 2 < m \le 3. \qquad (6.18)$$

Analogously, for $\boldsymbol{\eta}_{22}$ we obtain

$$\|\boldsymbol{\eta}_{22}\|_{(2+)} \le c|h|^{m-1}(\|K_{22}\|_{C(\overline{\Omega})} |\boldsymbol{u}|_{W_2^m(\Omega)} + \\ + |K_{22}|_{W_2^{m-1}(\Omega)} \|\boldsymbol{u}\|_{W_2^m(\Omega)} + |K_{22} D_2 \boldsymbol{u}|_{W_2^{m-1}(\Omega)}\big), \quad 2 < m \le 3. \qquad (6.19)$$

We now represent $\boldsymbol{\eta}_{12}$ in the form

$$\boldsymbol{\eta}_{12} = \ell_{12}^{(1)}(K_{12} D_2 \boldsymbol{u}) + \frac{h_2}{4}\, \ell_{12}^{(2)}(K_{12}) S_1 S_2 D_2^2 \boldsymbol{u} + \\ + 0.5 K_{12}\, \ell_{12}^{(3)}(\boldsymbol{u}) + \ell_{12}^{(4)}(\boldsymbol{u}) + \ell_{12}^{(5)} + 0.5 K_{12}(x_1 - h_1, x_2) \ell_{12}^{(6)}(\boldsymbol{u}),$$

where

$$\ell_{12}^{(4)}(\boldsymbol{u}) = \frac{h_1 h_2}{4}\, \big[S_1 S_2 (D_2 K_{12} D_2^2 \boldsymbol{u}) - S_1 S_2 D_2 K_{12}\, S_1 S_2 D_2^2 \boldsymbol{u}\big],$$

$$\ell_{12}^{(5)} = -\frac{h_1 h_2}{4}\, S_1 S_2 (D_1 K_{12} D_2^2 \boldsymbol{u}),$$

and $\ell_{12}^{(1)}$, $\ell_{12}^{(2)}$, $\ell_{12}^{(3)}$, $\ell_{12}^{(6)}$ are defined in Section 2. In the same manner as in obtaining the estimate (6.18), we find that

$$\|\boldsymbol{\eta}_{12}|_{(1+)}| \le c|h|^{m-1}\big(|K_{12} D_2 \boldsymbol{u}|_{W_2^{m-1}(\Omega)} + \|D_1 K_{12} D_2^2 \boldsymbol{u}\|_{L_{\frac{2}{4-m}}(\Omega)} +$$

$$+\|K_{12}\|_{W_2^{m-1}(\Omega)}|\boldsymbol{u}|_{W_2^m(\Omega)}\Big),\ \ 2<m\le 3. \tag{6.20}$$

Analogously,

$$\|\boldsymbol{\eta}_{21}|_{(2+)}|\le c|h|^{m-1}\big(|K_{21}D_1\boldsymbol{u}|_{W_2^{m-1}(\Omega)}+\|D_2K_{21}D_1^2\boldsymbol{u}\|_{L_{\frac{2}{4-m}}(\Omega)}+$$
$$+\|K_{21}\|_{W_2^{m-1}(\Omega)}|\boldsymbol{u}|_{W_2^m(\Omega)}\big),\ \ 2<m\le 3. \tag{6.21}$$

Next, in the same way as is done for η in Section 2, we obtain the estimate

$$\|\boldsymbol{\eta}\|\le c|h|^{m-1}\Big(|\boldsymbol{u}|_{W_2^m(\Omega)}+|K_0D_1\boldsymbol{u}|_{W_2^{m-2}(\Omega)}+|K_0D_2\boldsymbol{u}|_{W_2^{m-2}(\Omega)}+$$
$$+\|K_0D_1D_2\boldsymbol{u}\|_{L_{\frac{2}{4-m}}(\Omega)}+\|K_0D_1^2\boldsymbol{u}\|_{L_{\frac{2}{4-m}}(\Omega)}+\|K_0D_2^2\boldsymbol{u}\|_{L_{\frac{2}{4-m}}(\Omega)}\Big),\ \ 2<m\le 3. \tag{6.22}$$

Note that if for the elements $K^{\alpha\beta}$ of a variable matrix $K(x)$ and for the components v_β of a vector-function $\boldsymbol{v}(x)$ the inequalities $\|K^{\alpha\beta}v_\beta\|_{\sigma_1}\le c\|K^{\alpha\beta}\|_{\sigma_2}\|v_\beta\|_{\sigma_3}$ are valid, where $\|\cdot\|_{\sigma_i}$, $i=1,2,3$, are some norms, then

$$\|K\boldsymbol{v}\|_{\sigma_1}\le c\|K\|_{\sigma_2}\|\boldsymbol{v}\|_{\sigma_3}. \tag{6.23}$$

Indeed,

$$\|K\boldsymbol{v}\|_{\sigma_1}^2=\sum_{\alpha=1}^n\|(K\boldsymbol{v})_\alpha\|_{\sigma_1}^2=\sum_{\alpha=1}^n\Big\|\sum_{\beta=1}^n K^{\alpha\beta}v_\beta\Big\|_{\sigma_1}^2\le$$
$$\le\sum_{\alpha=1}^n\Big(\sum_{\beta=1}^n\|K^{\alpha\beta}v_\beta\|_{\sigma_1}\Big)^2\le c^2\sum_{\alpha=1}^n\Big(\sum_{\beta=1}^n\|K^{\alpha\beta}\|_{\sigma_2}\|v_\beta\|_{\sigma_3}\Big)^2\le$$
$$\le c^2\sum_{\alpha=1}^n\sum_{\beta=1}^n\|K^{\alpha\beta}\|_{\sigma_2}^2\sum_{\beta=1}^n\|v_\beta\|_{\sigma_3}^2=c^2\|K\|_{\sigma_2}^2\|\boldsymbol{v}\|_{\sigma_3}^2,$$

which was to be demonstrated.

Since

$$\big\|K_{ij}^{\alpha\beta}D_ju_\beta\big\|_{W_2^{m-1}(\Omega)}\le c\|K_{ij}^{\alpha\beta}\|_{W_2^{m-1}(\Omega)}\|u_\beta\|_{W_2^m(\Omega)},\ \ 2<m\le 3,$$
$$\big\|D_iK_{i,3-i}^{\alpha\beta}D_{3-i}^2u_\beta\big\|_{L_{\frac{2}{4-m}}(\Omega)}\le c\|K_{i,3-i}^{\alpha\beta}\|_{W_2^{m-1}(\Omega)}\|u_\beta\|_{W_2^m(\Omega)},\ \ 2<m\le 3,$$
$$\big\|K_0^{\alpha\beta}D_iD_ju_\beta\big\|_{L_{\frac{2}{4-m}}(\Omega)}\le c\|K_0^{\alpha\beta}\|_{W_2^{m-2}(\Omega)}\|u_\beta\|_{W_2^m(\Omega)},\ \ 2<m\le 3,$$
$$\sum_p b_p^t\le\Big(\sum_p b_p\Big)^t,\ \ \forall\, b_p\ge 0,\ \ t\ge 1,$$

using (6.18), (6.23) and the a priori estimate (6.10), and also taking into account the fact that $|\cdot|_{W_2^1(\omega)}$ and $\|\cdot\|_{W_2^1(\omega)}$ are equivalent, we can see that the following theorem is valid.

Theorem 6.1. *Let* $\boldsymbol{u} \in W_2^m(\Omega)$, $2 < m \le 3$, *be the solution of the problem* (6.1), *and the conditions* (6.2), (6.3) *be fulfilled. Then the convergence of the difference scheme* (6.4) *in the norm* $W_2^1(\omega)$ *is characterized by the estimate*

$$\|\boldsymbol{y} - \boldsymbol{u}\|_{W_2^1(\omega)} \le c|h|^{m-1}\|\boldsymbol{u}\|_{W_2^m(\Omega)}, \quad 2 < m \le 3, \tag{6.24}$$

with a constant $c > 0$ *independent of* h *and* $u(x)$.

Consider now the first boundary value problem for the system of plane deformation of an inhomogeneous elastic body:

$$\begin{gathered}
\frac{\partial}{\partial x_1}\Big((2\mu+\lambda)\frac{\partial u^1}{\partial x_1}+\lambda\frac{\partial u^2}{\partial x_2}\Big)+\frac{\partial}{\partial x_2}\Big(\mu\frac{\partial u^1}{\partial x_2}+\mu\frac{\partial u^2}{\partial x_1}\Big)=-f^1, \\
\frac{\partial}{\partial x_1}\Big(\mu\frac{\partial u^1}{\partial x_2}+\mu\frac{\partial u^2}{\partial x_2}\Big)+\frac{\partial}{\partial x_2}\Big(\lambda\frac{\partial u^1}{\partial x_1}+(2\mu+\lambda)\frac{\partial u^2}{\partial x_2}\Big)=-f^2, \quad x\in\Omega, \\
u^1\big|_\Gamma = u^2\big|_\Gamma = 0.
\end{gathered} \tag{6.25}$$

Under the assumptions

$$\begin{gathered}
\boldsymbol{f} = (f^1, f^2) \in L_2(\Omega), \quad \mu,\, \lambda \in W^1_{2+\varepsilon}(\Omega), \\
\forall\, \varepsilon > 0, \quad \lambda(x) \ge 0, \quad \mu(x) \ge \mu_0 = \text{const} > 0,
\end{gathered} \tag{6.26}$$

there exists a unique generalized solution $\boldsymbol{u} = (u^1, u^2)$ of the problem (6.25), belonging to the space $W_2^2(\Omega)$ ([58]).

We approximate the problem (6.25) by the difference scheme

$$\begin{gathered}
\big((2\mu^+ + \lambda^+)y^1_{x_1} + \lambda^+ y^2_{x_2}\big)_{\overline{x}_1} + \big((2\mu^- + \lambda^-)y^1_{\overline{x}_1} + \lambda^- y^2_{\overline{x}_2}\big)_{x_1} + \\
+(\mu^+ y^1_{x_2} + \mu^+ y^2_{x_1})_{\overline{x}_2} + (\mu^- y^1_{\overline{x}_2} + \mu^- y^2_{\overline{x}_1})_{x_2} = -2\varphi^1, \\
(\mu^+ y^1_{x_2} + \mu^+ y^2_{x_1})_{\overline{x}_1} + (\mu^- y^1_{\overline{x}_2} + \mu^- y^2_{\overline{x}_1})_{x_1} + \\
+\big(\lambda^+ y^1_{x_1} + (2\mu^+ + \lambda^+)y^2_{x_2}\big)_{\overline{x}_2} + \\
+\big(\lambda^- y^1_{\overline{x}_1} + (2\mu^- + \lambda^-)y^2_{\overline{x}_2}\big)_{x_2} = -2\varphi^2, \quad x \in \omega, \\
\varphi^\alpha = T_1T_2 f^\alpha, \quad y^1|_\gamma = y^2|_\gamma = 0,
\end{gathered} \tag{6.27}$$

where

$$\begin{gathered}
\mu^\pm(x) = S_1^\pm S_2^\pm \mu, \quad \lambda^\pm(x) = S_1^\pm S_2^\pm \lambda \ \text{ for } \ \lambda,\, \mu \in W^1_{2+\varepsilon}(\Omega), \\
\mu^\pm(x) = \mu\Big(x_1 \pm \frac{h_1}{2}, x_2 \pm \frac{h_2}{2}\Big), \\
\lambda^\pm(x) = \lambda\Big(x_1 \pm \frac{h_1}{2}, x_2 \pm \frac{h_2}{2}\Big) \ \text{ for } \ \lambda,\, \mu \in W^1_\infty(\Omega).
\end{gathered} \tag{6.28}$$

Let us now establish an a priori estimate of the solution of the problem (6.27). After some transformations we have

$$2W_h = (\varphi^1, y^1) + (\varphi^2, y^2), \tag{6.29}$$

where

$$W_h \equiv \frac{1}{2}\big(|y^1_{x_1}|^2, \mu^+\big)_{+\omega_1\times\omega_2} + \frac{1}{2}\big(|y^1_{\overline{x}_1}|^2, \mu^-\big)_{\omega_1^+\times\omega_2} +$$

$$+\frac{1}{2}\left(|y^2_{x_2}|^2,\mu^+\right)_{\omega_1\times^+\omega_2}+\frac{1}{2}\left(|y^2_{\overline{x}_2}|^2,\mu^-\right)_{\omega_1\times\omega_2^+}+$$
$$+\frac{1}{4}\left(|y^1_{x_2}+y^2_{x_1}|^2,\mu^+\right)_{+\omega_1\times^+\omega_2}+\frac{1}{4}\left(|y^1_{\overline{x}_2}+y^2_{\overline{x}_1}|^2,\mu^-\right)_{\omega_1^+\times\omega_2^+}+$$
$$+\frac{1}{4}\left(|y^1_{x_1}+y^2_{x_2}|^2,\lambda^+\right)_{+\omega_1\times^+\omega_2}+\frac{1}{4}\left(|y^1_{\overline{x}_1}+y^2_{\overline{x}_2}|^2,\lambda^-\right)_{\omega_1^+\times\omega_2^+} \qquad (6.30)$$

is the mesh analogue of elastic deformation energy (cf. [70], p. 329).

Omitting in (6.30) the summands involving $\lambda^\pm$, replacing $\mu^\pm$ by μ_0, and using the inequality $\left(|y^1_{x_2}+y^2_{x_1}|^2,1\right)_{+\omega_1\times^+\omega_2}+\left(|y^1_{\overline{x}_2}+y^2_{\overline{x}_1}|^2,1\right)_{\omega_1^+\times\omega_2^+}\geq 2\left[\|y^1_{\overline{x}_2}\|^2_{(2)}+\|y^2_{\overline{x}_1}\|^2_{(1)}-\|y^1_{\overline{x}_1}\|^2_{(1)}-\|y^2_{\overline{x}_2}\|^2_{(2)}\right]$, we can conclude that $W_h\geq(\mu_0/2)\|\nabla\boldsymbol{y}\|^2$, $\boldsymbol{y}=(y^1,y^2)$. Taking into account the Friedrichs inequality (6.9) the latter results in

$$W_h\geq c_1\|\boldsymbol{y}\|^2_{W_2^1(\omega)},\quad c_1=8\mu_0^2(\ell_0^2+16)^{-1}. \qquad (6.31)$$

On the basis of the definition of the norm $\|\cdot\|_{-1}$, we have $2W_h\leq\|\boldsymbol{\varphi}\|_{-1}\|\boldsymbol{y}\|_{W_2^1(\omega)}$, and hence from (6.29) and (6.31) we find that

$$\|\boldsymbol{y}\|_{W_2^1(\omega)}\leq c_2\|\boldsymbol{\varphi}\|_{-1},\quad \boldsymbol{\varphi}=(\varphi^1,\varphi^2),\quad c_2=1/(2c_1). \qquad (6.32)$$

It follows from the estimate (6.32) that the problem (6.27) is uniquely solvable.

Theorem 6.2. *The difference scheme* (6.27), (6.28) *converges in the norm* $W_2^1(\omega)$, *and the estimate*

$$\|\boldsymbol{y}-\boldsymbol{u}\|_{W_2^1(\omega)}\leq c|h|\|\boldsymbol{f}\|_{L_2(\Omega)} \qquad (6.33)$$

holds.

The proof of the above theorem does not, in fact, differ from that of Theorems 3.1 and 3.2.

Assume now that the coefficients λ, $\mu\in W_2^{m-1}(\Omega)$, $2<m\leq 3$ are smoother, and the solution of the problem (6.25) belongs to the space $W_2^m(\Omega)$, $2<m\leq 3$. We calculate the coefficients in (6.27) by the formulas

$$\lambda^\pm=\lambda,\quad \mu^\pm=\mu,\quad \varphi^\alpha=S_1S_2f^\alpha,\ \ \alpha=1,2. \qquad (6.34)$$

Analogously to Theorem 2.1 we prove the following

Theorem 6.3. *The difference scheme* (6.27), (6.34) *converges in the mesh norm* $W_2^1(\omega)$, *and the estimate*

$$\|\boldsymbol{y}-\boldsymbol{u}\|_{W_2^1(\omega)}\leq c|h|^{m-1}\|\boldsymbol{f}\|_{L_2(\Omega)},\ \ 2<m\leq 3, \qquad (6.35)$$

holds.

7. The Mixed Boundary Value Problem for Systems

History of the matter. In this section, for a system of statical theory of elasticity we consider the mixed boundary value problem. The results of Section 7 have been published in [24]. In the case of constant coefficients, analogous results were obtained in [10]; the difference scheme converging with the rate $O(|h|^2)$ for $\boldsymbol{u} \in C^4(\overline{\Omega})$ is studied in [70] (1976). In the case of variable coefficients, in [16] the difference schemes are verified and the consistent estimate of the convergence rate for $s = 1$, $m \in (2, 3]$ is established.

Let in a rectangle Ω we are required to find a solution of the following boundary value problem:

$$\begin{gathered} \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \Big(K_{ij}(x) \frac{\partial \boldsymbol{u}}{\partial x_j} \Big) = -\boldsymbol{f}, \quad x \in \Omega, \\ K_{11} \frac{\partial \boldsymbol{u}}{\partial x_1} + K_{12} \frac{\partial \boldsymbol{u}}{\partial x_2} = -\boldsymbol{g}, \quad x \in \Gamma_1, \quad \boldsymbol{u}(x) = 0, \quad x \in \Gamma_0 = \Gamma \setminus \Gamma_{-1}, \end{gathered} \tag{7.1}$$

where $K_{11} = \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{pmatrix}$, $K_{12} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$, $K_{21} = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix}$, $K_{22} = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix}$, $\boldsymbol{u} = (u^1, u^2)$, $\boldsymbol{f} = (f^1, f^2)$, $\boldsymbol{g} = (g^1, g^2)$.

Assume that the Lamé coefficients satisfy the conditions $\lambda(x) \geq 0$, $\mu(x) \geq \mu_0 = \text{const} > 0$, λ, $\mu \in W_2^{m-1}(\Omega)$ for $2 < m \leq 3$, λ, $\mu \in W_{2+\varepsilon}^1(\Omega)$ for $m = 2$, λ, $\mu \in W_{2/(m-1)}^{m-1+\varepsilon}(\Omega)$ for $1 < m < 2$, $i, j = 1, 2$, $\varepsilon > 0$ and the problem (7.1) with the right-hand side $\boldsymbol{f} \in W_2^m(\Omega)$, $\boldsymbol{g} \in W_2^{m-3/2}(\Gamma_{-1})$ is uniquely solvable in $W_2^m(\Omega)$, $1 < m \leq 3$. We approximate the problem (7.1) by the difference scheme

$$\boldsymbol{A}\boldsymbol{y} = \sum_{i,j=1}^{2} \boldsymbol{A}_{ij} \boldsymbol{y} = \boldsymbol{\varphi}, \quad x \in \omega \cup \gamma_{-1}, \quad \boldsymbol{y} = 0, \quad x \in \gamma_0. \tag{7.2}$$

where

$$\boldsymbol{A}_{11}\boldsymbol{y} = -\begin{cases} \dfrac{1}{2}\,(K_{11}\boldsymbol{y}_{x_1})_{\overline{x}_1} + \dfrac{1}{2}\,(K_{11}\boldsymbol{y}_{\overline{x}_1})_{x_1}, & x \in \omega, \\ \dfrac{2}{h_1}\, \dfrac{K_{11}(0, x_2) + K_{11}(h_1, x_2)}{2}\, \boldsymbol{y}_{x_1}(0, x_2), & x \in \gamma_{-1}, \end{cases}$$

$$\boldsymbol{A}_{12}\boldsymbol{y} = -\begin{cases} \dfrac{1}{2}\,(K_{12}\boldsymbol{y}_{x_2})_{\overline{x}_1} + \dfrac{1}{2}\,(K_{12}\boldsymbol{y}_{\overline{x}_2})_{x_1}, & x \in \omega, \\ \dfrac{K_{12}(0, x_2)}{h_1}\, \boldsymbol{y}_{x_2}(0, x_2) + \dfrac{K_{12}(h_1, x_2)}{h_1}\, \boldsymbol{y}_{\overline{x}_2}(h_1, x_2), & x \in \gamma_{-1}, \end{cases}$$

$$\boldsymbol{A}_{21}\boldsymbol{y} = -\begin{cases} \dfrac{1}{2}\,(K_{21}\boldsymbol{y}_{x_1})_{\overline{x}_2} + \dfrac{1}{2}\,(K_{21}\boldsymbol{y}_{\overline{x}_1})_{x_2}, & x \in \omega, \\ (K_{21}\boldsymbol{y}_{x_1})_{\overline{x}_2}, & x \in \gamma_{-1}, \end{cases}$$

$$\boldsymbol{A}_{22}\boldsymbol{y} = -\begin{cases} \dfrac{1}{2}\,(K_{22}\boldsymbol{y}_{x_2})_{\overline{x}_2} + \dfrac{1}{2}\,(K_{22}\boldsymbol{y}_{\overline{x}_2})_{x_2}, & x \in \omega, \\ (K_{22}\boldsymbol{y}_{x_2})_{\overline{x}_2}, & x \in \gamma_{-1}, \end{cases}$$

$$\boldsymbol{\varphi}(x)=\begin{cases}T_1T_2\boldsymbol{f}, & x\in\omega,\\ T_1^{+}T_2\boldsymbol{f}+\dfrac{2}{h_1}\,T_2\boldsymbol{g}, & x\in\gamma_{-1},\end{cases}$$

$$\delta(\gamma_{-1})=\begin{cases}\dfrac{2}{h_1}, & x\in\gamma_{-1},\\ 0, & x\notin\gamma_{-1}.\end{cases}$$

Let H be the space of mesh functions defined in Section 5.

In the space $\boldsymbol{H}=H\times H$ of two-dimensional mesh vector-functions we define the inner product and the norm $(\boldsymbol{y},\boldsymbol{v})=(y^1,v^1)+(y^2,v^2)$, $\|\boldsymbol{y}\|=(\boldsymbol{y},\boldsymbol{y})^{1/2}$.

Lemma 7.1. *The operator A is self-adjoint and positive definite in $\boldsymbol{H}$ for which the estimate $\|\boldsymbol{y}\|^2_{W_2^1(\omega)}\le c(\boldsymbol{A}\boldsymbol{y},\boldsymbol{y})$, $c=(1+\ell_1\ell_2/8)(2-\sqrt{2})/\mu_0$ is valid.*

Proof. Using the formulas of summation by parts, we find that for any $\boldsymbol{y},\boldsymbol{v}\in\boldsymbol{H}$:

$$(\boldsymbol{A}_{\alpha\alpha}\boldsymbol{y},\boldsymbol{v})=0.5\sum_{\omega^-}h_1h_2(\lambda+2\mu)y^\alpha_{x_\alpha}v^\alpha_{x_\alpha}+0.5\sum_{\omega^+}h_1h_2(\lambda+2\mu)y^\alpha_{\bar{x}_\alpha}v^\alpha_{\bar{x}_\alpha}+$$

$$+0.5\sum_{\omega^-}h_1h_2\mu y^\beta_{x_\alpha}v^\beta_{x_\alpha}+0.5\sum_{\omega^+}h_1h_2\mu y^\beta_{\bar{x}_\alpha}v^\beta_{\bar{x}_\alpha},\quad \beta=3-\alpha,\quad \alpha=1,2,\tag{7.3}$$

$$(\boldsymbol{A}_{\alpha\beta}\boldsymbol{y},\boldsymbol{v})=0.5\sum_{\omega}h_1h_2\lambda y^\beta_{\bar{x}_\beta}v^\alpha_{\bar{x}_\alpha}+0.5\sum_{\omega^-}h_1h_2\lambda y^\beta_{x_\beta}v^\alpha_{x_\alpha}+$$

$$+0.5\sum_{\omega}h_1h_2\mu y^\alpha_{\bar{x}_\beta}v^\beta_{\bar{x}_\alpha}+0.5\sum_{\omega^-}h_1h_2\mu y^\alpha_{x_\beta}v^\beta_{x_\alpha},\quad \beta=3-\alpha,\quad \alpha=1,2.\tag{7.4}$$

It follows from (7.3), (7.4) that $\boldsymbol{A}_{\alpha\alpha}=\boldsymbol{A}^*_{\alpha\alpha}$, $\boldsymbol{A}_{\alpha\beta}=\boldsymbol{A}^*_{\beta\alpha}$, $\beta=3-\alpha$, $\alpha=1,2$. Consequently, $\boldsymbol{A}=\boldsymbol{A}^*$ in $\boldsymbol{H}$.

Substituting $\boldsymbol{v}=\boldsymbol{y}$ in (7.3), (7.4), we obtain

$$(\boldsymbol{A}\boldsymbol{y},\boldsymbol{y})=I_1+I_2+I_3,\tag{7.5}$$

where $I_1=\big(\mu,(y^1_{\bar{x}_1})^2+(y^2_{\bar{x}_2})^2\big)_{\omega^+}+\big(\mu,(y^1_{x_1})^2+(y^2_{x_2})^2\big)_{\omega^-}$, $I_2=0.5\big(\lambda,(y^1_{\bar{x}_1}+y^2_{\bar{x}_2})^2\big)_{\omega^+}+0.5\big(\lambda,(y^1_{x_1}+y^2_{x_2})^2\big)_{\omega^-}\ge 0$, $I_3=0.5\big(\mu,(y^1_{\bar{x}_2}+y^2_{\bar{x}_1})^2\big)_{\omega^+}+0.5\big(\lambda,(y^1_{x_2}+y^2_{x_1})^2\big)_{\omega^-}\ge 0.5\mu_0\Big(\big((y^1_{\bar{x}_2}+y^2_{\bar{x}_1})^2,1\big)_{\omega^+}+\big((y^1_{x_2}+y^2_{x_1})^2,1\big)_{\omega^-}\Big)$.

Obviously,

$$I_1\ge 2\mu_0\big(\|y^1_{\bar{x}_1}\|^2_{(1+)}+\|y^2_{\bar{x}_2}\|^2_{(2+)}\big),\quad I_2\ge 0,\tag{7.6}$$

$$I_3\ge\mu_0\big(\|y^1_{\bar{x}_2}\|^2_{(2+)}+\|y^2_{\bar{x}_1}\|^2_{(1+)}-\|y^1_{\bar{x}_1}\|^2_{(1+)}-\|y^2_{\bar{x}_2}\|^2_{(2+)}\big)+\mu_0\widetilde{I}_3,\tag{7.7}$$

$$\widetilde{I}_3=\sum_{\omega_2^+}h_2\big(y^1(0,x_2)+y^1(0,x_2-h_2)\big)y^2_{\bar{x}_2}(0,x_2)$$

since

$$(y^1_{\bar{x}_2},y^2_{\bar{x}_1})_{\omega^+}=(y^1_{x_1},y^2_{x_2})_{\omega^-}+\sum_{\omega_2^+}h_2y^1(0,x_2-h_2)y^2_{\bar{x}_2}(0,x_2),$$

$$(y^1_{x_2}, y^2_{x_1})_{\omega^-} = (y^1_{\overline{x}_1}, y^2_{\overline{x}_2})_{\omega^+} + \sum_{\omega_2^+} h_2 y^1(0,x_2) y^2_{\overline{x}_2}(0,x_2).$$

Observing the proof of Theorem 6 in [70] (p. 342), we can conclude that for any $\boldsymbol{y} \in \boldsymbol{H}$ the estimate

$$\widetilde{I}_3 \le 2\big(\varepsilon\|y^1_{\overline{x}_2}\|^2_{(2+)} + (1/\varepsilon)\|y^1_{\overline{x}_1}\|^2_{(1+)}\big)^{1/2}\big((1/\varepsilon)\|y^2_{\overline{x}_2}\|^2_{(2+)} + \varepsilon\|y^2_{\overline{x}_1}\|^2_{(1+)}\big)^{1/2} \quad (7.8)$$

is valid.

Taking into account (7.8) from (7.7) it follows that

$$I_3 \ge (1-\varepsilon)\mu_0\big(\|y^1_{\overline{x}_2}\|^2_{(2+)} + \|y^2_{\overline{x}_1}\|^2_{(1+)}\big) - \Big(1+\frac{1}{\varepsilon}\Big)\mu_0\big(\|y^1_{\overline{x}_1}\|^2_{(1+)} + \|y^2_{\overline{x}_2}\|^2_{(2+)}\big). \quad (7.9)$$

Since $I_3 \ge 0$, we have $I_3 \ge tI_3$ for any $t \in [0,1]$, and from (7.5), (7.6), (7.9) we find that

$$(\boldsymbol{Ay}, \boldsymbol{y}) \ge \mu_0\Big(2 - t\Big(1 + \frac{1}{\varepsilon}\Big)\Big)\big(\|y^1_{\overline{x}_1}\|^2_{(1+)} + \|y^2_{\overline{x}_2}\|^2_{(2+)}\big) +$$
$$+ \mu_0 t(1-\varepsilon)\big(\|y^1_{\overline{x}_2}\|^2_{(2+)} + \|y^2_{\overline{x}_1}\|^2_{(1+)}\big).$$

Choosing $t = 2\varepsilon/(2\varepsilon - \varepsilon^2 + 1)$, $\varepsilon = \sqrt{2} - 1$, we obtain the estimate $(\boldsymbol{Ay}, \boldsymbol{y}) \ge \frac{2-\sqrt{2}}{2}\mu_0\|\nabla \boldsymbol{y}\|^2$ which together with the Friedrichs inequality $\|\boldsymbol{y}\| \le \sqrt{\frac{\ell_1\ell_2}{8}}\|\nabla \boldsymbol{y}\|$ completes the proof of the lemma. □

On the basis of Lemma 7.1 we can conclude that the difference scheme (7.2) is uniquely solvable.

Let $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{u}$ be the error of the method. Substituting $\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{u}$ in (7.2), we obtain for $\boldsymbol{z}$ the problem

$$\boldsymbol{Az} = \boldsymbol{\psi}, \quad x \in \omega \cup \gamma_{-1}, \quad \boldsymbol{z} \in \boldsymbol{H}, \quad (7.10)$$

where $\boldsymbol{\psi} = \boldsymbol{\varphi} - \boldsymbol{Au}$ is the approximation error. By analogy with Section 5, it can be represented as follows:

$$\boldsymbol{\psi} = \begin{cases} \boldsymbol{\eta}_{11x_1} + \boldsymbol{\eta}_{12x_1} + \boldsymbol{\eta}_{21x_2} + \boldsymbol{\eta}_{22x_2}, & x \in \omega, \\ \dfrac{2}{h_1}\big((\boldsymbol{\eta}_{11} + \boldsymbol{\eta}_{12})^{(+1_1)} + (\overline{\boldsymbol{\eta}}_{21} + \overline{\boldsymbol{\eta}}_{22})_{\overline{x}_2}\big), & x \in \gamma_{-1}, \end{cases}$$

where the components of the vectors $\boldsymbol{\eta}_{\alpha\beta} = (\eta^1_{\alpha\beta}, \eta^2_{\alpha\beta})$ and $\overline{\boldsymbol{\eta}}_{2\beta} = (\overline{\eta}^1_{2\beta}, \overline{\eta}^2_{2\beta})$ are defined by the equalities $\eta^1_{11} = \ell_{11}(\lambda + 2\mu, u^1)$, $\eta^2_{11} = \ell_{11}(\mu, u^2)$, $\overline{\eta}^1_{21} = \overline{\ell}_1(\mu, u^2)$, $\eta^1_{12} = \ell_{12}(\lambda, u^2)$, $\eta^2_{12} = \ell_{12}(\mu, u^1)$, $\overline{\eta}^2_{21} = \overline{\ell}_1(\lambda, u^1)$, $\eta^1_{21} = \ell_{21}(\mu, u^2)$, $\eta^2_{21} = \ell_{21}(\lambda, u^1)$, $\overline{\eta}^1_{22} = \overline{\ell}_2(\mu, u^1)$, $\eta^1_{22} = \ell_{22}(\mu, u^1)$, $\eta^2_{22} = \ell_{22}(\lambda + 2\mu, u^2)$, $\overline{\eta}^2_{22} = \overline{\ell}_2(\lambda + 2\mu, u^2)$, $\ell_{\alpha,\beta}(a, v) = 0.5\big(av_{\overline{x}_\beta} + (av_{x_\beta})^{(-1_\alpha)}\big) - S^-_\alpha T_{3-\alpha}(aD_\beta v)$, $\overline{\ell}_\beta(a, v) = \frac{h_1}{2}\big(av_{x_\beta} - T^+_1 S^+_2(aD_\beta v)\big)$.

We multiply both parts of (7.5) scalarly by $\boldsymbol{z}$ and use the estimate (7.4). Next, for the summands involving $\boldsymbol{\eta}_{\alpha\beta x_\alpha}$ we use the summation by parts and the Cauchy inequality and for those involving $\overline{\boldsymbol{\eta}}_{2\beta\overline{x}_2}$ we apply Lemma 5.2

and the Cauchy inequality. Thus we see that the following a priori estimate

$$\|\boldsymbol{z}\|_{W_2^1(\omega)} \leq c \sum_{\alpha,\beta=1}^{2} \big(\|\boldsymbol{\eta}_{\alpha\beta}\|_{\omega\cup\gamma_{+\alpha}} + \|\overline{\boldsymbol{\eta}}_{2\beta\overline{x}_{3-\alpha}}\|_{\omega\cup\gamma_{-\alpha}}\big) \tag{7.11}$$

is valid.

To obtain an estimate of the convergence rate, it is sufficient to estimate the norms of error functionals appearing in (7.6). It should only be noted that the estimates of $\ell_{\alpha,\beta}(a,v)$, $\overline{\ell}_\beta(a,v)$ differ in no way from those of $\eta_{\alpha\beta}$, $\overline{\eta}_{2\beta}$ from Section 5.

As a result, we arrive to the following statement.

Theorem 7.1. *The difference scheme* (7.2) *converges in the mesh norm* $W_2^1(\omega)$*, and the estimate*

$$\|\boldsymbol{y}-\boldsymbol{u}\|_{W_2^1(\omega)} \leq c|h|^{m-1}\,\|\boldsymbol{u}\|_{W_2^s(\Omega)}, \quad 1<m\leq 3, \tag{7.12}$$

is valid with a constant $c>0$ *independent of* h *and* $\boldsymbol{u}$*.*

8. The Third Boundary Value Problem of the Theory of Elasticity (The Case of Constant Coefficients)

History of the matter. The aim of the present section is to investigate difference schemes approximating the third boundary value problem of statical theory of elasticity (the problem on rigid contact) in a rectangle. The results of this section have been published in [17].

Difference schemes for the above-mentioned problem were considered in [5] and [60]. In [5] the convergence in the mesh norm W_2^1 is proved with the rate $O(|h|^2)$ to the exact solution from the class $C^4(\overline{\Omega})$. In [60], a consistent estimate with $s=0$ and $m=1,2$ is obtained for the difference scheme which is constructed upon introduction of different meshes for components of an unknown vector-function. Note that in this case the original problem should be continued outside of the domain Ω.

1^0. Statement of the problem. In the rectangle $\overline{\Omega}$ we consider the boundary value problem

$$\sum_{\alpha=1}^{2}\Big(\lambda\,\frac{\partial^2 u^\alpha}{\partial x_\alpha\partial x_\beta}+\mu\,\frac{\partial}{\partial x_\alpha}\Big(\frac{\partial u^\alpha}{\partial x_\beta}+\frac{\partial u^\beta}{\partial x_\alpha}\Big)\Big)+f^\beta(x)=0, \quad x\in\Omega, \quad \beta=1,2, \tag{8.1}$$

$$u^\alpha(x)=0, \quad \frac{\partial u^{3-\alpha}(x)}{\partial x_\alpha}=0, \quad x\in\Gamma, \quad x_\alpha=0,\ell_\alpha, \quad \alpha=1,2. \tag{8.2}$$

Here $\lambda,\mu=\text{const}$, $\mu>0$, $\lambda\geq-\mu$ are the Lamé coefficients, $\boldsymbol{u}=(u^1,u^2)$ is an unknown displacement vector, and $\boldsymbol{f}=(f^1,f^2)$ is the given vector.

As is mentioned in [60], if $f^\alpha(x)=f_1^\alpha(x)+\partial f_2^\alpha(x)/\partial x_\alpha$, $f_k^\alpha(x)\in L_2(\Omega)$, $\alpha,k=1,2$, then in $W_2^1(\Omega)$ there exists a unique solution of the problem (8.1), (8.2); while if $f_2^\alpha(x)\equiv 0$, $\alpha=1,2$, then this solution belongs to the space $W_2^2(\Omega)$. We approximate the problem (8.1), (8.2) by the difference scheme

$$
\begin{gathered}
(\lambda+2\mu)\Lambda_{11}y^1+(\lambda+\mu)\Lambda_{12}y^2+\mu\Lambda_{22}y^1+\varphi^1=0,\ \ x\in\overline{\omega}\setminus\gamma_1,\\
\mu\Lambda_{11}y^2+(\lambda+\mu)\Lambda_{12}y^1+(\lambda+2\mu)\Lambda_{22}y^2+\varphi^2=0,\ \ x\in\overline{\omega}\setminus\gamma_2,\\
y^\alpha(x)=0,\ \ x\in\gamma_\alpha,\ \ \varphi^\alpha(x)=T_1T_2f^\alpha(x),\ \ \alpha=1,2,
\end{gathered}
\tag{8.3}
$$

where

$$
\Lambda_{\alpha\alpha}v=\begin{cases}\dfrac{2}{h_\alpha}\,v_{x_\alpha}, & x\in\gamma_{-\alpha},\\ v_{\overline{x}_\alpha x_\alpha}, & x\in\overline{\omega}\setminus\gamma_\alpha,\\ -\frac{2}{h_\alpha}\,v_{\overline{x}_\alpha}, & x\in\gamma_{+\alpha},\end{cases}\qquad
\Lambda_{12}v=\begin{cases}v_{x_\alpha\overset{\circ}{x}_{3-\alpha}}, & x\in\gamma_{-\alpha},\\ v_{\overset{\circ}{x}_1\overset{\circ}{x}_2}, & x\in\omega;\\ v_{\overline{x}_\alpha\overset{\circ}{x}_{3-\alpha}}, & x\in\gamma_{+\alpha}.\end{cases}
$$

2^0. Solvability of the scheme. By $\overset{\circ}{H}_h$ we denote the set of two-dimensional mesh vector-functions whose components are defined on $x\in\gamma_\alpha$, $\alpha=1,2$, and vanish on $\overline{\omega}$, respectively. Let H_h be the set of two-dimensional mesh vector-functions whose components are defined on the meshes $\overline{\omega}\setminus\gamma_\alpha$, $\alpha=1,2$, respectively.

We write the difference scheme (8.3) in the operator form

$$
A\boldsymbol{y}=\boldsymbol{\varphi},\ \ \boldsymbol{y}\in\overset{\circ}{H}_h,\ \ \boldsymbol{\varphi}\in H_h,\tag{8.4}
$$

where $A=-\begin{pmatrix}(\lambda+2\mu)\Lambda_{11}+\mu\Lambda_{22} & (\lambda+\mu)\Lambda_{12}\\ (\lambda+\mu)\Lambda_{12} & \mu\Lambda_{11}+(\lambda+2\mu)\Lambda_{22}\end{pmatrix}$, $\boldsymbol{y}=(y^1,y^2)^T$, $\boldsymbol{\varphi}=(\varphi^1,\varphi^2)^T$.

Define in $\overset{\circ}{H}_h$ the inner product and the norms:

$$
\begin{aligned}
(\boldsymbol{y},\boldsymbol{v})&=(y^1,v^1)_{(1)}+(y^2,v^2)_{(2)},\quad \|\boldsymbol{y}\|=(\boldsymbol{y},\boldsymbol{y})^{1/2},\\
\|\boldsymbol{y}\|^2_{W_2^1(\omega)}&=\|\nabla\boldsymbol{y}\|^2=\|\nabla y^1\|^2+\|\nabla y^2\|^2,\\
\|\nabla y^1\|^2&=\sum_{\omega_1^+\times\overline{\omega}_2}h_1\hbar_2(y^1_{\overline{x}_1})^2+\sum_{\omega_1\times\omega_2^+}h_1h_2(y^1_{\overline{x}_2})^2,\\
\|\nabla y^2\|^2&=\sum_{\overline{\omega}_1\times\omega_2^+}\hbar_1h_2(y^2_{\overline{x}_2})^2+\sum_{\omega_1^+\times\omega_2}h_1h_2(y^2_{\overline{x}_1})^2,\\
\|\boldsymbol{y}\|^2_{W_2^2(\omega)}&=\sum_{\alpha=1}^2\big(\|\Lambda_{11}y^\alpha\|^2_{(\alpha)}+\|\Lambda_{22}y^\alpha\|^2_{(\alpha)}+2\|y^\alpha_{\overline{x}_1\overline{x}_2}\|^2_{\omega^+}\big).
\end{aligned}
$$

Let us show the basic properties of the operator A.

Lemma 8.1. *The operator $A:\overset{\circ}{H}_h\to H_h$ is self-adjoint, positive definite and the estimates*

$$
(A\boldsymbol{y},\boldsymbol{y})\geq c_1\|\boldsymbol{y}\|^2,\ \ c_1=\frac{8\mu}{\ell_1^2+\ell_2^2},\tag{8.5}
$$

$$
(A\boldsymbol{y},\boldsymbol{y})\geq\mu\|\nabla\boldsymbol{y}\|^2,\tag{8.6}
$$

$$\|A\boldsymbol{y}\| \geq \mu\|\boldsymbol{y}\|_{W_2^2(\omega)} \tag{8.7}$$

are valid.

Proof. The self-conjugacy follows from the identity

$$(A\boldsymbol{y}, \boldsymbol{v}) =$$

$$= \frac{\lambda+\mu}{4}\Big(\big(y^1_{\overline{x}_1}+y^2_{\overline{x}_2}, v^1_{\overline{x}_1}+v^2_{\overline{x}_2}\big)_{\omega_1^+\times\omega_2^+} + \big(y^1_{\overline{x}_1}+y^2_{x_2}, v^1_{\overline{x}_1}+v^2_{x_2}\big)_{\omega_1^+\times{}^+\omega_2} +$$
$$+\big(y^1_{x_1}+y^2_{\overline{x}_2}, v^1_{x_1}+v^2_{\overline{x}_2}\big)_{{}^+\omega_1\times\omega_2^+} + \big(y^1_{x_1}+y^2_{x_2}, v^1_{x_1}+v^2_{x_2}\big)_{{}^+\omega_1\times{}^+\omega_2}\Big) +$$
$$+\mu\Big(\sum_{\omega_1^+\times\overline{\omega}_2} h_1\hbar_2 y^1_{\overline{x}_1}v^1_{\overline{x}_1} + \sum_{\overline{\omega}_1\times\omega_2^+} \hbar_1 h_2 y^2_{\overline{x}_2}v^2_{\overline{x}_2} +$$
$$+\sum_{\omega_1\times\omega_2^+} h_1h_2 y^1_{\overline{x}_2}v^1_{\overline{x}_2} + \sum_{\omega_1^+\times\omega_2} h_1h_2 y^2_{\overline{x}_1}v^2_{\overline{x}_1}\Big),$$

which is verified by the summation by parts. From the above identity it follows that the estimate (8.6) is valid.

Since $y^\alpha(x)$ vanishes for $x_\alpha = 0, \ell_\alpha$, therefore ([69], p. 55) $\sum\limits_{\omega_\alpha^+} h_\alpha(y^\alpha_{\overline{x}_\alpha})^2 \geq \frac{8}{\ell_\alpha^2}\sum_{\omega_\alpha} h_\alpha(y^\alpha)^2$, $\alpha = 1,2$.

Consequently, $\sum\limits_{\omega_1^+\times\overline{\omega}_2} h_1\hbar_2(y^1_{\overline{x}_1})^2 + \sum\limits_{\overline{\omega}_1\times\omega_2^+} \hbar_1 h_2(y^2_{\overline{x}_2})^2 \geq \frac{8}{\ell_1^2+\ell_2^2}\|\boldsymbol{y}\|^2$, which together with (8.6) proves the estimate (8.5).

Let $\overset{\circ}{A} = -\begin{pmatrix}\Lambda_{11}+\Lambda_{22} & 0\\ 0 & \Lambda_{11}+\Lambda_{22}\end{pmatrix}$.

Then

$$(A\boldsymbol{y}, \overset{\circ}{A}\boldsymbol{y}) = \mu\|\overset{\circ}{A}\boldsymbol{y}\|^2 + (\lambda+\mu)(I_1+I_2), \tag{8.8}$$

where $I_\beta = \sum\limits_{\alpha=1}^{2}\big(\Lambda_{\beta\beta}y^\alpha, \Lambda_{\alpha\alpha}y^\alpha + \Lambda_{12}y^{3-\alpha}\big)_{(\alpha)}$, $\beta = 1,2$.

Let us show that $I_\alpha \geq 0$, $\alpha = 1,2$. Indeed, using the formulas of summation by parts, we obtain

$$4(\Lambda_{11}y^1, \Lambda_{12}y^2)_{(1)} = 4(\Lambda_{11}y^2, \Lambda_{12}y^1)_{(2)} =$$
$$= \sum_{x\in\omega_1\times\omega_2^+} h_1h_2\big(y^1_{\overline{x}_1x_1}y^2_{\overline{x}_1\overline{x}_2} + y^1_{\overline{x}_1x_1}y^2_{x_1\overline{x}_2}\big) +$$
$$+ \sum_{x\in\omega_1\times{}^+\omega_2} h_1h_2\big(y^1_{\overline{x}_1x_1}y^2_{x_1x_2} + y^1_{\overline{x}_1x_1}y^2_{\overline{x}_1x_2}\big),$$
$$4(\Lambda_{22}y^2, \Lambda_{12}y^1)_{(2)} = 4(\Lambda_{22}y^1, \Lambda_{12}y^2)_{(1)} =$$
$$= \sum_{x\in\omega_1^+\times\omega_2} h_1h_2\big(y^2_{\overline{x}_2x_2}y^1_{\overline{x}_1\overline{x}_2} + y^2_{\overline{x}_2x_2}y^1_{\overline{x}_1x_2}\big) +$$
$$+ \sum_{x\in{}^+\omega_1\times\omega_2} h_1h_2\big(y^2_{\overline{x}_2x_2}y^1_{x_1x_2} + y^2_{\overline{x}_2x_2}y^1_{x_1\overline{x}_2}\big),$$

$$(\Lambda_{11}y^\alpha, \Lambda_{22}y^\alpha)_{(\alpha)} = \|y^\alpha_{\overline{x}_1\overline{x}_2}\|^2_{\omega^+}, \quad \alpha = 1, 2.$$

Consequently, we can reduce $4I_1$ to the form

$$4I_1 = \sum_{x\in\omega_1\times\omega_2^+} h_1h_2\big(y^1_{\overline{x}_1x_1} + y^2_{\overline{x}_1\overline{x}_2}\big)^2 + \sum_{x\in\omega_1\times^+\omega_2} h_1h_2\big(y^1_{\overline{x}_1x_1} + y^2_{\overline{x}_1x_2}\big)^2 +$$
$$+ \sum_{x\in\omega_1\times\omega_2^+} h_1h_2\big(y^1_{\overline{x}_1x_1} + y^2_{x_1\overline{x}_2}\big)^2 + \sum_{x\in\omega_1\times^+\omega_2} h_1h_2\big(y^1_{\overline{x}_1x_1} + y^2_{x_1x_2}\big)^2 +$$
$$+ 2\sum_{x_2\in\omega_2^+} h_1h_2\left(\big(y^2_{x_1\overline{x}_2}(0,x_2)\big)^2 + \big(y^2_{x_1\overline{x}_2}(\ell_1,x_2)\big)^2\right),$$

whence $I_1 \geq 0$. Analogously, $I_2 \geq 0$, and from (8.8) we obtain

$$(A\boldsymbol{y}, \overset{\circ}{A}\boldsymbol{y}) \geq \mu\|\overset{\circ}{A}\boldsymbol{y}\|^2. \tag{8.9}$$

Further, we have $\|\overset{\circ}{A}\boldsymbol{y}\|^2 = \sum_{\alpha=1}^{2}\big((\Lambda_{11}y^\alpha + \Lambda_{22}y^\alpha)^2, 1\big)_{(\alpha)} = \|\boldsymbol{y}\|^2_{W_2^2(\omega)}$. Therefore it follows from (8.9) that the estimate (8.7) is valid. □

Due to the fact that the operator A is positive definite, it has the bounded inverse A^{-1}, hence the solution of the equation (8.4) (or of the difference scheme (8.3)) exists and is unique.

3^0. A priori error estimates. For the error vector $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{u}$ we have the problem

$$A\boldsymbol{z} = \boldsymbol{\psi}, \quad \boldsymbol{z} \in \overset{\circ}{H}_h, \quad \boldsymbol{\psi} = \boldsymbol{\varphi} - A\boldsymbol{u}. \tag{8.10}$$

If we denote $\eta^\beta_{12} = S_1^- S_2^- u^\beta - \frac{1}{4}\left(u^\beta + u^{\beta(-1_1)} + u^{\beta(-1_2)} + u^{\beta(-1_1,-1_2)}\right)$, $x \in \omega^+$, $\eta^\beta_{\alpha\alpha} = T_{3-\alpha}u^\beta - u^\beta$, $x \in \overline{\omega} \setminus \gamma_\beta$, $\alpha, \beta = 1, 2$, and

$$B_{12}y = \begin{cases} \dfrac{2}{h_\alpha}\, y^{(+1_\alpha)}_{x_{3-\alpha}}, & x \in \gamma_{-\alpha}, \\ y_{x_1x_2}, & x \in \omega, \qquad \alpha = 1, 2, \\ -\dfrac{2}{h_\alpha}\, y_{x_{3-\alpha}}, & x \in \gamma_{+\alpha} \end{cases}$$

then the approximation error $\boldsymbol{\psi}$ can be represented as

$$\boldsymbol{\psi} = (\lambda + 2\mu)\begin{pmatrix}\Lambda_{11} & 0 \\ 0 & \Lambda_{22}\end{pmatrix}\begin{pmatrix}\eta^1_{11} \\ \eta^2_{22}\end{pmatrix} + \mu\begin{pmatrix}\Lambda_{22} & 0 \\ 0 & \Lambda_{11}\end{pmatrix}\begin{pmatrix}\eta^1_{22} \\ \eta^2_{11}\end{pmatrix} +$$
$$+ (\lambda + \mu)\begin{pmatrix}0 & B_{12} \\ B_{12} & 0\end{pmatrix}\begin{pmatrix}\eta^1_{12} \\ \eta^2_{12}\end{pmatrix}.$$

Lemma 8.2. *For the solution of the difference problem* (8.10) *the a priori estimates*

$$\|\boldsymbol{z}\| \leq c_2\sum_{\alpha=1}^{2}\Big(\sum_{\beta=1}^{2}\|\eta^\alpha_{\beta\beta}\|_{(\alpha)} + \|\eta^\alpha_{12}\|_{\omega^+}\Big), \tag{8.11}$$

$$\|\nabla\boldsymbol{z}\| \leq c_2\Big(\|\eta^1_{11\overline{x}_1}\|_{\omega_1^+\times\overline{\omega}_2} + \|\eta^2_{22\overline{x}_2}\|_{\overline{\omega}_1\times\omega_2^+} + \|\eta^1_{22\overline{x}_2}\|_{\omega_1\times\omega_2^+} +$$

$$+\|\eta^2_{11\overline{x}_1}\|_{\omega_1^+\times\omega_2}+\|\eta^2_{12x_1}\|_{\omega_1\times\omega_2^+}+\|\eta^1_{12x_2}\|_{\omega_1^+\times\omega_2}), \qquad (8.12)$$

$$\|\boldsymbol{z}\|_{W_2^2(\omega)}\le c_2\sum_{\alpha=1}^{2}\Big(\sum_{\beta=1}^{2}\|\Lambda_{\beta\beta}\eta^\alpha_{\beta\beta}\|_{(\alpha)}+\|B_{12}\eta^\alpha_{12}\|_{(3-\alpha)}\Big) \qquad (8.13)$$

are valid, where $c_2=2+|\lambda|/\mu,$.

Proof. Sing the obtained from (8.7) inequalities $\|A\boldsymbol{v}\|\ge\mu\|\Lambda_{jj}v^i\|_{(i)}$, $i,j=1,2$, we obtain

$$\left\|A^{-1}\begin{pmatrix}\Lambda_{\alpha\alpha} & 0\\ 0 & \Lambda_{\beta\beta}\end{pmatrix}\begin{pmatrix}\eta^1_{\alpha\alpha}\\ \eta^2_{\beta\beta}\end{pmatrix}\right\|=\sup_{\|\boldsymbol{v}\|\ne0}\frac{\big|(\Lambda_{\alpha\alpha}\eta^1_{\alpha\alpha},v^1)_{(1)}+(\Lambda_{\beta\beta}\eta^2_{\beta\beta},v^2)_{(2)}\big|}{\|A\boldsymbol{v}\|}\le$$

$$\le\sup_{\|\boldsymbol{v}\|\ne0}\frac{\|\eta^1_{\alpha\alpha}\|_{(1)}\|\Lambda_{\alpha\alpha}v^1\|_{(1)}+\|\eta^2_{\beta\beta}\|_{(2)}\|\Lambda_{\beta\beta}v^2\|_{(2)}}{\|A\boldsymbol{v}\|}\le$$

$$\le\frac{1}{\mu}\big(\|\eta^1_{\alpha\alpha}\|_{(1)}+\|\eta^2_{\beta\beta}\|_{(2)}\big),\quad \beta=3-\alpha,\quad \alpha=1,2. \qquad (8.14)$$

Analogously, taking into account the equalities $(B_{12}\eta^\alpha_{12},v^\beta)_{(\beta)}=(\eta^\alpha_{12},v^\beta_{\overline{x}_1\overline{x}_2})_{\omega^+}$, $\beta=3-\alpha$, $\alpha=1,2$, and the obtained from (8.7) inequalities $\|A\boldsymbol{v}\|\ge\mu\|v^i\|_{\omega^+}$, $i=1,2$, we get

$$\left\|A^{-1}\begin{pmatrix}0 & B_{12}\\ B_{12} & 0\end{pmatrix}\begin{pmatrix}\eta^1_{12}\\ \eta^2_{12}\end{pmatrix}\right\|=\sup_{\|\boldsymbol{v}\|\ne0}\frac{\big|(B_{12}\eta^2_{12},v^1)_{(1)}+(B_{12}\eta^1_{12},v^2)_{(2)}\big|}{\|A\boldsymbol{v}\|}\le$$

$$\le\sup_{\|\boldsymbol{v}\|\ne0}\frac{\|\eta^2_{12}\|_{\omega^+}\|v^1_{\overline{x}_1\overline{x}_2}\|_{\omega^+}+\|\eta^1_{12}\|_{\omega^+}\|v^2_{\overline{x}_1\overline{x}_2}\|_{\omega^+}}{\|A\boldsymbol{v}\|}\le$$

$$\le\frac{1}{\mu}\big(\|\eta^1_{12}\|_{\omega^+}+\|\eta^2_{12}\|_{\omega^+}\big). \qquad (8.15)$$

Finally, by virtue of (8.14),(8.15), from (8.10) it follows the inequality (8.11).

To obtain the inequality (8.12), we have to multiply scalarly both parts of (8.10) by $\boldsymbol{z}$ and make use of the estimate (8.6) and the formulas

$$(\Lambda_{11}\eta^1_{11},z^1)_{(1)}=-\sum_{\omega_1^+\times\overline{\omega}_2}h_1\hbar_2\eta^1_{11\overline{x}_1}z^1_{\overline{x}_1},\quad (\Lambda_{11}\eta^2_{11},z^2)_{(2)}=-\sum_{\omega_1^+\times\omega_2}h_1h_2\eta^2_{11\overline{x}_1}z^2_{\overline{x}_1},$$

$$(\Lambda_{22}\eta^1_{22},z^1)_{(1)}=-\sum_{\omega_1\times\omega_2^+}h_1h_2\eta^1_{22\overline{x}_2}z^1_{\overline{x}_2},\quad (\Lambda_{22}\eta^2_{22},z^2)_{(2)}=-\sum_{\overline{\omega}_1\times\omega_2^+}\hbar_1h_2\eta^2_{22\overline{x}_2}z^2_{\overline{x}_2},$$

$$(B_{12}\eta^2_{12},z^1)_{(1)}=-\sum_{\omega_1\times\omega_2^+}h_1h_2\eta^2_{12x_1}z^1_{\overline{x}_2},\quad (B_{12}\eta^1_{12},z^2)_{(2)}=-\sum_{\omega_1^+\times\omega_2}h_1h_2\eta^1_{12x_2}z^2_{\overline{x}_1}.$$

The estimate (8.13) follows directly from (8.7). □

4⁰. Accuracy of the scheme. To estimate $\eta^\alpha_{\beta\beta}$, η^α_{12}, $\alpha,\beta=1,2$, and their difference ratios, we use the well-known method [71] involving the Bramble–Hilbert lemma. Here we indicate only some noteworthy points.

In order to estimate the expression $\eta_{11}^1 = T_2u^1 - u^1$ at the points $x \in \gamma_{\pm 2}$ for $\boldsymbol{u} \in W_2^m(\Omega)$, $1 < m < 1.5$, we write

$$|\eta_{11}^1|^2 \le 2\Big(T_2u^1 - u^1 \pm \frac{h_2}{3}\, S_1 S_2^{\mp} D_2 u^1\Big)^2 + 2\Big(\frac{h_2}{3}\, S_1 S_2^{\mp} D_2 u^1\Big)^2. \qquad (8.16)$$

Next, by $\widetilde{\Omega}$ we denote the domain which is obtained from the rectangle Ω by rounding its angles by the arcs of circumferences of radius $r = h_1/2$ circumscribed from the centers $O_1(r,r), O_2(\ell_1 - r, r)$, $O_3(\ell_1 - r, \ell_2 - r)$, $O_4(r, \ell_2 - r)$. Let Ω_ε be the inner strip of width $\varepsilon = h_1/4$ along the boundary of the domain $\widetilde{\Omega}$. Since the boundary of $\widetilde{\Omega}$ belongs to the class C^1, for the second summand in the right-hand side of (8.15) we can use Theorem 1.4, while for the first one we use the Bramble–Hilbert lemma. For $\boldsymbol{u} \in W_2^m(\Omega)$, $1.5 \le m \le 2$, in the boundary nodes $x \in \gamma_{\pm 2}$ we can write $\eta_{11}^1 = T_2u^1 - u^1 \pm \frac{h_2}{3}\, T_1 D_2 u^1$. Since $D_2u^1 \in W_2^{m-1.5}(\Gamma)$, the averaging $T_1D_2u^1$ makes sense and is equal to zero for every node $x \in \gamma_{\pm 2}$. In this case we can again apply the Bramble–Hilbert lemma. η_{22}^2 for $x \in \gamma_{\pm 1}$ is estimated analogously.

To estimate $\eta_{\alpha\alpha\overline{x}_\alpha}^\alpha$ at the points $x \in \gamma_{\pm(3-\alpha)}$ for $u \in W_2^m(\Omega)$, $2 < m \le 3$, we can write $\eta_{\alpha\alpha\overline{x}_\alpha}^\alpha = \big(T_{3-\alpha}u^\alpha - u^\alpha \pm \frac{h_{3-\alpha}}{3}\, D_{3-\alpha}u^\alpha\big)_{\overline{x}_\alpha}$, $\alpha = 1, 2$. To estimate $\Lambda_{\alpha\alpha}\eta_{\alpha\alpha}^\alpha$ at the points $x \in \gamma_{\pm(3-\alpha)}$, we represent it in the form $\Lambda_{\alpha\alpha}\eta_{\alpha\alpha}^\alpha = \Big(T_{3-\alpha}u^\alpha - u^\alpha \pm \frac{h_{3-\alpha}}{3}\, D_{3-\alpha}u^\alpha\Big)_{\overline{x}_\alpha x_\alpha}$, $\boldsymbol{u} \in W_2^m(\Omega)$, $2 < m \le 4$. Moreover, it should be noted that $\Lambda_{\alpha\alpha}\eta_{\alpha\alpha}^{3-\alpha} = (T_1T_2 - T_\alpha)D_\alpha^2 u^{3-\alpha}$, $x \in \omega \cup \gamma_{-\alpha} \cup \gamma_{+\alpha}$, $B_{12}\eta_{12}^\alpha = T_1T_2D_1D_2u^\alpha - (S_\alpha^{\mp} D_\alpha u^\alpha)_{\overset{\circ}{x}_{3-\alpha}}$, $x \in \gamma_{\pm\alpha}$.

It can be easily verified that $\Lambda_{\alpha\alpha}\eta_{\alpha\alpha}^\beta$ $(\alpha, \beta = 1, 2)$ vanishes on polynomials of the third degree, while $B_{12}\eta_{12}^\alpha$ for $x \in \gamma_{\pm\alpha}$ only on polynomials of the second degree. Therefore, for example, at the points $x \in \gamma_{\pm 2}$ for $\boldsymbol{u} \in W_2^m(\Omega)$, $3 < m < 3.5$, we write

$$|B_{12}\eta_{12}^2|^2 \le 2\Big(B_{12}\eta_{12}^2 \mp \frac{h_2}{6}\, S_1 S_2^{\mp} D_1 D_2^2 u^2\Big)^2 + 2\Big(\frac{h_2}{6}\, S_1 S_2^{\mp} D_1 D_2^2 u^2\Big)^2$$

and perform estimation analogously to η_{11}^1.

As a result, we arrive at the following

Theorem 8.1. *Let a solution $\boldsymbol{u}(x)$ of the problem* (8.1), (8.2) *belong to the space $W_2^m(\Omega)$, $m > 1$. Then the convergence rate of the difference scheme* (8.3) *in the mesh norm W_2^s is determined by the estimate*

$$\|\boldsymbol{y} - \boldsymbol{u}\|_{W_2^s(\omega)} \le c|h|^{m-s}\|\boldsymbol{u}\|_{W_2^m(\Omega)}, \quad s = 0, 1, 2, \quad \max(1; s) < m \le s + 2, \quad (8.17)$$

where the constant $c > 0$ is independent of h and $\boldsymbol{u}(x)$.

When $\boldsymbol{u}(x) \in W_2^1(\Omega$, we compare the mesh solution $\boldsymbol{y}(x)$ with some averaging $\overline{\boldsymbol{u}}(x) = (\overline{u}^1, \overline{u}^2)$ of the exact solution $\boldsymbol{u}(x)$ in the vicinity of the mesh nodes, for example,

$$\overline{u}^{\alpha}(x)=\begin{cases}0, & x\in\gamma_{\alpha},\\ T_{\alpha}u^{\alpha}, & x\in\gamma_{\pm(3-\alpha)},\\ T_1T_2u^{\alpha}, & x\in\omega.\end{cases}\quad \alpha=1,2.$$

In this case there takes place the following

Theorem 8.2. *Let the solution of the problem* (8.1), (8.2) *belong to the space* $W_2^1(\Omega)$. *Then the solution of the difference scheme (8.3) converges in the mesh norm* $L_2(\omega)$ *to the averaging* $\overline{\boldsymbol{u}}(x)$ *of the exact solution with the rate* $O(|h|)$, *and hence the estimate* $\|\boldsymbol{y}-\overline{\boldsymbol{u}}\|_{L_2(\omega)}\le c|h|\|\boldsymbol{u}\|_{W_2^1(\Omega)}$, *where the constant* h *is independent of* $\boldsymbol{u}(x)$, *is fulfilled.*

9. The Third Boundary Value Problem of the Theory of Elasticity (the Case of Variable Coefficients)

History of the matter. The goal of this section is to investigate difference schemes which approximate the third boundary value problem of the statical theory of elasticity (the problem on rigid contact) with variable coefficients, and also to obtain consistent estimates of the convergence rate in the mesh norm $W_2^1(\omega)$. The results of Section 9 have been published in [22], [28] and [27].

1^0. Statement of the problem. In the rectangle $\overline{\Omega}=\Omega\cup\Gamma$ we consider the boundary value problem (here and in the sequel, $\beta=3-\alpha$)

$$L_{\alpha\alpha}^{\alpha}u^{\alpha}+L_{\alpha\beta}^{\beta}u^{\beta}+L_{\beta\beta}^{\alpha}u^{\alpha}+L_{\beta\alpha}^{\beta}u^{\beta}+f^{\alpha}=0,\ \ \alpha=1,2, \tag{9.1}$$

$$u^{\alpha}(x)=0,\ \ \frac{\partial u^{\beta}(x)}{\partial x_{\alpha}}=0,\ \ x\in\Gamma,\ \ x_{\alpha}=0,\ell_{\alpha},\ \ \alpha=1,2, \tag{9.2}$$

where $\lambda(x),\mu(x)$ are the Lamé coefficients, $\boldsymbol{u}=(u^1,u^2)$ is an unknown vector of displacements, $\boldsymbol{f}=(f^1,f^2)$ is the given vector,

$$L_{\alpha\alpha}^{\alpha}u=\frac{\partial}{\partial x_{\alpha}}\Big((\lambda+2\mu)\,\frac{\partial u}{\partial x_{\alpha}}\Big),\quad L_{\beta\beta}^{\alpha}u=\frac{\partial}{\partial x_{\beta}}\Big(\mu\,\frac{\partial u}{\partial x_{\beta}}\Big),$$
$$L_{\beta\alpha}^{\beta}u=\frac{\partial}{\partial x_{\beta}}\Big(\mu\,\frac{\partial u}{\partial x_{\alpha}}\Big),\quad L_{\alpha\beta}^{\beta}u=\frac{\partial}{\partial x_{\alpha}}\Big(\lambda\,\frac{\partial u}{\partial x_{\beta}}\Big).$$

Let the Lamé coefficients λ, $\mu\in W_q^{m-1}(\Omega)$, where $q=2$ for $2<m\le3$, $q>2/(m-1)$ for $1<m\le2$, $\mu(x)\ge\mu_0=\text{const}>0$, $\lambda(x)\ge-\mu(x)$.

We can show that if $\mu(x)\ge\mu_0=\text{const}>0$, $\lambda(x)\ge-\mu(x)$, λ, $\mu\in L_{\infty}(\Omega)$, $f^{\alpha}(x)=f_1^{\alpha}(x)+\frac{\partial f_2^{\alpha}}{\partial x_{\alpha}}$, $f_k^{\alpha}(x)\in L_2(\Omega)$, $\alpha,k=1,2$, then there exists in $W_2^1(\Omega)$ a unique solution of the problem (9.1), (9.2); note that if $f_2^{\alpha}(x)\equiv0$, $\alpha=1,2$, λ, $\mu\in W_{2+\varepsilon}^1(\Omega)$, $\forall\,\varepsilon>0$, then the solution belongs to the space $W_2^2(\Omega)$. Under the imbedding $\boldsymbol{u}\in W_2^k$ it is meant that every component of the vector belongs to that space.

Denote $v^+(x)=S_1^+S_2^+v$, $v^-(x)=v^+(x_1-h_1,x_2-h_2)$.

Let $\overset{\circ}{H}_\alpha = \overset{\circ}{H}_\alpha(\overline{\omega})$ be the set of mesh functions defined on $\overline{\omega}$ and vanishing on γ_α, and $H_\alpha = H_\alpha(\overline{\omega}\setminus\gamma_\alpha)$ be the set of mesh functions defined on $\overline{\omega}\setminus\gamma_\alpha$, $\alpha = 1,2$. We introduce the spaces $\overset{\circ}{\boldsymbol{H}} = \overset{\circ}{H}_1 \times \overset{\circ}{H}_2$, and $\boldsymbol{H} = H_1 \times H_2$ of vector mesh functions and define the inner product and norms just in the same way as in Section 8.

The problem (9.1), (9.2) is approximated by the difference scheme

$$\Lambda\boldsymbol{y} + \boldsymbol{\varphi}(x) = 0, \ \ \boldsymbol{y} \in \overset{\circ}{\boldsymbol{H}}, \ \ \boldsymbol{\varphi} \in \boldsymbol{H}, \tag{9.3}$$

where

$$(\Lambda\boldsymbol{y})^\alpha = \Lambda^\alpha_{\alpha\alpha}y^\alpha + \Lambda^\beta_{\alpha\beta}y^\beta + \Lambda^\alpha_{\beta\beta}y^\alpha + \Lambda^\beta_{\beta\alpha}y^\beta, \ \ x \in \overline{\omega}\setminus\gamma_\alpha,$$

$$\varphi^\alpha(x) = T_1T_2f^\alpha(x), \ \ \alpha = 1,2,$$

$$\Lambda^\alpha_{\alpha\alpha}v = \begin{cases} 0.5\big((\lambda^+ + 2\mu^+)v_{x_\alpha}\big)_{\overline{x}_\alpha} + 0.5\big((\lambda^- + 2\mu^-)v_{\overline{x}_\alpha}\big)_{x_\alpha}, & x \in \omega, \\ \big((\lambda^+ + 2\mu^+)v_{x_\alpha}\big)_{\overline{x}_\alpha}, & x \in \gamma_{-\beta}, \\ \big((\lambda^- + 2\mu^-)v_{\overline{x}_\alpha}\big)_{x_\alpha}, & x \in \gamma_{+\beta}, \end{cases}$$

$$\Lambda^\alpha_{\beta\beta}v = \begin{cases} 0.5(\mu^+ v_{x_\beta})_{\overline{x}_\beta} + 0.5(\mu^- v_{\overline{x}_\beta})_{x_\beta}, & x \in \omega, \\ \dfrac{1}{h_\beta}\,(\mu^+ + I^+_\beta\mu^-)v_{x_\beta}, & x \in \gamma_{-\beta}, \\ -\dfrac{1}{h_\beta}\,(\mu^- + I^-_\beta\mu^+)v_{\overline{x}_\beta}, & x \in \gamma_{+\beta}, \end{cases}$$

$$\Lambda^\beta_{\alpha\beta}v = \begin{cases} 0.5(\lambda^+ v_{x_\beta})_{\overline{x}_\alpha} + 0.5(\lambda^- v_{\overline{x}_\beta})_{x_\alpha}, & x \in \omega, \\ (\lambda^+ v_{x_\beta})_{\overline{x}_\alpha}, & x \in \gamma_{-\beta}, \\ (\lambda^- v_{\overline{x}_\beta})_{x_\alpha}, & x \in \gamma_{+\beta}, \end{cases}$$

$$\Lambda^\beta_{\beta\alpha}v = \begin{cases} 0.5(\mu^+ v_{x_\alpha})_{\overline{x}_\beta} + 0.5(\mu^- v_{\overline{x}_\alpha})_{x_\beta}, & x \in \omega, \\ \dfrac{1}{h_\beta}\,I^+_\beta(\mu^- v_{\overline{x}_\alpha}), & x \in \gamma_{-\beta}, \\ -\dfrac{1}{h_\beta}\,I^-_\beta(\mu^+ v_{x_\alpha}), & x \in \gamma_{+\beta}. \end{cases}$$

2^0. The solvability of the scheme. As is seen, the domain of definition and that of values of the operator Λ ($\Lambda : \overset{\circ}{\boldsymbol{H}} \to \boldsymbol{H}$) do not coincide. We define an operator A and the vector mesh function $\boldsymbol{F}$ as follows:

$$(A\boldsymbol{y})^\alpha = \begin{cases} -(\Lambda\boldsymbol{y})^\alpha, & x \in \overline{\omega}\setminus\gamma_\alpha, \\ 0, & x \in \gamma_\alpha, \end{cases} \quad (\boldsymbol{F})^\alpha = \begin{cases} (\boldsymbol{\varphi})^\alpha, & x \in \overline{\omega}\setminus\gamma_\alpha, \\ 0, & x \in \gamma_\alpha, \end{cases} \quad \alpha = 1,2.$$

We write the difference scheme in the form of the operator equation

$$A\boldsymbol{y} = \boldsymbol{F}, \ \ \boldsymbol{y}, \ \boldsymbol{F} \in \overset{\circ}{\boldsymbol{H}}. \tag{9.4}$$

The operator A maps $\overset{\circ}{\boldsymbol{H}}$ onto $\overset{\circ}{\boldsymbol{H}}$ and is linear. Here we indicate the basic properties of the operator A.

Lemma 9.1. *The operator* $A : \overset{\circ}{\boldsymbol{H}} \to \overset{\circ}{\boldsymbol{H}}$ *is self-adjoint, positive definite and the estimates*

$$(A\boldsymbol{y}, \boldsymbol{y}) \geq c_1 \|\boldsymbol{y}\|^2, \ \ c_1 = \frac{8\mu_0}{\ell_1^2 + \ell_2^2}\,, \tag{9.5}$$

$$(A\boldsymbol{y}, \boldsymbol{y}) \geq \mu_0 \|\nabla \boldsymbol{y}\|^2 \tag{9.6}$$

are valid.

Proof. By the definition of the inner product,

$$(A\boldsymbol{y}, \boldsymbol{v}) = \sum_{\alpha=1}^{2} \sum_{x \in \overline{\omega} \setminus \gamma_\alpha} h_\alpha \hbar_\beta \big(\Lambda_{\alpha\alpha}^{\alpha} y^\alpha + \Lambda_{\alpha\beta}^{\beta} y^\beta + \Lambda_{\beta\beta}^{\alpha} y^\alpha + \Lambda_{\beta\alpha}^{\beta} y^\beta\big) v^\alpha. \tag{9.7}$$

Using the summation by parts, we can see that the equalities

$$\begin{aligned}
(\Lambda_{\alpha\alpha}^{\alpha} y^\alpha, v^\alpha)_{(\alpha)} &= -0.5\big((\lambda^- + 2\mu^-) y_{\overline{x}_\alpha}^{\alpha}, v_{\overline{x}_\alpha}^{\alpha}\big)_{\omega^+} - \\
&\quad - 0.5\big((\lambda^+ + 2\mu^+) y_{x_\alpha}^{\alpha}, v_{x_\alpha}^{\alpha}\big)_{\omega^-}, \\
(\Lambda_{\alpha\beta}^{\beta} y^\beta, v^\alpha)_{(\alpha)} &= -0.5(\lambda^- y_{\overline{x}_\beta}^{\beta}, v_{\overline{x}_\alpha}^{\alpha})_{\omega^+} - 0.5(\lambda^+ y_{x_\beta}^{\beta}, v_{x_\alpha}^{\alpha})_{\omega^-}, \\
(\Lambda_{\beta\beta}^{\alpha} y^\alpha, v^\alpha)_{(\alpha)} &= -0.5(\mu^- y_{\overline{x}_\beta}^{\alpha}, v_{\overline{x}_\beta}^{\alpha})_{\omega^+} - 0.5(\mu^+ y_{x_\beta}^{\alpha}, v_{x_\beta}^{\alpha})_{\omega^-}, \\
(\Lambda_{\beta\alpha}^{\beta} y^\beta, v^\alpha)_{(\alpha)} &= -0.5(\mu^- y_{\overline{x}_\alpha}^{\beta}, v_{\overline{x}_\beta}^{\alpha})_{\omega^+} - 0.5(\mu^+ y_{x_\alpha}^{\beta}, v_{x_\beta}^{\alpha})_{\omega^-}
\end{aligned}$$

are valid. Consequently, after some transformations, from (9.7) we have

$$\begin{gathered}
2(A\boldsymbol{y}, \boldsymbol{v}) = \\
= \big((\lambda^- + \mu^-)(y_{\overline{x}_1}^1 + y_{\overline{x}_2}^2), v_{\overline{x}_1}^1 + v_{\overline{x}_2}^2\big)_{\omega^+} + \big((\lambda^+ + \mu^+)(y_{x_1}^1 + y_{x_2}^2), v_{x_1}^1 + v_{x_2}^2\big)_{\omega^-} + \\
+ \big(\mu^-(y_{\overline{x}_1}^2 + y_{\overline{x}_2}^1), v_{\overline{x}_1}^2 + v_{\overline{x}_2}^1\big)_{\omega^+} + \big(\mu^+(y_{x_1}^2 + y_{x_2}^1), v_{x_1}^2 + v_{x_2}^1\big)_{\omega^-} + \\
+ \big(\mu^-(y_{\overline{x}_1}^1 - y_{\overline{x}_2}^2), v_{\overline{x}_1}^1 - v_{\overline{x}_2}^2\big)_{\omega^+} + \big(\mu^+(y_{x_1}^1 - y_{x_2}^2), v_{x_1}^1 - v_{x_2}^2\big)_{\omega^-}.
\end{gathered} \tag{9.8}$$

The equality (9.8) implies that the operator A is self-adjoint.

Assuming $\boldsymbol{v} = \boldsymbol{y}$ in (9.8), omitting the summands involving $(\lambda^\pm + \mu^\pm)$ and replacing $\mu^\pm$ by μ_0, we conclude that

$$\begin{gathered}
\frac{2}{\mu_0}\,(A\boldsymbol{y}, \boldsymbol{y}) \geq \big(1, (y_{\overline{x}_1}^2 + y_{\overline{x}_2}^1)^2\big)_{\omega^+} + \\
+ \big(1, (y_{x_1}^2 + y_{x_2}^1)^2\big)_{\omega^-} + \big(1, (y_{\overline{x}_1}^1 - y_{\overline{x}_2}^2)^2\big)_{\omega^+} + \big(1, (y_{x_1}^1 - y_{x_2}^2)^2\big)_{\omega^-}.
\end{gathered}$$

Taking into account the equalities $(y_{\overline{x}_1}^1, y_{\overline{x}_2}^2)_{\omega^+} = (y_{x_2}^1, y_{x_1}^2)_\omega$, $(y_{x_1}^1, y_{x_2}^2)_{\omega^-} = (y_{\overline{x}_2}^1, y_{\overline{x}_1}^2)_\omega$, we obtain

$$(A\boldsymbol{y}, \boldsymbol{y}) \geq \mu_0 \|\boldsymbol{y}\|_{W_2^1(\omega)}^2. \tag{9.9}$$

Since $y^\alpha(x)$ vanishes for $x_\alpha = 0, \ell_\alpha$, therefore ([69], p. 120)

$$\sum_{\omega_\alpha^+} h_\alpha (y_{\overline{x}_\alpha}^{\alpha})^2 \geq \frac{8}{\ell_\alpha^2} \sum_{\omega_\alpha} h_\alpha (y^\alpha)^2, \ \ \alpha = 1, 2.$$

Consequently,

$$\sum_{\omega_1^+\times\overline{\omega}_2} h_1\hbar_2(y^1_{\overline{x}_1})^2+\sum_{\overline{\omega}_1\times\omega_2^+}\hbar_1h_2(y^2_{\overline{x}_2})^2\geq\frac{8}{\ell_1^2+\ell_2^2}\,\|\boldsymbol{y}\|^2,$$

which together with (9.6) proves the estimate (9.5). □

Since the operator A is positive definite, it has its inverse A^{-1}, and hence the solution of the equation (9.4) (and of the difference scheme (9.3)) exists and is unique.

3^0. A priori estimate of error. Let $n_\beta=1$ for $x_\beta=0$, $n_\beta=-1$ for $x_\beta=\ell_\beta$, $n_\beta=0$, for $0<x_\beta<\ell_\beta$, $\beta=1,2$. First of all, we note that the following relations are valid:

$$T_1T_2L^\alpha_{\alpha\alpha}u^\alpha=\Lambda^\alpha_{\alpha\alpha}u^\alpha-(\psi^\alpha_{\alpha\alpha})_{x_\alpha}+\frac{2n_\beta}{h_\beta}\,\eta^\beta_{x_\alpha},\quad x_\alpha\in\omega_\alpha,\quad x_\beta\in\overline{\omega}_\beta,\qquad(9.10)$$

where

$$\psi^\alpha_{\alpha\alpha}=S^-_\alpha\overline{S}_\beta(2\mu+\lambda)u^\alpha_{\overline{x}_\alpha}-S^-_\alpha T_\beta\Big((2\mu+\lambda)\,\frac{\partial u^\alpha}{\partial x_\alpha}\Big)+\frac{2n_\beta}{h_\beta}\,\eta^\beta,$$

$$T_1T_2L^\beta_{\alpha\beta}u^\beta=\Lambda^\beta_{\alpha\beta}u^\beta-(\psi^\beta_{\alpha\beta})_{x_\alpha}+\frac{2n_\beta}{h_\beta}\,\chi^\beta_{x_\alpha},\quad x_\alpha\in\omega_\alpha,\quad x_\beta\in\overline{\omega}_\beta,\qquad(9.11)$$

where

$$\begin{aligned}\psi^\beta_{\alpha\beta}=\frac{1-n_\beta}{2}\,S^-_\alpha S^+_\beta\lambda I^-_\alpha u^\beta_{x_\beta}+\frac{1+n_\beta}{2}\,S^-_1S^-_2\lambda u^\beta_{\overline{x}_\beta}-\\-S^-_\alpha T_\beta\Big(\lambda\,\frac{\partial u^\beta}{\partial x_\beta}\Big)+\frac{2n_\beta}{h_\beta}\,\chi^\beta,\end{aligned}$$

$$T_1T_2L^\beta_{\beta\alpha}u^\beta=\Lambda^\beta_{\beta\alpha}u^\beta-B_\beta\psi^\beta_{\beta\alpha},\quad x_\alpha\in\omega_\alpha,\quad x_\beta\in\overline{\omega}_\beta,\qquad(9.12)$$

$$T_1T_2L^\alpha_{\beta\beta}u^\alpha=\Lambda^\alpha_{\beta\beta}u^\alpha-B_\beta\psi^\alpha_{\beta\beta},\quad x_\alpha\in\omega_\alpha,\quad x_\beta\in\overline{\omega}_\beta,\qquad(9.13)$$

where

$$\psi^\beta_{\beta\alpha}=0.5S^+_\alpha S^-_\beta\mu I^-_\beta u^\beta_{x_\alpha}+0.5S^-_1S^-_2\mu u^\beta_{\overline{x}_\alpha}-S^-_\beta T_\alpha\Big(\mu\,\frac{\partial u^\beta}{\partial x_\alpha}\Big),\ x_\alpha\in\omega_\alpha,\ x_\beta\in\omega^+_\beta,$$

$$\psi^\alpha_{\beta\beta}=\overline{S}_\alpha S^-_\beta\mu u^\alpha_{\overline{x}_\beta}-S^-_\beta T_\alpha\Big(\mu\,\frac{\partial u^\alpha}{\partial x_\beta}\Big),\quad x_\alpha\in\omega_\alpha,\quad x_\beta\in\omega^+_\beta,$$

$$B_\alpha v=\begin{cases}\dfrac{2}{h_\alpha}\,I^+_\alpha v, & x_\alpha=0,\\ v_{x_\alpha}, & x_\alpha\in\omega_\alpha,\quad \alpha=1,2.\\ -\dfrac{2}{h_\alpha}\,v, & x_\alpha=\ell_\alpha.\end{cases}$$

Depending on the smoothness of the exact solution $\boldsymbol{u} \in W_2^m(\Omega)$, we define concretely the functions η^β, χ^β appearing in (9.10) and (9.11) as follows:

$$\eta^\beta = \begin{cases} \dfrac{h_\beta^2}{12}\,\overline{S}_\beta\,\dfrac{\partial}{\partial x_\beta}\Big((\lambda+2\mu)\,\dfrac{\partial u^\alpha}{\partial x_\alpha}\Big), & m \in (2,3], \\ 0, & m \in (1,2], \end{cases}$$

$$\chi^\beta = \begin{cases} \dfrac{h_\beta^2}{12}\,\overline{S}_\beta\,\dfrac{\partial}{\partial x_\beta}\Big(\lambda\,\dfrac{\partial u^\beta}{\partial x_\beta}\Big) - \dfrac{h_1h_2}{4}\,S_\alpha\overline{S}_\beta\lambda\overline{S}_\beta\,\dfrac{\partial^2 u^\beta}{\partial x_1\partial x_2}, & m \in (2,3], \\ 0, & m \in (1,2]. \end{cases}$$

To study the question on the convergence and accuracy of the scheme (9.3), we consider the error of the method $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{u}$, where $\boldsymbol{y}$ is a solution of the problem (9.3), and $\boldsymbol{u} = \boldsymbol{u}(x)$ is a solution of the problem (9.1), (9.2).

Substituting $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{z}$ in (9.3) and taking into account the equalities (9.10)–(9.13), for the error $\boldsymbol{z}$ we obtain the problem

$$\Lambda\boldsymbol{z} + \boldsymbol{\psi} = 0, \quad \boldsymbol{z} \in \overset{\circ}{\boldsymbol{H}}, \quad \boldsymbol{\psi} \in \boldsymbol{H}, \tag{9.14}$$

where $\boldsymbol{\psi}=(\psi^1,\psi^2)$, $\psi^\alpha=(\psi^\alpha_{\alpha\alpha}+\psi^\beta_{\alpha\beta})_{x_\alpha} + B_\beta(\psi^\beta_{\beta\alpha}+\psi^\alpha_{\beta\beta}) - \frac{2n_\beta}{h_\beta}\,(\eta^\beta+\chi^\beta)_{x_\alpha}$.

The lemma below is of importance for obtaining a needed estimate for the error $\boldsymbol{z}$.

Lemma 9.2. *For any mesh function g^β defined on the mesh $\omega_\alpha^+ \times \overline{\omega}_\beta$, and for any $z^\alpha \in \overset{\circ}{H}_\alpha$, the inequalities*

$$\Big|\sum_{\gamma_{-\beta}} h_\alpha g^\alpha_{x_\alpha} z^\alpha\Big| \le J_\alpha^-(g^\beta)\|\nabla z^\alpha\|, \quad \Big|\sum_{\gamma_{+\beta}} h_\alpha g^\beta_{x_\alpha} z^\alpha\Big| \le J_\alpha^+(g^\beta)\|\nabla z^\alpha\|, \tag{9.15}$$

$$\alpha = 1,2,$$

are valid, where

$$J_\alpha^\pm(g) = \|g_{x_\alpha}\|_{\omega_\alpha\times\omega_\beta^\mp} + \|\partial_\beta^\mp g\|_{\omega_\alpha^+\times\omega_\beta} + \frac{1}{\ell_\beta}\,\|g\|_{\omega_\alpha^+\times\omega_\beta^\mp}, \quad \partial_\beta^+ g = g_{x_\beta}, \quad \partial_\beta^- g = g_{\overline{x}_\beta}.$$

Proof. Let $\rho_\beta^- = (x_\beta - \ell_\beta)/\ell_\beta$, $\rho_\beta^+ = x_\beta/\ell_\beta$. It is not difficult to verify that the equalities

$$2(z^1, g^2_{x_1})_{\gamma_{-2}} = R_1 - R_2, \quad 2(z^1, g^2_{x_1})_{\gamma_{+2}} = Q_1 - Q_2 \tag{9.16}$$

are valid, in which the expressions

$$R_1 = (\rho_2^- z^1_{x_2}, g^2_{x_1})_{\omega_1\times\omega_2^-} + (\rho_2^- z^1_{\overline{x}_2}, g^2_{x_1})_\omega,$$
$$R_2 = (z^1_{\overline{x}_1}, (\rho_2^- g^2)_{\overline{x}_2})_{\omega^+} + (z^1_{\overline{x}_1}, (\rho_2^- g^2)_{x_2})_{\omega_1^+\times\omega_2^-},$$
$$Q_1 = (\rho_2^+ z^1_{x_2}, g^2_{x_1})_\omega + (\rho_2^+ z^1_{\overline{x}_2}, g^2_{x_1})_{\omega_1\times\omega_2^+},$$
$$Q_2 = (z^1_{\overline{x}_1}, (\rho_2^+ g^2)_{\overline{x}_2})_{\omega^+} + (z^1_{\overline{x}_1}, (\rho_2^+ g^2)_{x_2})_{\omega_1^+\times\omega_2^-}$$

are estimated as follows:

$$|R_1| \le 2\|z^1_{\overline{x}_2}\|_{\omega_1\times\omega_2^+}\|g^2_{x_1}\|_{\omega_1\times\omega_2^-}, \quad |Q_1| \le 2\|z^1_{\overline{x}_2}\|_{\omega_1\times\omega_2^+}\|g^2_{x_1}\|_{\omega_1\times\omega_2^+},$$

$$|R_2| \le 2\Big(\sum_{\omega_1^+\times\overline{\omega}_2} h_1\hbar_2|z^1_{\overline{x}_1}|^2\Big)^{1/2}\|g^2_{\overline{x}_2}\|_{\omega_1^+\times\omega_2} + \frac{2}{\ell_2}\Big(\sum_{\omega_1^+\times\overline{\omega}_2} h_1\hbar_2|z^1_{\overline{x}_1}|^2\Big)^{1/2}\|g^2\|_{\omega_1^+\times\overline{\omega}_2},$$

$$|Q_2| \le 2\Big(\sum_{\omega_1^+\times\overline{\omega}_2} h_1\hbar_2|z^1_{\overline{x}_1}|^2\Big)^{1/2}\|g^2_{x_2}\|_{\omega_1^+\times\omega_2} + \frac{2}{\ell_2}\Big(\sum_{\omega_1^+\times\overline{\omega}_2} h_1\hbar_2|z^1_{\overline{x}_1}|^2\Big)^{1/2}\|g^2\|_{\omega}.$$

Therefore for $\alpha = 1$, from (9.16) follows (9.15). For $\alpha = 2$, the validity of (9.15) is proved analogously. □

Remark 9.1. The estimates (9.15) are the difference analogues of the inequalities for functions of a continuous argument

$$\Big|\int_0^{\ell_\alpha} \widetilde{z}^\alpha \frac{\partial\widetilde{g}}{\partial x_\alpha}\Big|_{x_\beta=0,\ell_\beta} dx_\alpha\Big| \le \|\nabla\widetilde{z}^\alpha\|\Big(\Big\|\frac{\partial\widetilde{g}}{\partial x_1}\Big\| + \Big\|\frac{\partial\widetilde{g}}{\partial x_2}\Big\| + \frac{1}{\ell_\beta}\|\widetilde{g}\|\Big), \qquad (9.17)$$

where $\widetilde{z}^\alpha(x) = 0$ for $x_\alpha = 0$, $x_\alpha = \ell_\alpha$, $\alpha = 1, 2$.

Lemma 9.3. *For the solution of the difference scheme* (9.14) *the estimate*

$$\mu_0\|\nabla\boldsymbol{z}\| \le \sum_{\alpha=1}^{2}\big(\|\psi^\alpha_{\alpha\alpha}\|_{\omega_\alpha^+\times\overline{\omega}_\beta} + \|\psi^\beta_{\alpha\beta}\|_{\omega_\alpha^+\times\overline{\omega}_\beta} + \|\psi^\beta_{\beta\alpha}\|_{\omega_\alpha\times\omega_\beta^+} +$$
$$+ \|\psi^\alpha_{\beta\beta}\|_{\omega_\alpha\times\omega_\beta^+} + J_\alpha(\eta^\beta) + J_\alpha(\chi^\beta)\big) \qquad (9.18)$$

is valid, where $J_\alpha(g) = 2\|g_{x_\alpha}\|_{\omega_\alpha\times\overline{\omega}_\beta} + 2\|g_{\overline{x}_\beta}\|_{\omega^+} + \frac{2}{\ell_\beta}\|g\|_{\omega_\alpha^+\times\overline{\omega}_\beta}$.

Proof. We multiply both parts of (9.14) scalarly by $\boldsymbol{z}$ and use the estimate (9.6). As a result, we obtain

$$\mu_0\,\|\nabla\boldsymbol{z}\|^2 \le |(\boldsymbol{\psi}, \boldsymbol{z})|. \qquad (9.19)$$

Taking into account the structure of $\boldsymbol{\psi}$, we find that

$$(\boldsymbol{\psi}, \boldsymbol{z}) = \sum_{\alpha=1}^{2}\Big[\sum_{\gamma_{-\beta}} h_\alpha(\eta^\beta + \chi^\beta)_{x_\alpha} z^\alpha - \sum_{\gamma_{+\beta}} h_\alpha(\eta^\beta + \chi^\beta)_{x_\alpha} z^\alpha\Big] +$$

$$= \sum_{\alpha=1}^{2}\Big(\sum_{\omega_\alpha\times\overline{\omega}_\beta} h_\alpha\hbar_\beta(\psi^\alpha_{\alpha\alpha} + \psi^\beta_{\alpha\beta})_{x_\alpha} z^\alpha + \sum_{\omega_\alpha\times\overline{\omega}_\beta} h_\alpha\hbar_\beta B_\beta(\psi^\beta_{\beta\alpha} + \psi^\alpha_{\beta\beta}) z^\alpha\Big). \qquad (9.20)$$

Bearing in mind that $(B_\beta v, z^\alpha)_{(\alpha)} = -\sum\limits_{\substack{x_\alpha\in\overline{\omega}_\alpha \\ x_\beta\in\omega_\beta^+}} h_1 h_2 v z^\alpha_{\overline{x}_\beta}$, using in the right-hand side of (9.20) the formulas of summation by parts, the Cauchy inequality and Lemma 9.2, we obtain

$$|(\boldsymbol{\psi}, \boldsymbol{z})| \le \Big(\sum_{\alpha=1}^{2}\big(\|\psi^\alpha_{\alpha\alpha} + \psi^\beta_{\alpha\beta}\|_{\omega_\alpha^+\times\overline{\omega}_\beta} + \|\psi^\beta_{\beta\alpha} + \psi^\alpha_{\beta\beta}\|_{\omega_\alpha\times\omega_\beta^+} +$$
$$+ J_\alpha(\eta^\beta) + J_\alpha(\chi^\beta)\big)\Big)\|\nabla\boldsymbol{z}\|,$$

which together with (9.19) proves the unknown estimate (9.18). □

4^0. Accuracy of the scheme. To find the convergence rate of the difference scheme (9.3) by means of Lemma 9.3, it is sufficient in the right-hand side of (9.18) to estimate norms of the expressions ψ_{ij}^α, η^α, χ^α. In this connection, we have to group together the summands.

For example, represent η^2 as the sum $\eta^2(u) = \eta'((\lambda+2\mu)D_1u^1) + \eta''$, where $\eta'(v) = \frac{h_2^2}{12}\big(\overline{S}_2D_2v - \overline{S}_1\overline{S}_2D_2v\big)$, $\eta'' = \frac{h_2^2}{12}\,\overline{S}_1\overline{S}_2D_2\big((\lambda+2\mu)D_1u^1\big)$.

$\eta'(v)$ is, in fact, a linear functional which is bounded in W_2^{m-1}, $m > 2$, and vanishes for $v \in \boldsymbol{\pi}_1$. Therefore on the basis of the Bramble–Hilbert lemma we obtain for the above functional the estimate $|\eta'(v)| \le \frac{c|h|^{m-1}}{(h_1h_2)^{1/2}}\,|v|_{W_2^{m-1}(e)}$, $2 < m \le 3$, i.e. $|\eta'((\lambda+2\mu)D_1u^1)| \le \frac{c|h|^{m-1}}{(h_1h_2)^{1/2}}\,\big|(\lambda+2\mu)D_1u^1\big|_{W_2^{m-1}(e)}$, $2 < m \le 3$. For the summand η'' we easily get $|\eta''| \le c|h|\big(\|D_1u^1\|_{C(\overline{\Omega})}|\lambda+2\mu|_{W_2^1(e)} + \|\lambda+2\mu\|_{C(\overline{\Omega})}|u|_{W_2^2(e)}\big)$. The above inequalities result in the estimate

$$
\begin{aligned}
\|\eta^2\| \le c|h|^2\big(\|D_1u^1\|_{C(\overline{\Omega})}|\lambda+2\mu|_{W_2^1(e)} + \|\lambda+2\mu\|_{C(\overline{\Omega})}|u|_{W_2^2(e)}\big) &+ \\
+\, c|h|^{m-1}|(\lambda+2\mu)D_1u^1|_{W_2^{m-1}(\Omega)},\quad 2 < m \le 3.&
\end{aligned}
$$

Taking here into account the imbeddings $W_2^m \subset C^1$, $W_2^{m-1} \subset C$ and Lemma 1.1, for η^2 we finally obtain the estimate $\|\eta^2\| \le c|h|^{m-1}\|\lambda+2\mu\|_{W_2^{m-1}(\Omega)}\|u\|_{W_2^m(\Omega)}$, $2 < m \le 3$. We now represent $\psi_{\alpha\alpha}^\alpha$ for $2 < m \le 3$ at the boundary points $x \in \gamma_{\pm\beta}$ as follows:

$$
\begin{aligned}
\psi_{\alpha\alpha}^\alpha = {}& S_\alpha^-\overline{S}_\beta a\Big[u^\alpha - \overline{S}_\beta u^\alpha - \frac{n_\beta h_\beta}{2}\,D_\beta u^\alpha\Big]_{\overline{x}_\alpha} + \\
&+ \big[S_\alpha^-\overline{S}_\beta a\,S_\alpha^-\overline{S}_\beta D_\alpha u^\alpha - S_\alpha^-\overline{S}_\beta(aD_\alpha u^\alpha)\big] + \\
&+ \Big[S_\alpha^-\overline{S}_\beta(aD_\alpha u^\alpha) - S_\alpha^-T_\beta\Big(aD_\alpha u^\alpha) + \frac{2n_\beta}{h_\beta}\,\eta^\beta\Big],\quad a = \lambda+2\mu,\\
\psi_{\alpha\beta}^\beta = {}& \frac{1-n_\beta}{2}\,S_\alpha^-\overline{S}_\beta\lambda\Big[I_\alpha^-u_{x_\beta}^\beta - S_\alpha^-\overline{S}_\beta D_\beta u^\beta + \frac{h_\alpha}{2}\,\overline{S}_\beta D_1D_2u^\beta\Big] + \\
&+ \frac{1+n_\beta}{2}\,S_\alpha^-\overline{S}_\beta\lambda\Big[u_{\overline{x}_\beta}^\beta - S_\alpha^-\overline{S}_\beta D_\beta u^\beta - \frac{h_\alpha}{2}\,\overline{S}_\beta D_1D_2u^\beta\Big] + \\
&+ \big[S_\alpha^-\overline{S}_\beta\lambda S_\alpha^-\overline{S}_\beta D_\beta u^\beta - S_\alpha^-\overline{S}_\beta(\lambda D_\beta u^\beta)\big] + \\
&+ \Big[S_\alpha^-\overline{S}_\beta(\lambda D_\beta u^\beta) - S_\alpha^-T_\beta(\lambda D_\beta u^\beta) + \frac{n_\beta h_\beta}{6}\,\overline{S}_\beta D_\beta(\lambda D_\beta u^\beta)\Big].
\end{aligned}
$$

The estimation of individual summands is performed by the well-known method. As a result, we arrive at the following

Theorem 9.1. *Let the solution $\boldsymbol{u}(x)$ of the problem* (9.1), (9.2) *belong to the space $W_2^m(\Omega)$, $m > 1$. Then the convergence rate of the difference scheme* (9.4) *in the mesh norm W_2^1 is defined by the estimate*

$$
\|\boldsymbol{y}-\boldsymbol{u}\|_{W_2^1(\omega)} \le c|h|^{s-1}\|\boldsymbol{u}\|_{W_2^{m-1}(\Omega)},\quad m \in (1,3], \tag{9.21}
$$

where the constant c does not depend on h and $\boldsymbol{u}(x)$.

Remark 9.2. The estimate (9.21) could have been obtained without applying Lemma 9.2, but only for $m \in (1, 2.5)$. This could have been realized by the method used in [71] (Ch. IV, § 2(2)) by representing the approximation error on γ as a sum of two summands, one of which is estimated by the Bramble–Hilbert lemma and the other one by the inequality providing us with the estimate of a function's L_2-norm in a strip along the boundary through its norm W_2^m in the domain.

Remark 9.3. The difference scheme (2.3) is advisable to be employed in the case of rapidly varying coefficients of the differential problem. In other cases, since, by the assumption, the Lamé coefficients are continuous, in the difference scheme (9.4) the averaged coefficients $\lambda^{\pm}$, $\mu^{\pm}$ can be replaced respectively by λ, μ without affecting the validity of all the above results.

10. The First Boundary Value Problem for Elliptic Equation of Fourth Order

History of the matter. Here we consider the question of obtaining a consistent estimate of the convergence rate of the difference scheme approximating the first boundary value problem for elliptic equation of the fourth order with variable coefficients. In [59], for the equation with variable coefficients the convergence of the difference scheme with the rate $O(|h|^{m-2.5})$ in the mesh norm W_2^2 is proved under the condition that the exact solution belongs to the space $W_2^m(\Omega)$, $m = 3, 4$. In the case of biharmonic equation, the consistent estimate of the convergence rate for $s = 2$, $m \in (2.5, 3.5)$ has been obtained in [71]. In [54], the author investigated the difference scheme for the fourth order equation with variable coefficients and proved the convergence in W_2^2 with the rate

$$O\big(h^{\min(m-2;1.5)}|\ln h|^{1-|\operatorname{sgn}(m-3.5)|}\big), \quad m \in (2.5, 4].$$

Note that in the above-mentioned as well as in a number of works (see, e.g., [40], [41]), devoted to difference schemes for problems with fourth order equations, the solution of the difference problem $y(x)$ is defined not only at the mesh points $\overline{\omega}$ belonging to the closed domain $\overline{\Omega}$, but also at the nodes lying beyond the contour. Therefore when investigating the error $z = y - u$, there arises the need to extend the solution of the initial problem $u(x)$ outside the domain $\overline{\Omega}$, but preserving the smoothness.

In the present section we choose another approach: the error $z(x)$ is assumed to be the restriction on $\overline{\omega}$ of some mesh function, and without extension of the unknown solution $u(x)$ we obtained a consistent estimate for the convergence rate. The results of this section have been published in [21]. Analogous results for the biharmonic equation are obtained in [15].

1^0. Statement of the problem. We consider the problem

$$Lu \equiv D_1^2 M_1 + 2D_1 D_2 M_3 + D_2^2 M_2 = f(x), \quad x \in \Omega, \tag{10.1}$$

$$u(x) = 0, \quad x \in \Gamma, \quad D_\alpha u(x) = 0, \quad x \in \Gamma_{\pm\alpha}, \quad \alpha = 1, 2. \tag{10.2}$$

Here, $M_\alpha = a_\alpha D_\alpha^2 u + a_0 D_{3-\alpha}^2 u$, $\alpha = 1, 2$, $M_3 = a_3 D_1 D_2 u$. The following sufficient conditions of ellipticity $0 < c_1 \le a_\alpha \le c_2$, $\alpha = 1, 2, 3$, $0 < c_0 \le 1 - \frac{|a_0|}{\sqrt{a_1 a_2}}$, $x \in \Omega$, are assumed to be fulfilled. Moreover, let $f \in W_2^{m-4}(\Omega)$, $a_\alpha \in W_p^{m-2}(\Omega)$, $\alpha = 0, 1, 2, 3$, $p = 2$ for $3 < m \le 4$, $p > \frac{2}{m-2}$ for $2.5 < m \le 3$.

Everywhere in this section $q = 2p/(p-2)$. We assume that the problem (10.1), (10.2) is uniquely solvable in the class $W_2^m(\Omega)$, $m \in (2.5, 4]$.

For approximation of the problem (10.1), (10.2), we use the scheme [54]

$$L_h y \equiv m_1(y)_{\overline{x}_1 x_1} + 2m_3(y)_{x_1 x_2} + m_2(y)_{\overline{x}_2 x_2} = \varphi(x), \quad x \in \omega, \tag{10.3}$$

$$y(x) = 0, \quad x \in \gamma, \quad y_{\overset{\circ}{x}_\alpha}(x) = 0, \quad x \in \gamma_{\pm\alpha}, \quad \alpha = 1, 2, \tag{10.4}$$

where $m_1(y) = a_1 y_{\overline{x}_1 x_1} + a_0 y_{\overline{x}_2 x_2}$, $m_2(y) = a_0 y_{\overline{x}_1 x_1} + a_2 y_{\overline{x}_2 x_2}$, $m_3(y) = a_3(\check{x}) y_{\overline{x}_1 \overline{x}_2}$, $\check{x} = (x_1 - 0.5h_1, x_2 - 0.5h_2)$, $\varphi(x) = T_1 T_2 f$. Omitting from (10.3), (10.4) the values $y(x)$ lying beyond the contour (defined on γ^-), we can represent the scheme in the form

$$\begin{aligned} &\overset{\circ}{L}_h y \equiv \overset{\circ}{m}_1(y)_{\overline{x}_1 x_1} + 2m_3(y)_{x_1 x_2} + \overset{\circ}{m}_2(y)_{\overline{x}_2 x_2} = \varphi(x), \quad x \in \omega, \\ &y(x) = 0, \quad x \in \gamma, \end{aligned} \tag{10.5}$$

where

$$\overset{\circ}{m}_\alpha(y) = a_0 y_{\overline{x}_{3-\alpha} x_{3-\alpha}} + a_\alpha \Lambda_\alpha y, \quad \Lambda_\alpha y = \begin{cases} \dfrac{2}{h_\alpha}\, y_{x_\alpha}, & x_\alpha = 0, \\ y_{\overline{x}_\alpha x_\alpha}, & x_\alpha \in \omega_\alpha, \quad \alpha = 1, 2. \\ -\dfrac{2}{h_\alpha}\, y_{\overline{x}_\alpha}, & x_\alpha = \ell_\alpha, \end{cases}$$

Let H be the space of mesh functions defined on $\overline{\omega} \cup \gamma^-$ and satisfying the conditions (10.4), $\overset{\circ}{H}$ be the space of mesh functions given on $\overline{\omega}$ and equal to zero on γ, with the inner product $(y, v) = (y, v)_\omega$ and the norm $\|y\| = (y, y)^{1/2}$. The norms $\|\Delta_h y\|$, $\|y\|_{W_2^2(\omega)}$ were defined in Section 1 (the case of homogeneous Dirichlet conditions). Denote also $|y|^2_{2,\overline{\omega}} = \|y_{\overline{x}_1 x_1}\|^2_{(2)} + \|y_{\overline{x}_2 x_2}\|^2_{(1)}$, $|y|^2_{W_2^2(\overline{\omega})} = |y|^2_{2,\overline{\omega}} + 2\|y_{\overline{x}_1 \overline{x}_2}\|^2_{\omega^+}$.

Using the formulas of summation by parts, it is not difficult to establish that the operator L_h in H (or the operator $\overset{\circ}{L}_h$ in $\overset{\circ}{H}$) is self-conjugate and positive definite. Consequently, the problem (10.3), (10.4) (or (10.5)) is uniquely solvable. Note that the following lemma is valid.

Lemma 10.1. *Let $v \in H$, and y be any mesh function defined on $\overline{\omega} \cup \gamma^-$ and vanishing on γ. Then*

$$c_0 c_1 |v|^2_{W_2^2(\overline{\omega})} \le (L_h v, v), \quad |(L_h y, v)| \le \sqrt{5}\, c_2 |y|_{2,\overline{\omega}} |v|_{W_2^2(\overline{\omega})}.$$

Proof. The identity

$$(L_h y, v) = \sum_{\omega_{(1)}} \hbar_1 h_2 (a_1 y_{\overline{x}_1 x_1} + a_0 y_{\overline{x}_2 x_2}) v_{\overline{x}_1 x_1} +$$
$$+ \sum_{\omega_{(2)}} h_1 \hbar_2 (a_0 y_{\overline{x}_1 x_1} + a_2 y_{\overline{x}_2 x_2}) v_{\overline{x}_1 x_1} + 2 \sum_{\omega^+} h_1 h_2 a_3 y_{\overline{x}_1 \overline{x}_2} v_{\overline{x}_1 \overline{x}_2} \qquad (10.6)$$

is valid. Note that by the conditions of ellipticity we have $|a_0| \le \sqrt{a_1 a_2} \le c_2$. Therefore

$$\frac{1}{c_2} |(L_h y, v)| \le \|v_{\overline{x}_1 x_1}\|_{(2)} \big(\|y_{\overline{x}_1 x_1}\|_{(2)} + \|y_{\overline{x}_2 x_2}\|_{(2)} \big) +$$
$$+ \|v_{\overline{x}_2 x_2}\|_{(1)} \big(\|y_{\overline{x}_1 x_1}\|_{(1)} + \|y_{\overline{x}_2 x_2}\|_{(1)} \big) + 2 \|v_{\overline{x}_1 \overline{x}_2}\|_{\omega^+} \|y_{\overline{x}_1 \overline{x}_2}\|_{\omega^+},$$

and since $\|y_{\overline{x}_1 x_1}\|_{(1)} = \|y_{\overline{x}_1 x_1}\| \le \|y_{\overline{x}_1 x_1}\|_{(2)}$, $\|y_{\overline{x}_2 x_2}\|_{(2)} = \|y_{\overline{x}_2 x_2}\| \le \|y_{\overline{x}_2 x_2}\|_{(1)}$, $\|y_{\overline{x}_1 \overline{x}_2}\|^2_{\omega^+} = (y_{\overline{x}_1 x_1}, y_{\overline{x}_2 x_2}) \le \|y_{\overline{x}_1 x_1}\| \, \|y_{\overline{x}_2 x_2}\| \le \|y_{\overline{x}_1 x_1}\|_{(2)} \|y_{\overline{x}_2 x_2}\|_{(1)}$, $2\|y_{\overline{x}_1 \overline{x}_2}\|_{\omega^+} \le \|y_{\overline{x}_1 x_1}\|_{(2)} + \|y_{\overline{x}_2 x_2}\|_{(1)}$, therefore

$$\frac{1}{c_2} |(L_h y, v)| \le \big(\|y_{\overline{x}_1 x_1}\|_{(2)} + \|y_{\overline{x}_2 x_2}\|_{(1)} \big) \times$$
$$\times \big(\|v_{\overline{x}_1 x_1}\|_{(2)} + \|v_{\overline{x}_2 x_2}\|_{(1)} + \|v_{\overline{x}_1 \overline{x}_2}\|_{\omega^+} \big) = \vec{y} \cdot \vec{v}, \qquad (10.7)$$

where $\vec{y} = \big(y^1, y^2, y^1, y^2, \frac{1}{\sqrt{2}} y^1, \frac{1}{\sqrt{2}} y^2 \big)$, $\vec{v} = \big(v^1, v^1, v^2, v^2, \sqrt{2} v^{12}, \sqrt{2} v^{12} \big)$, $y^\alpha = \|y_{\overline{x}_\alpha x_\alpha}\|_{(3-\alpha)}$, $v^\alpha = \|v_{\overline{x}_\alpha x_\alpha}\|_{(3-\alpha)}$, $\alpha = 1, 2$, $v^{12} = \|v_{\overline{x}_1 \overline{x}_2}\|_{\omega^+}$.

Estimating the inner product of the vectors

$$\vec{y} \cdot \vec{v} \le \Big(\frac{5}{2} \big((y^1)^2 + (y^2)^2 \big) \Big)^{1/2} \Big(2 \big((v^1)^2 + (v^2)^2 + 2 (v^{12})^2 \big) \Big)^{1/2},$$

from (10.7) we obtain the second inequality of our lemma.

Noticing now that $a_1 t_1^2 + a_2 t_2^2 + 2 a_0 t_1 t_2 \ge a_1 t_1^2 \big(1 - \frac{|a_0|}{\sqrt{a_1 a_2}} \big) + a_2 t_2^2 \big(1 - \frac{|a_0|}{\sqrt{a_1 a_2}} \big) \ge c_1 c_0 (t_1^2 + t_2^2)$ and substituting $y = v$ in (10.6), we obtain the first inequality of the lemma. □

2^0. A priori error estimate. Consider the problem for the error $z = y - u$, $x \in \overline{\omega}$,

$$\overset{\circ}{L}_h z = \psi, \quad x \in \omega, \quad z \in \overset{\circ}{H}. \qquad (10.8)$$

Here $\psi = T_1 T_2 f - \overset{\circ}{L}_h u$ is the approximation error which with regard for the properties of the operators $T_\alpha (\alpha = 1, 2)$, $T_\alpha D_\alpha^2 u = \Lambda_\alpha u$, $T_1 T_2 D_1 D_2 u = S_1^- S_2^- u_{x_1 x_2}$, can be reduced to the form

$$\psi = \eta_{1 \overline{x}_1 x_1} + \eta_{2 \overline{x}_2 x_2} + 2 \eta_{3 x_1 x_2} +$$
$$+ \frac{2}{h_1^3} \big[\delta(h_1, x_1) a_1 \overset{1}{g}(0, x_2) - \delta(\ell_1 - h_1, x_1) a_1 \overset{1}{g}(\ell_1, x_2) \big] +$$
$$+ \frac{2}{h_2^3} \big[\delta(h_2, x_2) a_2 \overset{2}{g}(x_1, 0) - \delta(\ell_2 - h_2, x_2) a_2 \overset{2}{g}(x_1, \ell_2) \big], \qquad (10.9)$$

where $\eta_1 = \eta_{11} + \eta_{10}$, $\eta_2 = \eta_{22} + \eta_{20}$,

$$\eta_{\alpha\alpha} = \begin{cases} T_{3-\alpha}(a_\alpha D_\alpha^2 u) - a_\alpha \Lambda_\alpha u, & x \in \omega, \\ T_{3-\alpha}(a_\alpha D_\alpha^2 u) - a_\alpha \Lambda_\alpha u \pm \dfrac{2}{h_\alpha}\, a_\alpha \overset{\alpha}{g}, & x \in \gamma_{\pm\alpha}, \end{cases} \tag{10.10}$$

$$\eta_{\alpha 0} = T_{3-\alpha}(a_0 D_{3-\alpha}^2 u) - a_0 u_{\overline{x}_{3-\alpha} x_{3-\alpha}}, \quad x \in \omega \cup \gamma_{\pm\alpha}, \quad \alpha = 1,2, \tag{10.11}$$

$$\eta_3 = S_1^- S_2^- (a_3 D_1 D_2 u) - a_3(\check{x}) u_{\overline{x}_1 \overline{x}_2}, \quad x \in \omega^+, \tag{10.12}$$

and an arbitrary function $\overset{\alpha}{g}(x)$ appearing in (10.9) will be assumed to be defined in the form

$$\overset{\alpha}{g}(x) = \begin{cases} -\dfrac{h_\alpha^2}{6}\, T_\alpha D_\alpha^3 u, & x \in \overline{\omega} \ \text{ for } \ u \in W_2^m(\Omega), \ \ 3 < m \le 4, \\ 0 & \text{for } \ u \in W_2^m(\Omega), \ \ 2.5 < m \le 3. \end{cases}$$

To obtain the needed a priori estimate of the function $z(x)$, we consider the auxiliary problems

$$\begin{gathered} \overset{\alpha}{v}_{\overline{x}_1 x_1 \overline{x}_1 x_1} + \overset{\alpha}{v}_{\overline{x}_2 x_2 \overline{x}_2 x_2} = 0, \quad x \in \omega, \quad \overset{\alpha}{v}(x) = 0, \quad x \in \gamma, \\ \overset{\alpha}{v}_{\overset{\circ}{x}_\alpha}(x) = \overset{\alpha}{g}(x), \quad x \in \gamma_{\pm\alpha}, \quad \overset{\alpha}{v}_{\overset{\circ}{x}_{3-\alpha}}(x) = 0, \quad x \in \gamma_{\pm(3-\alpha)}, \quad \alpha = 1,2, \end{gathered} \tag{10.13}$$

and

$$\begin{gathered} L_h \overset{\circ}{v} = \eta_{1\overline{x}_1 x_1} + \eta_{2\overline{x}_2 x_2} + 2\eta_{3 x_1 x_2} - L_h(\overset{1}{v} + \overset{2}{v}), \quad x \in \omega, \\ \overset{\circ}{v}(x) = 0, \quad x \in \gamma, \quad \overset{\circ}{v}_{\overset{\circ}{x}_\alpha}(x) = 0, \quad x \in \gamma_{\pm\alpha}. \end{gathered} \tag{10.14}$$

Note at once that since

$$\begin{aligned} L_h \overset{1}{v} - \overset{\circ}{L}_h \overset{1}{v} &= \frac{2}{h_1^3}\, \delta(x_1, \ell_1 - h_1) a_1 \overset{1}{g}(\ell_1, x_2) - \frac{2}{h_1^3}\, \delta(x_1, h_1) a_1 \overset{1}{g}(0, x_2), \\ L_h \overset{2}{v} - \overset{\circ}{L}_h \overset{2}{v} &= \frac{2}{h_2^3}\, \delta(x_2, \ell_2 - h_2) a_2 \overset{2}{g}(x_1, \ell_2) - \frac{2}{h_2^3}\, \delta(x_2, h_2) a_2 \overset{2}{g}(x_1, 0), \\ L_h \overset{\circ}{v} &= \overset{\circ}{L}_h \overset{\circ}{v}, \end{aligned}$$

we have $\overset{\circ}{L}_h(\overset{\circ}{v} + \overset{1}{v} + \overset{2}{v}) = \psi$. Consequently, the solution $z(x)$, $x \in \overline{\omega}$, of the problem (10.8) is the restriction of the function $(\overset{\circ}{v} + \overset{1}{v} + \overset{2}{v})$ defined on $\overline{\omega} \cup \gamma^-$, i.e.,

$$z(x) = \overset{\circ}{v}(x) + \overset{1}{v}(x) + \overset{2}{v}(x) \ \text{ for } \ x \in \overline{\omega}. \tag{10.15}$$

Let λ_k and μ_k be, respectively, eigenvalues and eigenfunctions of the problem ([80], [1])

$$\begin{gathered} (\mu_k(x_2))_{\overline{x}_2 x_2 \overline{x}_2 x_2} = \lambda_k \mu_k(x_2), \quad x_2 \in \omega_2, \\ \mu_k(0) = \mu_k(\ell_2) = 0, \quad \mu_{k\overset{\circ}{x}_2}(0) = \mu_{k\overset{\circ}{x}_2}(\ell_2) = 0, \end{gathered}$$

where λ_k are positive and satisfy the relation $\lambda_k \ge \lambda_1 > \overset{\circ}{\lambda}_1 = \left(\frac{2}{h_2} \sin \frac{\pi h_2}{2\ell_2}\right)^4 > \frac{64}{\ell_2^4}$, and the functions μ_k form orthonormal basis in the sense of the inner product $(u,v)_{\omega_2} = \sum\limits_{x_2 \in \omega_2} h_2 uv$.

Consider the following problem:

$$\begin{gathered} v_{k\overline{x}_1x_1\overline{x}_1x_1}(x_1) + \lambda_k v_k(x_1) = 0, \ \ x_1 \in \omega_1, \\ v_k(0) = v_k(\ell_1) = 0, \ \ v_{k\overset{\circ}{x}_1}(0) = g_k(0), \quad v_{k\overset{\circ}{x}_1}(\ell_1) = g_k(\ell_1). \end{gathered} \tag{10.16}$$

Lemma 10.2. *For the solution of the problem* (10.16) *the estimate* $v^2_{k\overline{x}_1x_1}(0) + v^2_{k\overline{x}_1x_1}(\ell_1) \le 4c_3^2\lambda_k^{1/2}\big(g_k^2(0) + g_k^2(\ell_1)\big)$, $c_3 = 10 + \frac{3\ell_2}{\ell_1}$, *is valid.*

Proof. The energetic identity for the problem (10.16) has the form

$$\sum_{\overline{\omega}_1} \hbar_1 v^2_{k\overline{x}_1x_1} + \lambda_k \sum_{\omega_1} h_1 v_k^2 = g_k(\ell_1)v_{k\overline{x}_1x_1}(\ell_1) - g_k(0)v_{k\overline{x}_1x_1}(0).$$

Consequently,

$$\begin{gathered} \sum_{\overline{\omega}_1} \hbar_1 v^2_{k\overline{x}_1x_1} + \lambda_k \sum_{\omega_1} h_1 v_k^2 \le \\ \le \big(g_k^2(0) + g_k^2(\ell_1)\big)^{1/2}\big(v^2_{k\overline{x}_1x_1}(0) + v^2_{k\overline{x}_1x_1}(\ell_1)\big)^{1/2}. \end{gathered} \tag{10.17}$$

For every mesh function defined on the mesh $\overline{\omega}_1$, the estimate

$$\max_{\overline{\omega}_1} |y|^2 \le \varepsilon\ell_1^3 \sum_{\omega_1} h_1 y^2_{\overline{x}_1x_1} + \frac{1}{\ell_1}\Big(9\sqrt[3]{\varepsilon} + 6\Big)\sum_{\overline{\omega}_1} \hbar_1 y^2, \ \ \forall\, \varepsilon > 0,$$

is valid ([1], formula (3.7). Using the given estimate for $y = v_{k\overline{x}_1x_1}$ and taking into account the obtained from (10.16) equality $\sum\limits_{\omega_1} h_1 v^2_{k\overline{x}_1x_1\overline{x}_1x_1} = \lambda_k^2 \sum\limits_{\omega_1} h_1 v_k^2$, we find that

$$\begin{gathered} \max_{\overline{\omega}_1} |v_{k\overline{x}_1x_1}|^2 \le \\ \le c_4\Big(\sum_{\overline{\omega}_1} \hbar_1 v^2_{k\overline{x}_1x_1} + \lambda_k \sum_{\omega_1} h_1 v_k^2\Big), \ \ c_4 = \varepsilon\ell_1^3\lambda_k + \frac{9\varepsilon^{-1/3} + 6}{\ell_1}. \end{gathered} \tag{10.18}$$

Choosing now $\varepsilon = \ell_1^{-3}\lambda_k^{-3/4}$ and noticing that $3\ell_2\lambda_k^{1/4} > 6$, we have $c_4 < c_3\lambda_k^{1/4}$. Finally, with regard for the estimate (10.17), from (10.18) it follows that $\max_{\overline{\omega}_1} |v_{k\overline{x}_1x_1}|^2 \le c_3\lambda_k^{1/4}\big(g_k^2(0) + g_k^2(\ell_1)\big)^{1/2}\big(v^2_{k\overline{x}_1x_1}(0) + v^2_{k\overline{x}_1x_1}(\ell_1)\big)^{1/2}$. This completes the proof of the lemma. □

Theorem 10.1. *For the solution of the problem* (10.13)*, the a priori estimates* $|\overset{\alpha}{v}|_{2,\overline{\omega}} \le c_5\big(\|\overset{\alpha}{g}_{\overline{x}_\alpha}\|_{\omega\cup\gamma_{+\alpha}} + \|\overset{\alpha}{g}_{\overline{x}_\beta}\|_{(\beta)}\big)$, $\beta = 3 - \alpha$, $\alpha = 1, 2$, *where* $c_5^2 = (10 + 3l)(l + \sqrt{\ell^2 + 8})/\sqrt{2}$, $\ell = \max(\ell_1/\ell_2; \ell_2/\ell_1)$, *are valid.*

Proof. Since $(\overset{1}{v}_{\overline{x}_1x_1\overline{x}_1x_1}, \overset{1}{v}) = \|\overset{1}{v}_{\overline{x}_1x_1}\|^2_{(2)} + \sum\limits_{\omega_2} h_2 \overset{1}{v}_{\overline{x}_1x_1}(0, x_2)\overset{1}{g}(0, x_2) - \sum\limits_{\omega_2} h_2 \overset{1}{v}_{\overline{x}_1x_1}(\ell_1, x_2)\overset{1}{g}(\ell_1, x_2)$, $(\overset{1}{v}_{\overline{x}_2x_2\overline{x}_2x_2}, \overset{1}{v}) = \|\overset{2}{v}\|^2_{(1)}$, from (10.13) ($\alpha = 1$)

it follows that

$$|\overset{1}{v}|^2_{2,\overline{\omega}} = \sum_{\omega_2} h_2 \overset{1}{v}_{\overline{x}_1 x_1}(\ell_1, x_2)\overset{1}{g}(\ell_1, x_2) - \sum_{\omega_2} h_2 \overset{1}{v}_{\overline{x}_1 x_1}(0, x_2)\overset{1}{g}(0, x_2). \quad (10.19)$$

We expand $\overset{1}{v}(x)$ and $\overset{1}{g}(x)$ with respect to the eigenfunctions $\{\mu_k\}$:

$$\overset{1}{v}(x) = \sum_{k=1}^{N_2-1} v_k(x_1)\mu_k(x_2), \quad \overset{1}{g}(x) = \sum_{k=1}^{N_2-1} g_k(x_1)\mu_k(x_2), \quad (10.20)$$

Then from (10.19) we obtain

$$\begin{aligned} |\overset{1}{v}|^2_{2,\overline{\omega}} &= \sum_{k=1}^{N_2-1} \big[g_k(\ell_1)v_{k\overline{x}_1 x_1}(\ell_1) - g_k(0)v_{k\overline{x}_1 x_1}(0)\big] \le \\ &\le \Big(\sum_{k=1}^{N_2-1} \lambda_k^{1/4}\big(g_k^2(0) + g_k^2(\ell_1)\big)\Big)^{1/2} \times \\ &\quad \times \Big(\sum_{k=1}^{N_2-1} \lambda_k^{-1/4}\big(v^2_{k\overline{x}_1 x_1}(0) + v^2_{k\overline{x}_1 x_1}(\ell_1)\big)\Big)^{1/2}. \end{aligned} \quad (10.21)$$

Substituting the expansions (10.20) in (10.13) ($\alpha = 1$) and taking into account orthonormality $\{\mu_k\}$, we can see that $v_k(x_1)$ is a solution of the problem (10.16). Therefore by virtue of Lemma 10.2, from (10.21) we find that $|\overset{1}{v}|^2_{2,\overline{\omega}} \le 2c_3 \sum\limits_{k=1}^{N_2-1} \lambda_k^{1/4}\big(g_k^2(0) + g_k^2(\ell_1)\big)$. But ([70], p. 290) $g_k^2(0) + g_k^2(\ell_1) \le \varepsilon_k \sum\limits_{\omega_1^+} h_1 g^2_{k\overline{x}_1} + \big(\frac{1}{\varepsilon_k} + \frac{2}{\ell_1}\big) \sum\limits_{\overline{\omega}_1} \hbar_1 g_k^2$, $\forall\, \varepsilon_k > 0$, and hence

$$|\overset{1}{v}|^2_{2,\overline{\omega}} \le 2c_3 \sum_{k=1}^{N_2-1} \varepsilon_k \lambda_k^{1/4} \sum_{\omega_1^+} h_1 g^2_{k\overline{x}_1} + 2c_3 \sum_{k=1}^{N_2-1} \Big(\frac{1}{\varepsilon_k} + \frac{2}{\ell_1}\Big)\lambda_k^{1/4} \sum_{\overline{\omega}_1} \hbar_1 g_k^2.$$

Choosing here $\varepsilon_k = \big(\frac{1}{\ell_1} + \sqrt{\frac{1}{\ell_1^2} + \lambda_k^{1/2}}\big)\lambda_k^{-1/2}$, we obtain

$$|\overset{1}{v}|^2_{2,\overline{\omega}} \le 2c_3 \sum_{k=1}^{N_2-1} \phi(\lambda_k)\Big(\sum_{\omega_1^+} h_1 g^2_{k\overline{x}_1} + \lambda_k^{1/2} \sum_{\overline{\omega}_1} \hbar_1 g_k^2\Big),$$

where $\phi(\lambda_k) = \big(\frac{1}{\ell_1} + \sqrt{\frac{1}{\ell_1^2} + \lambda_k^{1/2}}\big)\lambda_k^{-1/4}$. Since $\phi(\lambda_k)$ is decreasing and $\lambda_k > 64/\ell_2^4$, therefore $\phi(\lambda_k) < \Big(\frac{\ell_2}{\ell_1} + \sqrt{\big(\frac{\ell_2}{\ell_1}\big)^2 + 8}\Big)2^{-3/2}$. Consequently,

$$|\overset{1}{v}|^2_{2,\overline{\omega}} \le c_5^2 \Big(\sum_{\omega_1^+} h_1 \sum_{k=1}^{N_2-1} g^2_{k\overline{x}_1} + \sum_{\overline{\omega}_1} \hbar_1 \sum_{k=1}^{N_2-1} \lambda_k^{1/2} g_k^2\Big),$$

and finally we have $|\overset{1}{v}|^2_{2,\overline{\omega}} \le c_5^2\big(\|\overset{1}{g}_{\overline{x}_1}\|^2_{\omega\cup\gamma_{+1}} + \|\overset{1}{g}_{\overline{x}_2}\|^2_{(2+)}\big)$. Thus we have proved the first estimate of Theorem 10.1. The second estimate is obtained analogously. □

Theorem 10.2. *For the solution of the problem* (10.14)*, the estimate*

$$|\overset{\circ}{v}|_{W_2^2(\overline{\omega})} \le c_6\big(\|\eta_1\|_{(2)} + \|\eta_2\|_{(1)} + 2\|\eta_3\|_{\omega^+} + \sqrt{5}\,c_2|\overset{1}{v} + \overset{2}{v}|_{2,\overline{\omega}}\big), \quad c_6 = \frac{1}{c_0c_1},$$

is valid.

Proof. Indeed, it follows from (10.14) that

$$(L_h\overset{\circ}{v}, \overset{\circ}{v}) = \sum_{\alpha=1}^{2}\sum_{\omega_{(\alpha)}} \hbar_\alpha h_{3-\alpha}\eta_\alpha \overset{\circ}{v}_{\overline{x}_\alpha x_\alpha} + 2\sum_{\omega^+} h_1h_2\eta_3\overset{\circ}{v}_{\overline{x}_1\overline{x}_2} - \big(L_h(\overset{1}{v} + \overset{2}{v}), \overset{\circ}{v}\big),$$

after which using Lemma 10.1 we obtain the required statement. □

Theorem 10.3. *For the solution of the problem* (10.8)*, the a priori estimate*

$$\|\Delta_h z\| \le c_6\big[\|\eta_1\|_{(2)} + \|\eta_2\|_{(1)} + 2\|\eta_3\|_{\omega^+}\big] + \big(\sqrt{5}\,c_2 + \sqrt{2}c_0c_1\big)\times$$
$$\times c_5\big[\|\overset{1}{g}_{\overline{x}_1}\|_{\omega\cup\gamma_{+1}} + \|\overset{1}{g}_{\overline{x}_2}\|_{(2+)} + \|\overset{2}{g}_{\overline{x}_1}\|_{(1+)} + \|\overset{2}{g}_{\overline{x}_2}\|_{\omega\cup\gamma_{+2}}\big] \qquad (10.22)$$

is valid.

Proof. From the equality (10.15) it follows $\|\Delta_h z\| \le \|\Delta_h\overset{\circ}{v}\| + \|\Delta_h\overset{1}{v}\| + \|\Delta_h\overset{2}{v}\| \le \|\Delta_h\overset{\circ}{v}\| + \sqrt{2}\big(|\overset{1}{v}|_{2,\overline{\omega}} + |\overset{2}{v}|_{2,\overline{\omega}}\big)$. Using the obvious inequality $\|\Delta_h\overset{\circ}{v}\| \le |\overset{\circ}{v}|_{W_2^2(\overline{\omega})}$ and Theorems 10.1 and 10.2, we obtain the estimate (10.22). □

3^0. Estimation of the convergence rate. Let $e(x) = \big\{\xi = (\xi_1, \xi_2) : |x_\alpha - \xi_\alpha| \le h_\alpha,\ \alpha = 1,2\big\} \cap \Omega$, $x \in \overline{\omega}$. By $\widetilde{u}(x)$ we denote the function obtained from $u(\xi)$ by the change of the variables $\xi_\alpha = x_\alpha + t_\alpha h_\alpha$, $\alpha = 1,2$, which maps the domain $e(x)$ into $\widetilde{e}$.

To obtain estimates of the convergence rate of the difference scheme (10.7), it is sufficient to estimate the summands in the right-hand side of (10.22). We will consider two separate cases.

(a) $u \in W_2^m(\Omega)$, $3 < m \le 4$. We reduce η_α, $\alpha = 1,2$, to the form

$$\eta_\alpha = \ell^{(\alpha)}(M_\alpha) + a_\alpha\ell^{(6-\alpha)}(D_\alpha^2 u) - a_0\ell^{(\alpha)}(D_{3-\alpha}^2 u), \quad x \in \overline{\omega}_{(\alpha)}, \quad \alpha = 1,2,$$
$$\eta_3 = \ell^{(3)}(a_3D_1D_2u) - a_3(\check{x})\ell^{(3)}(D_1D_2u), \quad x \in \omega^+,$$

where

$$\ell^{(6-\alpha)}(v) = \begin{cases} v - T_\alpha v, & x \in \omega, \\ v - T_\alpha v \mp \dfrac{h_\alpha}{3}\,T_\alpha D_\alpha v, & x \in \gamma_{\pm\alpha}, \end{cases}$$
$$\ell^{(\alpha)}(v) = T_{3-\alpha}v - v, \quad \alpha = 1,2, \quad \ell^{(3)}(v) = S_1^-S_2^-v - v(\check{x}).$$

It can be easily verified that the linear functionals $\ell^{(\alpha)}(v)$, $\alpha = 1,2,\ldots,5$, are bounded for $v \in W_2^{m-2}(\Omega)$ and vanish on $\boldsymbol{\pi}_1$. Taking this fact into account and using the generalized Bramble–Hilbert lemma, after simple calculations we obtain

$$\|\eta_\alpha\|_{(3-\alpha)} \le c|h|^{m-2}\big(|M_\alpha|_{W_2^{m-2}(\Omega)} + \|a_\alpha\|_{C(\overline{\Omega})}|D_\alpha^2 u|_{W_2^{m-2}(\Omega)} + \\ + \|a_0\|_{C(\overline{\Omega})}|D_{3-\alpha}^2 u|_{W_2^{m-2}(\Omega)}\big), \quad \alpha = 1,2,$$

$$\|\eta_3\|_{\omega^+} \le c|h|^{m-2}\big(|a_3 D_1 D_2 u|_{W_2^{m-2}(\Omega)} + \|a_3\|_{C(\overline{\Omega})}|D_1 D_2 u|_{W_2^{m-2}(\Omega)}\big).$$

As to the functionals $\overset{\alpha}{g}_{\overline{x}_\beta}$, it is sufficient to notice that they are bounded for $u \in W_2^m$ and vanish on $\boldsymbol{\pi}_3$. Therefore the norms of these functionals are estimated from above by means of $|h|^{m-2}\|u\|_{W_2^m(\Omega)}$. Moreover, taking into account the imbedding $W_2^{m-2} \subset C$ and the inequality $|av|_{W_2^{m-2}(\Omega)} \le c\|a\|_{W_2^{m-2}(\Omega)}\|v\|_{W_2^{m-2}(\Omega)}$, from (10.22) we obtain the estimate

$$\|\Delta_h z\| \le c|h|^{m-2}\|u\|_{W_2^m(\Omega)} \max_{0\le\alpha\le 3}\|a_\alpha\|_{W_2^{m-2}(\Omega)}, \quad 3 < m \le 4. \tag{10.23}$$

(b) $u \in W_2^m(\Omega)$, $2.5 < m \le 3$. In this case, by definition, $\overset{\alpha}{g}(x) \equiv 0$, and the estimate (10.22) takes the form

$$\|\Delta_h z\| \le c_6\big(\|\eta_1\|_{(1)} + \|\eta_2\|_{(2)} + 2\|\eta_3\|_{\omega^+}\big), \tag{10.24}$$

and moreover, instead of (10.10) we have

$$\eta_{\alpha\alpha} = \big(T_{3-\alpha}(a_\alpha D_\alpha^2 u) - T_{3-\alpha}a_\alpha T_\alpha D_\alpha^2 u\big) + T_\alpha D_\alpha^2 u(T_{3-\alpha}a_\alpha - a_\alpha) = \\ = \eta'_{\alpha\alpha} + \eta''_{\alpha\alpha}. \tag{10.25}$$

For every (fixed) function $a_1 \in C(\overline{\Omega})$, the linear with respect to $u(x)$ functional $\eta'_{11} = \eta'_{11}(u)$ is bounded since

$$|\eta'_{11}| \le \|a_1\|_{C(\overline{\Omega})}\frac{1}{h_1^2}\bigg(\int_{-1}^{1}|D_1^2\widetilde{u}(0,t_2)|\,dt_2 + \int_{-1}^{1}|D_1^2\widetilde{u}(t_1,0)|dt_1\bigg) \le \\ \le c|h|^{-2}\|a_1\|_{C(\overline{\Omega})}\|\widetilde{u}\|_{W_2^m(\widetilde{e})}.$$

Therefore using again the generalized Bramble–Hilbert lemma, we have $|\eta'_{11}| \le c|h|^{-2}\|a_1\|_{C(\overline{\Omega})}|\widetilde{u}|_{W_2^m(\widetilde{e})} \le c|h|^{m-2}(h_1h_2)^{-1/2}\|a_1\|_{C(\overline{\Omega})}|u|_{W_2^m(e)}$, whence

$$\|\eta'_{11}\|_{(2)} \le c|h|^{m-2}\|a_1\|_{C(\overline{\Omega})}|u|_{W_2^m(\Omega)}. \tag{10.26}$$

$\eta''_{11} = \eta''_{11}(a_1)$ is a linear (with respect to $a_1(x)$, for fixed $u(x)$) bounded functional,

$$|\eta''_{11}| \le c(h_1h_2)^{-1/q}|u|_{W_q^2(e)}\|\widetilde{a}_1\|_{C(\widetilde{e})} \le c(h_1h_2)^{-1/q}|u|_{W_q^2(e)}\|\widetilde{a}_1\|_{W_p^{m-2}(\widetilde{e})},$$

which vanishes on $\boldsymbol{\pi}_0$. Consequently,

$$|\eta''_{11}| \le c(h_1h_2)^{-1/q}|u|_{W_q^2(e)}|\widetilde{a}_1|_{W_p^{m-2}(\widetilde{e})} \le c(h_1h_2)^{-1/2}|u|_{W_q^2(e)}|a_1|_{W_p^{m-2}(e)},$$

and

$$\|\eta''_{11}\|_{(2)} \le c|h|^{m-2}|a_1|_{W_p^{m-2}(\Omega)}|u|_{W_q^2(\Omega)}. \tag{10.27}$$

Taking into account (10.26), (10.27) and analogous estimates for η'_{22} and η''_{22}, as well as the imbeddings $W_p^{m-2} \subset C$, $W_2^m \subset W_q^2$, from (10.25) we get

$$\|\eta_{\alpha\alpha}\|_{(3-\alpha)} \le c|h|^{m-2}\|a_\alpha\|_{W_p^{m-2}(\Omega)}\|u\|_{W_2^m(\Omega)}. \tag{10.28}$$

Further, representing $\eta_{\alpha 0}$ in the form $\eta_{\alpha 0} = (T_{3-\alpha}(a_0 D_{3-\alpha}^2 u) - T_{3-\alpha}a_0 T_{3-\alpha}D_{3-\alpha}^2 u) + T_{3-\alpha}D_{3-\alpha}^2 u(T_{3-\alpha}a_0 - a_0)$, similarly to (10.25) we have

$$\|\eta_{\alpha 0}\|_{(3-\alpha)} \le c|h|^{m-2}\|a_0\|_{W_p^{m-2}(\Omega)}\|u\|_{W_2^m(\Omega)}. \tag{10.29}$$

From (10.12) it follows $|\eta_3| \le 2\|a_3\|_{C(e)}(h_1h_2)^{-1}\int_e |D_1D_2u|\,d\xi \le c(h_1h_2)^{-1/q}\|\widetilde{a}_3\|_{W_p^{m-2}(\tilde{e})}|u|_{W_q^2(e)}$. Since η_3 is a linear bounded functional with respect to $a_3 \in W_p^{m-2}$ and vanishes on $\boldsymbol{\pi}_0$,

$$|\eta_3| \le c(h_1h_2)^{-1/q}|a_3|_{W_p^{m-2}(\tilde{e})}|u|_{W_q^2(e)} \le$$
$$\le c|h|^{m-2}(h_1h_2)^{-1/2}|a_3|_{W_p^{m-2}(e)}|u|_{W_q^2(e)},$$

whence $\|\eta_3\|_{\omega^+} \le c|h|^{m-2}|a_3|_{W_p^{m-2}(\Omega)}|u|_{W_q^2(\Omega)}$. The latter together with the imbedding $W_2^m \subset W_q^2$ results in

$$\|\eta_3\|_{\omega^+} \le c|h|^{m-2}|a_3|_{W_p^{m-2}(\Omega)}|u|_{W_2^m(\Omega)}. \tag{10.30}$$

Relying now on the estimate (10.24) and the inequalities (10.28), (10.29) and (10.30), we can write

$$\|\Delta_h z\| \le c|h|^{m-2}|u|_{W_2^m(\Omega)} \max_{0\le\alpha\le 3}\|a_\alpha\|_{W_p^{m-2}(\Omega)}, \quad 2.5 < m \le 3. \tag{10.31}$$

Combining the results (10.23), (10.31) and taking into account the fact that the norms $\|\Delta_h z\|$ and $\|z\|_{W_2^2(\omega)}$ are equivalent in the space $\overset{\circ}{H}$ (see [L1]), we obtain the following

Theorem 10.4. *Let the coefficients of the problem* (10.1), (10.2) *satisfy the conditions* (10.3), (10.4)*, and the solution* $u(x) \in W_2^m(\Omega)$*. Then the convergence rate of the difference scheme* (10.7) *is characterized by the estimate*

$$\|y-u\|_{W_2^2(\omega)} \le c|h|^{m-2}\max_{0\le\alpha\le 3}\|a_\alpha\|_{W_p^{m-2}(\Omega)}\|u\|_{W_2^m(\Omega)}, \quad 2.5 < m \le 4, \tag{10.32}$$

where the constant $c > 0$ *does not depend on* $|h|$ *and* $u(x)$.

CHAPTER 3

Schemes of Higher Accuracy

To avoid cumbersome calculations, it is desirable for the difference scheme to be sufficiently good on rough meshes, i.e. to have higher order accuracy. The problem of increasing the accuracy of a method without increasing standard pattern of difference schemes has always been topical.

In this chapter we suggest difference schemes of higher accuracy for: elliptic equations with the mixed derivatives and lowest derivatives (Sections 11, 12); the problem of bending of an orthotropic plate simply supported over the contour (Section 15). In the case we fail in obtaining such schemes, it is reasonable to obtain an approximate solution of higher accuracy by using the method due to Richardson in which solution of a difference scheme is given on a sequence of meshes (Section 14).

In the problem of bending of an orthotropic plate we suggest a new method of decomposition.

11. Elliptic Equation with a Mixed Derivative

History of the matter. The results of Section 11 have been published in [68]. Analogous estimates for difference schemes of more complicated structure than in the present paragraph were established in [11]. The schemes converging with the rate $O(h^4)$ to the solution $u(x) = C^6(\overline{\Omega})$ of the original problem, were suggested and investigated in [70].

In [51] consistent estimates were obtained for the difference schemes (under the condition that the differential equation is satisfied outside of the boundaries, and the solution $u(x)$ preserves the required smoothness).

A difference scheme with the estimate (0.1) for $s = 2$, $m \in (4, 6]$, is considered in [81].

1^0. Statement of the problem. Here we suggest and investigate difference schemes of higher accuracy which approximate the problem

$$\frac{\partial^2 u}{\partial x_1^2} + 2a\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} = -f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \Gamma, \qquad (11.1)$$

with the solutions $u(x) \in W_2^m(\Omega)$, $m > 1$. By Ω is denoted a rectangle with the boundary Γ, the constant a satisfies the condition $|a| < 1$ for which the operator (11.1) is elliptic.

Suppose that the lengths ℓ_1 and ℓ_2 of the rectangle sides are commensurable.

Let ω be a quadratic mesh, i.e. $h_1 = h_2 = h$.

We introduce the space H of the mesh functions defined on $\overline{\omega}$ and equal to zero on γ, with the inner product $(y, v) = (y, v)_\omega$ and with the norm $\|y\| = (y, y)^{1/2}$.

On the mesh $\overline{\omega}$ we approximate the problem (11.1) by the difference scheme

$$\Lambda y = -\varphi(x), \quad x \in \omega, \quad y(x) = 0, \quad x \in \gamma, \tag{11.2}$$

$$\varphi(x) = T_1 T_2 f + \frac{ah^2}{6}\,(S_1^- S_2^- f)_{x_1 x_2}, \tag{11.3}$$

where $\Lambda y = y_{\overline{x}_1 x_1} + a(y_{\overline{x}_1 x_2} + y_{x_1 \overline{x}_2}) + y_{\overline{x}_2 x_2} + \frac{h^2}{6}\,(1 + 3a + 2a^2) y_{\overline{x}_1 x_1 \overline{x}_2 x_2}$.

The existence of a unique solution of the problem (11.2) for any right-hand side $\varphi(x)$ is proved in [70].

2^0. A priori estimates of error. The error $z = y - u$ is a solution of the problem

$$\Lambda z = -\psi(x), \quad x \in \omega, \quad z(x) = 0, \quad x \in \gamma. \tag{11.4}$$

Here $\psi = \Lambda u + \varphi$ is the approximation error which can be transformed to $\psi = \eta^{(1)}_{\overline{x}_1 x_1} + a\eta^{(3)}_{x_1 x_2} + \eta^{(2)}_{\overline{x}_2 x_2}$, where $\eta^{(3-\alpha)} = u - T_\alpha u + \frac{h^2}{12}\, u_{\overline{x}_\alpha x_\alpha}$, $\alpha = 1, 2$, $\eta^{(3)} = u^{(-1_1)} + u^{(-1_2)} + \frac{h^2}{2}\, u_{\overline{x}_1 \overline{x}_2} - 2S_1^- S_2^- \big(u + \frac{h^2}{12}\,\Delta u\big)$.

Theorem 11.1. *For a solution z of the problem* (11.4) *the estimates*

$$\|z\|_{W_2^s(\omega)} \le \nu^{-1} J_s, \quad \nu = \frac{2}{3}\,(1 - |a|), \quad s = 0, 1, 2, \tag{11.5}$$

hold, where $J_0 = \|\eta^{(1)}\| + \|\eta^{(2)}\| + |a|\|\eta^{(3)}\|_{\omega^+}$, $J_2 = \|\eta^{(1)}_{\overline{x}_1 x_1}\| + \|\eta^{(2)}_{\overline{x}_2 x_2}\| + |a|\|\eta^{(3)}_{x_1 x_2}\|$, $J_1 = \|\eta^{(1)}_{\overline{x}_1}\|_{\omega_1^+ \times \omega_2} + \|\eta^{(2)}_{\overline{x}_2}\|_{\omega_1 \times \omega_2^+} + |a|\|\eta^{(3)}_{x_1}\|_{\omega_1 \times \omega_2^+}$.

Proof. The validity of Theorem 12.1 for the case $s = 0$ follows from the inequality $\|z\| \le \big\|\Lambda^{-1}\eta^{(1)}_{\overline{x}_1 x_1}\big\| + \big\|\Lambda^{-1}\eta^{(2)}_{\overline{x}_2 x_2}\big\| + |a|\,\big\|\Lambda^{-1}\eta^{(3)}_{x_1 x_2}\big\|$, whose summands in the right-hand side are estimated analogously to [83]. To prove the theorem for $s = 1, 2$ we use the formulas of summation by parts, the estimates $\nu\|z\|_1^2 \le (-\Lambda z, z)$ and $\nu\|z\|_2 \le \|-\Lambda z\|$ established in [70] (pp. 320, 321), and the Cauchy–Buniakowski's inequality. □

3^0. Accuracy of the scheme. It is not difficult to verify that $\eta^{(\alpha)}$, $\alpha = 1, 2, 3$, being linear functionals of the function $u(x)$, vanish for $u(x) \in \boldsymbol{\pi}_3$ and are bounded in $W_2^m(\Omega)$, $m > 1$. The first and the second difference derivatives of $\eta^{(\alpha)}$ are likewise bounded in $W_2^m(\Omega)$, $m > 1$, and vanish, respectively, for $u(x) \in \boldsymbol{\pi}_4$ and $u(x) \in \boldsymbol{\pi}_5$.

Using the generalized Bramble–Hilbert lemma and the above-mentioned properties of the functionals $\eta^{(\alpha)}$, we establish upper bounds for the functional norms, their first and second difference derivatives, respectively through the values $h^m|u|_{m,\Omega}$, $m \in (1, 4]$, $h^{m-1}|u|_{m,\Omega}$, $m \in (1, 5]$, $h^{m-2}|u|_{m,\Omega}$, $m \in (1, 6]$. Using these estimates in the inequalities (11.5), we can see that the following theorem is valid.

Theorem 11.2. *If a solution of the problem* (11.1) *is sufficiently smooth, i.e.* $m > 1$*, then the convergence of the scheme* (11.2), (11.3) *is characterized by the estimates*

$$\|y - u\|_{W_2^s(\omega)} \leq ch^{m-s}|u|_{W_2^m(\Omega)}, \quad m \in (3, 4+s], \quad s = 0, 1, 2. \tag{11.6}$$

4^0. Modification of the scheme. Consider the scheme (11.2) with the right-hand side

$$\widetilde{\varphi} = T_1 T_2 f + \frac{ah^2}{12}\,(f_{\overline{x}_1 x_2} + f_{x_1 \overline{x}_2}). \tag{11.7}$$

The following theorem is valid.

Theorem 11.3. *If a solution of the problem* (11.2) *is sufficiently smooth, i.e.* $m > 1$*, then the convergence of the scheme* (11.2), (11.7) *is characterized by the estimates*

$$\|\widetilde{y} - u\|_{W_2^s(\omega)} \leq ch^{m-s}|u|_{W_2^m(\Omega)}, \quad m \in (3, 4+s], \quad s = 0, 1, 2. \tag{11.8}$$

Proof. Let y and $\widetilde{y}$ be, respectively, solutions of the discrete problems (11.2), (11.3) and (11.2), (11.7). For the difference $\widetilde{y} - y$ we formulate the following problem:

$$\begin{gathered} \Lambda(\widetilde{y} - y) = \frac{ah^2}{12}\left(2S_1^- S_2^- f - f^{(-1_1)} - f^{(-1_2)}\right)_{x_1 x_2}, \quad x \in \omega, \\ \widetilde{y} - y = 0, \quad x \in \gamma. \end{gathered} \tag{11.9}$$

For a solution of the problem (11.9) it is not difficult to obtain an estimate through the semi-norm of the right-hand side of the problem (11.1), $\|\widetilde{y} - y\|_{W_2^s(\omega)} \leq ch^{2+\beta-s}|f|_{W_2^\beta(\Omega)}$, $\beta \in (1, 2+s]$, $s = 0, 1, 2$, or through the semi-norm of a solution of the problem (11.1), $\|\widetilde{y} - y\|_{W_2^s(\omega)} \leq ch^{m-s}|u|_{W_2^m(\Omega)}$, $m \in (3, 4+s]$, $s = 0, 1, 2$.

To see that (11.8) is valid, it remains to make use of the triangle inequality $\|\widetilde{y} - u\|_{W_2^s(\omega)} \leq \|\widetilde{y} - y\|_{W_2^s(\omega)} + \|y - u\|_{W_2^s(\omega)}$, whose second summand is estimated by Theorem 11.2. □

Remark 11.1. Using the imbedding $W_2^1(\omega)$ in $C(\Omega)$ for the mesh functions of two variables with the multiplier $|\ln h|^{0.5}$ ([63]), we obtain for the schemes (11.2), (11.3) and (11.2), (11.7) the estimates

$$\|y - u\|_{C(\omega)} \leq c|\ln h|^{0.5} h^{m-1}|u|_{W_2^m(\Omega)} \tag{11.10}$$

with $m \in (1, 5]$ and $m \in (3, 5]$, respectively.

Using the imbedding $W_2^2(\omega)$ in $C(\omega)$, for the both schemes we obtain $\|y - u\|_{C(\omega)} \leq ch^4|u|_{W_2^6(\Omega)}$.

12. Elliptic Equation with the Lowest Derivatives

History of the matter. The results of this section have been published in [18]. In [47], for the problem (12.1) on the square mesh a difference scheme possessing the fourth order accuracy is considered. This scheme is a particular case of the constructed and investigated earlier in [3] scheme of higher accuracy, converging with the rate $O(|h|^4)$ to the solution $u(x) \in C^6(\overline{\Omega})$.

1^0. Statement of the problem. In the rectangle Ω we consider the Dirichlet problem

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \lambda_1 \frac{\partial u}{\partial x_1} + \lambda_2 \frac{\partial u}{\partial x_2} = -f(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \Gamma, \quad (12.1)$$

where λ_1 and λ_2 are constant coefficients. It is assumed that $f \in W_2^{m-2}(\Omega)$ and $W_2^m(\Omega)$, $m \geq 2$.

Suppose that lengths ℓ_1 and ℓ_2 of the sides of the rectangle Ω are commensurable. Let ω be a quadratic mesh, i.e. $h_1 = h_2 = h$.

Consider the space H of the mesh functions given on ω, with the inner product $(y, v) = (y, v)_\omega$ and with the norm $\|y\| = \|y\|_\omega$.

In H, the designations $\|\cdot\|_{(\alpha+)}$ and $\|\cdot\|_{(\alpha)}$ take the form $\|v\|^2_{(1+)} = \sum\limits_{\omega_1^+ \times \omega_2} h_1 h_2 v^2$, $\|v\|^2_{(2+)} = \sum\limits_{\omega_1 \times \omega_2^+} h_1 h_2 v^2$, $\|v\|_{(\alpha)} = \|v\|$.

In the space H we define the operators Λ_α, ∂_α, $\alpha = 1, 2$, as follows: $\Lambda_\alpha y = \overset{\circ}{y}_{\overline{x}_\alpha x_\alpha}$, $\partial_\alpha y = \overset{\circ}{y}_{\overset{\circ}{x}_\alpha}$, $y \in H$ and $\overset{\circ}{y}(x) = y(x)$ for $x \in \omega$, $\overset{\circ}{y}(x) = 0$ for $x \in \gamma$.

Let $\Lambda = -\Lambda_1 - \Lambda_2$. It is known (see, e.g., [69]) that the operator Λ is self-conjugate, positive definite in H, and

$$\Big(\frac{8}{\ell_1^2} + \frac{8}{\ell_2^2}\Big) E \leq \Lambda \leq \Big(\frac{4}{h_1^2} + \frac{4}{h_2^2}\Big) E, \quad Ey \equiv y. \quad (12.2)$$

It is not difficult to verify that

$$|y|_{W_2^1(\omega)} = (\Lambda y, y)^{1/2}, \quad |y|_{W_2^2(\omega)} = \|\Lambda y\|. \quad (12.3)$$

We approximate the problem (12.1) by the difference scheme

$$L_h y \equiv (A + B + C) y = \varphi, \quad x \in \omega, \quad y \in H, \quad (12.4)$$

where

$$\varphi(x) = T_1 T_2 \Big(f + \frac{h^2}{12} \sum_{\alpha=1}^{2} \lambda_\alpha \frac{\partial f}{\partial x_\alpha}\Big), \quad (12.5)$$

$$A = -\sum_{\alpha=1}^{2} \Lambda_\alpha \Big(E + \frac{h^2}{12} \Lambda_{3-\alpha}\Big), \quad C = \sum_{\alpha=1}^{2} \lambda_\alpha \partial_\alpha \Big(E + \frac{h^2}{6} \Lambda_{3-\alpha}\Big),$$

$$B = -\frac{h^2}{12} \big(\lambda_1^2 \Lambda_1 + \lambda_2^2 \Lambda_2 + 2\lambda_1 \lambda_2 \partial_1 \partial_2\big).$$

2^0. A priori estimates. The following lemma is valid.

Lemma 12.1. *The operator L_h is positive definite in H and the estimate*

$$L_h \geq \frac{2}{3}\,\Lambda, \tag{12.6}$$

$$\|\Lambda y\| \leq c_0 \|L_h y\|, \tag{12.7}$$

$$\|L_h^{-1} y\| \leq c_0 \|\Lambda^{-1} y\| \tag{12.8}$$

is valid with $c_0 = 1.5(1 + 3\sqrt{2\ell_1\ell_2}\lambda/8)$, $\lambda = \max(|\lambda_1|; |\lambda_2|)$.

Proof. The estimate (12.6) follows from the easily verifiable relations $A = A^* \geq \frac{2}{3}\,\Lambda$, $B = B^* \geq 0$, $C = -C^*$, and on the basis of (12.2) it is obvious that the operator L_h is positive definite.

Next, taking into account the relation $\|y_{\overset{\circ}{x}_1\overset{\circ}{x}_2}\| \leq \|y_{\overline{x}_1\overline{x}_2}\|_{\omega^+} = (\Lambda_1 y, \Lambda_2 y)$, we have

$$(By, -\Lambda_\alpha y) \geq$$

$$\geq \frac{h^2}{12}\Big(\lambda_\alpha^2 \|\Lambda_\alpha y\|^2 + \lambda_{3-\alpha}^2 \|y_{\overline{x}_1\overline{x}_2}\|_{\omega^+}^2 - 2|\lambda_1\lambda_2|\,\|\Lambda_\alpha y\|\,\|y_{\overline{x}_1\overline{x}_2}\|_{\omega^+}\Big) \geq 0.$$

Moreover, $(Ay, -\Lambda_\alpha y) \geq \frac{2}{3}\,\|\Lambda_\alpha y\|^2 + \frac{2}{3}\,(\Lambda_1 y, \Lambda_2 y)$, $\alpha = 1, 2$. Consequently, $((A+B)y, \Lambda y) \geq (2/3)\|\Lambda y\|$, so

$$\|\Lambda y\| \leq \frac{3}{2}\,\|(A+B)y\|. \tag{12.9}$$

Obviously,

$$\|Cy\| \leq \lambda \sum_{\alpha=1}^{2} \Big\|\Big(E + \frac{h^2}{6}\,\Lambda_{3-\alpha}\Big)\partial_\alpha y\Big\| \leq$$

$$\leq \lambda \sum_{\alpha=1}^{2} \Big\|\Big(E + \frac{h^2}{6}\,\Lambda_{3-\alpha}\Big)\Big\|\,\|\partial_\alpha y\| \leq \lambda\big(\|\partial_1 y\| + \|\partial_2 y\|\big),$$

and hence

$$\|Cy\| \leq \sqrt{2}\,\lambda |y|_{W_2^1(\omega)}. \tag{12.10}$$

Using the inequality (12.6) and the difference analogue of the Fridrichs inequality, we obtain

$$|y|_{W_2^1(\omega)} \leq \frac{3\sqrt{\ell_1\ell_2}}{8}\,\|L_h y\|, \tag{12.11}$$

and from (12.10) it follows that the estimate

$$\|Cy\| \leq \frac{3\sqrt{2\ell_1\ell_2}}{8}\,\lambda\|L_h y\| \tag{12.12}$$

is valid.

Replacing in (12.9) $A + B = L_h - C$ and taking into account (12.12), we arrive at (12.7).

Analogously to (12.7), we can see that the estimate

$$\|\Lambda y\| \leq c\|L_h^* y\|, \ \ \forall\, y \in H, \tag{12.13}$$

is valid.

Further, the operator $L_h^* L_h$ is self-conjugate and positive definite in H. Therefore

$$\|L_h^{-1} y\| = \big\|(L_h^* L_h)^{-1} L_h^* y\big\| =$$
$$= \sup_{v\neq 0} \frac{|(L_h^* y, v)|}{\|L_h^* L_h v\|} = \sup_{v\neq 0} \frac{|(\Lambda^{-1} y, \Lambda L_h v)|}{\|L_h^* L_h v\|} \le \|\Lambda^{-1} y\| \sup_{v\neq 0} \frac{\|\Lambda L_h v\|}{\|L_h^* L_h v\|},$$

whence on the basis of (12.13) we obtain the inequality (12.8). □

The error $z = y - u$ of the difference scheme (12.4), (12.5) is a solution of the problem

$$L_h z = \psi, \quad z \in H, \tag{12.14}$$

where the approximation error is representable in the form of the sum

$$\psi = \sum_{\alpha=1}^{2} \big(\Lambda_\alpha \eta_{\alpha 1} + \lambda_\alpha \Lambda_{3-\alpha} \eta_{\alpha 2} + \lambda_\alpha \eta_{\alpha 3 x_\alpha} + \lambda_\alpha^2 \Lambda_\alpha \eta_{\alpha 4}\big) + \lambda_1 \lambda_2 \eta_{x_1 x_2}. \tag{12.15}$$

Here $\eta_{\alpha 1} = u + \frac{h^2}{12} \Lambda_{3-\alpha} u - T_{3-\alpha} u$, $\eta_{\alpha 2} = \frac{h^2}{6} \big(\frac{1}{2}(u + u^{(-1_\alpha)}) - S_\alpha^- u\big)_{x_\alpha}$, $\eta_{\alpha 3} = \frac{1}{2}(u + u^{(-1_\alpha)}) + \frac{h^2}{12} S_\alpha^- \Lambda_{3-\alpha} u - S_\alpha^- T_{3-\alpha} u - \frac{h^2}{12} T_{3-\alpha} \big(\frac{\partial u}{\partial x_\alpha}\big)_{\overline{x}_\alpha}$, $\eta_{\alpha 4} = \frac{h^2}{12}(u - T_{3-\alpha} u)$, $\eta = \frac{h^2}{6}\big(\frac{1}{4}\big(u + u^{(-1_1)} + u^{(-1_2)} + u^{(-1_1,-1_2)}\big) - S_1^- S_2^- u\big)$.

Since the operator L_h is positive definite, this implies that the scheme (12.4), (12.5) and the problem (12.14) are uniquely solvable.

Lemma 12.2. *For the solution of the difference boundary value problem* (12.14) *the a priori estimates*

$$\|z\| \le$$
$$\le c\Big(\sum_{\alpha=1}^{2}\Big(\|\eta_{\alpha 1}\| + \frac{\lambda \ell_\alpha}{\sqrt{8}} \|\eta_{\alpha 3}\|_{(\alpha+)} + \lambda^2 \|\eta_{\alpha 4}\| + \lambda \|\eta_{\alpha 2}\|\Big) + \lambda^2 \|\eta\|_{\omega^+}\Big), \tag{12.16}$$

$$|z|_{W_2^1(\omega)} \le \lambda^2 \|\eta_{x_1}\|_{(2+)} + \frac{2}{3}\Big(\sum_{\alpha=1}^{2} \|\eta_{\alpha 1 \overline{x}_\alpha}\|_{(\alpha+)} + \lambda \|\eta_{\alpha 3}\|_{(\alpha+)} +$$
$$+ \lambda^2 \|\eta_{\alpha 4 \overline{x}_\alpha}\|_{(\alpha+)} + \lambda \|\eta_{\alpha 2 \overline{x}_\beta}\|_{(\beta+)}\Big), \quad \beta = 3 - \alpha, \tag{12.17}$$

$$\|\Lambda z\| \le \lambda^2 \|\eta_{x_1 x_2}\| +$$
$$+ c\Big(\sum_{\alpha=1}^{2} \|\Lambda_\alpha \eta_{\alpha 1}\| + \lambda \|\eta_{\alpha 3 x_\alpha}\| + \lambda^2 \|\Lambda_\alpha \eta_{\alpha 4}\| + \lambda \|\Lambda_{3-\alpha} \eta_{\alpha 2}\|\Big) \tag{12.18}$$

are valid.

Proof. The inequality (12.16) is obtained from the estimate (12.8). To prove the estimate (12.17), we have to multiply both parts of (12.14) scalarly by z and make use of the formulas of summation by parts and the inequality (12.6). The estimate (12.18) follows directly from (12.14) by using (12.7). □

3^0. Accuracy of the scheme. The norms $|\cdot|_{W_2^s(\omega)}$ and $\|\cdot\|_{W_2^s(\omega)}$, $s=1,2$, are equivalent. Therefore the inequalities (12.15)–(12.17) together with the Bramble-Hilbert lemma allow one to establish that the following statement on the accuracy of the difference scheme (12.4), (12.5) is valid.

Theorem 12.1. *Let the solution of the problem* (12.1) *belong to the space* $W_2^m(\Omega)$*. Then the convergence rate of the difference scheme* (12.4), (12.5) *is defined by the estimate*

$$\|y-u\|_{W_2^s(\omega)} \le ch^{m-s}\,\|u\|_{W_2^m(\Omega)}, \quad m\in[2,4+s], \quad s=0,1,2. \qquad (12.19)$$

Let now the right-hand side in (12.4) be defined by the equality

$$\varphi = f + \frac{h^2}{12}\big(\Lambda_1+\Lambda_2+\lambda_1\partial_1+\lambda_2\partial_2\big)f. \qquad (12.20)$$

The difference scheme (12.4), (12.20) is, in fact, the scheme from [2], [47].

The following statement is valid.

Theorem 12.2. *Let the solution of the problem* (12.1) *belong to the space* $W_2^m(\Omega)$*,* $m>3$*. Then the convergence rate of the different scheme* (12.4), (12.20) *is defined by the estimate*

$$\|y-u\|_{W_2^s(\omega)} \le ch^{m-s}\,\|u\|_{W_2^m(\Omega)}, \quad s=2, \quad m\in(3,6]. \qquad (12.21)$$

As is mentioned in [47], the equations of the type (12.1) often appear in the problems of hydrodynamics upon linearization of the equation of motion, and it is desirable for the corresponding difference scheme to possess good accuracy for sufficiently large values of λ_α. Therefore of special interest is to write out the estimate for the convergence rate with regard for λ. Such estimate, for example, in the norm $W_2^2(\omega)$ for the scheme (12.4), (12.5) has the form $\|y-u\|_{W_2^2(\omega)} \le ch^{m-2}(1+\lambda h^{\varkappa_1}+\lambda^2h^{\varkappa_2})\|u\|_{W_2^m(\Omega)}$, $m\in(3,6]$, where $\varkappa_1=\min(1;6-m)$, $\varkappa_2=\min(2;6-m)$.

Remark 12.1. Using the imbedding of $W_2^1(\omega)$ in $C(\omega)$ for the mesh functions of two variables with the multiplier $|\ln h|^{0.5}$ ([63]), for the schemes (12.4), (12.5) and (12.4), (12.20) we obtain the estimates $\|y-u\|_{C(\omega)} \le c|\ln h|^{0.5}h^{m-1}\|u\|_{W_2^m(\Omega)}$ with $m\in(2,5]$ and $m\in(3,5]$, respectively.

Using the imbedding $W_2^2(\omega)$ in $C(\omega)$, for both schemes we obtain $\|y-u\|_{C(\omega)} \le ch^{m-2}\|u\|_{W_2^m(\Omega)}$ with $m\in(2,6]$ and $m\in(3,6]$, respectively.

13. Richardson's Method of Extrapolation

History of the matter. One of the methods of constructing approximate solutions of higher accuracy is Richardson's method of extrapolation in which the use is made of the solution of difference schemes on a sequence of meshes. For the Poisson equation, this method is justified in the works by E. A. Volkov (see, e.g., [82]) and considered in detail in [61]. However, the above-mentioned works essentially use Taylor's formula, and this results in too strict smoothness requirements imposed on the coefficients and solutions of the original problem.

The present section is devoted to obtaining consistent estimates for the convergence rate in Richardson's method of extrapolation for elliptic equations with mixed derivatives and variable coefficients. The results of Section 13 have been published in [14]. In the case of constant coefficients, analogous results were obtained in [9].

1^0. Statement of the problem. In the domain Ω we consider the Dirichlet problem

$$Lu \equiv -\sum_{\alpha=1,2}^{2} \frac{\partial}{\partial x_\alpha}\Big(a_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\Big) + a_0 u = f,\ \ x \in \Omega, \quad u(x) = 0,\ \ x \in \Gamma. \tag{13.1}$$

It is assumed that

$$\begin{gathered} a_{\alpha\beta} \in W_2^{m-1}(\Omega)\ \ (\alpha,\beta = 1,2),\ \ 0 \le a_0(x) \in W_2^{m-2}(\Omega), \\ f(x) \in W_2^{m-2}(\Omega),\ \ u(x) \in W_2^{m}(\Omega),\ \ m \in (3,4] \end{gathered} \tag{13.2}$$

and let the condition of uniform ellipticity

$$\sum_{\alpha,\beta=1,2}^{2} a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \ge \nu(\xi_1^2 + \xi_2^2),\ \ \nu = \text{const} > 0,\ \ x \in \Omega, \tag{13.3}$$

be fulfilled.

We approximate the problem (13.1) by the difference scheme

$$Ay \equiv -\sum_{\alpha,\beta=1,2}^{2} \Lambda_{\alpha\beta} y + ay = \varphi,\ \ x \in \omega, \quad y(x) = 0,\ \ x \in \gamma, \tag{13.4}$$

where $\Lambda_{\alpha\beta} y = 0.5\big((a^+_{\alpha\beta} y_{x_\beta})_{\overline{x}_\alpha} + (a^-_{\alpha\beta} y_{\overline{x}_\beta})_{x_\alpha}\big)$, $\varphi(x) = T_1T_2 f$, $a^+_{\alpha\beta}(x) = S_1^+S_2^+ a_{\alpha\beta}$, $a^-_{\alpha\beta}(x) = a^+_{\alpha\beta}(x_1 - h_1, x_2 - h_2)$, $a(x) = T_1T_2 a_0$.

Let H be the space of the mesh functions defined on $\overline{\omega}$ and equal to zero on γ, with the inner product $(y,v) = (y,v)_\omega$ and the norm $\|y\| = \|y\|_\omega$. The notation $\|\cdot\|_{(\alpha+)}$ takes the form $\|v\|^2_{(1+)} = \sum\limits_{\omega_1^+\times\omega_2} h_1h_2v^2$, $\|v\|^2_{(2+)} = \sum\limits_{\omega_1\times\omega_2^+} h_1h_2v^2$.

The operator A is positive definite (and for $a_{12}(x) \equiv a_{21}(x)$ is self-conjugate as well) in H, hence the problem (13.4) is uniquely solvable, and the estimate (see (3.6))

$$\|y\|_{W_2^1(\omega)} \le \frac{1}{\nu}\Big(1 + \frac{\ell_1^2\ell_2^2}{8(\ell_1^2+\ell_2^2)}\Big)(Ay,y),\ \ \forall\, y \in H, \tag{13.5}$$

is valid.

Let $y^h(x)$ be a solution of the difference scheme (13.4) on the mesh $\omega_h = \omega$, and $y^{h/2}(x)$ be a solution on the mesh $\omega_{h/2}$ with the steps $h_1/2$ and $h_2/2$.

We will show that the linear combination

$$\widetilde{y}(x)=\frac{4y^{h/2}(x)-y^h(x)}{3}\,,\quad x\in\overline{\omega}_h \tag{13.6}$$

has higher accuracy as $|h|\to 0$.

2^0. Expansion of the difference solution. In the course of our investigation we will consider the following auxiliary differential problems:

$$\begin{gathered}Lv_{(\alpha)}=\frac{\partial}{\partial x_\alpha}\,(\Phi_{\alpha\alpha}-F_{\alpha\alpha})+\frac{\partial}{\partial x_{3-\alpha}}\,(\Phi_{3-\alpha}-F_{3-\alpha,\alpha})+F_\alpha,\quad x\in\Omega,\\ v_{(\alpha)}\big|_\Gamma=0,\quad \alpha=1,2,\\ Lv_{(3)}=-\frac{\partial}{\partial x_1}\,\Phi_{12}-\frac{\partial}{\partial x_2}\,\Phi_{21},\quad x\in\Omega,\quad v_{(3)}(x)=0,\quad x\in\Gamma,\end{gathered} \tag{13.7}$$

where

$$F_{\alpha\beta}=\frac{\partial a_{\alpha\alpha}}{\partial x_\beta}\,\frac{\partial^2 u}{\partial x_\alpha\partial x_\beta}+ +|\alpha-\beta|\Big(a_{\alpha\alpha}\,\frac{\partial^3 u}{\partial x_\alpha\partial x_\beta^2}+\frac{\partial a_{\alpha\alpha}}{\partial x_\beta}\,\frac{\partial^2 u}{\partial x_\alpha\partial x_\beta}-\frac{\partial^2 a_{\alpha\alpha}}{\partial x_\beta^2}\,\frac{\partial u}{\partial x_\alpha}\Big),$$

$$\Phi_{\alpha\beta}=a_{\alpha,3-\alpha}\,\frac{\partial^3 u}{\partial x_\beta^2\partial x_{3-\beta}}-(-1)^{\alpha+\beta}\,\frac{\partial a_{\alpha,3-\alpha}}{\partial x_\beta}\,\frac{\partial^2 u}{\partial x_1\partial x_2}\,,\quad \alpha,\beta=1,2,$$

$$F_\alpha=a_0\,\frac{\partial^2 u}{\partial x_\alpha^2}+2\,\frac{\partial a_0}{\partial x_\alpha}\,\frac{\partial u}{\partial x_\alpha}\,,$$

$$\Phi_\alpha=a_{\alpha\beta}\,\frac{\partial^3 u}{\partial x_\beta^3}+\frac{\partial a_{\alpha\beta}}{\partial x_\beta}\,\frac{\partial^2 u}{\partial x_\beta^2}+\frac{\partial^2 a_{\alpha\beta}}{\partial x_\beta^2}\,\frac{\partial u}{\partial x_\beta},\quad \beta=3-\alpha,\quad \alpha=1,2.$$

The fulfilment of the assumptions (13.2) and (13.3) implies the existence of a unique (from the class $W_2^\lambda(\Omega)$, $1<\lambda\le 2$) solution of the problem (13.1), satisfying the a priori estimates ([71], p. 172)

$$\begin{gathered}\|v_{(\alpha)}\|_{m-2,\Omega}\le c\Big(\|\Phi_{\alpha\alpha}-F_{\alpha\alpha}\|_{m-3,\Omega}+\\ +\|\Phi_{3-\alpha}-F_{3-\alpha,\alpha}\|_{m-3,\Omega}+\|F_\alpha\|_{m-3,\Omega}\Big),\quad \alpha=1,2,\\ \|v_{(3)}\|_{m-2,\Omega}\le c\Big(\|\Phi_{12}\|_{m-3,\Omega}+\|\Phi_{21}\|_{m-3,\Omega}\Big),\quad m\in(3,4].\end{gathered} \tag{13.8}$$

Let

$$\begin{gathered}\eta_0=T_1T_2(a_0u)-T_1T_2a_0u-T_1T_2\Big(\frac{h_1^2}{12}\,F_1+\frac{h_2^2}{12}\,F_2\Big),\\ \eta_{\alpha\alpha}=S_\alpha^-\overline{S}_\beta a_{\alpha\alpha}u_{\overline{x}_\alpha}-S_\alpha^-T_\beta\Big(a_{\alpha\alpha}D_\alpha u-\frac{h_\alpha^2}{12}\,F_{\alpha\alpha}-\frac{h_\beta^2}{12}\,F_{\alpha\beta}\Big),\\ \eta_{\alpha\beta}=0.5\big(S_1^-S_2^-a_{\alpha\beta}u_{\overline{x}_\beta}+S_\alpha^-S_\beta^+a_{\alpha\beta}u_{x_\beta}^{(-1_\alpha)}\big)-\\ -S_\alpha^-T_\beta\Big(a_{\alpha\beta}D_\beta u+\frac{h_\alpha^2}{12}\,\Phi_{\alpha\alpha}+\frac{h_\beta^2}{12}\,\Phi_\alpha-\frac{h_1h_2}{4}\,\Phi_{\alpha\beta}\Big),\ \beta=3-\alpha,\ \alpha=1,2.\end{gathered} \tag{13.9}$$

Then after some transformations we will have

$$\Lambda_{\alpha\alpha}u = T_1T_2\Big(D_\alpha(a_{\alpha\alpha}D_\alpha u) - \frac{h_\alpha^2}{12}\,D_\alpha F_{\alpha\alpha} - \frac{h_\beta^2}{12}\,D_\alpha F_{\alpha\beta}\Big) + (\eta_{\alpha\alpha})_{x_\alpha},$$

$$\Lambda_{\alpha\beta}u = T_1T_2\Big(D_\alpha(a_{\alpha\beta}D_\beta u) + \frac{h_\alpha^2}{12}\,D_\alpha\Phi_{\alpha\alpha} + \frac{h_\beta^2}{12}\,D_\alpha\Phi_\alpha - \frac{h_1h_2}{4}\,D_\alpha\Phi_{\alpha\beta}\Big) + \\ +(\eta_{\alpha\beta})_{x_\alpha},\ \ \beta = 3-\alpha,\ \ \alpha = 1,2,$$

and hence

$$Au = T_1T_2(Lu) + \frac{h_1^2}{12}\,T_1T_2\big(D_1(F_{11}-\Phi_{11}) + D_2(F_{21}-\Phi_2) - F_1\big) + \\ + \frac{h_2^2}{12}\,T_1T_2\big(D_1(F_{12}-\Phi_1) + D_2(F_{22}-\Phi_{22}) - F_2\big) + \\ + \frac{h_1h_2}{4}\,T_1T_2\big(D_1\Phi_{12} + D_2\Phi_{21}\big) - (\eta_{11}+\eta_{12})_{x_1} - (\eta_{21}+\eta_{22})_{x_2} - \eta_0. \quad (13.10)$$

Introduce the notation

$$\eta_0^{(j)} = T_1T_2(a_0v_{(j)}) - T_1T_2a_0v_{(j)},$$
$$\eta_{\alpha\alpha}^{(j)} = S_\alpha^-\overline{S}_\beta a_{\alpha\alpha}(v_{(j)})_{\overline{x}_\alpha} - S_\alpha^-T_\beta\big(a_{\alpha\alpha}D_\alpha v_{(j)}\big),$$
$$\eta_{\alpha\beta}^{(j)} = 0.5\big(S_1^-S_2^-a_{\alpha\beta}(v_{(j)})_{\overline{x}_\beta} + S_\alpha^-S_\beta^+a_{\alpha\beta}(v_{(j)}^{(-1_\alpha)})_{x_\beta}\big) - \\ - S_\alpha^-T_\beta\Big(a_{\alpha\beta}\,\frac{\partial v_{(j)}}{\partial x_\beta}\Big),\ \ \beta = 3-\alpha,\ \ \alpha = 1,2,\ \ j = 1,2,3. \quad (13.11)$$

It is not difficult to notice that

$$\Lambda_{\alpha\beta}v_{(j)} = T_1T_2D_\alpha\big(a_{\alpha\beta}D_\beta v_{(j)}\big) + (\eta_{\alpha\beta}^{(j)})_{x_\alpha},\ \ \alpha,\beta = 1,2,\ \ j = 1,2,3,$$

and thus

$$Av_{(j)} = T_1T_2\Big(Lv_{(j)} - \sum_{\alpha,\beta=1}^{2}(\eta_{\alpha\beta}^{(j)})_{x_\alpha} - \eta_0^{(j)}\Big),\ \ j = 1,2,3. \quad (13.12)$$

The functions $y(x)$, $u(x)$, $v_j(x)$, $j = 1,2,3$, are defined on the mesh $\overline{\omega}$. Therefore the function $z = y - u - \frac{h_1^2}{12}\,v_{(1)} - \frac{h_2^2}{12}\,v_{(2)} - \frac{h_1h_2}{4}\,v_{(3)}$ will likewise be defined on the mesh $\overline{\omega}$.

Substituting in (13.4) the obtained from the above inequality function

$$y(x) = u(x) + \frac{h_1^2}{12}\,v_{(1)}(x) + \frac{h_2^2}{12}\,v_{(2)}(x) + \frac{h_1h_2}{4}\,v_{(3)}(x) + z(x),\ \ x \in \overline{\omega}, \quad (13.13)$$

and taking into account (13.1), (13.7), (13.10), (13.12), we obtain for z the problem

$$Az = \psi_{1x_1} + \psi_{2x_2} + \psi_0,\ \ x \in \omega,\quad z(x) = 0,\ \ x \in \gamma, \quad (13.14)$$

where

$$\psi_0 = \eta_0 + \frac{h_1^2}{12}\,\eta_0^{(1)} + \frac{h_2^2}{12}\,\eta_0^{(2)} + \frac{h_1h_2}{4}\,\eta_0^{(3)}, \quad (13.15)$$

$$\psi_\alpha = \sum_{\beta=1}^{2} \Big(\eta_{\alpha\beta} + \frac{h_1^2}{12}\,\eta_{\alpha\beta}^{(1)} + \frac{h_2^2}{12}\,\eta_{\alpha\beta}^{(2)} + \frac{h_1h_2}{4}\,\eta_{\alpha\beta}^{(3)}\Big), \ \ \alpha = 1,2. \qquad (13.16)$$

Using the inequality (13.5), it is not difficult to get an a priori estimate for a solution of the problem (13.14): $\|z\|_{W_2^1(\omega)} \le c\big(\|\psi_1\|_{(1+)} + \|\psi_2\|_{(2+)} + \|\psi_0\|\big)$, so taking into account (13.15), (13.16), we find

$$\|z\|_{W_2^1(\omega)} \le$$
$$\le c\Big(\|\eta_0\| + \sum_{\alpha,\beta=1}^{2} \|\eta_{\alpha\beta}\|_{(\alpha+)} + |h|^2 \sum_{j=1}^{3}\Big(\|\eta_0^{(j)}\| + \sum_{\alpha,\beta=1}^{2} \|\eta_{\alpha\beta}^{(j)}\|_{(\alpha+)}\Big)\Big). \qquad (13.17)$$

Thus we have proved the following

Lemma 13.1. *Let $u(x)$ and $v_{(j)}(x)$, $j = 1,2,3$, be solutions of the problem* (13.1), (13.7)*, respectively, and let the assumptions* (13.2), (13.3) *be valid. Then for the solution $y(x)$ of the difference scheme* (13.4) *the expansion* (13.13) *is valid, and for the mesh function $z(x)$ the estimate* (13.17) *is valid.*

3^0. Estimation of a solution of the problem (13.4). To obtain an estimate for the convergence rate of the extrapolation solution (13.6), we will need an estimate of the mesh function $z(x)$ appearing in the right-hand side of the inequality (13.17).

Let $e = e(x) = (x_1 - h_1, x_1 + h_1) \times (x_2 - h_2, x_2 + h_2)$, $e_\alpha = e_\alpha(x) = (x_1 - h_1, x_1 + (\alpha - 1)h_1) \times (x_2 - h_2, x_2 + (2 - \alpha)h_2)$, $\alpha = 1,2$.

We prove the following

Lemma 13.2. *Let $u \in W_2^m(\Omega)$, $a_0 \in W_2^{m-2}(\Omega)$, $a_{\alpha\beta} \in W_2^{m-1}(\Omega)$, $\alpha,\beta = 1,2$, $m \in (3,4]$. Then for the expressions $\eta_{\alpha\beta}$ $(\alpha,\beta = 1,2)$, η_0, defined by the formulas* (13.3) *the estimates*

$$|\eta_{\alpha\beta}| \le c|h|^{m-1}(h_1h_2)^{-1/2} \times$$
$$\times\Big(|a_{\alpha\beta}D_\beta u|_{m-1,e_\alpha} + \|a_{\alpha\beta}\|_{m-1,\Omega}\|u\|_{m,e_\alpha} + \|a_{\alpha\beta}\|_{m-1,e_\alpha}\|u\|_{m,\Omega}\Big), \quad (13.18)$$

$$|\eta_0| \le c|h|^{m-1}(h_1h_2)^{-1/2}\Big(\|a_0\|_{m-2,\Omega}|u|_{m-1,e} +$$
$$+ \sum_{\alpha=1}^{2}\Big(\sum_{\beta=1}^{2} |a_0 D_\alpha D_\beta u|_{m-3,e} + |a_0 D_\alpha u|_{m-2,e}\Big)\Big) \qquad (13.19)$$

are fulfilled.

Proof. Using the notation

$$\ell^{(1)}(v) = S_1^- v - \overline{v} - \frac{h_1^2}{24}\,S_1^- T_2 D_1^2 u,$$
$$\ell^{(2)}(v) = \overline{v} - S_1T_2\Big(v - \frac{h_1^2}{24}\,D_1^2 v - \frac{h_2^2}{12}\,D_2^2 v\Big), \ \ \overline{v}(x) \equiv v\Big(x_1 - \frac{h_1}{2}, x_2\Big),$$

$$\ell^{(3)}(v) = S_1^- \overline{S}_2 v - \overline{v} - S_1^- T_2\Big(\frac{h_1^2}{24}\, D_1^2 v + \frac{h_2^2}{6}\, D_2^2 v\Big),$$
$$\ell^{(4)}(v,w) = S_1^- T_2 v S_1^- \overline{S}_2 w - S_1^- T_2(vw),$$
$$\ell^{(5)}(v,w) = S_1^- T_2 \overline{w} - S_1^- T_2(vw),$$

we represent η_{11} in the form

$$\eta_{11} = S_1^- \overline{S}_2 a_{11} \ell^{(1)}(D_1 u) + \ell^{(2)}(a_{11} D_1 u) + D_1 \overline{u} \ell^{(3)}(a_{11}) + \\ + \ell^{(4)}\Big(\frac{h_1^2}{24}\, D_1^2 a_{11} + \frac{h_2^2}{6}\, D_2^2 a_{11}, D_1 u\Big) + \frac{h_1^2}{24}\, \ell^{(5)}(D_1^3 u, a_{11}). \quad (13.20)$$

It can be easily verified that the functionals $\ell^{(i)}(v)$, $i = 1,2,3$, are bounded in $W_2^2(\Omega)$ and vanish on polynomials of second degree. Therefore using the Bramble–Hilbert lemma, we obtain

$$|\ell^{(i)}(v)| \le c|h|^m (h_1 h_2)^{-1/2} |v|_{m,e_1}, \quad m \in [2,3], \quad i = 1,2,3. \quad (13.21)$$

If $w \in W_2^m(\Omega)$, $m > 2$, then $w \in C^1(\overline{\Omega})$, and hence the direct checking results in the estimate

$$|\ell^{(i)}(v,w)| \le c|h|(h_1 h_2)^{-1/2} \|v\|_{0,e_1} \|w\|_{m,\Omega}, \quad m > 2, \quad i = 4,5. \quad (13.22)$$

Using the inequalities (13.21) and (13.22), we see from (13.20) that the estimate (13.18) for $\alpha = \beta = 1$ is valid. The functional η_{22} is estimated analogously.

Estimate now the functional η_{12}. Towards this end, we transform it as follows:

$$\eta_{12} = \ell^{(2)}(a_{12} D_2 u) + D_2 \overline{u} \ell^{(3)}(a_{12}) + \ell^{(4)}\Big(\frac{h_1^2}{24}\, D_1^2 a_{12} + \frac{h_2^2}{6}\, D_2^2 a_{12}, D_2 u\Big) + \\ + \ell^{(5)}\Big(\frac{h_1^2}{8}\, D_1^2 D_2 u + \frac{h_2^2}{6}\, D_2^3 u - \frac{h_1 h_2}{4}\, D_1 D_2^2 u, a_{12}\Big) + \\ + \ell^{(5)}\Big(D_2 a_{12}, \frac{h_2^2}{4}\, D_2^2 u - \frac{h_1 h_2}{4}\, D_1 D_2 u\Big) + \\ + \frac{h_2}{4}\, S_1^- T_2 D_2 a_{12}\, \ell^{(6)}(D_2 u) + S_1^- \overline{S}_2 a_{12}\, \ell^{(7)}(D_2 u), \quad (13.23)$$

where

$$\ell^{(6)}(v) = S_2^+ v^{(-1_1)} - S_2^- v + h_1 S_1^- \overline{S}_2 D_1 v - h_2 S_1^- \overline{S}_2 D_2 v,$$
$$\ell^{(7)}(v) = 0.5\big(S_2^- v + S_2^+ v^{(-1_1)}\big) - \overline{v} - \\ - S_1^- T_2\Big(\frac{h_1^2}{8}\, D_1^2 v + \frac{h_2^2}{6}\, D_2^2 v - \frac{h_1 h_2}{4}\, D_1 D_2 v\Big).$$

The first four summands in (13.23) we estimate by means of (13.21) and (13.22), and the rest of the summands we estimate by the inequality

$$|\ell^{(5)}(v,w)| \le c|h|^\lambda (h_1 h_2)^{-1/2} \|v\|_{1+\lambda,\Omega} \|w\|_{\lambda,e_1},$$
$$|\ell^{(6)}(v) \le c|h|^{1+\lambda} (h_1 h_2)^{-1/2} |v|_{1+\lambda,e_1},$$
$$|\ell^{(7)}(v) \le c|h|^{2+\lambda} (h_1 h_2)^{-1/2} |v|_{2+\lambda,e_1}, \quad \lambda \in (0,1],$$

obtained by virtue of Theorem 1.2 and the Bramble–Hilbert lemma.

As a result, we arrive at (13.18) for the case $\alpha = 1$, $\beta = 2$. The functional η_{21} is estimated analogously.

Finally, we represent η_0 as

$$\eta_0 = \ell_0(u) + \widetilde{\ell}_0(a_0 D_1 D_2 u) + \sum_{\alpha=1}^{2} \big(\ell_\alpha(a_0 D_\alpha u) + \widetilde{\ell}_\alpha(a_0 D_\alpha^2 u)\big), \qquad (13.24)$$

where

$$\begin{aligned}
\ell_0(u) = T_1T_2\Big[a_0(\zeta)\Big(&u(\zeta) - u(x) - (\zeta_1 - x_1)\,\frac{\partial u(\zeta)}{\partial \zeta_1} - (\zeta_2 - x_2)\,\frac{\partial u(\zeta)}{\partial \zeta_2} + \\
&+ \frac{(\zeta_1 - x_1)^2}{2}\,\frac{\partial^2 u(\zeta)}{\partial \zeta_1^2} + \frac{(\zeta_2 - x_2)^2}{2}\,\frac{\partial^2 u(\zeta)}{\partial \zeta_2^2} - \\
&- (\zeta_1 - x_1)(\zeta_2 - x_2)\,\frac{\partial^2 u(\zeta)}{\partial \zeta_1 \partial \zeta_2}\Big)\Big],\\
\widetilde{\ell}_0(v) = T_1T_2\big(&(\zeta_1 - x_1)(\zeta_2 - x_2)v(\zeta)\big),\\
\ell_\alpha(v) = T_1T_2\big(&(\zeta_\alpha - x_\alpha)v(\zeta)\big) - \frac{h_\alpha^2}{6}\,T_1T_2\,\frac{\partial v(\zeta)}{\partial \zeta_\alpha}\,,\\
\widetilde{\ell}_\alpha(v) = \frac{h_\alpha^2}{12}\,&T_1T_2 v - \frac{1}{2}\,T_1T_2\big((\zeta_\alpha - x_\alpha)^2 v(\zeta)\big), \quad \alpha = 1, 2,
\end{aligned}$$

ζ_α is a variable with respect to which we perform integration in T_α, $\alpha = 1, 2$, $\zeta = (\zeta_1, \zeta_2)$.

Note that: the linear functionals $\ell_\alpha(v)$ $(\alpha = 1, 2)$ are bounded in the space $W_2^1(\Omega)$ and vanish on the polynomials of first degree; $\widetilde{\ell}_\alpha(v)$ $(\alpha = 0, 1, 2)$ are bounded in $L_2(\Omega)$ and vanish on the constants; $\ell_0(u)$, being a linear functional of $u(x)$ (for the given function $a_0 \in C(\overline{\Omega})$)), is bounded in $W_2^2(\Omega)$ and vanishes on polynomials of second degree.

Then using Theorem 1.2 and the Bramble–Hilbert lemma, we obtain the estimates

$$\begin{aligned}
&|\ell_\alpha(v)| \le c|h|^{2+\lambda}(h_1h_2)^{-1/2}|v|_{1+\lambda,e}, \quad \alpha = 1, 2,\\
&|\widetilde{\ell}_\alpha(v)| \le c|h|^{2+\lambda}(h_1h_2)^{-1/2}|v|_{\lambda,e}, \quad \alpha = 0, 1, 2,\\
&|\ell_0(u)| \le c|h|^{2+\lambda}(h_1h_2)^{-1/2}|u|_{2+\lambda,e}\|a_0\|_{C(\overline{\Omega})}, \quad 0 < \lambda \le 1,
\end{aligned}$$

by means of which from (13.24) it follows that the estimate (13.19) is valid. □

We rewrite $\eta_0^{(j)}$ in the form $\eta_0^{(j)} = \big(T_1T_2a_0T_1T_2v_{(j)} - T_1T_2a_0v_{(j)}\big) + \big(T_1T_2(a_0v_{(j)}) – T_1T_2a_0\,T_1T_2v_{(j)}\big)$, and $\eta_{\alpha,\beta}^{(j)}$ in the form indicated in Section 3. Estimating each group of summands by the Bramble–Hilbert lemma, we prove that the following lemma is valid.

Lemma 13.3. *Let* $v_{(j)} \in W_2^{m-2}(\Omega)$ $(j = 1, 2, 3)$*,* $a_0 \in W_2^{m-2}(\Omega)$*,* $a_{\alpha\beta} \in W_2^{m-1}(\Omega)$ $(\alpha, \beta = 1, 2)$*,* $m \in (3, 4]$*. Then for* $\eta_{\alpha\beta}^{(j)}$ *and* $\eta_0^{(j)}$ *defined in*

(13.11) *the estimates*

$$|\eta_{\alpha\beta}^{(j)}| \le c|h|^{m-3}(h_1h_2)^{-1/2}\big(\|a_{\alpha\beta}\|_{C(\overline{\Omega})}\,|v_{(j)}|_{m-2,e} + \big|a_{\alpha\beta}D_\beta v_{(j)}\big|_{m-3,e}\big),$$
$$|\eta_0^{(j)}| \le c|h|^{m-3}(h_1h_2)^{-1/2}\|a_0\|_{C(\overline{\Omega})}\|v_{(j)}\|_{m-2,e},\quad \alpha,\beta=1,2,\quad j=1,2,3,$$

are fulfilled.

Using Lemma 1.1, from (13.18) and (13.19) we obtain

$$\|\eta_0\| + \sum_{\alpha,\beta=1}^{2}\|\eta_{\alpha\beta}\|_{(\alpha+)} \le c|h|^{m-1}\|u\|_{m,\Omega},\quad m\in(3,4]. \tag{13.25}$$

Analogously, by means of Lemmas 1.2 and 13.3, we find that $\|\eta_0^{(j)}\| + \sum\limits_{\alpha,\beta=1}^{2}\|\eta_{\alpha\beta}^{(j)}\|_{(\alpha+)} \le c|h|^{m-3}\|v_{(j)}\|_{m-2,\Omega}$. But on the basis of Lemmas 1.1 and 1.2, from (13.8) we obtain $\|v_{(j)}\|_{m-2,\Omega} \le c\|u\|_{m,\Omega}$. Hence

$$\|\eta_0^{(j)}\| + \sum_{\alpha,\beta=1}^{2}\|\eta_{\alpha\beta}^{(j)}\|_{(\alpha+)} \le c|h|^{m-3}\|u\|_{m,\Omega},\quad j=1,2,3,\quad m\in(3,4]. \tag{13.26}$$

Taking now into account the inequalities (13.25) and (13.26), from the estimate (13.17) we conclude that

$$\|z\|_{W_2^1(\omega)} \le c|h|^{m-1}\|u\|_{m,\Omega},\quad m\in(3,4]. \tag{13.27}$$

4^0. Convergence of the improved solution. Let $y^h(x)$ and $y^{h/2}(x)$ be solutions of the difference scheme (13.4) respectively on the meshes ω_h and $\omega_{h/2}$.

By Lemma 13.1, the expansions

$$y^h(x) = u(x) + \frac{h_1^2}{12}v_{(1)}(x) + \frac{h_2^2}{12}v_{(2)}(x) + \frac{h_1h_2}{4}v_{(3)}(x) + z^h(x),\quad x\in\overline{\omega}_h, \tag{13.28}$$

$$y^{h/2}(x) = u(x) + \frac{h_1^2}{48}v_{(1)}(x) + \frac{h_2^2}{48}v_{(2)}(x) + \frac{h_1h_2}{16}v_{(3)}(x) + z^{h/2}(x),\quad x\in\overline{\omega}_{h/2}, \tag{13.29}$$

are valid for them.

Here z^h and $z^{h/2}$ are solutions of the difference problem (13.14) on the meshes ω_h and $\omega_{h/2}$, respectively, for which, according to (13.27), the estimates

$$\|z^h\|_{W_2^1(\omega_h)} \le c|h|^{m-1}\|u\|_{m,\Omega},\quad \|z^{h/2}\|_{W_2^1(\omega_{h/2})} \le c|h|^{m-1}\|u\|_{m,\Omega} \tag{13.30}$$

are valid.

From (13.28), (13.29) and (13.6) it follows

$$\|u-\widetilde{y}\|_{W_2^1(\omega_h)} \le \frac{4}{3}\|z^{h/2}\|_{W_2^1(\omega_h)} + \frac{1}{3}\|z^h\|_{W_2^1(\omega_h)}. \tag{13.31}$$

But for any mesh function $z^{h/2}$ defined on the mesh $\omega_{h/2}$ we can write

$$\|z^{h/2}\|^2_{\omega_h} = 4\sum_{x\in\omega_h} \frac{h_1}{2}\,\frac{h_2}{2}\,|z^{h/2}(x)|^2 \le 4\sum_{x\in\omega_{h/2}} \frac{h_1}{2}\,\frac{h_2}{2}\,|z^{h/2}(x)|^2 = 4\|z^{h/2}\|_{\omega_{h/2}}.$$

Estimating analogously the first differences of $z^{h/2}$, we obtain $\|z^{h/2}\|_{W_2^1(\omega_h)} \le 2\|z^{h/2}\|_{W_2^1(\omega_{h/2})}$, and hence from (13.30) and (13.31) we can conclude that

$$\|u - \widetilde{y}\|_{W_2^1(\omega)} \le c|h|^{m-1}\|u\|_{m,\Omega}, \quad m \in (3,4]. \tag{13.32}$$

Thus we have proved the following

Theorem 13.1. *Let the coefficients of the differential equation in the Dirichlet problem* (13.1) *satisfy the condition* (13.3) *of ellipticity. Moreover, let* $0 \le a_0 \in W_2^{m-2}(\Omega)$, $a_{\alpha\beta} \in W_2^{m-1}$ $(\alpha,\beta = 1,2)$, *and the solution of the problem* (13.1) *belong to the space* $W_2^m(\Omega)$, $m \in (3,4]$. *Then the convergence of the Richardson-extrapolated solution* (13.6) *is characterized by the estimate* (13.32).

14. The Problem of Bending of Orthotropic Plate

History of the matter. For a numerical solution of the problem of bending of an isotropic plate, many authors (see, e.g., [62], [32], [46]) apply the method of reduction of a biharmonic equation to two Poisson equations (the method due to Marcus). The method suggested in [2] can be considered as a generalization of the above-mentioned expansion to the case of orthotropic plates. In [8], the authors suggest a decomposition of another type. In the present section, relying on the methods of decomposition ([2], [8]), we construct schemes of higher accuracy and obtain consistent estimates for the convergence rate. Difference schemes for the biharmonic equation (free from expansion) with solutions from the Sobolev space have been studied in [71], [59], [42], [40], [50]. The results of this section have been published in [8], [19].

1^0. As is known,the equation of elastic equilibrium of a homogeneous orthotropic plate has the form

$$\mathcal{D}_1\,\frac{\partial^4 w}{\partial x_1^4} + 2\mathcal{D}_3\,\frac{\partial^4 w}{\partial x_1^2\partial x_2^2} + \mathcal{D}_2\,\frac{\partial^4 w}{\partial x_2^4} = q(x), \tag{14.1}$$

where w is the midsurface deflection of the plate; q is the intensity of transversal load; $\mathcal{D}_i$, $i = 1,2,3$, are the constants depending on the Young modulus, plate thickness and Poisson coefficients ν_1, ν_2 for the principal directions. In the case of an isotropic plate, $\nu_1 = \nu_2 = \nu$, $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \mathcal{D}$.

Let us consider an isotropic plate simply supported over the contour. The axes x_1 and x_2 are directed along its sides whose lengths we denote by ℓ_1 and ℓ_2. Thus in the rectangle $\Omega = \{(x_1,x_2):\ 0 < x_i < \ell_i,\ i = 1,2\}$ with the boundary Γ we seek for a solution of the equation (14.1) satisfying the boundary conditions

$$w(x) = \frac{\partial^2 w(x)}{\partial x_1^2} = \frac{\partial^2 w(x)}{\partial x_2^2} = 0, \quad x \in \Gamma. \tag{14.2}$$

By $\varkappa_1$ and $\varkappa_2$ we denote the midsurface curvatures of the plate in the directions x_1 and x_2, respectively. A characteristic peculiarity of the method is that instead of the problem (14.1), (14.2) we approximate successively two problems: the system of differential equations

$$B_1 \frac{\partial^2 \varkappa}{\partial x_1^2} + B_2 \frac{\partial^2 \varkappa}{\partial x_2^2} = -Q(x), \quad x \in \Omega, \quad \varkappa(x) = 0, \quad x \in \Gamma, \tag{14.3}$$

where $B_1 = \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}$, $B_2 = \begin{pmatrix} 2\mathcal{D}_3 & \mathcal{D}_2 \\ -\mathcal{D}_2 & 0 \end{pmatrix}$, $\varkappa = \begin{pmatrix} \varkappa_1 \\ \varkappa_2 \end{pmatrix}$, $Q = \begin{pmatrix} q \\ 0 \end{pmatrix}$ and the one-dimensional equation

$$\frac{\partial^2 w}{\partial x_1^2} = -\varkappa_1(x) \ \left(\text{or } \frac{\partial^2 w}{\partial x_2^2} = -\varkappa_2(x)\right), \quad x \in \Omega, \quad w(x) = 0, \quad x \in \Gamma. \tag{14.4}$$

The second equation of the system (14.3) is, in fact, the condition of compatibility of deformations, while the first one is the equation of statics (14.1).

The bending moments $\mathcal{M}_1$ and $\mathcal{M}_2$ and the transversal forces $\mathcal{N}_1$ and $\mathcal{N}_2$ are defined by the formulas $\mathcal{M}_1 = \mathcal{D}_1(\varkappa_1 + \nu_2\varkappa_2)$, $\mathcal{M}_2 = \mathcal{D}_2(\nu_1\varkappa_1 + \varkappa_2)$, $\mathcal{N}_1 = \frac{\partial}{\partial x_1}(\mathcal{D}_1\varkappa_1 + \mathcal{D}_3\varkappa_2)$, $\mathcal{N}_2 = \frac{\partial}{\partial x_2}(\mathcal{D}_3\varkappa_1 + \mathcal{D}_2\varkappa_2)$.

2^0. Let $\overline{\Lambda}_\alpha = \Lambda_\alpha\big(E + \frac{h_{3-\alpha}^2}{12}\Lambda_{3-\alpha}\big)$, $Ey = y$, $\Lambda_\alpha y = y_{\bar{x}_\alpha x_\alpha}$, $\alpha = 1, 2$.

We approximate the problem (14.3) by the difference scheme

$$B_1\overline{\Lambda}_1\widetilde{\varkappa} + B_2\overline{\Lambda}_2\widetilde{\varkappa} = -\widetilde{Q}, \quad x \in \omega, \quad \widetilde{\varkappa}(x) = 0, \quad x \in \gamma, \tag{14.5}$$

while the problem (14.4) by the scheme

$$\Lambda_1\widetilde{w} = -\Big(E - \frac{h_1^2}{12}\Lambda_1\Big)\widetilde{\varkappa}_1, \quad x \in \omega, \quad \widetilde{w}(x) = 0, \quad x \in \gamma. \tag{14.6}$$

Here, $\widetilde{\varkappa} = (\widetilde{\varkappa}_1, \widetilde{\varkappa}_2)^T$, $\widetilde{Q} = (T_1T_2q, 0)^T$, and the mesh functions $\widetilde{\varkappa}_1$, $\widetilde{\varkappa}_2$, $\widetilde{w}$ are approximate values of the functions $\varkappa_1$, $\varkappa_2$, w in the mesh nodes.

The scheme (14.6) is, in fact, a one-dimensional problem. Its second argument is taken as a parameter.

Below we will prove that the problems (14.5), (14.6) are uniquely solvable. Having found a solution of the problem (14.5), we can find approximate values of the bending moments and transversal forces in the nodes of the mesh ω by the formulas

$$\widetilde{\mathcal{M}}_1 = \mathcal{D}_1(\widetilde{\varkappa}_1 + \nu_2\widetilde{\varkappa}_2), \quad \widetilde{\mathcal{M}}_2 = \mathcal{D}_2(\nu_1\widetilde{\varkappa}_1 + \widetilde{\varkappa}_2),$$

$$\widetilde{\mathcal{N}}_1 = (\mathcal{D}_1\widetilde{\varkappa}_1 + \mathcal{D}_3\widetilde{\varkappa}_2)_{\overset{\circ}{x}_1}, \quad \widetilde{\mathcal{N}}_2 = (\mathcal{D}_3\widetilde{\varkappa}_1 + \mathcal{D}_2\widetilde{\varkappa}_2)_{\overset{\circ}{x}_2} \quad \text{for } w \in W_2^m(\Omega), \quad 3 < m \leq 5,$$

$$\widetilde{\mathcal{N}}_1 = \Big(\mathcal{D}_1\widetilde{\varkappa}_1 + \mathcal{D}_3\widetilde{\varkappa}_2 + \frac{h_1^2}{6}\Lambda_2(\mathcal{D}_2\widetilde{\varkappa}_2 + \mathcal{D}_3\widetilde{\varkappa}_1) + \frac{h_1^2}{6}q\Big)_{\overset{\circ}{x}_1},$$

$$\widetilde{\mathcal{N}}_2 = \Big(\mathcal{D}_3\widetilde{\varkappa}_1 + \mathcal{D}_2\widetilde{\varkappa}_2 + \frac{h_2^2}{6}\,\Lambda_1(\mathcal{D}_1\widetilde{\varkappa}_1 + \mathcal{D}_3\widetilde{\varkappa}_2) + \frac{h_2^2}{6}q\Big)_{\overset{\circ}{x}_2}$$
$$\text{for } \ w \in W_2^m(\Omega), \ \ 5 < m \le 7.$$

3^0. By $\overset{\circ}{H}$ we denote the set of the mesh functions defined on $\overline{\omega}$ and vanishing on γ. Let H be the set of the mesh functions defined on the mesh ω, with the inner product $(y,v)=(y,v)_\omega$ and with the norm $\|y\|=\|y\|_\omega$.

In H, the designations $(\cdot\,,\cdot)_{(\alpha_+)}$, $\|\cdot\|_{(\alpha+)}$, $\|\cdot\|_{(\alpha)}$ take the form

$$(y,v)_{(1+)} = \sum_{\omega_1^+\times\omega_2} h_1h_2yv, \quad (y,v)_{(2+)} = \sum_{\omega_1\times\omega_2^+} h_1h_2yv,$$
$$\|v\|_{(\alpha+)} = (v,v)_{(\alpha+)}^{1/2}, \quad \|v\|_{(\alpha)} = \|v\|.$$

In the space H we define the linear operators A_α, $\overline{A}_\alpha$, $\alpha=1,2$, as follows: $\overline{A}_\alpha y = -\overline{\Lambda}_\alpha \overset{\circ}{y}$, $A_\alpha y = -\Lambda_\alpha\overset{\circ}{y}$, $y\in H_h$, $\overset{\circ}{y}\in\overset{\circ}{H}_h$, $\overset{\circ}{y}(x)=y(x)$ for $x\in\omega$. Let $A=A_1+A_2$.

The operators A_α (and hence $\overline{A}_\alpha, A$) are self-conjugate, positive definite and permutational for which the estimates (see, e.g., [69], p. 274)

$$\frac{8}{\ell_\alpha^2}\,E \le A_\alpha \le \frac{4}{h_\alpha^2}\,E, \ \ \frac{2}{3}\,A_\alpha \le \overline{A}_\alpha \le A_\alpha \tag{14.7}$$

are valid.

We write the scheme (14.5), (14.6) in the operator form

$$L_h\widetilde{\varkappa} \equiv B_1\overline{A}_1\widetilde{\varkappa} + B_2\overline{A}_2\widetilde{\varkappa} = \widetilde{Q}, \ \ \widetilde{\varkappa},\, \widetilde{Q}\in H\times H, \tag{14.8}$$

$$A_1\widetilde{w} = \Big(E - \frac{h_1^2}{12}\,A_1\Big)\widetilde{\varkappa}_1, \ \ \widetilde{w},\, \varkappa_1\in H. \tag{14.9}$$

Not giving rise to misunderstanding, for the inner product and for the norms of the mesh functions and vector-functions the use will be made of the same notation: $(\boldsymbol{y},\boldsymbol{v}) = (y_1,v_1)+(y_2,v_2)$, $|\boldsymbol{y}\| = (\boldsymbol{y},\boldsymbol{y})^{1/2}$, $\boldsymbol{y}$, $\boldsymbol{v}\in H\times H$.

Lemma 14.1. *The operator L_h is positive definite in $H\times H$ and the estimate $L_h\ge c_1E$ is valid, where $c_1=\min\big((8/\ell_1^2)\mathcal{D}_1+(16/\ell_2^2)\mathcal{D}_3;(8/\ell_1^2)\mathcal{D}_2\big)$.*

Proof. Indeed, this follows from the inequality $(L_h\boldsymbol{y},\boldsymbol{y}) = \mathcal{D}_1(\overline{A}_1y_1,y_1) + \mathcal{D}_2(\overline{A}_1y_2,y_2) + 2\mathcal{D}_3(\overline{A}_2y_1,y_1) \ge \big(\frac{8}{\ell_1^2}\,\mathcal{D}_1 + \frac{16}{\ell_2^2}\,\mathcal{D}_3\big)\|y_1\|^2 + \frac{8}{\ell_1^2}\,\|y_2\|^2$. □

Owing to the positive definiteness of the operators L_h and A_1, the equations (14.8), (14.9) (or the difference scheme (14.5), (14.6)) are uniquely solvable.

Lemma 14.2. *Let L_h^* be the operator adjoint to L_h. Then there exists a constant $c_2>0$ such that $\|A\boldsymbol{y}\|\le c_2\|L_h\boldsymbol{y}\|$, $\|A\boldsymbol{y}\|\le c_2\|L_h^*\boldsymbol{y}\|$ for every $\boldsymbol{y}\in H\times H$.*

Proof. Let $B_3 = \begin{pmatrix}\sqrt{\mathcal{D}_2} & 0\\ 0 & \sqrt{\mathcal{D}_1}\end{pmatrix}$.

Since

$$B_3L_h^* = B_3B_1\overline{A}_1 + B_3B_2^T\overline{A}_2 = \begin{pmatrix} \mathcal{D}_1\sqrt{\mathcal{D}_2}\,\overline{A}_1 + 2\mathcal{D}_3\sqrt{\mathcal{D}_2}\,\overline{A}_2 & -\mathcal{D}_2\sqrt{\mathcal{D}_2}\,\overline{A}_2 \\ \mathcal{D}_2\sqrt{\mathcal{D}_1}\,\overline{A}_2 & \mathcal{D}_2\sqrt{\mathcal{D}_1}\,\overline{A}_1 \end{pmatrix},$$

we have

$$\|B_3L_h^*\boldsymbol{y}\|^2 = \mathcal{D}_1^2\mathcal{D}_2\|\overline{A}_1y_1\|^2 + \mathcal{D}_1\mathcal{D}_2^2\|\overline{A}_1y_2\|^2 + (4\mathcal{D}_3^2\mathcal{D}_2 + \mathcal{D}_2^2\mathcal{D}_1)\|\overline{A}_2y_1\|^2 +$$
$$+ \mathcal{D}_2^3\|\overline{A}_2y_2\|^2 + 4\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3(\overline{A}_1y_1, \overline{A}_2y_1) + 2\varepsilon\mathcal{D}_1\mathcal{D}_2^2(\overline{A}_1y_2, \overline{A}_2y_2) -$$
$$- 2\varepsilon\mathcal{D}_1\mathcal{D}_2^2(\overline{A}_1y_2, \overline{A}_2y_2) - 4\mathcal{D}_2^2\mathcal{D}_3(\overline{A}_2y_1, \overline{A}_2y_2).$$

Applying to the last two summands Cauchy–Buniakowski inequality, as well as the ε-inequality, we obtain

$$\|B_3L_h^*\boldsymbol{y}\|^2 \geq \mathcal{D}_1^2\mathcal{D}_2\|\overline{A}_1y_1\|^2 + \mathcal{D}_1\mathcal{D}_2^2(1 - 2\varepsilon\varepsilon_1)\|\overline{A}_1y_2\|^2 +$$
$$+ (4\mathcal{D}_3^2\mathcal{D}_2 + \mathcal{D}_2^2\mathcal{D}_1 - 4\varepsilon_2\mathcal{D}_2^2\mathcal{D}_3)\|\overline{A}_2y_1\|^2 +$$
$$+ \left(\mathcal{D}_2^3 - \frac{\varepsilon}{2\varepsilon_1}\mathcal{D}_1\mathcal{D}_2^2 - \frac{1}{\varepsilon_2}\mathcal{D}_2^2\mathcal{D}_3\right)\|\overline{A}_2y_2\|^2 +$$
$$+ 4\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3(\overline{A}_1y_1, \overline{A}_2y_1) + 2\varepsilon\mathcal{D}_1\mathcal{D}_2^2(\overline{A}_1y_2, \overline{A}_2y_2). \qquad (14.10)$$

Let $\varepsilon_1 = \frac{1-\varepsilon}{2\varepsilon}$, $\varepsilon_2 = \frac{\mathcal{D}_3(1-\varepsilon)}{\mathcal{D}_2 - \varepsilon(\mathcal{D}_1 + \mathcal{D}_2)}$, $0 < \varepsilon < \frac{\mathcal{D}_2}{\mathcal{D}_1 + \mathcal{D}_2}$.

Then from (14.10) it follows that

$$\frac{1}{\mathcal{D}_1\mathcal{D}_2}\|B_3L_h^*\boldsymbol{y}\|^2 \geq \mathcal{D}_1\|\overline{A}_1y_1\|^2 + \frac{\mathcal{D}_2^2 - \varepsilon(\mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2^2 + 4\mathcal{D}_3^2)}{\mathcal{D}_2 - \varepsilon(\mathcal{D}_1 + \mathcal{D}_2)}\|\overline{A}_2y_1\|^2 +$$
$$+ 4\mathcal{D}_3(\overline{A}_1y_1, \overline{A}_2y_1) + \varepsilon\mathcal{D}_2(\|\overline{A}_1y_2\|^2 + 2(\overline{A}_1y_2, \overline{A}_2y_2) + \|\overline{A}_2y_2\|^2). \qquad (14.11)$$

Taking into account the restriction imposed earlier on ε, from the requirement for the coefficient of $\|\overline{A}_2y_1\|$ to be positive, it follows that $0 < \varepsilon < \mathcal{D}_2^2(\mathcal{D}_2(\mathcal{D}_1 + \mathcal{D}_2) + 4\mathcal{D}_3^2)^{-1}$. We choose, for example, $\varepsilon = \mathcal{D}_2^2(\mathcal{D}_2(\mathcal{D}_1 + 2\mathcal{D}_2) + 4\mathcal{D}_3^2)^{-1}$. Then

$$\frac{\mathcal{D}_2^2 - \varepsilon(\mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2^2 + 4\mathcal{D}_3^2)}{\mathcal{D}_2 - \varepsilon(\mathcal{D}_1 + \mathcal{D}_2)} = \frac{\mathcal{D}_2^3}{\mathcal{D}_2^2 + 4\mathcal{D}_3^2} > \varepsilon\mathcal{D}_2,$$

and from the inequality (14.11) we obtain

$$\frac{1}{\mathcal{D}_1\mathcal{D}_2}\|B_3L_h^*\boldsymbol{y}\|^2 \geq \mathcal{D}_1\|\overline{A}_1y_1\|^2 + \frac{\mathcal{D}_2^3}{4\mathcal{D}_3^2 + \mathcal{D}_2(\mathcal{D}_1 + 2\mathcal{D}_2)}\|\overline{A}_2y_1\|^2 +$$
$$+ 4\mathcal{D}_3(\overline{A}_1y_1, \overline{A}_2y_1) + \frac{\mathcal{D}_2^3}{4\mathcal{D}_3^2 + \mathcal{D}_2(\mathcal{D}_1 + 2\mathcal{D}_2)}\|\overline{A}y_2\|^2, \quad \overline{A} = \overline{A}_1 + \overline{A}_2.$$

Under the notation $c_3 = \min\left(\mathcal{D}_1; 2\mathcal{D}_3; \frac{\mathcal{D}_2^3}{4\mathcal{D}_3^2 + \mathcal{D}_2(\mathcal{D}_1 + 2\mathcal{D}_2)}\right)$, we find that $\|B_3L_h^*\boldsymbol{y}\|^2 \geq c_3\mathcal{D}_1\mathcal{D}_2\|\overline{A}\boldsymbol{y}\|^2$, or, taking into account (14.7),

$$\max(\mathcal{D}_1; \mathcal{D}_2)\|L_h^*\boldsymbol{y}\|^2 \geq \|B_3L_h^*\boldsymbol{y}\|^2 \geq \frac{4}{9}c_3\mathcal{D}_1\mathcal{D}_2\|A\boldsymbol{y}\|^2. \qquad (14.12)$$

From (14.12) it follows that the second inequality of the lemma is valid. The validity of the first inequality is verified analogously. □

4^0. If in every node $x \in \omega$ we apply to the system (14.3) the averaging operator T_1T_2, and to the equation (14.4) the operator T_1, then we obtain

$$B_1A_1T_2\varkappa + B_2A_2T_1\varkappa = \widetilde{Q}, \tag{14.13}$$

$$A_1w = T_1\varkappa_1. \tag{14.14}$$

Introduce the notation $\psi^\alpha = (\eta_1^\alpha, \eta_2^\alpha)^T$, $\eta_\beta^\alpha = T_{3-\alpha}\varkappa_\beta - \big(E - \frac{h_{3-\alpha}^2}{12} A_{3-\alpha}\big)\varkappa_\beta$. Then (14.13), (14.14) can be rewritten as $L_h\varkappa = \widetilde{Q} - B_1A_1\psi^1 - B_2A_2\psi^2$, $A_1w = \big(E - \frac{h_1^2}{12} A_1\big)\varkappa_1 + \eta_1^2$.

This and (14.8), (14.9) imply that the errors $z = \widetilde{\varkappa} - \varkappa$ and $\chi = \widetilde{w} - w$ are solutions of the problems

$$L_hz = \psi, \ \text{ where } \ \psi = B_1A_1\psi^1 + B_2A_2\psi^2, \ \ z, \psi \in H \times H, \tag{14.15}$$

$$A_1\chi = \Big(E - \frac{h_1^2}{12} A_1\Big)z_1 - \eta_1^2. \tag{14.16}$$

Lemma 14.3. *For the solution of the problem* (14.15)*, the following a priori estimates are valid:*

$$\|z\| \le c_4 \sum_{\alpha,\beta=1,2}^{2} \|\eta_\beta^\alpha\|, \tag{14.17}$$

$$\|z\|_{W_2^1(\omega)} \le c_5 \sum_{\alpha,\beta=1}^{2} \|\eta_{\beta\overline{x}_\alpha}^\alpha\|_{(\alpha+)}, \tag{14.18}$$

$$\|z\|_{W_2^2(\omega)} \le c_6 \sum_{\alpha,\beta=1}^{2} \|A_\alpha\eta_\beta^\alpha\|. \tag{14.19}$$

Proof. From (14.15) it follows

$$\|z\| = \big\|(L_h^*L_h)^{-1}L_h^*\psi\big\| = \sup_{v\in H} \frac{|(L_h^*\psi, v)|}{\|L_h^*L_hv\|} =$$

$$= \sup_{v\in H} \frac{|(A^{-1}\psi, AL_hv)|}{\|L_h^*L_hv\|} \le \|A^{-1}\psi\| \sup_{v\in H} \frac{\|AL_hv\|}{\|L_h^*L_hv\|}\,,$$

so applying Lemma 14.2,

$$\|z\| \le c_2\|A^{-1}\psi\| \le c\big(\|B_1\psi^1\| + \|B_2\psi^2\|\big),$$

which proves the estimate (14.17).

Denote $B_4 = \begin{pmatrix} 0 & -\dfrac{\mathcal{D}_1}{\mathcal{D}_2} \\ 1 & \dfrac{2\mathcal{D}_3}{\mathcal{D}_2} \end{pmatrix}$. Then

$$\big((L_h + B_4L_h)z, z\big) =$$

$$= \sum_{\alpha=1}^{2} \big(\mathcal{D}_\alpha(\overline{A}_\alpha z^\alpha, z^\alpha) + (\mathcal{D}_\beta + 2\mathcal{D}_3)(\overline{A}_\alpha z^\beta, z^\beta)\big), \ \ \beta = 3-\alpha,$$

and taking into account (14.7), we have $\big((L_h+B_4L_h)z,z\big)\ge c_7|z|^2_{W_2^1(\omega)}$, $c_7=\frac{2}{3}\min(\mathcal{D}_1;\mathcal{D}_2;\mathcal{D}_1+2\mathcal{D}_3;\mathcal{D}_2+2\mathcal{D}_3)$. Using this inequality, from (14.15) we find

$$c_7|z|^2_{W_2^1(\omega)}\le(\psi+B_4\psi,z)=$$

$$=\sum_{\alpha=1}^{2}\Big\{\big(A_\alpha(\mathcal{D}_\alpha\eta^\alpha_\alpha-\mathcal{D}_\alpha\eta^\alpha_\beta),z^\alpha\big)+\big(A_\alpha(\mathcal{D}_\alpha\eta^\alpha_\alpha+(\mathcal{D}_\beta+2\mathcal{D}_3)\eta^\alpha_\beta),z^\beta\big)\Big\}=$$

$$=\sum_{\alpha=1}^{2}\Big\{\big((\mathcal{D}_\alpha\eta^\alpha_\beta-\mathcal{D}_\alpha\eta^\alpha_\alpha)_{\overline{x}_\alpha},z^\alpha_{\overline{x}_\alpha}\big)_{(\alpha+)}-\big((\mathcal{D}_\alpha\eta^\alpha_\alpha+(\mathcal{D}_\beta+2\mathcal{D}_3)\eta^\alpha_\beta)_{\overline{x}_\alpha},z^\beta_{\overline{x}_\alpha}\big)_{(\alpha+)}\Big\}\le$$

$$\le\sum_{\alpha=1}^{2}\Big\{\big\|(\mathcal{D}_\alpha\eta^\alpha_\beta-\mathcal{D}_\alpha\eta^\alpha_\alpha)_{\overline{x}_\alpha}\big\|_{(\alpha+)}\|z^\alpha_{\overline{x}_\alpha}\|_{(\alpha+)}+$$

$$+\big\|(\mathcal{D}_\alpha\eta^\alpha_\alpha+(\mathcal{D}_\beta+2\mathcal{D}_3)\eta^\alpha_\beta)_{\overline{x}_\alpha}\big\|_{(\alpha+)}\|z^\beta_{\overline{x}_\alpha}\|_{(\alpha+)}\big\}\le$$

$$\le\Big\{\sum_{\alpha=1}^{2}\Big(\big\|(\mathcal{D}_\alpha\eta^\alpha_\beta-\mathcal{D}_\alpha\eta^\alpha_\alpha\big)_{\overline{x}_\alpha}\|^2_{(\alpha+)}+$$

$$+\big\|\big(\mathcal{D}_\alpha\eta^\alpha_\alpha+(\mathcal{D}_\beta+2\mathcal{D}_3)\eta^\alpha_\beta\big)_{\overline{x}_\alpha}\big\|^2_{(\alpha+)}\Big)\Big\}^{1/2}|z|_{W_2^1(\omega)},$$

so

$$c_7|z|^2_{W_2^1(\omega)}\le$$

$$\le\sum_{\alpha=1}^{2}\Big(\big\|(\mathcal{D}_\alpha\eta^\alpha_\beta-\mathcal{D}_\alpha\eta^\alpha_\alpha)_{\overline{x}_\alpha}\big\|^2_{(\alpha+)}+\big\|\big(\mathcal{D}_\alpha\eta^\alpha_\alpha+(\mathcal{D}_\beta+2\mathcal{D}_3)\eta^\alpha_\beta\big)_{\overline{x}_\alpha}\big\|^2_{(\alpha+)}\Big)\le$$

$$\le\Big\{\sum_{\alpha=1}^{2}\Big(\big\|(\mathcal{D}_\alpha\eta^\alpha_\beta-\mathcal{D}_\alpha\eta^\alpha_\alpha)_{\overline{x}_\alpha}\big\|_{(\alpha+)}+\big\|\big(\mathcal{D}_\alpha\eta^\alpha_\alpha+(\mathcal{D}_\beta+2\mathcal{D}_3)\eta^\alpha_\beta\big)_{\overline{x}_\alpha}\big\|_{(\alpha+)}\Big)\Big\}\le$$

$$\le 4c_8^2\Big(\sum_{\alpha=1}^{2}\big(\|\eta^\alpha_{\alpha\overline{x}_\alpha}\|_{(\alpha+)}+\|\eta^\alpha_{\beta\overline{x}_\alpha}\|_{(\alpha+)}\big)\Big)^2,\quad\beta=3-\alpha.$$

Hence with regard for the inequality $\|z\|_{W_2^1(\omega)}\le c_9|z|_{W_2^1(\omega)}$, we obtain the estimate (14.18) with $c_5=2c_8c_9/c_7$.

It is not difficult to verify that $|z|_{W_2^2(\omega)}\le\|Az\|$. Therefore owing to $\|z\|_{W_2^2(\omega)}\le c_{10}|z|_{W_2^2(\omega)}$ and the first inequality of Lemma 14.2, from (14.15) it follows $|z|_{W_2^2(\omega)}\le c_2\|\psi\|\le c_2\big(\mathcal{D}_1\|A_1\eta^1_1\|+(\mathcal{D}_2+2\mathcal{D}_3)\|A_2\eta^2_1\|+\mathcal{D}_2\|A_2\eta^2_2\|\big)$ and hence we arrive at the estimate (14.19) with $c_6=c_2c_{10}$. □

Lemma 14.4. *For the solution of the problem* (14.16) *the a priori estimate* $|\chi|_{W_2^2(\omega)}\le\|z_1\|+\|z_2\|+\|\eta^2_1\|+\|\eta^1_2\|$ *is valid.*

Proof. First of all, we notice that by virtue of (14.19) and the second inequality of the system (14.8), we have $A_2\widetilde{w}=\big(E-\frac{h_2^2}{12}A_2\big)\widetilde{\varkappa}_2$. Therefore performing some transformations, we can show that χ satisfies the analogous to (14.16) equation

$$A_2\chi = \Big(E - \frac{h_1^2}{12}A_2\Big)z_2 - \eta_2^1. \tag{14.20}$$

Since $|\chi|_{W_2^2(\omega)} = \|A\chi\| \le \|A_1\chi\| + \|A_2\chi\|$, the statement of the lemma follows directly from (14.16), (14.20). □

Applying the Bramble–Hilbert lemma, for the functionals η_β^α we obtain the estimates

$$\|\eta_\beta^\alpha\| \le c|h|^s\|\varkappa\|_{W_2^s(\Omega)}, \quad s \in (1,4],$$

that is,

$$\|\eta_\beta^\alpha\| \le c|h|^{m-2}\|w\|_{W_2^m(\Omega)}, \quad m \in (3,6],$$

while for their differences we get

$$\|\eta^\alpha_{\beta\overline{x}_\alpha}\| \le c|h|^{m-3}\|w\|_{W_2^m(\Omega)}, \quad m \in (3,7], \quad \beta = 3-\alpha,$$

$$\|A_\alpha\eta_\beta^\alpha\| \le c|h|^{m-4}\|w\|_{W_2^m(\Omega)}, \quad m \in (3,8], \quad \beta = 3-\alpha,$$

which together with Lemmas 14.3 and 14.4 prove the following convergence theorem.

Theorem 14.1. *Let the solution of the problem* (14.1), (14.2) *belong to the Sobolev space* $W_2^m(\Omega)$, $m > 3$. *Then for the difference scheme* (14.8), (14.9) *the following estimates of the convergence rate*

$$\|\varkappa - \widetilde{\varkappa}\|_{W_2^s(\omega)} \le c|h|^{m-s-2}\|w\|_{W_2^m(\Omega)}, \quad m \in (3, s+6], \quad s = 0,1,2,$$

$$\|w - \widetilde{w}\|_{W_2^2(\omega)} \le c|h|^{m-2}\|w\|_{W_2^m(\Omega)}, \quad m \in (3,6],$$

are valid with the constant $c > 0$, *independent of* h *and* $w(x)$.

Find now the accuracy of the approximate values of bending moments and transversal forces.

Theorem 14.2. *Let the solution of the problem* (14.1), (14.2) *belong to the Sobolev space* $W_2^m(\Omega)$, $m > 3$. *Then the estimates*

$$\|\mathcal{M}_\alpha - \widetilde{\mathcal{M}}_\alpha\|_{W_2^s(\omega)} \le c|h|^{m-s-2}\|w\|_{W_2^m(\omega)}, \quad m \in (3, s+6], \quad s = 0,1, \quad \alpha = 1,2, \tag{14.21}$$

$$\|\mathcal{N}_\alpha - \widetilde{\mathcal{N}}_\alpha\| \le c|h|^{m-3}\|w\|_{W_2^m(\omega)}, \quad m \in (3,7], \quad \alpha = 1,2, \tag{14.22}$$

are valid with the positive constant $c > 0$, *independent of* h *and* $w(x)$.

Proof. Since $\mathcal{M}_\alpha - \widetilde{\mathcal{M}}_\alpha = \mathcal{D}_\alpha\big((\varkappa_\alpha - \widetilde{\varkappa}_\alpha) + \nu_\beta(\varkappa_\beta - \widetilde{\varkappa}_\beta)\big)$, $\beta = 3-\alpha$, $\alpha = 1,2$, by Theorem 14.1, we obtain (14.21).

The estimate (14.22) for $w \in W_2^m(\omega)$, $3 < m \le 5$ follows from the equality $\mathcal{N}_\alpha - \widetilde{\mathcal{N}}_\alpha = \big(\mathcal{D}_\alpha(\varkappa_\alpha - \widetilde{\varkappa}_\alpha) + \mathcal{D}_3(\varkappa_\beta - \widetilde{\varkappa}_\beta)\big)_{\overset{\circ}{x}_\alpha}$, $\beta = 3-\alpha$, with regard for Theorem 14.1.

For $w \in W_2^m(\omega)$, $5 < m \leq 7$, we have

$$\mathcal{N}_\alpha - \widetilde{\mathcal{N}}_\alpha = \big(\mathcal{D}_\alpha(\varkappa_\alpha - \widetilde{\varkappa}_\alpha) + \mathcal{D}_3(\varkappa_\beta - \widetilde{\varkappa}_\beta)\big)_{\overset{\circ}{x}_\alpha} + \\ + \frac{h_\alpha^2}{6}\,\Lambda_\beta\big(\mathcal{D}_\beta(\varkappa_\beta - \widetilde{\varkappa}_\beta) + \mathcal{D}_3(\varkappa_\alpha - \widetilde{\varkappa}_\alpha)\big)_{\overset{\circ}{x}_\alpha} + \mathcal{D}_\alpha \ell_\alpha^{(1)}(\varkappa_\alpha) + \\ + \mathcal{D}_3 \ell_\alpha^{(1)}(\varkappa_\beta) + \frac{h_\alpha^2}{6}\,\mathcal{D}_\beta \ell_\alpha^{(2)}(\varkappa_\beta) + \frac{h_\alpha^2}{6}\,\mathcal{D}_3 \ell_\alpha^{(2)}(\varkappa_\alpha), \tag{14.23}$$

where $\ell_\alpha^{(1)}(v) = \frac{\partial v}{\partial x_\alpha} - \big(v - \frac{h_\alpha^2}{6}\frac{\partial^2 v}{\partial x_\alpha^2}\big)_{\overset{\circ}{x}_\alpha}$, $\ell_\alpha^{(2)}(v) = \big(\frac{\partial^2 v}{\partial x_\beta^2} - \Lambda_\beta v\big)_{\overset{\circ}{x}_\alpha}$, $\beta = 3 - \alpha$.

Estimating $\ell_\alpha^{(1)}$ and $\ell_\alpha^{(2)}$ by the Bramble–Hilbert lemma and taking into account Theorem 14.1, from (14.23) we obtain the estimate (14.22) in the case under consideration. □

5^0. We assume that

$$\mathcal{D}_3^2 \geq \mathcal{D}_1\mathcal{D}_2 \tag{14.24}$$

and consider a decomposition of the problem (14.1), (14.2) of the type ([2])

$$\frac{\partial^2 u}{\partial x_1^2} + b_1\,\frac{\partial^2 u}{\partial x_2^2} = \frac{q}{\mathcal{D}_1}\,, \quad x \in \Omega, \quad u = 0, \quad x \in \Gamma, \tag{14.25}$$

$$\frac{\partial^2 w}{\partial x_1^2} + b_2\,\frac{\partial^2 w}{\partial x_2^2} = u, \quad x \in \Omega, \quad w = 0, \quad x \in \Gamma, \tag{14.26}$$

where $b_{1,2} = \frac{\mathcal{D}_3 \pm \sqrt{\mathcal{D}_3^2 - \mathcal{D}_1\mathcal{D}_2}}{\mathcal{D}_1}$.

On the mesh $\overline{\omega}$, we approximate the problem (14.25), (14.26) by the difference scheme

$$\overline{\Lambda}_1\widetilde{u} + b_1\overline{\Lambda}_2\widetilde{u} = \varphi_1, \quad x \in \Omega, \quad \widetilde{u} = 0, \quad x \in \gamma, \quad \varphi_1 = T_1T_2\,\frac{q}{D_1}\,, \tag{14.27}$$

$$\overline{\Lambda}_1\widetilde{w} + b_1\overline{\Lambda}_2\widetilde{w} = \varphi_2, \quad x \in \Omega, \\ \widetilde{w} = 0, \quad x \in \gamma, \quad \varphi_2 = \widetilde{u} + \frac{h_1^2}{12}\,\Lambda_1\widetilde{u} + \frac{h_2^2}{12}\,\Lambda_2\widetilde{u}, \tag{14.28}$$

or in the operator form

$$L_{1,h}\widetilde{u} = -\varphi_1, \quad \widetilde{u}, \varphi_1 \in H_h, \tag{14.29}$$

$$L_{2,h}\widetilde{w} = -\varphi_2, \quad \widetilde{w}, \varphi_2 \in H_h, \tag{14.30}$$

where $L_{\alpha,h} = \overline{A}_1 + b_\alpha\overline{A}_2$.

It is easy to see that the difference scheme (14.29), (14.30) is uniquely solvable. If in every node $x \in \omega$ we apply to (14.25), (14.26) the averaging operator T_1T_2, then we obtain

$$L_{1,h}u = -A_1\eta_1(u) - b_1A_2\eta_2(u) - \varphi_1, \tag{14.31}$$

$$L_{2,h}w = -A_1\eta_1(w) - b_2A_2\eta_2(w) - T_1T_2u, \tag{14.32}$$

where $\eta_\alpha(v) = T_\beta v - v - \frac{h_\beta^2}{12}\,\Lambda_\beta v$, $\beta = 3 - \alpha$, $\alpha = 1, 2$.

From (14.29)–(14.32) it follows that the errors $z=\widetilde{u}-u$ and $\chi=\widetilde{w}-w$ are solutions of the problems

$$L_{1,h}z=A_1\eta_1(u)+b_1A_2\eta_2(u), \tag{14.33}$$

$$L_{2,h}\chi=A_1\eta_1(w)+b_2A_2\eta_2(w)-\Big(z+\frac{h_1^2}{12}\,\Lambda_1z+\frac{h_2^2}{12}\,\Lambda_2z\Big)+\eta(u), \tag{14.34}$$

where

$$\eta(u)=T_1T_2u-u-\frac{h_1^2}{12}\,\Lambda_1u-\frac{h_2^2}{12}\,\Lambda_2u.$$

Using for the estimation of solutions of the problems (14.33), (14.34) the well-known method from [71], we obtain

$$\begin{aligned}&\|\widetilde{u}-u\|_{W_2^s(\omega)}\le c|h|^{m-s-2}\|w\|_{W_2^m(\Omega)},\ \ m\in(3,6+s],\ \ s=0,1,2,\\ &\|\widetilde{w}-w\|_{W_2^2(\omega)}\le c|h|^{m-2}\|w\|_{W_2^m(\Omega)},\ \ m\in(3,6].\end{aligned} \tag{14.35}$$

Thus the following theorem is valid.

Theorem 14.3. *Let the condition* (14.24) *be fulfilled and the solution of the problem* (14.1), (14.2) *belong to* $W_2^m(\Omega)$, $m>3$*. Then the convergence rate of the approximate value of bending function defined by means of the difference scheme* (14.27), (14.28) *is characterized by the estimate* (14.35)*.*

CHAPTER 4

Nonlocal Boundary Value Problems

The aim of this chapter is to study the solvability of nonlocal boundary value problems in weighted Sobolev spaces and to construct the corresponding difference schemes.

15. On the Solvability of a Nonlocal Bitsadze–Samarskiĭ Boundary Value Problem in Sobolev Spaces

History of the matter. The generalization of the Bitsadze–Samarskiĭ type nonlocal problem [31] has been investigated by many authors. Theorems on the existence and uniqueness of the classical solution of the abovementioned problem for uniformly elliptic equations were established in [43] and [44]. For the Poisson operators this problem is studied in [49]. The Fredholmity of the problem is stated in [74]. The results of this section are published in [20] and [23].

1^0. The notion of a nonlocal trace of a function. When investigating nonlocal Bitsadze–Samarskiĭ type boundary value problems in Sobolev spaces, there naturally arises the question what is meant under the nonlocal boundary values. Below we will prove the theorem allowing one, unlike the traditional approach (when the trace is defined by the limiting passage along the normal of the contour), to determine a nonlocal trace of the function. The idea of applying the weighted Sobolev spaces to our purposes originates from the works of D. Gordeziani (see, e.g., [43]) in which the weighted inner product and the corresponding norm are used for proving the uniqueness of the classical solution of the nonlocal Bitsadze-Samarskiĭ boundary value problem.

Let $\Omega = \Omega_0 = \{(x_1, x_2) :\ 0 < x_k < 1,\ k = 1, 2\}$ be a square with the boundary Γ; $\alpha_1, \alpha_2, \dots, \alpha_m$ be arbitrary real numbers; $\xi_1, \xi_2, \dots, \xi_{m+1}$ be fixed points from $[0, 1]$ with $0 < \xi_1 < \xi_1 < \cdot < \xi_m < \xi_{m+1} = 1$; $\Gamma_{(i)} = \big\{(\xi_i, x_2) :\ 0 < x_2 < 1\big\}$, $i = 1, \dots, m+1$, $\Gamma_1 = \Gamma_{(m+1)}$, $\Gamma_* = \Gamma \backslash \Gamma_1$.

We assume that the weight function $r(x) = 1 - x_1$.

As an immediate corollary of Theorem 1.2, $W_2^k(\Omega, r)$ can be defined as the closure of the set $C^\infty(\overline{\Omega})$ in the norm $W_2^k(\Omega, r)$.

Let us define a subspace of the space $W_2^1(\Omega, r)$ which is obtained by the closure of the set

$$\overset{*}{C}{}^\infty(\overline{\Omega}) =$$

$$= \left\{ u \in C^\infty(\overline{\Omega}) : \ \operatorname{supp} u \cap \Gamma_* = \varnothing, \ u(1, x_2) = \sum_{i=1}^m \alpha_i u(\xi_i, x_2), \ 0 < x_2 < 1 \right\}$$

in the norm of the space $W_2^1(\Omega, r)$. We denote it by $\overset{*}{W}{}_2^1(\Omega, r)$.

Let $W_{2,*}^2(\Omega, r) = \overset{*}{W}{}_2^1(\Omega, r) \cap W_2^2(\Omega, r)$.

Lemma 15.1. *$\overset{*}{C}{}^\infty(\overline{\Omega})$ is dense in the space $W_{2,*}^2(\Omega, r)$.*

Proof. Indeed, the inclusion $\overset{*}{C}{}^\infty(\overline{\Omega}) \subset W_{2,*}^2(\Omega, r)$ follows from the definition of these spaces, and since $\overset{*}{C}{}^\infty(\overline{\Omega})$ is dense in $\overset{*}{W}{}_2^1(\Omega, r)$, it is likewise dense in $\overset{*}{W}{}_2^1(\Omega, r) \cap W_2^2(\Omega, r) = W_{2,*}^2(\Omega, r)$. □

Lemma 15.2. *For any function $u(x) \in \overset{*}{C}{}^\infty(\overline{\Omega})$ the inequalities $\|u\|_{L_2(\Gamma_1)} \le c|u|_{1,\Omega,r}$, $|u|_{1/2,\Gamma_1} \le c|u|_{1,\Omega,r}$ are valid, where the constant $c > 0$ is independent of $u(x)$.*

Theorem 15.1. *There exists a unique bounded operator $T : \overset{*}{W}{}_2^1(\Omega, r) \to W_2^{1/2}(\Gamma_1)$, which for any $u(x) \in \overset{*}{C}{}^\infty(\overline{\omega})$ satisfiers the equality $Tu(x) = u(1, x_2)$.*

Proof. Let $u(x) \in \overset{*}{W}{}_2^1(\Omega, r)$. Then since the set $\overset{*}{C}{}^\infty(\overline{\Omega})$ is dense in $\overset{*}{W}{}_2^1(\Omega, r)$, there exists a sequence of functions $u_n(x) \in \overset{*}{C}{}^\infty(\overline{\Omega})$ which converges to $u(x)$ in the norm $W_2^1(\Omega, r)$. On the basis of Lemma 15.1, it is not difficult to show that the sequence $\{u_n(1, x_2)\}$ is fundamental in $W_2^{1/2}(0, 1)$. Taking into account the fact that the space $W_2^{1/2}(0, 1)$ is dense, there exists a function $v(x_1) \in W_2^{1/2}(0, 1)$ to which the sequence $\{u_n(1, x_2)\}$ converges as $n \to \infty$. The function $v(x_2)$ does not depend on the choice of the sequence $\{u_n(x)\}$. Consequently, for $u \in \overset{*}{W}{}_2^1(\Omega, r)$ the operator T is defined as follows: $Tu = v(x_2)$, where $v(x_2) = \lim\limits_{n \to \infty} u_n(1, x_2)$ in the norm of $W_2^{1/2}(0, 1)$. Moreover, $\|Tu\|_{1/2,\Gamma_1} \le c\|u\|_{1,\Omega,r}$. Thus the theorem is proved. □

By Theorem 15.1, we can find a nonlocal trace of the function $u \in \overset{*}{W}{}_2^1(\Omega, r)$ on Γ_1, as $u(1, x_2) = Tu$. In addition, for almost all $x_2 \in (0, 1)$ the equality $u(x)\big|_{\Gamma_1} - \sum_{i=1}^m \alpha_i u(x)\big|_{\Gamma_{(i)}} = 0$, $\forall\, u \in \overset{*}{W}{}_2^1(\Omega, r)$ is valid.

Analogously we prove that for any function $u(x) \in W_{2,*}^2(\Omega, r)$ there exists a nonlocal trace $u(x)|_{\Gamma_1} \in W_2^{3/2}(\Gamma_1)$ satisfying for almost all $x_2 \in (0, 1)$ the equality

$$\frac{\partial u(x)}{\partial x_2}\Big|_{\Gamma_1} - \sum_{i=1}^m \alpha_i \frac{\partial u(x)}{\partial x_2}\Big|_{\Gamma_{(i)}} = 0, \ \ \forall\, u \in \overset{*}{W}{}_2^2(\Omega, r).$$

2^0. Statement of the problem. We consider a nonlocal Bitsadze–Samarskiĭ type boundary value problem [43] for the elliptic equation with constant coefficients

$$Lu \equiv \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - a_0 u = f(x), \;\; x \in \Omega, \tag{15.1}$$

$$u(x) = 0, \;\; x \in \Gamma_*, \quad \sum_{k=1}^{m} \alpha_k u(\xi_k, x_2) = u(1, x_2), \;\; 0 < x_2 < 1. \tag{15.2}$$

Assume that the conditions

$$\sum_{i,j=1}^{2} a_{ij} t_i t_j \ge \nu_1 (t_1^2 + t_2^2), \;\; \nu_1 > 0, \;\; a_0 \ge 0, \tag{15.3}$$

are fulfilled.

We introduce the weight function

$$\rho(x_1) = \begin{cases} \rho_i(x_1), & \xi_i \le x_1 < \xi_{i+1}, \;\; i = 0, \dots, m-1, \\ r(x_1), & \xi_m \le x_1 \le 1, \end{cases} \tag{15.4}$$

where $\rho_i(x_1) = r(x_1) - \varkappa \sum\limits_{k=i+1}^{m} \frac{|\alpha_k|}{\sqrt{\xi_k}} r_k(x_1)$, $r(x_1) = 1 - x_1$, $r_k(x_1) = \xi_k - x_1$, $\varkappa = \sum\limits_{k=1}^{m} |\alpha_k| \sqrt{\xi_k}$.

Let $\varkappa < 1$.

Lemma 15.3. *The function* $\rho(x_1)$ *defined in* (15.4) *is continuous on the segment* $[0, 1]$, *and* $(1 - \varkappa^2) r(x_1) \le \rho(x_1) \le r(x_1)$.

We say that a function $u(x) \in W_{2,*}^2(\Omega, r)$ is a strong solution of the nonlocal boundary value problem (15.1)–(15.3) if the relation

$$a(u, v) = \ell(v), \;\; \forall\, v \in W_{2,*}^2(\Omega, \rho) \tag{15.5}$$

is fulfilled, where $a(u, v) = (Lu, \Delta v)_{\Omega,\rho}$, $\ell(v) = (f, \Delta v)_{\Omega,\rho}$.

3^0. The solvability. Before we proceed to proving the basic theorem, let us establish that some inequalities are valid. Let $(u, v)_{\Omega,r} = \int\limits_{\Omega} r(x_1) u(x) v(x)\, dx$, and the symbols $|\cdot|_{k,\Omega,\rho}$ and $\|\cdot\|_{k,\Omega,\rho}$ have the same meaning as $|\cdot|_{k,\Omega,r}$ and $\|\cdot\|_{k,\Omega,r}$, respectively.

Lemma 15.4. *If* $u \in W_{2,*}^2(\Omega, r)$, *then* $(-\Delta u, u)_{\Omega,\rho} \ge |u|_{1,\Omega,\rho}^2$.

Proof. Since $\overset{*}{C}{}^{\infty}(\overline{\Omega})$ is dense in $W_{2,*}^2(\Omega, r)$, it suffices to prove the lemma for $u \in \overset{*}{C}{}^{\infty}(\overline{\Omega})$. Using integration by parts, we find that

$$I = \int_0^1 \rho(x_1) \frac{\partial^2 u}{\partial x_1^2} u\, dx_1 = - \int_0^1 \rho(x_1) \Big| \frac{\partial u}{\partial x_1} \Big|^2 dx_1 - \int_0^1 \rho'(x_1) \frac{\partial u}{\partial x_1} u\, dx_1$$

and since $\frac{\partial u}{\partial x_1}\,u = \frac{1}{2}\,\frac{\partial (u)^2}{\partial x_1}$, therefore

$$I = -\int_0^1 \rho(x_1)\Big|\frac{\partial u}{\partial x_1}\Big|^2\,dx_1 + \frac{1}{2}\,\Big(u^2(1,x_2) + \sum_{i=1}^{m} u^2(\xi_i,x_2)(\rho_i' - \rho_{i-1}')\Big).$$

But $\rho_i' - \rho_{i-1}' = -|\alpha_i|/\sqrt{\xi_i}\,\varkappa$. Hence

$$I = -\int_0^1 \rho(x_1)\Big|\frac{\partial u}{\partial x_1}\Big|^2\,dx_1 + \frac{1}{2}\,\Big(u^2(1,x_2) - \varkappa\sum_{i=1}^{m} u^2(\xi_i,x_2)\,\frac{|\alpha_i|}{\sqrt{\xi_i}}\Big). \tag{15.6}$$

It follows from the nonlocal condition (15.2) that

$$u^2(1,x_2) = \Big(\sum_{k=1}^{m} \frac{\sqrt{|\alpha_k|}}{\sqrt[4]{\xi_k}}\,u(\xi_k,x_2)\sqrt[4]{\alpha_k^2\xi_k}\Big)^2 \le \varkappa\sum_{k=1}^{m}\frac{|\alpha_k|}{\sqrt{\xi_k}}\,u^2(\xi_k,x_2),$$

by virtue of which from (15.6) we obtain

$$\Big(-\frac{\partial^2 u}{\partial x_1^2},u\Big)_{\Omega,\rho} \ge \Big\|\frac{\partial u}{\partial x_1}\Big\|_{\Omega,\rho}^2. \tag{15.7}$$

On the other hand, it is obvious that

$$\Big(-\frac{\partial^2 u}{\partial x_2^2},u\Big)_{\Omega,\rho} = \Big\|\frac{\partial u}{\partial x_2}\Big\|_{\Omega,\rho}^2, \tag{15.8}$$

which together with (15.7) completes the proof of the lemma. □

Lemma 15.5. *The semi-norms* $|\cdot|_{k,\Omega,\rho}$ *are equivalent respectively to the norms* $\|\cdot\|_{k,\Omega,\rho}$, $k=1,2$*:*

$$|u|_{1,\Omega,\rho} \le \|u\|_{1,\Omega,\rho} \le \frac{\sqrt{5}}{2}\,|u|_{1,\Omega,\rho}, \quad \forall\, u\in \overset{*}{W}{}_2^1(\Omega,r), \tag{15.9}$$

$$|u|_{2,\Omega,\rho} \le \|u\|_{2,\Omega,\rho} \le \frac{\sqrt{21}}{4}\,|u|_{2,\Omega,\rho}, \quad \forall\, u\in W_{2,*}^2(\Omega,r). \tag{15.10}$$

Proof. Since $\overset{*}{C}{}^\infty(\overline{\Omega})$ is dense in $\overset{*}{W}{}_2^1(\Omega)$ and $W_{2,*}^2(\Omega)$, it suffices to prove (15.9), (15.10) for the functions $u(x)$ from the class $\overset{*}{C}{}^\infty(\overline{\Omega})$. The left inequalities in (15.9), (15.10) are obvious. Since $u(x) = \int_0^{x_2} \frac{\partial u(x_1,\tau)}{\partial\tau}\,d\tau$, $u(x) = -\int_{x_2}^1 \frac{\partial u(x_1,\tau)}{\partial\tau}\,d\tau$, therefore $2|u(x)| \le \int_0^1 \big|\frac{\partial u(x_1,\tau)}{\partial\tau}\big|\,d\tau$, $4u^2(x) \le \int_0^1 \big|\frac{\partial u(x)}{\partial x_2}\big|^2 dx_2$. Consequently, the estimate

$$\|u\|_{\Omega,\rho} \le 0.5|u|_{1,\Omega,\rho} \tag{15.11}$$

is valid and allows us to prove the right inequality in (15.9).

Analogously to (15.11), we have $\big\|\frac{\partial u}{\partial x_1}\big\|_{\Omega,\rho} \le \frac{1}{2}\big\|\frac{\partial^2 u}{\partial x_1\partial x_2}\big\|_{\Omega,\rho}$. Moreover,

$$\Big\|\frac{\partial u}{\partial x_2}\Big\|_{\Omega,\rho}^2 = -\Big(\frac{\partial^2 u}{\partial x_2^2},u\Big)_{\Omega,\rho} \le \frac{1}{8}\,\Big\|\frac{\partial^2 u}{\partial x_2^2}\Big\|_{\Omega,\rho}^2 + 2\|u\|_{\Omega,\rho}^2 \le \frac{1}{8}\,\Big\|\frac{\partial^2 u}{\partial x_2^2}\Big\|_{\Omega,\rho}^2 + \frac{1}{2}\,|u|_{1,\Omega,\rho}^2.$$

Consequently, $|u|_{1,\Omega,\rho} \le (1/2)|u|_{2,\Omega,\rho}$, which together with (15.9) proves the right inequality in (15.10). □

Lemma 15.6. *If* $u \in W^2_{2,*}(\Omega, r)$, *then* $\|\Delta u\|_{\Omega,\rho} \ge |u|_{2,\Omega,\rho}$.

Proof. Let $u \in \overset{*}{C}{}^{\infty}(\overline{\omega})$. Denote $J = \big(\frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}\big)_{\Omega,\rho}$.

Integration by parts yields

$$J = -\int_0^1\int_0^1 \rho(x_1)\,\frac{\partial u}{\partial x_1}\,\frac{\partial^3 u}{\partial x_1 \partial x_2^2}\,dx_1\,dx_2 - \sum_{k=0}^{m}\rho'_k \int_{\xi_k}^{\xi_{k+1}}\int_0^1 \frac{\partial u}{\partial x_1}\,\frac{\partial^2 u}{\partial x_2^2}\,dx_2\,dx_1, \quad (15.12)$$

and since

$$\int_0^1 \frac{\partial u}{\partial x_1}\,\frac{\partial^3 u}{\partial x_1 \partial x_2^2}\,dx_2 = -\int_0^1 \Big(\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big)^2\,dx_2,$$

$$\int_0^1 \frac{\partial u}{\partial x_1}\,\frac{\partial^2 u}{\partial x_2^2}\,dx_2 = -\frac{1}{2}\int_0^1 \frac{\partial}{\partial x_1}\Big(\frac{\partial u}{\partial x_2}\Big)^2\,dx_2,$$

from (15.12) we obtain

$$J = \Big|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|^2_{\Omega,\rho} + \frac{1}{2}\int_0^1 \Big(\varkappa\sum_{k=1}^{m}\frac{|\alpha_k|}{\sqrt{\xi_k}}\Big|\frac{\partial u(\xi_k, x_2)}{\partial x_2}\Big|^2 - \Big|\frac{\partial u(1, x_2)}{\partial x_2}\Big|^2\Big)\,dx_2. \quad (15.13)$$

Using the consequence from the nonlocal condition (15.2),

$$\Big|\frac{\partial u(1, x_2)}{\partial x_2}\Big|^2 = \Big(\sum_{k=1}^{m}\alpha_k\,\frac{\partial u(\xi_k, x_2)}{\partial x_2}\Big)^2 \le \varkappa\sum_{k=1}^{m}\frac{|\alpha_k|}{\sqrt{\xi_k}}\Big(\frac{\partial u(\xi_k, x_2)}{\partial x_2}\Big)^2,$$

from (15.13) we have

$$J \ge \Big|\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big|^2_{\Omega,\rho}. \quad (15.14)$$

Thus $\|\Delta u\|^2_{\Omega,\rho} = \big\|\frac{\partial^2 u}{\partial x_1^2}\big\|^2_{\Omega,\rho} + \big\|\frac{\partial^2 u}{\partial x_2^2}\big\|^2_{\Omega,\rho} + 2\big(\frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}\big)_{\Omega,\rho} \ge |u|^2_{2,\Omega,\rho}$. Using this inequality and the standard argument of density, we can see that the lemma is valid. □

Theorem 15.2. *Let* $f(x) \in L_2(\Omega, r)$. *Then the nonlocal boundary value problem* (15.1)–(15.3) *has a unique strong solution. Moreover, there exists an independent of* f *positive constant* c_1 *such that*

$$\|u\|_{2,\Omega,r} \le c_1\|f\|_{\Omega,r}. \quad (15.15)$$

Proof. To prove this theorem, we use the Lax–Milgram lemma (see, e.g., [36], p. 19). Performing some transformations, we obtain

$$a(u,u)=\sum_{k=1}^{2}\int\limits_{\Omega}\rho(x_1)\sum_{i,j=1}^{2}a_{ij}\,\frac{\partial^2 u}{\partial x_i\partial x_k}\,\frac{\partial^2 u}{\partial x_j\partial x_k}\,dx+ \\ +(a_{11}+a_{22})\int\limits_{\Omega}\rho(x_1)\Big(\frac{\partial^2 u}{\partial x_1^2}\,\frac{\partial^2 u}{\partial x_2^2}-\Big|\frac{\partial^2 u}{\partial x_1\partial x_2}\Big|^2\Big)\,dx-a_0(u,\Delta u)_{\Omega,\rho}.$$

whence using the ellipticity condition (15.3), Lemma 15.4 and the inequality (15.14), we see that $a(u,u)\geq \nu_1|u|^2_{2,\Omega,\rho}+a_0|u|^2_{1,\Omega,\rho}$, that is, by virtue of (15.10) and the equivalence of the norms $\|\cdot\|_{2,\Omega,\rho}$, $\|\cdot\|_{2,\Omega,r}$,

$$a(u,u)\geq c_2\|u\|_{2,\Omega,r}, \tag{15.16}$$

which means the W_2^2-ellipticity of the bilinear form $a(u,v)$.

Obviously, $\|Lu\|_{\Omega,\rho}\leq \nu_2|u|_{2,\Omega,\rho}+a_0\|u\|_{\Omega,\rho}$. Therefore $|a(u,v)|\leq \|Lu\|_{\Omega,\rho}\|\Delta v\|_{\Omega,\rho}\leq\sqrt{\nu_2^2+a_0^2}\,\|u\|_{2,\Omega,\rho}\|v\|_{2,\Omega,\rho}$, $\nu_2=\max_{i,j}|a_{i,j}|$, so $|a(u,v)|\leq c_3\|u\|_{2,\Omega,r}\|v\|_{2,\Omega,r}$.

Hence the bilinear form $a(\cdot\,,\cdot):W^2_{2,*}(\Omega,r)\times W^2_{2,*}(\Omega,r)\to\mathbf{R}$ is continuous. Moreover,

$$|\ell(v)|\leq\|f\|_{\Omega,\rho}\|\Delta v\|_{\Omega,\rho}\leq c_4\|v\|_{2,\Omega,r}, \tag{15.17}$$

i.e., the linear form $\ell(v)$ is continuous in $W^2_{2,*}(\Omega,r)$.

Thus all the conditions of the Lax–Milgram lemma are fulfilled. This guarantees the existence of a unique solution $u\in W^2_{2,*}(\Omega,r)$. The estimate (15.15) follows directly from (15.16), (15.17). □

16. Difference Scheme for a Nonlocal Bitsadze–Samarskiĭ Type Boundary Value Problem

History of the matter. In this section we will consider difference approximation of a nonlocal Bitsadze–Samarskiĭ type boundary value problem for the second order elliptic equation with constant coefficients. The results of the present section have been published in [29]. In [49], in the case of Poisson equation, the difference scheme is investigated which converges in the mesh norm W_2^2 with the rate $O(h^2)$ to the exact solution of the class $C^4(\overline{\Omega})$.

1^0. Statement of the problem. We consider the nonlocal boundary value problem (15.1), (15.2). Assume that $f(x)\in W_2^{m-2}(\Omega)$, and the problem (15.1), (15.2) is uniquely solvable in the class $W_2^m(\Omega)$, $1<m\leq 4$.

Introduce the mesh domains with the step $h=1/N$.

Let $\gamma_0=\Gamma_0\cap\overline{\omega}$, $\omega_{1,k}=\{x_1:\ x_1=ih,\ i=1,2,\dots,n_k\}$, $\xi_k=(n_k+\theta_k)h$, $0\leq\theta_k<1$, $k=1,2,\dots,m$, where n_k are nonnegative integers, $0\leq n_1\leq n_2\leq\cdots\leq n_m<N$, among which there exists equality if in the corresponding subinterval (between adjacent points of the mesh ω_1) there are more than one point ξ_k.

Assume that

$$\frac{h}{2} \le 1 - \xi_m - \nu, \tag{16.1}$$

where $\nu > 0$ is a fixed number.

Define the following projection operators:

$$G_k u = (1-\theta_k)\int_0^{\theta_k} t u(n_k h + th, x_2)\,dt + \theta_k \int_{\theta_k}^1 (1-t)u(n_k h + th, x_2)\,dt,$$

$$k = 1, 2, \dots, n_m.$$

Let T_α, $S_\alpha^\pm$ be the averaging operators defined in Section 1 for which the identities

$$T_\alpha \frac{\partial^2 u}{\partial x_\alpha^2} = u_{\overline{x}_\alpha x_\alpha}, \quad T_\alpha \frac{\partial u}{\partial x_\alpha} = S_\alpha^+ u_{\overline{x}_\alpha} = S_\alpha^- u_{x_\alpha}, \quad \alpha = 1, 2,$$

$$G_k \frac{\partial^2 u}{\partial x_1^2} = \frac{1}{h^2}\big((1-\theta_k)u(n_k h, x_2) + \theta_k u(n_k h + h, x_2) - u(\xi_k, x_2)\big),$$

$$k = 1, 2, \dots, n_m,$$

are valid.

By $Y_k(x_2)$, $Z_k(x_2)$, $\overline{Z}_k(x_2)$, $\widetilde{Z}(x_2)$ we denote the expressions

$$Y_k(x_2) = (1-\theta_k)y(n_k h, x_2) + \theta_k y(n_k h + h, x_2), \ \text{ and so on.}$$

We approximate the problem (15.1), (15.2) by the difference scheme

$$L_h y = \varphi(x), \quad x \in \omega, \quad \varphi = T_1 T_2 f, \tag{16.2}$$

$$y = 0, \quad x \in \gamma_0, \quad y(1, x_2) = \sum_{k=1}^m \alpha_k Y_k(x_2), \quad x_2 \in \omega_2, \tag{16.3}$$

where $L_h y = a_{11} y_{\overline{x}_1 x_1} + a_{12}(y_{\overline{x}_1 x_2} + y_{x_1 \overline{x}_2}) + a_{22} y_{\overline{x}_2 x_2} - a_0 y$.

2^0. The correctness of the difference scheme. Let H be the space of the mesh functions defined on $\overline{\omega}$ and satisfying the conditions (16.3), with the inner product and the norm $(y, v)_r = \sum_\omega h^2 r(x_1) y(x) v(x)$, $\|y\|_r^2 = (y, y)_r$, $r(x_1) = 1 - x_1$. Let, moreover,

$$\|y]|_r^2 = \sum_{\omega_1^+ \times \omega_2} h^2 \overline{r} y^2, \quad \|y|]_r^2 = \sum_{\omega_1 \times \omega_2^+} h^2 r y^2, \quad \|y]]_r^2 = \sum_{\omega_1^+ \times \omega_2^+} h^2 \overline{r} y^2,$$

$$|y|_{1,\omega,r}^2 = \|y_{\overline{x}_1}]|_r^2 + \|y_{\overline{x}_2}|]_r^2, \quad \|y\|_{1,\omega,r}^2 = \|y\|_r^2 + |y|_{1,\omega,r}^2,$$

$$|y|_{2,\omega,r}^2 = \|y_{\overline{x}_1 x_1}\|_r^2 + \|y_{\overline{x}_2 x_2}|]_r^2 + 2\|y_{\overline{x}_1 \overline{x}_2}]]_r^2, \quad \|y\|_{2,\omega,r}^2 = |y|_{2,\omega,r}^2 + \|y\|_{1,\omega,r}^2,$$

$$\|y\|^2 = \sum_\omega h^2 y^2, \quad \|y]|^2 = \sum_{\omega_1^+ \times \omega_2} h^2 y^2, \quad \|y|]^2 = \sum_{\omega_1 \times \omega_2^+} h^2 y^2,$$

$$\|y]]^2 = \sum_{\omega_1^+ \times \omega_2^+} h^2 y^2, \quad \|y\|_*^2 = \sum_{\omega_2} h y^2, \quad \|y|]_*^2 = \sum_{\omega_2^+} h y^2,$$

$$\overline{r}(x_1)=\frac{r(x_1)+r(x_1-h)}{2}\,.$$

We introduce the auxiliary weight function

$$\rho(x_1)=\begin{cases}\rho_i(x_1), & \xi_i\le x_1<\xi_{i+1},\quad i=0,1,2,\dots,m-1,\\ r(x_1), & \xi_m\le x_1\le 1,\end{cases}\tag{16.4}$$

where $\rho_i(x_1)=r(x_1)-\varkappa\sum\limits_{k=i+1}^{m}\frac{|\alpha_k|}{\sqrt{\xi_k}}\,r_k(x_1)$, $r_k(x_1)=\xi_k-x_1$, $\varkappa=\sum\limits_{k=1}^{m}|\alpha_k|\sqrt{\xi_k}$.

Suppose $\varkappa<1$. Then

$$(1-\varkappa^2)r(x_1)\le\rho(x_1)\le r(x_1).\tag{16.5}$$

In what follows, we will assume that the inner product and the norm containing the index ρ have the same meaning as the expression with the index r.

By $\Phi(y)$ we denote the functional of the type

$$\Phi(y)=\frac{1}{2}\sum_{\omega_2}h\Big(\sum_{k=1}^{m}\varkappa\,\frac{|\alpha_k|}{\sqrt{\xi_k}}\,Y_k^2(x_2)-y^2(1,x_2)\Big).\tag{16.6}$$

To apply the obtained in this section results in the sequel for a priori estimate of error of the method in the case where the nonlocal condition will not be homogeneous, the estimates for the function $y(x)$ will be obtained in the form, where the nonlocal condition is not taken into account.

Lemma 16.1. *For every mesh function $y(x)$ defined on $\overline{\omega}$ and vanishing for $x_1=0$ the estimate*

$$(-y_{\overline{x}_1x_1},y)_\rho\ge\|y_{\overline{x}_1}\|_\rho^2+\Phi(y)\tag{16.7}$$

is valid.

Proof. We represent the weight function in the form $\rho(x_1)=1-x_1-\sum\limits_{k=1}^{m}\varkappa\frac{|\alpha_k|}{\sqrt{\xi_k}}\,\chi(\xi_k-x_1)$, where $\chi(t)=\begin{cases}t, & \text{for } t\ge 0,\\ 0, & \text{for } t<0.\end{cases}$ This implies that $\rho_{\overline{x}_1x_1}=-\sum\limits_{k=1}^{m}\varkappa\frac{|\alpha_k|}{\sqrt{\xi_k}}\,\big(\chi(\xi_k-x_1)\big)_{\overline{x}_1x_1}$.

It can be easily verified that

$$h\big(\chi(\xi_k-x_1)\big)_{\overline{x}_1x_1}=\begin{cases}0, & \text{if } i\le n_k-1 \text{ or } i\ge n_k+2,\\ (1-\theta_k), & \text{if } i=n_k,\\ \theta_k, & \text{if } i=n_k+1,\end{cases}$$

so

$$\big(\chi(\xi_k-x_1)\big)_{\overline{x}_1x_1}=\frac{1}{h}\,\big((1-\theta_k)\,\delta(n_k,i)+\theta_k\delta(n_k+1,i)\big),$$

where $\delta(\cdot\,,\cdot)$ is the Kronecker symbol. Consequently,

$$\rho_{\overline{x}_1x_1}(ih)=-\sum_{k=1}^{m}\frac{\varkappa|\alpha_k|}{h\sqrt{\xi_k}}\,\big((1-\theta_k)\delta(n_k,i)+\theta_k\delta(n_k+1,i)\big).\tag{16.8}$$

Using summation by parts, we obtain

$$-\sum_{\omega_1} h\rho y_{\overline{x}_1 x_1} y = \sum_{\omega_1^+} h\overline{\rho} y_{\overline{x}_1}^2 - \frac{1}{2}\, y^2(1,x_2) - \sum_{\omega_1} \frac{h}{2}\, y^2 \rho_{\overline{x}_1 x_1},$$

so taking into account (16.8), we find

$$-\sum_{\omega_1} h\rho y_{\overline{x}_1 x_1} y = \sum_{\omega_1^+} h\overline{\rho} y_{\overline{x}_1}^2 - \frac{1}{2}\, y^2(1,x_2) + \\ + \frac{1}{2}\sum_{k=1}^{m} \varkappa \frac{|\alpha_k|}{\sqrt{\xi_k}} \big((1-\theta_k) y^2(n_k h, x_2) + \theta_k y^2(n_k h + h, x_2)\big).$$

Replacing here $(1-\theta_k)y^2(n_kh,x_2)+\theta_k y^2(n_kh+h,x_2) = Y_k^2(x_2)+h^2\theta_k(1-\theta_k)y_{x_1}^2(n_kh,x_2) \ge Y_k^2(x_2)$, we see that Lemma 16.1 is valid. □

Lemma 16.2. *For every mesh function $y(x)$ defined on $\overline{\omega}$ and vanishing on γ_*, the estimate*

$$(y_{\overline{x}_1 x_1}, y_{\overline{x}_2 x_2})_\rho \ge \|y_{\overline{x}_1\overline{x}_2}]|_\rho^2 + \Phi(y_{\overline{x}_2}) \tag{16.9}$$

is valid.

Proof. Let $J_h(y) = (y_{\overline{x}_1 x_1}, y_{\overline{x}_2 x_2})_\rho$. Using summation by parts, we obtain

$$J_h(y) = \sum_{\omega^+} h^2 \overline{\rho} y_{\overline{x}_1\overline{x}_2}^2 - \sum_{\omega_2^+} \frac{h}{2}\, y_{\overline{x}_2}^2(1,x_2) - \frac{1}{2}\sum_{\omega_1\times\omega_2^+} h^2 \rho_{\overline{x}_1 x_1} y_{\overline{x}_2}^2,$$

whence with regard for (16.8) we find that

$$J_h(y) = \sum_{\omega^+} h^2 \overline{\rho} y_{\overline{x}_1\overline{x}_2}^2 - \sum_{\omega_2^+} \frac{h}{2} \Big(y_{\overline{x}_2}^2(1,x_2) - \\ - \sum_{k=1}^{m} \frac{\varkappa|\alpha_k|}{\sqrt{\xi_k}} \Big((1-\theta_k) y_{\overline{x}_2}^2(n_k h, x_2) + \theta_k y_{\overline{x}_2}^2(n_k h + h, x_2)\Big)\Big).$$

Replacing here $(1-\theta_k)y_{\overline{x}_2}^2(n_kh,x_2)+\theta_k y_{\overline{x}_2}^2(n_kh+h,x_2) = (Y_{k\overline{x}_2})_k^2+h^2\theta_k(1-\theta_k)y_{x_1\overline{x}_2}^2(n_kh,x_2) \ge (Y_{k\overline{x}_2})^2$, we see that the inequality (16.9) is valid. Thus the lemma is proved. □

Lemma 16.3. *For every mesh function $y(x)$ defined on $\overline{\omega}$ and vanishing on γ_*, the estimates*

$$a_{11}\Phi(y) + \frac{4\nu_1}{5}\, \|y\|_{1,\omega,\rho}^2 \le (-L_h y, y)_\rho, \tag{16.10}$$

$$c_1\Phi(y) + c_2\Phi(y_{\overline{x}_2}) + c_3\|y\|_{2,\omega,\rho}^2 \le \|L_h y\|_\rho^2, \quad c_1, c_2, c_3 = \text{const} > 0, \tag{16.11}$$

are valid.

In addition, if $y(x)$ satisfies the nonlocal condition (16.3), then the appearing in the left-hand sides of (16.10), (16.11) summands with the functional Φ can be omitted.

Proof. If we multiply $(-L_h y)$ scalarly by y and make use of the estimate (16.7), after simple transformations we obtain

$$(-L_h y, y)_\rho \geq$$
$$\geq \frac{1}{2}\sum_{\omega^+} h^2\rho \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta} y_{\overline{x}_\alpha} y_{\overline{x}_\beta} + \frac{1}{2}\sum_{\omega^-} h^2\rho \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta} y_{x_\alpha} y_{x_\beta} + a_{11}\Phi(y),$$

so taking into account the ellipticity condition (15.3),

$$a_{11}\Phi(y) + \nu_1 |y|^2_{1,\omega,\rho} \leq (-L_h y, y)_\rho. \tag{16.12}$$

Further, $2|y(x)| \leq \sum_{\omega_2^+} h|y_{\overline{x}_2}|$. Therefore $4\|y\|^2_\rho \leq |y|^2_{1,\omega,\rho}$, which together with (16.12) proves the estimate (16.10).

An obvious consequence of (16.10) is the inequality

$$\Phi(y) + |y|^2_{1,\omega,\rho} \leq (-\Delta_h y, y)_\rho, \tag{16.13}$$

where $\Delta_h y = y_{\overline{x}_1 x_1} + y_{\overline{x}_2 x_2}$.

For y we write the identity

$$(L_h y, \Delta_h y)_\rho = I_1(y) + I_2(y) + I_3(y), \tag{16.14}$$

where

$$I_1(y) = \sum_\omega h^2\rho\Big(a_{11} y^2_{\overline{x}_1 x_1} + a_{12}(y_{\overline{x}_1 x_2} + y_{x_1\overline{x}_2}) y_{\overline{x}_1 x_1} + a_{22}\Big(\frac{y_{\overline{x}_1 x_2} + y_{x_1\overline{x}_2}}{2}\Big)^2\Big) +$$
$$+ \sum_\omega h^2\rho\Big(a_{22} y^2_{\overline{x}_2 x_2} + a_{12}(y_{\overline{x}_1 x_2} + y_{x_1\overline{x}_2}) y_{\overline{x}_2 x_2} + a_{11}\Big(\frac{y_{\overline{x}_1 x_2} + y_{x_1\overline{x}_2}}{2}\Big)^2\Big),$$
$$I_2(y) = (a_{11} + a_{22}) \sum_\omega h^2\rho\Big(y_{\overline{x}_1 x_1} y_{\overline{x}_2 x_2} - \Big(\frac{y_{\overline{x}_1 x_2} + y_{x_1\overline{x}_2}}{2}\Big)^2\Big),$$
$$I_3(y) = -a_0 \sum_\omega h^2 \rho y \Delta_h y.$$

By the ellipticity condition (15.3), we get $I_1(y) \geq \nu_1\big(\|y_{\overline{x}_1 x_1}\|^2_\rho + \|y_{\overline{x}_2 x_2}\|^2_\rho\big)$. Next,

$$\sum_\omega h^2\rho\Big(\frac{y_{\overline{x}_1 x_2} + y_{x_1\overline{x}_2}}{2}\Big)^2 \leq$$
$$\leq \frac{1}{2}\sum_{\omega_1\times\omega_2^+} h^2\rho y^2_{\overline{x}_1\overline{x}_2} + \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\rho y^2_{\overline{x}_1\overline{x}_2} \leq \sum_{\omega^+} h^2\overline{\rho} y^2_{\overline{x}_1\overline{x}_2},$$

which together with (16.9) yields $I_2(y) \geq (a_{11} + a_{22})\Phi(y_{\overline{x}_2})$, and by virtue of (16.13) we have $I_3(y) \geq a_0\Phi(y)$.

Consequently, from (16.14) we obtain

$$\nu_1\big(\|y_{\overline{x}_1 x_1}\|^2_\rho + \|y_{\overline{x}_2 x_2}\|^2_\rho\big) + (a_{11} + a_{22})\Phi(y_{\overline{x}_2}) + a_0\Phi(y) \leq$$
$$\leq (L_h y, \Delta_h y)_\rho. \tag{16.15}$$

But since

$$(L_h y, \Delta_h y)_\rho \le \frac{1}{\nu_1}\,\|L_h y\|_\rho^2 + \frac{\nu_1}{2}\,\big(\|y_{\overline{x}_1 x_1}\|_\rho^2 + \|y_{\overline{x}_2 x_2}\|_\rho^2\big),$$

from (16.15) we have

$$\frac{\nu_1}{2}\,\big(\|y_{\overline{x}_1 x_1}\|_\rho^2 + \|y_{\overline{x}_2 x_2}\|_\rho^2\big) + (a_{11}+a_{22})\Phi(y_{\overline{x}_2}) + a_0\Phi(y) \le \frac{1}{\nu_1}\,\|L_h y\|_\rho^2. \quad (16.16)$$

On the other hand, by (16.9) $\Phi(y_{\overline{x}_2}) + 2\|y_{\overline{x}_1\overline{x}_2}\|_\rho^2 \le \|y_{\overline{x}_1 x_1}\|_\rho^2 + \|y_{\overline{x}_2 x_2}\|_\rho^2$, $2\Phi(y_{\overline{x}_2}) + |y|_{2,\omega,\rho}^2 \le 2\big(\|y_{\overline{x}_1 x_1}\|_\rho^2 + \|y_{\overline{x}_2 x_2}\|_\rho^2\big)$. Therefore from (16.16) we find that

$$\nu_1\big((a_{11}+a_{22})\Phi(y_{\overline{x}_2}) + a_0\Phi(y)\big) + \frac{\nu_1^2}{2}\,\Phi(y_{\overline{x}_2}) + \frac{\nu_1^2}{4}\,|y|_{2,\omega,\rho}^2 \le \|L_h y\|_\rho^2. \quad (16.17)$$

Further, from (16.12) we derive

$$a_{11}\Phi(y) + \nu_1|y|_{1,\omega,\rho}^2 \le \frac{1}{8\nu_1}\,\|L_h y\|_\rho^2 + 2\nu_1\|y\|_\rho^2. \quad (16.18)$$

Summing up $(3\nu_1/5)(4\|y\|_\rho^2 - |y|_{1,\omega,\rho}^2) \le 0$ and (16.18), we arrive at

$$8a_{11}\nu_1\Phi(y) + \frac{16\nu_1^2}{5}\,\|y\|_{1,\omega,\rho}^2 \le \|L_h y\|_\rho^2, \quad (16.19)$$

which together with (16.17) results in the inequality (16.11).

If $y(x)$ satisfies also the nonlocal condition (16.3), then

$$y^2(1,x_2) = \Big(\sum_{k=1}^m (\xi_k\alpha_k^2)^{1/4}\Big(\frac{\alpha_k^2}{\xi_k}\Big)^{1/4} Y_k(x_2)\Big)^2 \le \sum_{k=1}^m \varkappa\,\frac{|\alpha_k|}{\sqrt{\xi_k}}\,Y_k^2(x_2)$$

and analogously $(y_{\overline{x}_2}(1,x_2))^2 \le \sum\limits_{k=1}^m \varkappa\,\frac{|\alpha_k|}{\sqrt{\xi_k}}\,(Y_{k\overline{x}_2}(x_2))^2$. By virtue of the above inequalities we have $\Phi(y) \ge 0$, $\Phi(y_{\overline{x}_2}) \ge 0$, and the summands in (16.10), (16.11) containing these values can be neglected. Thus Lemma 16.3 is proved completely. □

On the basis of (16.11), for the solution of the difference scheme (16.2), (16.3) we obtain $\|y\|_{2,\omega,r} \le c\|\varphi\|_r$, which implies that the difference scheme in the metric of the space $W_2^2(\omega,r)$ is correct.

3^0. A priori estimate of error of the method. To study the question of convergence and accuracy of the difference scheme (16.2), (16.3), we consider the error $z = y - u$ of the method, where y is a solution of the difference scheme, and $u = u(x)$ is a solution of the initial problem. Substituting $y = z + u$ in (16.2), (16.3), we obtain

$$L_h z = \psi,\ \ x\in\omega,\ \ z = 0,\ \ x\in\gamma_0,\ \ z(1,x_2) = \sum_{k=1}^m \alpha_k Z_k + R,\ \ x_2\in\omega_2, \quad (16.20)$$

where $\psi = a_{11}\eta_{11\overline{x}_1x_1} + a_{12}\eta_{12\overline{x}_1x_2} + a_{22}\eta_{22\overline{x}_2x_2} + a_0\eta_0$, $R = \sum\limits_{k=1}^{m}\alpha_k R_k$, $R_k = h^2 G_k \frac{\partial^2 u}{\partial x_1^2}$, $\eta_{\alpha\alpha} = T_{3-\alpha}u - u$, $\alpha = 1,2$, $\eta_{12} = 2S_1^+S_2^- u(x) - u(x) - u(x_1+h, x_2-h)$, $\eta_0 = u - T_1T_2u$.

A solution z of the problem (16.20) we represent in the form of the sum $z = \overline{z} + \widetilde{z}$ of solutions of the following two problems:

$$L_h\overline{z}=0,\ \ x\in\omega,\ \ \overline{z}=0,\ \ x\in\gamma_0,\ \ \overline{z}(1,x_2)=\sum_{k=1}^{m}\alpha_k\overline{Z}_k+R,\ \ x_2\in\omega_2, \tag{16.21}$$

$$L_h\widetilde{z}=\psi,\ \ x\in\omega,\ \ \widetilde{z}=0,\ \ x\in\gamma_0,\ \ \widetilde{z}(1,x_2)=\sum_{k=1}^{m}\alpha_k\widetilde{Z}_k,\ \ x_2\in\omega_2. \tag{16.22}$$

Lemma 16.4. *For the functional* Φ *defined in* (16.6)*, the estimates*

$$-2\Phi(\overline{z}) \le \Big(1 + \frac{\varkappa}{\varepsilon_1\sqrt{\nu}}\Big)\|R\|_*^2 + \frac{\varkappa\varepsilon_1}{\sqrt{\nu}}\,\|\overline{z}\|_{1,\omega,r}^2, \quad \forall\,\varepsilon_1 > 0, \tag{16.23}$$

$$-2\Phi(\overline{z}_{\overline{x}_2}) \le \Big(1 + \frac{\varkappa}{\varepsilon_2\sqrt{\nu}}\Big)\|R_{\overline{x}_2}\|_*^2 + \frac{\varkappa\varepsilon_2}{\sqrt{\nu}}\,\|\overline{z}\|_{2,\omega,r}^2, \quad \forall\,\varepsilon_2 > 0, \tag{16.24}$$

are valid, where $\overline{z}$ *is a solution of the problem* (16.21).

Proof. First of all, we note that the inequalities

$$|\overline{Z}_k| \le \sqrt{\frac{\xi_k}{\nu}}\Big(\sum_{\omega_1^+} h\overline{r}\,\overline{z}_{\overline{x}_1}^2\Big)^{1/2}, \quad |\overline{Z}_{k\overline{x}_2}| \le \sqrt{\frac{\xi_k}{\nu}}\Big(\sum_{\omega_1^+} h\overline{r}\,\overline{z}_{\overline{x}_1\overline{x}_2}^2\Big)^{1/2} \tag{16.25}$$

are valid.

Indeed,

$$\begin{aligned}
|Z_k(x_2)| &\le \Big|\sum_{i=1}^{n_k} h\overline{z}_{\overline{x}_1}(ih,x_2)\Big| + \theta_k h\big|\overline{z}_{\overline{x}_1}(n_kh+h,x_2)\big| \le \\
&\le \sqrt{n_kh}\Big(\sum_{i=1}^{n_k} h\overline{z}_{\overline{x}_1}^2(ih,x_2)\Big)^{1/2} + \sqrt{\theta_kh}\big(h\overline{z}_{\overline{x}_1}^2(n_kh+h,x_2)\big)^{1/2} \le \\
&\le (n_kh+\theta_kh)^{1/2}\Big(\sum_{i=1}^{n_k+1} h\overline{z}_{\overline{x}_1}^2(ih,x_2)\Big)^{1/2},
\end{aligned} \tag{16.26}$$

and since by (16.1), $\overline{r}(ih) \ge \big(1 - ih + \frac{h}{2}\big) \ge \big(1 - (n_m+1)h + \frac{h}{2}\big) > \nu$, $i = 1,2,\dots,n_m+1$, from (16.26) we obtain the first of the inequalities (16.25); the second inequality is proved analogously.

We now make use of the nonlocal condition (16.21). Then

$$-2\Phi(\overline{z}) \le \sum_{\omega_2} h\Big(R^2 + 2R\sum_{k=1}^{m}\alpha_k\overline{Z}_k\Big),$$

$$-2\Phi(\overline{z}_{\overline{x}_2}) \le \sum_{\omega_2^+} h\Big((R_{\overline{x}_2})^2 + 2R_{\overline{x}_2}\sum_{k=1}^{m}\alpha_k\overline{Z}_{k\overline{x}_2}\Big).$$

Therefore by virtue of (16.25),

$$-2\Phi(\overline{z}) \le \|R\|_*^2 + \frac{2\varkappa}{\sqrt{\nu}}\,\|R\|_*\|\overline{z}_{\overline{x}_1}]|_r, \quad -2\Phi(\overline{z}_{\overline{x}_2}) \le \|R_{\overline{x}_2}|]_*^2 + \frac{2\varkappa}{\sqrt{\nu}}\,\|R_{\overline{x}_2}|]_*\|\overline{z}_{\overline{x}_1\overline{x}_2}]]_r,$$

whence follow (1.23) and (16.24). □

Lemma 16.5. *For the solution of the difference scheme* (16.20) *the a priori estimate*

$$\|z\|_{1,\omega,r} \le c\big(\|R\|_* + \|\eta_{11\overline{x}_1}]| + \|\eta_{12\overline{x}_1}\| + \|\eta_{22\overline{x}_2}|] + \|\eta_0\|\big), \quad c = \text{const} > 0, \quad (16.27)$$

is valid.

Proof. Relying on (16.5) and (16.10), for the solution of the problem (16.21) we obtain $a_{11}\Phi(\overline{z}) + \frac{4\nu_1(1-\varkappa^2)}{5}\,\|\overline{z}\|_{1,\omega,r}^2 \le 0$, which together with (16.23) (and ε_1 chosen appropriately) yields

$$\|\overline{z}\|_{1,\omega,r} \le c\|R\|_*. \quad (16.28)$$

Using the inequality (16.10) and taking into account (16.5), for the solution of the problem (16.22) we obtain

$$\frac{4\nu_1(1-\varkappa^2)}{5}\,\|\widetilde{z}\|_{1,\omega,r}^2 \le$$
$$\le \nu_2\big(|(\eta_{11\overline{x}_1x_1}, \widetilde{z})_\rho| + |(\eta_{12\overline{x}_1x_2}, \widetilde{z})_\rho| + |(\eta_{22\overline{x}_2x_2}, \widetilde{z})_\rho|\big) + a_0(\eta_0, \widetilde{z})_\rho. \quad (16.29)$$

It is not difficult to estimate the last two summands in the right-hand side of (16.28):

$$|(\eta_{\alpha 2\overline{x}_\alpha x_2}, \widetilde{z})_\rho| \le \|\eta_{\alpha 2\overline{x}_\alpha}|]\;\|\widetilde{z}_{\overline{x}_2}|]_r, \quad \alpha = 1, 2. \quad (16.30)$$

If for the second summand in the right-hand side of the inequality

$$(\eta_{11\overline{x}_1x_1}, \widetilde{z})_\rho = (\eta_{11\overline{x}_1x_1}, \widetilde{z})_r - \sum_{k=1}^m \varkappa\,\frac{|\alpha_k|}{\sqrt{\xi_k}} \sum_{\omega_{1,k}\times\omega_2} h^2 r_k \eta_{11\overline{x}_1x_1}\widetilde{z} \quad (16.31)$$

we use the formulas of summation by parts, we will get

$$\sum_{\omega_{1,k}\times\omega_2} h^2 r_k \eta_{11\overline{x}_1x_1}\widetilde{z} = Q_1 + Q_2, \quad (16.32)$$

where

$$Q_1 = -\sum_{\omega_{1,k}\times\omega_2} h^2 r_k \eta_{11\overline{x}_1}\widetilde{z}_{\overline{x}_1},$$
$$Q_2 = \sum_{\omega_{1,k-1}\times\omega_2} h^2 \eta_{11x_1}\widetilde{z} + \sum_{\omega_2} \theta_k h^2 \eta_{11x_1}(n_k h, x_2)\widetilde{z}(n_k h, x_2).$$

Taking into account the relations $r_k < \xi_k r < \xi_k$ and using the Cauchy inequality, after certain transformations we conclude that

$$|Q_1| \le \xi_k\Big(\sum_{\omega_{1,k}\times\omega_2} h^2 r\widetilde{z}_{\overline{x}_1}^2\Big)^{1/2}\Big(\sum_{\omega_{1,k}\times\omega_2} h^2 \eta_{11\overline{x}_1}^2\Big)^{1/2},$$

$$|Q_2| \le \Big| \sum_{i=1}^{n_k-1} \sum_{j=1}^{i} \sum_{\omega_2} h^3 \eta_{11x_1}(ih,x_2)\widetilde{z}_{\overline{x}_1}(jh,x_2)\Big| +$$
$$+ \Big| \sum_{i=1}^{n_k} \sum_{\omega_2} \theta_k h^3 \eta_{11x_1}(n_k h,x_2)\widetilde{z}_{\overline{x}_1}(ih,x_2)\Big| \le$$
$$\le \Big(\sum_{\omega_{1,k-1}\times\omega_2} h^2(n_k h - x_1)\widetilde{z}^2_{\overline{x}_1}\Big)^{1/2} \Big(\xi_k \sum_{\omega_{1,k-1}\times\omega_2} h^2 \eta^2_{11x_1}\Big)^{1/2} +$$
$$+ \Big(\sum_{\omega_{1,k}\times\omega_2} h^3 \theta_k \widetilde{z}^2_{\overline{x}_1}\Big)^{1/2} \Big(\xi_k \sum_{\omega_2} h^2 \eta^2_{11x_1}(n_k,x_2)\Big)^{1/2} \le$$
$$\le \Big(\sum_{\omega_{1,k}\times\omega_2} h^2 r_k \widetilde{z}^2_{\overline{x}_1}\Big)^{1/2} \Big(\xi_k \sum_{\omega_{1,k}\times\omega_2} h^2 \eta^2_{11x_1}\Big)^{1/2},$$

so

$$|Q_2| \le \xi_k \Big(\sum_{\omega_{1,k}\times\omega_2} h^2 r \widetilde{z}^2_{\overline{x}_1}\Big)^{1/2} \Big(\sum_{\omega_{1,k}\times\omega_2} h^2 \eta^2_{11x_1}\Big)^{1/2},$$

whence from (16.31) follows

$$\Big| \sum_{\omega_{1,k}\times\omega_2} h^2 r_k \eta_{11\overline{x}_1x_1} \widetilde{z}\Big| \le 2\xi_k \|\widetilde{z}_{\overline{x}_1}]|_r \, \|\eta_{11\overline{x}_1}]|. \tag{16.33}$$

It is not difficult to show that the estimate

$$|(\eta_{11\overline{x}_1x_1}, \widetilde{z})_r| \le 2\|\widetilde{z}_{\overline{x}_1}]|_r \|\eta_{11\overline{x}_1}]| \tag{16.34}$$

is valid.

Finally, with regard for (16.33), (16.34), from (16.31) we find that

$$|(\eta_{11\overline{x}_1x_1}, \widetilde{z})_\rho| \le 2(1+\varkappa^2)\|\widetilde{z}_{\overline{x}_1}]|_r \|\eta_{11\overline{x}_1}]| \tag{16.35}$$

since $r(x_1) < \overline{r}(x_1)$. Using (16.29), (16.30) and (16.35) we can see that the estimate

$$\frac{4\nu_1(1-\varkappa^2)}{5} \|\widetilde{z}\|_{1,\omega,r} \le$$
$$\le \nu_2\big(2(1+\varkappa^2)\|\eta_{11\overline{x}_1}]| + \|\eta_{12\overline{x}_1}|] + \|\eta_{22\overline{x}_2}|]\big) + a_0\|\eta_0\|_\rho \tag{16.36}$$

is valid.

The estimate (16.27) is the direct consequence of (16.28), (16.36). □

Lemma 16.6. *For the solution of the difference problem* (16.20) *the a priori estimate*

$$\|z\|_{2,\omega,r} \le c\big(\|R\|_* + \|R_{\overline{x}_2}|]_* + \|\eta_{11\overline{x}_1x_1}\| + \|\eta_{12\overline{x}_1x_2}\| + \|\eta_{22\overline{x}_2x_2}\| + \|\eta_0\|\big) \tag{16.37}$$

is valid.

Proof. We multiply (16.23) and (16.24) respectively by $c_1/2$ and $c_2/2$ and add the inequality (16.11) which was written for $\overline{z}$. In the right-hand side of the obtained inequality we choose ε_1 and ε_2 sufficiently small and find that

$$\|\overline{z}\|_{2,\omega,r} \le c\big(\|R\|_* + \|R_{\overline{x}_2}|]_*\big). \tag{16.38}$$

Next, by virtue of (16.11), for the solution of the problem (16.22) we have $c_3\|\widetilde{z}\|_{2,\omega,r} \le \nu_2\big(\|\eta_{11\overline{x}_1x_1}\| + \|\eta_{12\overline{x}_1x_2}\| + \|\eta_{22\overline{x}_2x_2}\|\big) + a_0\|\eta_0\|$, which together with (16.38) completes the proof of the lemma. □

4^0. Estimation of the convergence rate. Lemmas 16.5 and 16.6 show that to obtain estimates for the convergence rate of the difference scheme (16.2),(16.3) it is sufficient to estimate the corresponding norms of R and η_{11}, η_{12}, η_{22}, η_0. Let us show that

$$\|R\|_* \le ch^{m-1}\|u\|_{W_2^m(\Omega)}, \quad m \in (1,3]. \tag{16.39}$$

Let $e_k = (n_kh, n_kh+h)\times(x_2-h/2, x_2+h/2)$, $\Omega_k = (n_kh, n_kh+h)\times(0,1)$. We represent R_k as the sum $R_k = \big(h^2G_k\,\frac{\partial^2 u}{\partial x_1^2} - h^2G_kS_2^-\frac{\partial^2 u}{\partial x_1^2}\big) + h^2G_kS_2^-\frac{\partial^2 u}{\partial x_1^2} = R'_k + R''_k$, $k = 1,2,\dots,n_m$. Note that R'_k vanishes on the second degree polynomials and is bounded in $W_2^m(\Omega)$, $m > 1$. As a consequence, using the Bramble–Hilbert lemma, we obtain

$$|R'_k| \le ch^{m-1}|u|_{W_2^m(e_k)}, \quad \|R'_k\|_* \le ch^{m-1}|u|_{W_2^m(\Omega)}, \quad m \in (1,3]. \tag{16.40}$$

R''_k vanishes on the first degree polynomials and is bounded in $W_2^m(\Omega)$, $m > 1$. Therefore using the Bramble–Hilbert lemma, we obtain the estimate

$$\|R''_k\|_* \le ch^{m-1}|u|_{W_2^m(\Omega)}, \quad s \in (1,2.5]. \tag{16.41}$$

For $m > 2.5$, we write

$$\|R''_k\|_*^2 \le c\sum_{\omega_2} h^3 \int\limits_{e_k} \Big|\frac{\partial^2 u}{\partial x_1^2}\Big|^2 dx \le ch^3\Big\|\frac{\partial^2 u}{\partial x_1^2}\Big\|^2_{L_2(\Omega_k)}$$

and since $\partial^2u/\partial x_1^2 \in W_2^{m-2}(\Omega)$, $m-2 > 0.5$, we may use an estimate for L_2-norm of a function in a strip along the boundary in terms of W_2^{m-2}-norm in the domain (Theorem 1.5): $\big\|\frac{\partial^2 u}{\partial x_1^2}\big\|_{L_2(\Omega_k)} \le ch^{1/2}\big\|\frac{\partial^2 u}{\partial x_1^2}\big\|_{W_2^{m-2}(\Omega)}$, $0.5 < m-2 \le 1$. Consequently, we obtain the estimate $\|R''_k\|_* \le ch^2|u|_{W_2^m(\Omega)}$, $m \in (2.5,3]$, which together with (16.40) and (16.41) proves the validity of the inequality (16.39).

It is not also difficult to see that

$$\|R_{\overline{x}_2}|]_* \le ch^{m-2}|u|_{W_2^m(\Omega)}, \quad s \in (2,4]. \tag{16.42}$$

Taking into account the well-known estimates for η_{11}, η_{12}, η_{22}, η_0 and for their differences, on the basis of the estimates (16.27), (16.37) we convince ourselves that the following statement is valid.

Theorem 16.1. *Let the solution of the problem* (16.1), (16.2) *belong to the class $W_2^m(\Omega)$, $m \in (1,4]$. Then the convergence rate of the difference scheme* (16.2), (16.3) *is defined by the estimate*

$$\|y-u\|_{W_2^s(\omega,r)} \le ch^{m-s}\|u\|_{W_2^m(\Omega)}, \quad m \in (s, s+2], \quad s = 1,2, \tag{16.43}$$

where the positive constant c does not depend on $u(x)$ and h.

Remark 16.1. The obtained results are likewise valid both for the inhomogeneous boundary value problems and for the nonlocal conditions.

17. Difference Scheme in the Case of Variable Coefficients

History of the matter. In this section we consider difference approximation of a nonlocal Bitsadze–Samarskiĭ type boundary value problem for the second order elliptic equation with variable coefficients. The results have been published in [26].

1^0. Statement of the problem. We consider the nonlocal boundary value problem (15.1), (15.2), where $a_{1j} = a_{1j}(x_{3-j})$, $a_{2j} = a_{2j}(x)$ $(j = 1, 2)$, $a_0 = a_0(x)$, $a_{1j} \in W_p^{m-1}(0,1)$, $p > \max(1/(m-1); 2)$ for $m \in (1,2]$, $p = 2$ for $m \in (2,3]$, $a_{2j} \in W_q^{m-1}(\Omega)$, $q > 2/(m-1)$ for $m \in (1,2]$, $q = 2$ for $m \in (2,3]$, $a_0 \in L_{2+\varepsilon}(\Omega)$, $1 < m \le 2$, $a_0 \in W_2^{m-2}(\Omega)$, $2 < m \le 3$. Let, moreover, $f(x) \in W_2^{m-2}(\Omega)$, and the problem (15.1), (15.2) be uniquely solvable in the class $W_2^m(\Omega)$, $1 < m \le 3$.

For the mesh domains and mesh functions we use the notation of Section 16.

The problem (15.1), (15.2) is approximated by the difference scheme

$$Ay \equiv \sum_{i,j=1}^{2} A_{ij}y + a_0 y = -\varphi(x), \quad x \in \omega, \quad \varphi = T_1T_2 f, \tag{17.1}$$

$$y = 0, \quad x \in \gamma_0, \quad y(1, x_2) = \sum_{k=1}^{m} \alpha_k Y_k(x_2), \quad x_2 \in \omega_2, \tag{17.2}$$

where $A_{ij}y = -0.5(a_{ij}^{(-0.5_1)} y_{\overline{x}_j})_{x_i} - 0.5(a_{ij}^{(+0.5_1)} y_{x_j})_{\overline{x}_i}$.

2^0. The correctness of the difference scheme. Let H be the space of mesh functions introduced on Section 16.

By $\Phi(a_{11}, y)$ we denote the functional

$$\Phi(a_{11}, y) = \frac{1}{2} \sum_{\omega_2} h a_{11}(x_2) \Big(\varkappa \sum_{k=1}^{m} \frac{|\alpha_k|}{\sqrt{\xi_k}} Y_k^2(x_2) - y^2(1, x_2) \Big). \tag{17.3}$$

Lemma 17.1. *For every mesh function $y(x)$ defined on $\overline{\omega}$ and vanishing on γ_0 the estimate*

$$(Ay, y)_\rho \ge \frac{4\nu_1}{5} \|y\|^2_{1,\omega,\rho} + \Phi(a_{11}, y) \tag{17.4}$$

is valid. In addition, if $y(x)$ satisfies also the nonlocal condition (17.2)*, then in the right-hand side* (17.4) *the second summand can be omitted.*

Proof. Analogously to the inequality (16.7), we can show that

$$(A_{11}y,y)_\rho \geq \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\rho a_{11}y_{\overline{x}_1}^2 + \frac{1}{2}\sum_{\omega_1^-\times\omega_2} h^2\rho a_{11}y_{x_1}^2 + \Phi(a_{11},y).$$

Next, summing by parts, we find that

$$(A_{12}y,y)_\rho = \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\rho a_{12}^{(-0.5_1)}y_{\overline{x}_1}y_{\overline{x}_2} + \frac{1}{2}\sum_{\omega_1^-\times\omega_2} h^2\rho a_{12}^{(+0.5_1)}y_{x_1}y_{x_2} + I,$$

where

$$I=\frac{1}{2}\sum_{\omega_1^+} h\rho_{\overline{x}_1} a_{12}^{(-0.5_1)}\sum_{\omega_2} h(y_{\overline{x}_2}y^{(-1_1)} + y_{x_2}^{(-1_1)}y) = 0,$$

$$(A_{2j}y,y)_\rho=\frac{1}{2}\sum_{\omega_1\times\omega_2^+} h^2\rho a_{2j}^{(-0.5_1)}y_{\overline{x}_j}y_{\overline{x}_2}+\frac{1}{2}\sum_{\omega_1\times\omega_2^-} h^2\rho a_{2j}^{(+0.5_1)}y_{x_j}y_{x_2}, \quad j=1,2.$$

Consequently,

$$(Ay,y)_\rho \geq \frac{1}{2}\sum_{\omega^+} h^2\rho \sum_{i,j=1,2}^{2} a_{ij}^{(-0.5_1)}y_{\overline{x}_i}y_{\overline{x}_j} + \\ + \frac{1}{2}\sum_{\omega^-} h^2\rho \sum_{i,j=1,2}^{2} a_{ij}^{(+0.5_1)}y_{x_i}y_{x_j} + (a_0,y^2)_\rho + \Phi(a_{11},y).$$

Using the condition of ellipticity, we obtain the inequality $(Ay,y)_\rho \geq |y|_{1,\omega,\rho}+$ $\Phi(a_{11},y)$, which together with $2\|y\|_\rho \leq |y|_{1,\omega,\rho}$ results in (17.4).

If $y(x)$ satisfies also the nonlocal condition (17.2), then $y^2(1,x_2) \leq \sum\limits_{k=1}^{m} \varkappa\frac{|\alpha_k|}{\sqrt{\xi_k}}\,Y_k^2(x_2)$ and $\Phi(a_{11},y) \geq 0$. Thus the proof of the lemma is complete. □

By Lemma 17.1, the operator A is positive definite in the space H. Hence the difference scheme (17.1), (17.2) is uniquely solvable.

3^0. A priori error estimate of the method. To study the question of the convergence and accuracy of the difference scheme (17.1), (17.2), we consider the error $z = y-u$ of the method, where y is a solution of the difference scheme, and $u = u(x)$ is a solution of the initial problem. Substituting $y = z + u$ in (17.1), (17.2), we find that

$$Az=\psi, \quad x\in\omega, \quad z=0, \quad x\in\gamma_0, \quad z(1,x_2)=\sum_{k=1}^{m}\alpha_k Z_k+R, \quad x_2\in\omega_2, \tag{17.5}$$

where $\psi = \sum\limits_{i,j=1}^{2}(\eta_{ij})_{x_i} + \eta_0$, $R = \sum\limits_{k=1}^{m}\alpha_k R_k$, $R_k = h^2 G_k\,\frac{\partial^2 u}{\partial x_1^2}$, $\eta_0 = T_1T_2(a_0u) -$ $T_1T_2a_0u$, $\eta_{ij}=\frac{1}{2}\,a_{ij}^{(-0.5_1)}u_{\overline{x}_j}+\frac{1}{2}\,a_{ij}^{(+0.5_1,-1_i)}u_{x_j}^{(-1_i)}-S_i^-T_{3-i}\big(a_{ij}\,\frac{\partial u}{\partial x_j}\big)$, $i,j=1,2$.

We now represent the solution z of the problem (17.5) in the form of the sum $z=\overline{z}+\widetilde{z}$ of solutions of the following two problems:

$$A\overline{z}=0,\ \ x\in\omega,\ \ \overline{z}=0,\ \ x\in\gamma_0,\ \ \overline{z}(1,x_2)=\sum_{k=1}^{m}\alpha_k\overline{Z}_k+R,\ \ x_2\in\omega_2, \tag{17.6}$$

$$A\widetilde{z}=\psi,\ \ x\in\omega,\ \ \widetilde{z}=0,\ \ x\in\gamma_0,\ \ \widetilde{z}(1,x_2)=\sum_{k=1}^{m}\alpha_k\widetilde{Z}_k,\ \ x_2\in\omega_2. \tag{17.7}$$

Lemma 17.2. *For the functional* Φ *defined in* (17.3) *the estimate*

$$-2\Phi(a_{11},\overline{z})\le\Big(1+\frac{\varkappa}{\varepsilon\sqrt{\nu}}\Big)\|R\|_*^2+\frac{\varkappa\varepsilon}{\sqrt{\nu}}\,\|\overline{z}\|_{1,\omega,r}^2,\quad\forall\,\varepsilon>0, \tag{17.8}$$

where $\overline{z}$ *is a solution of the problem* (17.6)*, is valid.*

Proof. Using the nonlocal condition (17.6), we find that $-2\Phi(a_{11},\overline{z})\le\sum\limits_{\omega_2}h\big(R^2+2R\sum\limits_{k=1}^{m}\alpha_k\overline{Z}_k\big)$. Therefore by (16.25), $-2\Phi(a_{11},\overline{z})\le\|R\|_*^2+\frac{2\varkappa}{\sqrt{\nu}}\,\|R\|_*\|\overline{z}_{\overline{x}_1}]|_r$, whence we obtain (17.8). □

Lemma 17.3. *For the solution of the difference problem* (17.5) *the a priori estimate*

$$\|z\|_{1,\omega,r}\le c\big(\|R\|_*+\|\eta_{11}]|+\|\eta_{12}]|+\|\eta_{21}|]+\|\eta_{22}|]+\|\eta_0|]\big) \tag{17.9}$$

is valid.

Proof. On the basis of (16.85), (17.4), for the solution of the problem (17.6) we obtain $\Phi(a_{11},\overline{z})+\frac{4\nu_1(1-\varkappa^2)}{5}\,\|\overline{z}\|_{1,\omega,r}^2\le 0$, which together with (17.8) (with ε chosen appropriately) yields

$$\|\overline{z}\|_{1,\omega,r}\le c\|R\|_*. \tag{17.10}$$

Using Lemma 17.1 for the solution of the problem (17.7) and taking into account (16.5), we get

$$\frac{4\nu_1(1-\varkappa^2)}{5}\,\|\widetilde{z}\|_{1,\omega,r}^2\le\sum_{i,j=1}^{2}\big((\eta_{ij})_{x_i},\widetilde{z}\big)_\rho+(\eta_0,\widetilde{z})_\rho. \tag{17.11}$$

It is not difficult to estimate the last summand in the right-hand side (17.11):

$$(\eta_0,\widetilde{z})_\rho\le\frac{1}{2}\,\|\eta_0\|_r\|\widetilde{z}\|_{1,\omega,r}. \tag{17.12}$$

Analogously to (16.35), (16.30), we have

$$\big((\eta_{1j})_{x_1},\widetilde{z}\big)_\rho\le 2(1+\varkappa^2)\|\widetilde{z}_{\overline{x}_1}]|_r\|\eta_{1j}]|,\ \ j=1,2, \tag{17.13}$$

$$\big((\eta_{2j})_{x_2},\widetilde{z}\big)_\rho\le\|\widetilde{z}_{\overline{x}_2}|]_r\|\eta_{2j}|],\ \ j=1,2. \tag{17.14}$$

Applying (17.12)–(17.14), from (17.11) we obtain that

$$\frac{4\nu_1(1-\varkappa^2)}{5}\,\|\widetilde{z}\|_{1,\omega,r}\le$$

$$\leq 2(1+\varkappa^2)\big(\|\eta_{11}]| + \|\eta_{12}]|\big) + \|\eta_{21}|] + \|\eta_{22}|] + \frac{1}{2}\,\|\eta_0\|_r. \qquad (17.15)$$

The estimate (17.9) is a direct consequence of (17.10), (17.15). □

4⁰. Estimate of the convergence rate. Lemma 17.3 shows that in order to obtain the needed estimate of the convergence rate of the difference scheme (17.1), (17.2) it is necessary to have estimates for summands appearing in the right-hand side (17.9).

The estimate of the norm $\|R\|_*$ is given in (16.39).

Let us show that for η_{ij} defined in (17.5) the estimates

$$\|\eta_{1j}\| \leq ch^{m-1}\|u\|_{W_2^m(\Omega)}, \quad \|\eta_{2j}\| \leq ch^{m-1}\|u\|_{W_2^m(\Omega)}, \quad 1 < m \leq 3, \qquad (17.16)$$

are valid.

For $1 < m \leq 2$ we represent η_{ij} as the sum

$$\eta_{ij} = \eta'_{ij} + 0.5a_{ij}^{(-0.5_1)}\eta''_{ij}\Big(\frac{\partial u}{\partial x_j}\Big) + 0.5a_{ij}^{(+0.5_1,-1_i)}\eta'''_{ij}\Big(\frac{\partial u}{\partial x_j}\Big), \qquad (17.17)$$

where

$$\eta'_{ij} = 0.5\big(a_{ij}^{(-0.5_1)} + a_{ij}^{(+0.5_1,-1_i)}\big)S_i^- T_{3-i}\,\frac{\partial u}{\partial x_j} - S_i^- T_{3-i}\Big(a_{ij}\,\frac{\partial u}{\partial x_j}\Big),$$

$$\eta''_{ij}(v) = S_j^- v - S_i^- T_{3-i}v, \quad \eta'''_{ij}(v) = (S_j^+ v)^{(-1_i)} - S_i^- T_{3-i}v.$$

η'_{2j}, being a functional with respect to a_{2j}, vanishes for $a_{2j} \in \boldsymbol{\pi}_0$. By the Bramble–Hilbert lemma for that functional we obtain

$$|\eta'_{2j}| \leq ch^{-2}|\widetilde{a}_{2j}|_{W_q^{m-1}(\tilde{e})}\int\limits_e \Big|\frac{\partial u}{\partial x_j}\Big|\,dx \leq ch^{m-3-2/q}|a_{2j}|_{W_q^{m-1}(e)}\int\limits_e \Big|\frac{\partial u}{\partial x_j}\Big|\,dx.$$

But

$$\int\limits_e \Big|\frac{\partial u}{\partial x_j}\Big|\,dx \leq \bigg(\int\limits_e \Big|\frac{\partial u}{\partial x_j}\Big|^{\frac{2q}{q-2}}\,dx\bigg)^{\frac{q-2}{2q}} h^{1+2/q}.$$

Consequently, $|\eta'_{2j}| \leq ch^{m-2}|a_{2j}|_{W_q^{m-1}(e)}|u|_{W^1_{\frac{2q}{q-2}}(e)}$ and

$$\|\eta'_{2j}|] \leq ch^{m-1}\|a_{2j}\|_{W_q^{m-1}(\Omega)}\|u\|_{W^1_{\frac{2q}{q-2}}(\Omega)}. \qquad (17.18)$$

Analogously, $|\eta'_{1j}| \leq ch^{m-2+1/p}|a_{1j}|_{W_p^{m-1}(e_1)}|u|_{W^1_{\frac{2p}{p-2}}(e)}$ and hence

$$\|\eta'_{1j}]| \leq ch^{m-1}\|a_{1j}\|_{W_p^{m-1}(0,1)}\|u\|_{W^1_{\frac{2p}{p-2}}(\Omega)}. \qquad (17.19)$$

Since $W^1_{2q/(q-2)}$, $W^1_{2p/(p-2)} \subset W_2^{m-2}$, therefore using (17.18), (17.19) and the analogous inequalities for η''_{ij}, η'''_{ij}, from (17.17) we obtain (17.16) for $1 < m \leq 2$.

For $2 < m \leq 3$ we write

$$\eta_{i1} = \ell_1\Big(a_{i1}\,\frac{\partial u}{\partial x_1}\Big) + 0.5a_{i1}\ell_2\Big(\frac{\partial u}{\partial x_1}\Big) + 0.5a_{i1}^{(+0.5_1,-1_i)}\ell_3\Big(\frac{\partial u}{\partial x_1}\Big), \qquad (17.20)$$

where $\ell_1(v) = 0.5v^{(-0.5_1)} + 0.5v^{(+0.5_1,-1_i)} - S_i^- T_{3-i}v$, $\ell_2(v) = S_1^- v - v^{(-0.5_1)}$, $\ell_3(v) = S_1^+ v - v^{(+0.5_1)}$ and

$$\eta_{i2} = \ell_4\Big(a_{i2}\,\frac{\partial u}{\partial x_2}\Big) + a_{i2}^{(-0.5_i)}\ell_5\Big(\frac{\partial u}{\partial x_2}\Big) + 0.5u_{\overline{x}_2}\ell_6(a_{i2}), \tag{17.21}$$

where $\ell_4(v) = v^{(-0.5_i)} - S_i^- T_{3-i}v$, $\ell_5(v) = 0.5S_2^- v + 0.5S_2^+ v^{(-1_i)} - v^{(-0.5_i)}$, $\ell_6(v) = v^{(-0.5_1)} - 2v^{(-0.5_i)} + v^{(+0.5_1,-1_i)}$.

Note that $\ell_j(v)$, $j = 1,2,3,4,5,6$, vanish for $v \in \boldsymbol{\pi}_1$ and are bounded in W_2^m, $2 < m \le 3$. Therefore using the Bramble–Hilbert lemma, for the above values we obtain $|\ell_j(v)| \le ch^{m-2}|v|_{W_2^{m-1}(e)}$, $2 < m \le 3$, on the basis of which from (17.20), (17.21) we easily obtain (17.16) for $2 < m \le 3$.

For the norm of η_0 we have $\|\eta_0\| \le ch^{m-1}\|u\|_{W_2^m(\Omega)}$, $1 < m \le 3$.

Finally, the above-obtained estimates and Lemma 17.3 imply that the following statement is valid.

Theorem 17.1. *Let the solution of the problem* (15.1), (15.2) *belong to the class* $W_2^m(\Omega)$, $m \in (1,3]$. *Then the convergence rate of the difference scheme* (17.1), (17.2) *is defined by the estimate*

$$\|y-u\|_{W_2^1(\omega,r)} \le ch^{m-1}\|u\|_{W_2^m(\Omega)}, \quad m \in (1,3], \tag{17.22}$$

where the positive constant c *does not depend on* $u(x)$ *and* h.

18. Difference Scheme for The Poisson Equation with Integral Restriction

History of the matter. The boundary value problem for differential equations with nonlocal conditions are encountered in many applications. Various problems with integral conditions have been considered, for example, in [72], [73] and [45]. In this section we consider nonlocal boundary value problems with integral restriction for the Poisson equation. The results of Section 18 are published in [25].

1^0. Statement of the problem. Let $\Omega = \{(x_1,x_2):\ 0 < x_k < \ell_\alpha,\ \alpha = 1,2\}$ be a rectangle with the boundary Γ, $\Gamma_* = \{(0,x_2):\ 0 < x_2 < \ell_2\}$, $\Gamma_0 = \Gamma \setminus \Gamma_*$.

We consider the nonlocal boundary value problem

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = -f(x), \quad x \in \Omega, \tag{18.1}$$

$$u(x) = 0, \quad x \in \Gamma_0, \quad \int_0^{\ell_1} u(t,x_2)\,dt = 0, \quad 0 \le x_2 \le \ell_2. \tag{18.2}$$

Assume that $f(x) \in W_2^{m-2}(\Omega)$ and the problem (18.1), (18.2) is uniquely solvable in the class $W_2^m(\Omega)$, $1 < m \le 3$.

As usual, on $\overline{\Omega}$ we introduce mesh domains, and on the set of mesh functions we define the operator

$$Gy = x_1 y - Sy, \quad x \in \overline{\omega}, \tag{18.3}$$

where

$$(Sy)_{ij} = \sum_{k=0}^{i} h_1 y_{kj} - \frac{h_1}{2}(y_{ij} + y_{0j}), \tag{18.4}$$

$$i = 0, 1, 2, \ldots, N_1, \quad j = 0, 1, 2, \ldots, N_2.$$

We approximate the problem (18.1), (18.2) by the difference scheme

$$\Lambda y \equiv y_{\overline{x}_1 x_1} + y_{\overline{x}_2 x_2} = -\varphi(x), \quad x \in \omega, \quad \varphi = T_1 T_2 f, \tag{18.5}$$

$$y(x) = 0, \quad x \in \gamma_0, \quad Sy(\ell_1, x_2) = 0, \quad x_2 \in \overline{\omega}_2. \tag{18.6}$$

2^0. The correctness of the difference scheme. Let H be the space of mesh functions defined on $\overline{\omega}$ and satisfying the conditions (18.6), with the inner product and the norm $(y, v) = \sum\limits_{\omega_1^- \times \omega_2} \hbar_1 h_2 y(x) v(x)$, $\|y\| = (y, y)^{1/2}$. Let, moreover,

$$\|y\|_1^2 = \|y\|^2 + \|\nabla y\|^2, \quad \|\nabla y\|^2 = \|y_{\overline{x}_1}\|_{(1)}^2 + \|y_{\overline{x}_2}\|_{(2)}^2,$$

$$\|y_{\overline{x}_1}\|_{(1)}^2 = \sum_{\omega_1^+ \times \omega_2} h_1 h_2 r_1 (y_{\overline{x}_1})^2, \quad \|y_{\overline{x}_2}\|_{(2)}^2 = \sum_{\omega_1^- \times \omega_2^+} h_1 h_2 r_2 \, (y_{\overline{x}_2})^2,$$

$$r_1 = x_1 - \frac{h_1}{2}, \quad r_2 = x_1 \ \text{ for } \ x_1 \in \omega_1, \quad r_2 = \frac{h_1}{8} \ \text{ for } \ x_1 = 0,$$

$$\|y]|^2 = \sum_{\omega_1^+ \times \omega_2} h_1 h_2 y^2, \quad \|y|]^2 = \sum_{\omega_1 \times \omega_2^+} h_1 h_2 y^2,$$

$$\|y]]^2 = \sum_{\omega_1^+ \times \omega_2^+} h_1 h_2 y^2, \quad \|y\|_*^2 = \sum_{\omega_2} h_2 y^2, \quad \|y|]_*^2 = \sum_{\omega_2^+} h_2 y^2,$$

Lemma 18.1. *If the mesh function v defined on $\overline{\omega}$ satisfies the conditions $v = 0$, $Sv = 0$ for $x_1 = \ell_1$, then*

$$\sum_{\omega_1} h_1 (Sv)^2 \le 4 \sum_{\omega_1} h_1 x_1^2 v^2, \tag{18.7}$$

$$\sum_{\omega_1} h_1 (Gv)^2 \le 9 \sum_{\omega_1} h_1 x_1^2 v^2, \tag{18.8}$$

$$\sum_{\omega_1} h_1 Sv \, v = -\frac{h_1^2}{8} \, v^2(0, x_2). \tag{18.9}$$

Proof. It is not difficult to verify that

$$\sum_{\omega_1} h_1 (Sv)^2 = -\sum_{\omega_1^-} h_1 x_1 (Sv)_{x_1} \big((Sv)^{(+1_1)} + Sv\big),$$

from which owing to $(Sv)_{x_1} = \frac{1}{2}\,(v^{(+1_1)} + v)$, we obtain

$$\sum_{\omega_1} h_1(Sv)^2 \le$$
$$\le \frac{1}{2}\Big(\sum_{\omega_1^-} h_1x_1^2(v^{(+1_1)}+v)^2\Big)^{1/2}\Big(\sum_{\omega_1^-} h_1\big((Sv)^{(+1_1)}+Sv\big)^2\Big)^{1/2}. \qquad (18.10)$$

If we notice that

$$\sum_{\omega_1^-} h_1x_1^2(v^{(+1_1)}+v)^2 \le 4\sum_{\omega_1} h_1x_1^2\,v^2,$$
$$\sum_{\omega_1^-} h_1\big((Sv)^{(+1_1)}+Sv\big)^2 \le 4\sum_{\omega_1} h_1(Sv)^2,$$

then from (18.10) it follows the estimate (18.7). According to (18.3),

$$\sum_{\omega_1} h_1(Gv)^2 \le$$
$$\le \sum_{\omega_1} h_1x_1^2v^2 + \sum_{\omega_1} h_1(Sv)^2 + 2\Big(\sum_{\omega_1} h_1x_1^2v^2\Big)^{1/2}\Big(\sum_{\omega_1} h_1(Sv)^2\Big)^{1/2},$$

which together with (18.7) proves the estimate (18.8).

The relation (18.9) follows from the easily verifiable identity $\sum\limits_{\omega_1} h_1Svv = \sum\limits_{\omega_1} h_1(Sv)_{\mathring{x}_1}Sv - \frac{h_1^2}{8}\,v^2(0,x_2)$. Thus the lemma is complete. □

Lemma 18.2. *For any $y \in H$ the identity*

$$-(\Lambda y, Gy)_\omega = \|\nabla y\|^2 \qquad (18.11)$$

holds.

Proof. By (18.9) we have $\sum\limits_{\omega_1} h_1v\,Gv = \sum\limits_{\omega_1} h_1x_1v^2 + \frac{h_1^2}{8}\,v^2(0,x_2)$.

Substituting $v = y_{\overline{x}_2}$, we obtain

$$(y_{\overline{x}_2x_2}, Gy)_\omega = -\sum_{\omega_1\times\omega_2^+} h_1h_2y_{\overline{x}_2}Gy_{\overline{x}_2} = -\|y_{\overline{x}_2}\|^2_{(2)}. \qquad (18.12)$$

The summation by parts yields $(y_{\overline{x}_1x_1}, Gy)_\omega = -\sum\limits_{\omega_1^+\times\omega_2} h_1h_2y_{\overline{x}_1}(Gy)_{\overline{x}_1}$ and since

$$(Gy)_{\overline{x}_1} = \Big(x_1 - \frac{h_1}{2}\Big)y_{\overline{x}_1}, \qquad (18.13)$$

therefore

$$(y_{\overline{x}_1x_1}, Gy)_\omega = -\|y_{\overline{x}_1}\|^2_{(1)}. \qquad (18.14)$$

The equalities (18.12), (18.14) complete the proof of the lemma. □

Lemma 18.3. *For any $y \in H$ the estimate*

$$\|y\|_1^2 \le (1+4\ell_1)\|\nabla y\|^2 \qquad (18.15)$$

is valid.

Proof. We have

$$\sum_{\omega_1^-} \hbar_1 y^2 = -\sum_{\omega_1^+} h_1 r_1 \,(y^2)_{\overline{x}_1} = -\sum_{\omega_1^+} h_1 r_1 (y + y^{(-1_1)}) y_{\overline{x}_1} \le$$
$$\le 2\sum_{\omega_1^+} h_1 r_1^2 \,(y_{\overline{x}_1})^2 + \frac{1}{2}\sum_{\omega_1^+} h_1 (y + y^{(-1_1)})^2 \le$$
$$\le 2\sum_{\omega_1^+} h_1 r_1^2 \,(y_{\overline{x}_1})^2 + \frac{1}{2}\sum_{\omega_1^-} \hbar_1 y^2.$$

Therefore $\sum\limits_{\omega_1^-\times\omega_2} \hbar_1 h_2 y^2 \le 4 \sum\limits_{\omega_1^+\times\omega_2} h_1 h_2 r_1^2 (y_{\overline{x}_1})^2 \le 4\ell_1 \|\nabla y\|^2$, which proves the estimate (18.15). □

From (18.11), (18.15) it follows

$$\|y\|_1^2 \le -(1 + 4\ell_1)(\Lambda y, Gy)_\omega. \tag{18.16}$$

Thus if $\Lambda y \equiv 0$ for $x \in \omega$, then $y(x) \equiv 0$ for $x \in \overline{\omega}$. This means that the solution of the inhomogeneous problem (18.5), (18.6) exists and is unique.

3^0. A priori estimate of error of the method. For the error $z = y - u$, where y is a solution of the difference scheme (18.5), (18.6), and $u = u(x)$ is the exact solution, we obtain the problem

$$\Lambda z = \psi, \quad x \in \omega, \tag{18.17}$$

$$z(x) = 0, \quad x \in \gamma_0, \quad Sz = \chi(x_2), \quad x_1 = \ell_1, \quad x_2 \in \overline{\omega}_2, \tag{18.18}$$

where $\psi = \eta_{1\overline{x}_1 x_1} + \eta_{2\overline{x}_2 x_2}$, $\chi(x_2) = \sum\limits_{\omega_1^+} h_1 \eta$, $\eta_\alpha = T_{3-\alpha} u - u$, $\alpha = 1, 2$, $\eta = S_1^- u(x) - \frac{1}{2}\,(u(x) + u(x_1 - h_1, x_2))$.

If in (18.17), (18.18) we pass to the new unknown function

$$w(x) = z(x) - \frac{2}{\ell_1^2}\,(\ell_1 - x_1)\chi(x_2), \tag{18.19}$$

then for that function we will obtain the problem

$$\Lambda w = \psi - \frac{2}{\ell_1^2}\,(\ell_1 - x_1)\chi_{\overline{x}_2 x_2}, \quad x \in \omega, \quad w \in H. \tag{18.20}$$

Relying on (18.16), for the solution of the problem (18.20) we have

$$\|w\|_1^2 \le (1 + 4\ell_1)\times$$
$$\times\Big(\big|(\eta_{1\overline{x}_1 x_1}, Gw)_\omega\big| + \big|(\eta_{2\overline{x}_2 x_2}, Gw)_\omega\big| + \frac{2}{\ell_1^2}\,\big|\big((\ell_1 - x_1)\chi_{\overline{x}_2 x_2}, Gw\big)_\omega\big|\Big). \tag{18.21}$$

Using summation by parts, the formula (18.13) and the Cauchy–Bunjakovski's inequality, we obtain

$$\big|(\eta_{1\overline{x}_1x_1}, Gw)_\omega\big| = \Big|\sum_{\omega_1^+\times\omega_2} h_1h_2r_1\eta_{1\overline{x}_1}w_{\overline{x}_1}\Big| \le \sqrt{\ell_1}\|\eta_{1\overline{x}_1}]|\,\|w_{\overline{x}_1}\|_{(1)}. \quad (18.22)$$

The summation by parts, the Cauchy–Buniakowski inequality and the estimate (18.8) result in

$$\big|(\eta_{2\overline{x}_2x_2}, Gw)_\omega\big| = \Big|\sum_{\omega_1\times\omega_2^+} h_1h_2\eta_{2\overline{x}_2}Gw_{\overline{x}_2}\Big| \le 3\sqrt{\ell_1}\|w_{\overline{x}_2}\|_{(2)}\|\eta_{2\overline{x}_2}|], \quad (18.23)$$

and analogously,

$$\big|\big((\ell_1-x_1)\chi_{\overline{x}_2x_2}, Gw\big)_\omega\big| \le \ell_1^2\|\chi_{\overline{x}_2}|]_*\|w_{\overline{x}_2}\|_{(2)}. \quad (18.24)$$

Substituting (18.22)–(18.24) in (18.21), we obtain

$$\|w\|_1 \le c_1\big(\|\eta_{1\overline{x}_1}\|_{(1)} + \|\eta_{2\overline{x}_2}\|_{(2)} + \|\chi_{\overline{x}_2}|]_*\big). \quad (18.25)$$

According to (18.19), for the error of the method we can write

$$\|z\|_1 \le \|w\|_1 + c_2\big(\|\chi\|_* + \|\chi_{\overline{x}_2}|]_*\big). \quad (18.26)$$

From the definition of χ it immediately follows that

$$\|\chi\|_* \le \sqrt{\ell_1}\|\eta]|, \quad \|\chi_{\overline{x}_2}|]_* \le \sqrt{\ell_1}\|\eta_{\overline{x}_2}]]. \quad (18.27)$$

Substituting (18.25) in (18.26) and taking into account (18.27), we arrive at the following

Lemma 18.4. *For the solution of the difference problem* (18.17)*,* (18.18) *the a priori estimate*

$$\|z\|_1 \le c_3\big(\|\eta_{1\overline{x}_1}]| + \|\eta_{2\overline{x}_2}|] + \|\eta]| + \|\eta_{\overline{x}_2}]]\big) \quad (18.28)$$

is valid.

2^0. Estimation of the convergence rate. Lemma 18.4 shows that to obtain an estimate of the convergence rate of the difference scheme (18.5),(18.6) it is sufficient to estimate the norms of the summands in the right-hand side (18.28).

Note that η vanishes on $\boldsymbol{\pi}_1$, while $\eta_{\overline{x}_2}$, $\eta_{1\overline{x}_1}$ and $\eta_{2\overline{x}_2}$ vanish on $\boldsymbol{\pi}_2$. Using then the well-known techniques of estimation which is based on the Bramble–Hilbert lemma, we can see that the following theorem is valid.

Theorem 18.1. *The convergence rate of the difference scheme* (18.5)*,* (18.6) *is defined by the estimate*

$$\|y-u\|_1 \le c|h|^{m-1}\|u\|_{W_2^m(\Omega)}, \ \ m\in(1,3], \quad (18.29)$$

where the positive constant c does not depend on $u(x)$ and h.

19. Difference Scheme for a System of Statical Theory of Elasticity

History of the matter. In the present section the results obtained in Section 18 for the Poisson equation are generalized to a system of the statical theory of elasticity. The results were published in [25].

1^0. Statement of the problem. Consider the nonlocal boundary value problem

$$\begin{aligned}
&(\lambda+2\mu)\,\frac{\partial^2 u^1}{\partial x_1^2}+(\lambda+\mu)\,\frac{\partial^2 u^2}{\partial x_1\partial x_2}+\mu\,\frac{\partial^2 u^1}{\partial x_2^2}=-f^1,\\
&\mu\,\frac{\partial^2 u^2}{\partial x_1^2}+(\lambda+\mu)\,\frac{\partial^2 u^1}{\partial x_1\partial x_2}+(\lambda+2\mu)\,\frac{\partial^2 u^2}{\partial x_2^2}=-f^2,\ \ x\in\Omega,
\end{aligned}\tag{19.1}$$

$$\begin{gathered}
u^1(x)=0,\ \ x\in\Gamma_0,\\
\int_0^{\ell_1} u^1(t,x_2)\,dt=0,\ \ 0\le x_2\le \ell_2,\quad u^2(x)=0,\ \ x\in\Gamma,
\end{gathered}\tag{19.2}$$

where $\lambda,\mu=\mathrm{const}$ are the Lamé coefficients.

It is assumed that $\mu>0$, $\lambda+\mu\ge 0$ and the problem (19.1) is uniquely solvable in $W_2^m(\Omega)$, $m\in(1,3]$.

We approximate the problem (19.1), (19.2) by the difference scheme

$$\Lambda\boldsymbol{y}+\boldsymbol{\varphi}=0,\ \ x\in\omega,\tag{19.3}$$

$$y^1(x)=0,\ \ x\in\gamma_0,\ \ Sy^1=0,\ \ x_1=\ell_1,\ \ x_2\in\overline{\omega}_2,\ \ y^2(x)=0,\ \ x\in\gamma,\tag{19.4}$$

where $\boldsymbol{y}=(y^1,y^2)$, $\boldsymbol{\varphi}=(\varphi^1,\varphi^2)$, $\varphi^\alpha=T_1T_2f^\alpha$, $\alpha=1,2$, $(\Lambda\boldsymbol{y})^1=(\lambda+2\mu)y^1_{\overline{x}_1x_1}+0.5(\lambda+\mu)(y^2_{\overline{x}_1x_2}+y^2_{x_1\overline{x}_2})+\mu y^1_{\overline{x}_2x_2}$, $(\Lambda\boldsymbol{y})^2=\mu y^2_{\overline{x}_1x_1}+0.5(\lambda+\mu)(y^1_{\overline{x}_1x_2}+y^1_{x_1\overline{x}_2})+(\lambda+2\mu)y^2_{\overline{x}_2x_2}$.

2^0. The correctness of the difference scheme. Let $\|\boldsymbol{y}\|^2_{W_2^1(\omega,r)}=\|\boldsymbol{y}\|_1^2=\|\nabla\boldsymbol{y}\|^2+\|\boldsymbol{y}\|^2$, where $\|\nabla\boldsymbol{y}\|^2=\|\nabla y^1\|^2+\|\nabla y^2\|^2$, $\|\boldsymbol{y}\|^2=\|y^1\|^2+\|y^2\|^2$. Other notation not defined in this section is the same as in Section 18. We investigate the solvability of the problem (19.3), (19.4). As a result of the summation by parts we obtain

$$(y^1_{\overline{x}_1x_1},Gy^1)_\omega=-\|y^1_{\overline{x}_1}\|^2_{(1)},\tag{19.5}$$

$$(y^1_{\overline{x}_2x_2},Gy^1)_\omega=-\|y^1_{\overline{x}_2}\|^2_{(2)},\tag{19.6}$$

$$(y^2_{\overline{x}_1x_2},Gy^1)_\omega=-\sum_\omega h_1h_2\Big(x_1+\frac{h_1}{2}\Big)y^1_{x_1}y^2_{x_2},\tag{19.7}$$

$$(y^2_{x_1\overline{x}_2},Gy^1)_\omega=-\sum_\omega h_1h_2\Big(x_1-\frac{h_1}{2}\Big)y^1_{\overline{x}_1}y^2_{\overline{x}_2},\tag{19.8}$$

$$(y^2_{\overline{x}_2x_2},x_1y^2)_\omega=-\sum_{\omega^+} h_1h_2x_1|y^2_{\overline{x}_2}|^2,\tag{19.9}$$

$$(y^1_{\bar{x}_1x_2}, x_1y^2)_\omega = -\sum_{\omega^+} h_1h_2x_1y^1_{\bar{x}_1}y^2_{\bar{x}_2}, \tag{19.10}$$

$$e(y^1_{x_1\bar{x}_2}, x_1y^2)_\omega = -\sum_{\omega} h_1h_2x_1y^1_{x_1}y^2_{x_2}, \tag{19.11}$$

$$(y^2_{\bar{x}_1x_1}, x_1y^2)_\omega = -\sum_{\omega^+} h_1h_2\Big(x_1 - \frac{h_1}{2}\Big)|y^2_{\bar{x}_1}|^2. \tag{19.12}$$

After some transformations, from (19.8), (19.10) it follows that

$$(y^2_{x_1\bar{x}_2}, Gy^1)_\omega + (y^1_{\bar{x}_1x_2}, x_1y^2)_\omega = -2\sum_{\omega^+} h_1h_2\Big(x_1 - \frac{h_1}{4}\Big)y^1_{\bar{x}_1}y^2_{\bar{x}_2}, \tag{19.13}$$

from (19.7), (19.11) it follows

$$(y^2_{\bar{x}_1x_2}, Gy^1)_\omega + (y^1_{x_1\bar{x}_2}, x_1y^2)_\omega = -2\sum_{\omega^-} h_1h_2\Big(x_1 + \frac{h_1}{4}\Big)y^1_{x_1}y^2_{x_2}, \tag{19.14}$$

from (19.5) it follows

$$(y^1_{\bar{x}_1x_1}, Gy^1)_\omega =$$

$$= -\frac{1}{2}\sum_{\omega^+} h_1h_2\Big(x_1 - \frac{h_1}{4}\Big)|y^1_{\bar{x}_1}|^2 - \frac{1}{2}\sum_{\omega^-} h_1h_2\Big(x_1 + \frac{h_1}{4}\Big)|y^1_{x_1}|^2, \tag{19.15}$$

from (19.9) follows

$$(y^2_{\bar{x}_2x_2}, x_1y^2)_\omega =$$

$$= -\frac{1}{2}\sum_{\omega^+} h_1h_2\Big(x_1 - \frac{h_1}{4}\Big)|y^2_{\bar{x}_2}|^2 - \frac{1}{2}\sum_{\omega^-} h_1h_2\Big(x_1 + \frac{h_1}{4}\Big)|y^2_{x_2}|^2. \tag{19.16}$$

Multiplying scalarly (19.3) by Gy^1 and (19.4) by x_1y^2, summing up the obtained results and taking into account the formulas (19.6), (19.12)–(19.16), we obtain

$$2W_h = (\varphi^1, Gy^1)_\omega + (\varphi^2, x_1y^2)_\omega, \tag{19.17}$$

where

$$W_h \equiv \frac{\mu}{2}\,\|\nabla \boldsymbol{y}\|^2 +$$

$$+\frac{\lambda+\mu}{4}\,\Big(\sum_{\omega^+} h_1h_2\Big(x_1 - \frac{h_1}{4}\Big)(y^1_{\bar{x}_1} + y^2_{\bar{x}_2})^2 + \sum_{\omega^-} h_1h_2\Big(x_1 + \frac{h}{4}\Big)(y^1_{x_1} + y^2_{x_2})^2\Big)$$

is the mesh analogue of energy of elastic deformation.

Similarly to Lemma 18.3, we prove that $\|\boldsymbol{y}\|_1^2 \le (1+4\ell_1)\|\nabla \boldsymbol{y}\|^2$. Therefore from (19.17) we get

$$\|\boldsymbol{y}\|_1^2 \le (1 + 4\ell_1)\big((\varphi^1, Gy^1)_\omega + (\varphi^2, x_1y^2)_\omega\big). \tag{19.18}$$

Thus if $\boldsymbol{\varphi} = 0$, $x \in \omega$, then $\boldsymbol{y} \equiv 0$. This means that the solution of the problem (19.3), (19.4) exists and is unique.

3^0. The problem for the error of the method. Let $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{u}$. Substituting $\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{u}$ in (19.3), (19.4), for the error $\boldsymbol{z}$ we obtain the problem

$$\Lambda \boldsymbol{z} + \boldsymbol{\psi} = 0, \quad x \in \omega,$$
$$z^1(x) = 0, \quad x \in \gamma_0, \quad S z^1(\ell_1, x_2) = \chi(x_2), \quad x_2 \in \overline{\omega}_2, \quad z^2(x) = 0, \quad x \in \gamma, \tag{19.19}$$

where $\boldsymbol{\psi} = \boldsymbol{\varphi} + \lambda \boldsymbol{u}$ is the approximation error of the equation (19.1), and $\chi = -Su^1(\ell_1, x_2)$ is the approximation error of the nonlocal condition. Let $\eta = S_1^- u^1 - \frac{1}{2}\left(u^1 + u^{1(-1_1)}\right)$, $\eta^\beta_{\alpha\alpha} = T_{3-\alpha} u^\beta - u^\beta$, $\eta^\beta_{12} = S_1^- S_2^- u^\beta - \frac{1}{2}\left(u^{\beta(-1_1)} + u^{\beta(-1_2)}\right)$, $\alpha, \beta = 1, 2$.

Then $T_1 T_2 \frac{\partial^2 u^\beta}{\partial x_\alpha^2} = u^\beta_{\overline{x}_\alpha x_\alpha} + (\eta^\beta_{\alpha\alpha})_{\overline{x}_\alpha x_\alpha}$, $T_1 T_2 \frac{\partial^2 u^\beta}{\partial x_1 \partial x_2} = \frac{1}{2}\left(u^\beta_{\overline{x}_1 x_2} + u^\beta_{x_1 \overline{x}_2}\right) + \eta^\beta_{12 x_1 x_2}$, $\int\limits_0^{\ell_1} u^1(t, x_2)\, dt - Su^1(\ell_1, x_2) = \chi(x_2)$ and the components of the approximation error can be represented as

$$\psi^\alpha = (\lambda + 2\mu)(\eta^\alpha_{\alpha\alpha})_{\overline{x}_\alpha x_\alpha} +$$
$$+ \frac{\lambda + \mu}{2}(\eta^\beta_{12})_{x_1 x_2} + \mu(\eta^\alpha_{\beta\beta})_{\overline{x}_\beta x_\beta}, \quad \beta = 3 - \alpha, \quad \alpha = 1, 2, \quad \chi = \sum_{\omega_1^+} h_1 \eta.$$

Let

$$w^1 = z^1 - \frac{2}{\ell_1^2}(\ell_1 - x_1)\chi(x_2), \quad w^2 = z^2, \quad \boldsymbol{w} = (w^1, w^2). \tag{19.20}$$

Then for $\boldsymbol{w}$ we obtain the problem with the homogeneous nonlocal condition

$$\Lambda \boldsymbol{w} + \widetilde{\boldsymbol{\psi}} = 0, \quad x \in \omega,$$
$$w^1(x) = 0, \quad x \in \gamma_0, \quad S w^1(\ell_1, x_2) = 0, \quad x_2 \in \overline{\omega}_2, \quad w^2(x) = 0, \quad x \in \gamma, \tag{19.21}$$

where $\widetilde{\psi}^1 = \psi^1 + \frac{2\mu}{\ell_1^2}(\ell_1 - x_1)\chi_{\overline{x}_2 x_2}$, $\widetilde{\psi}^2 = \psi^2 + \frac{2(\lambda+\mu)}{\ell_1^2}\chi_{\overset{\circ}{x}_2}$.

4^0. Estimation of the convergence rate.

Lemma 19.1. *For the solution of the problem* (19.19) *the a priori estimate* $\|\boldsymbol{z}\|_1 \le cJ(\boldsymbol{u})$, *is valid, where* $J(\boldsymbol{u}) = \|\eta^1_{11\overline{x}_1}]| + \|\eta^1_{12x_1}|] + \|\eta^2_{11\overline{x}_1}]| + \|\eta^2_{12x_2}]| + \|\eta^1_{22\overline{x}_2}|] + \|\eta^2_{22\overline{x}_2}\| + \|\chi\|_* + \|\chi_{\overline{x}_2}|]_*$, *and the constant* c *does not depend on* $\boldsymbol{u}$ *and* h.

Proof. By virtue of (19.18), for the solution of the problem (19.21) we have

$$\|\boldsymbol{w}\|_1^2 \le (1 + 4\ell_1)\left((\widetilde{\psi}^1, Gw^1)_\omega + (\widetilde{\psi}^2, x_1 w^2)_\omega\right). \tag{19.22}$$

Using summation by parts and the Cauchy–Buniakowski inequality, we find that

$$(\widetilde{\psi}^1, Gw^1)_\omega \le (\lambda + 2\mu)\sqrt{\ell_1}\, \|\eta^1_{11\overline{x}_1}]|\, \|w^1_{\overline{x}_1}\|_{(1)} + \frac{\lambda + \mu}{2}\sqrt{\ell_1}\, \|\eta^2_{12x_2}]|\, \|w^1_{\overline{x}_1}\|_{(1)} +$$
$$+ 3\mu\sqrt{\ell_1}\, \|\eta^1_{22\overline{x}_2}|]\, \|w^1_{\overline{x}_2}\|_{(2)} + 2\mu\|\chi_{\overline{x}_2}|]_* \|w^1_{\overline{x}_2}\|_{(2)}. \tag{19.23}$$

Analogously,

$$(\widetilde{\psi}^2, x_1 w^2)_\omega \le (\lambda + 2\mu)\sqrt{\ell_1}\, \|\eta^2_{22\overline{x}_2}|]\, \|w^2_{\overline{x}_2}\|_{(2)} + \\ + \frac{\lambda+\mu}{2}\sqrt{\ell_1}\, \|\eta^1_{12x_1}|]\, \|w^2_{\overline{x}_2}\|_{(2)} + \\ + 3\mu\sqrt{\ell_1}\, \|w^2_{\overline{x}_1}\|_{(1)}\, \|\eta^2_{11\overline{x}_1}]| + \frac{2(\lambda+\mu)}{\ell_1}\, \|\chi\|_* \|w^2_{\overline{x}_2}\|_{(2)}. \quad (19.24)$$

Note that when deducing the inequality (19.24) we have used the estimate

$$(\eta^2_{11\overline{x}_1 x_1}, x_1 w^2)_\omega = \\ = -\sum_{\omega_1^+\times\omega_2} h_1 h_2\Big(x_1 - \frac{h_1}{2}\Big)\eta^2_{11\overline{x}_1} w^2_{\overline{x}_1} - \frac{1}{2}\sum_{\omega_1\times\omega_2} h_1 h_2 \eta^2_{11\overline{x}_1}(w^2 + w^{2(-1_1)}) \le \\ \le \sqrt{\ell_1}\, \|\eta^2_{11\overline{x}_1}\|\, \|w^2_{\overline{x}_1}\|_{(1)} + \|\eta^2_{11\overline{x}_1}]|\, \|w^2\| \le 3\sqrt{\ell_1}\, \|\eta^2_{11\overline{x}_1}]|\, \|w^2_{\overline{x}_1}\|_{(1)}.$$

Taking into account (19.23), (19.24), from (19.22) we obtain

$$\|\boldsymbol{w}\|_1 \le cJ(\boldsymbol{u}). \quad (19.25)$$

Moreover, according to (19.20),

$$\|\boldsymbol{z}\|_1 \le \|\boldsymbol{w}\|_1 + \frac{2}{\ell_1^2}\, \big\|(\ell_1 - x_1)\chi\big\|_{(1)} \le \|\boldsymbol{w}\|_1 + c\big(\|\chi\|_* + \|\chi_{\overline{x}_2}|]_*\big). \quad (19.26)$$

The inequalities (19.25), (19.26) complete the proof of the lemma. □

From Lemma 19.1, using the Bramble–Hilbert lemma we obtain the following

Theorem 19.1. *The difference scheme* (19.3), (19.4) *converges in the mesh norm* $W_2^1(\omega, r)$*, and the convergence rate is determined by the estimate* $\|\boldsymbol{y} - \boldsymbol{u}\|_{W_2^1(\omega,r)} \le c|h|^{m-1}\|\boldsymbol{u}\|_{W_2^m(\Omega)}$, $1 < m \le 3$*, where the positive constant* c *does not depend on* $\boldsymbol{u}$ *and* h.

20. A Nonlocal Problem with the Integral Condition for a Two-Dimensional Elliptic Equation

History of the matter. In this section we consider a nonlocal boundary value problem with integral restriction for second order elliptic equation with constant coefficients. The existence and uniqueness of a weak solution of the problem in a weighted Sobolev space are proved in the first part of this section. The second part is devoted to the construction and investigation of the corresponding difference scheme. The a priori estimate of the convergence rate,

$$\|y - u\|_{W_2^1(\omega,\rho)} \le ch^{m-1}\|u\|_{W_p^m(\Omega)}, \quad m \in (1, 3], \quad (20.1)$$

is obtained; here $p = 2$ for $\varepsilon \in (0.5, 1)$ and $p > 1/\varepsilon$ for $\varepsilon \in (0, 0.5]$.

The results of Section 20 were published in [28].

1^0. The solvability of the nonlocal problem. Let $\Omega = \Omega_0 = \{(x_1, x_2) : 0 < x_k < 1,\ k = 1, 2\}$ be a square with the boundary Γ; $\Gamma_* = \Gamma \setminus \Gamma_{-1}$. Here we consider the nonlocal boundary value problem with integral restriction for the second order elliptic equation with constant coefficients

$$Lu = f(x),\quad x \in \Omega, u(x) = 0,\quad x \in \Gamma_*,\quad \ell(u) = 0,\quad 0 < x_2 < 1, \tag{20.2}$$

where

$$Lu \equiv -\sum_{i,j=1}^{2} a_{ij}\,\frac{\partial^2 u}{\partial x_i \partial x_j} + a_0 u,$$

$$\ell(u) \equiv \int\limits_0^1 \beta(x_1) u(x)\, dx_1,\quad \beta(t) = \varepsilon t^{\varepsilon - 1},\quad \varepsilon \in (0, 1),$$

and the coefficients satisfy the conditions

$$\sum_{i,j=1}^{2} a_{ij} t_i t_j \geq \nu_1 (t_1^2 + t_2^2),\quad \nu_1 > 0,\quad a_0 \geq 0. \tag{20.3}$$

We choose the weight function $\rho(x)$ as follows: $\rho(x) = \rho(x_1) = \int\limits_0^{x_1} \beta(t)\, dt = x_1^\varepsilon$. Define a subspace of the space $W_2^1(\Omega, \rho)$ which is obtained by closing the set $\overset{*}{C}{}^\infty(\overline{\Omega}) = \big\{u \in C^\infty(\overline{\Omega}) :\ \operatorname{supp} u \cap \Gamma_* = \varnothing,\ \int\limits_0^1 \beta(x_1) u(x)\, dx_1 = 0,\ 0 < x_2 < 1\big\}$ in the norm $\|\cdot\|_{W_2^1(\Omega,\rho)}$. Denote it by $\overset{*}{W}{}_2^1(\Omega, \rho)$.

Let the right-hand side $f(x)$ in the equation (20.2) be a linear continuous functional on $\overset{*}{W}{}_2^1(\Omega, \rho)$ which is representable as

$$f = f_0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\,,\quad f_k(x) \in L_2(\Omega, \rho),\quad k = 0, 1, 2. \tag{20.4}$$

The function $u \in \overset{*}{W}{}_2^1(\Omega, \rho)$ is said to be a weak solution of the problem (20.2)–(20.4) if the identity

$$a(u, v) = \langle f, v \rangle,\quad \forall\, v \in \overset{*}{W}{}_2^1(\Omega, \rho), \tag{20.5}$$

is fulfilled; here

$$a(u, v) = \int\limits_\Omega \Big(a_{11} x_1^\varepsilon\, \frac{\partial u}{\partial x_1}\, \frac{\partial v}{\partial x_1} + (a_{12} + a_{21}) x_1^\varepsilon\, \frac{\partial u}{\partial x_2}\, \frac{\partial v}{\partial x_1} + {} + a_{22}\, \frac{\partial u}{\partial x_2}\, G\, \frac{\partial v}{\partial x_2} + a_0 u G v \Big)\, dx, \tag{20.6}$$

$$\langle f, v \rangle = \int\limits_\Omega f_0 G v\, dx - \int\limits_\Omega x_1^\varepsilon f_1\, \frac{\partial v}{\partial x_1}\, dx - \int\limits_\Omega f_2 G\, \frac{\partial v}{\partial x_2}\, dx, \tag{20.7}$$

$$Gv(x) = \rho v(x) - \int_0^{x_1} \beta(t) v(t, x_2)\, dt. \tag{20.8}$$

(The equality (20.5) is obtained formally from $(Lu - f, Gv) = 0$ by integration by parts.)

To prove the existence of a unique solution of the problem (20.5) (a weak solution of the problem (20.2)–(20.4)), the use will be made of the Lax–Milgram lemma ([s1]). First of all, we establish some auxiliary results.

Lemma 20.1. *Let $u, v \in L_2(\Omega, \rho)$, and $v(x)$ satisfy the condition $\ell(v) = 0$. Then*

$$|(u, Gv)| \le \frac{1+\varepsilon}{1-\varepsilon} \|u\|_{L_2(\Omega,\rho)} \|v\|_{L_2(\Omega,\rho)}, \tag{20.9}$$

$$\|v\|^2_{L_2(\Omega,\rho)} \le (v, Gv), \tag{20.10}$$

$$\|v\|_{L_2(\Omega,\rho^2)} \le \|Gv\| \le (2\varepsilon + 1)\|v\|_{L_2(\Omega,\rho^2)}. \tag{20.11}$$

Proof. Since $C^\infty(\overline{\Omega})$ is dense in $L_2(\Omega, \rho)$, it is sufficient to prove the lemma for arbitrary functions from $C^\infty(\overline{\Omega})$.

Using the Cauchy inequality, we obtain

$$|(u, Gv)| \le \|u\|_{L_2(\Omega,\rho)} \big(\|v\|_{L_2(\Omega,\rho)} + \varepsilon\, J_1(v)\big), \tag{20.12}$$

where

$$J_1^2(v) = \int_\Omega x_1^{-\varepsilon} \bigg(\int_0^{x_1} t^{\varepsilon-1} v(t, x_2)\, dt \bigg)^2 dx =$$

$$= -\frac{2}{1-\varepsilon} \int_\Omega v(x) \int_0^{x_1} t^{\varepsilon-1} v(t, x_2)\, dt\, dx \le \frac{2}{1-\varepsilon} \|v\|_{L_2(\Omega,\rho)} \cdot J_1(v).$$

Thus $J_1(v) \le 2(1-\varepsilon)^{-1} \|v\|_{L_2(\Omega,\rho)}$, and from (20.12) it follows (20.9).

The inequality (20.10) is the consequence of the identity $(v, Gv) = \|v\|^2_{L_2(\Omega,\rho)} + 0.5\varepsilon(1-\varepsilon) J_1^2(v)$.

Let left-hand side of (20.11) follows from the identity

$$\|Gv\|^2 = \int_\Omega x_1^{2\varepsilon} v^2(x)\, dx + (\varepsilon^2 + \varepsilon)(J_2(v))^2,$$

$$(J_2(v))^2 = \int_\Omega \bigg(\int_0^{x_1} t^{\varepsilon-1} v(t, x_2)\, dt \bigg)^2 dx.$$

To prove the validity of the right-hand side of (20.11), it suffices to notice that

$$(J_2(v))^2 = -2 \int_\Omega x_1^\varepsilon v(x) \int_0^{x_1} t^{\varepsilon-1} v(t, x_2)\, dt\, dx \le 2\|v\|_{L_2(\Omega,\rho^2)} J_2(v),$$

i.e., $(J_2(v))^2 \le 4\|v\|^2_{L_2(\Omega,\rho^2)}$. Thus the proof of the lemma is complete. □

Lemma 20.2. *If* $u \in \overset{*}{W}{}^1_2(\Omega,\rho)$, *then*

$$|u|_{W_2^1(\Omega,\rho)} \le \|u\|_{W_2^1(\Omega,\rho)} \le c_1 |u|_{W_2^1(\Omega,\rho)}, \quad c_1 = (4(1+\varepsilon)^{-2}+1)^{1/2}.$$

Proof. Since $\overset{*}{C}{}^\infty(\overline{\Omega})$ is dense in $\overset{*}{W}{}^1_2(\Omega,\rho)$, it is sufficient to prove the lemma for an arbitrary $u \in \overset{*}{C}{}^\infty(\overline{\Omega})$.

The left inequality of the lemma is obvious. Integration by parts results in

$$\int\limits_\Omega x_1^\varepsilon u^2(x)\,dx = -\int\limits_\Omega \Big(\varepsilon x_1^\varepsilon u^2(x) + 2x_1^{\varepsilon+1} u(x)\,\frac{\partial u}{\partial x_1}\Big)\,dx.$$

Consequently,

$$(1+\varepsilon)\int\limits_\Omega x_1^\varepsilon u^2(x)\,dx =$$

$$= -2\int\limits_\Omega x_1^{\varepsilon+1} u\,\frac{\partial u}{\partial x_1}\,dx \le 2\|u\|_{L_2(\Omega,\rho)}\bigg(\int\limits_\Omega x_1^{\varepsilon+2}\Big|\frac{\partial u}{\partial x_1}\Big|^2\,dx\bigg)^{1/2},$$

so

$$\|u\|_{L_2(\Omega,\rho)} \le \frac{2}{1+\varepsilon}\bigg(\int\limits_\Omega x_1^{\varepsilon+2}\Big|\frac{\partial u}{\partial x_1}\Big|^2\,dx\bigg)^{1/2}.$$

Application of this inequality completes the proof of the lemma. Using Lemmas 20.1, 20.2 and the conditions (20.3), from (20.6) follows the continuity: $|a(u,v)| \le c_2\|u\|_{W_2^1(\Omega,\rho)}\|v\|_{W_2^1(\Omega,\rho)}$, $c_2 > 0$, $\forall\, u,\, v \in \overset{*}{W}{}^1_2(\Omega,\rho)$ and W_2^1-ellipticity: $a(u,u) \ge c_3\|u\|^2_{W_2^1(\Omega,\rho)}$, $c_3 > 0$, $\forall\, u \in \overset{*}{W}{}^1_2(\Omega,\rho)$ of the bilinear form $a(u,v)$.

Again, using Lemmas 20.1 and 20.2, from (20.7) follows the continuity of the linear form $\langle f,v\rangle$, $|\langle f,v\rangle| \le c_4\|v\|_{W_2^1(\Omega,\rho)}$, $c_4>0$, $\forall\, v \in \overset{*}{W}{}^1_2(\Omega,\rho)$.

Thus all the Lax–Milgram conditions are fulfilled, and hence the following theorem is valid. □

Theorem 20.1. *The problem* (20.2)–(20.4) *has a unique weak solution from* $\overset{*}{W}{}^1_2(\Omega,\rho)$.

2^0. The difference scheme. Here we introduce mesh domains with the step $h = 1/N$. Let $\gamma_* = \Gamma_* \cap \overline{\omega}$, $\omega_{1,k} = \big\{x_1 :\ x_1 = ih,\ i = 1,2,\dots,k\big\}$.

Denote $\beta^+ = T_1^+\beta$, $\beta^- = T_1^-\beta$, $\beta_k = \frac{1}{2}\,(\beta^+(kh) + \beta^-(kh))$, $\beta_0^- = \beta_N^+ = 0$, $\rho_i = \sum\limits_{k=0}^{i} h\beta_k - \frac{h}{2}\,\beta_i^+$. It is not difficult to notice that $\rho_i = \int\limits_0^{ih}\beta(t)\,dt = \rho(ih)$.

Let $\rho^+ \equiv \rho + \frac{h}{2}\,\beta^+ = S_1^+\rho$, $\rho^- \equiv \rho - \frac{h}{2}\,\beta^- = S_1^-\rho$, $\overline{\rho} = \frac{1}{2}\,(\rho^+ + \rho^-)$. The difference analogue of the operator G from (20.8) is defined as follows:

$$G_h y = \overline{\rho} y - Py, \quad Py(ih, x_2) = \sum_{k=0}^{i} \beta_k y(kh, x_2) - \frac{h}{2}\,\beta_i y(ih, x_2). \tag{20.13}$$

By H we denote the set of the mesh functions defined on $\overline{\omega}$ and satisfying the conditions

$$y = 0, \;\; x \in \gamma_*, \quad \ell_h(y) \equiv \sum_{k=0}^{N} \beta_k y(kh, x_2) = 0, \;\; x_2 \in \omega_2. \tag{20.14}$$

Let

$$(y, v) = \sum_{\omega_1^- \times \omega_2} \hbar h y v, \;\; \|y\| = (y, y)_0^{1/2}, \;\; \|y\|_\rho^2 = \sum_{\omega_1^- \times \omega_2} \hbar h \overline{\rho} y^2, \;\; \|y]|_\rho^2 = \sum_{\omega_1^- \times \omega_2^+} \hbar h \overline{\rho} y^2,$$

$$\|y\|_1^2 = \|y\|^2 + \|\nabla y\|^2, \;\; \|\nabla y\|^2 = \|y_{\overline{x}_1}\|_{(1)}^2 + \|y_{\overline{x}_2}\|_{(2)}^2,$$

$$\|y_{\overline{x}_1}\|_{(1)}^2 = (\rho^- y_{\overline{x}_1}, y_{\overline{x}_1})_{\omega_1^+ \times \omega_2}, \quad \|y_{\overline{x}_2}\|_{(2)}^2 = \|y_{\overline{x}_2}]|_\rho^2,$$

$$\|y\|_*^2 = \sum_{\omega_2} h y^2, \quad \|y]|_*^2 = \sum_{\omega_2^+} h y^2.$$

We approximate the problem (20.2)–(20.4) by the difference scheme

$$L_h y = -a_{11} y_{\overline{x}_1 x_1} - 2a_{12} y_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} - a_{22} y_{\overline{x}_2 x_2} + a_0 y = \varphi(x), \;\; x \in \omega, \;\; y \in H, \tag{20.15}$$

where $\varphi = T_1 T_2 f_0 + (S_1^- T_2 f_1)_{x_1} + (T_1 S_2^- f_2)_{x_2}$.

Lemma 20.3. *For the mesh functions $y(x)$ satisfying the conditions $\ell_h(x) = 0$, $y(1, x_2) = 0$, $x_2 \in \omega_2$, the estimates $(y, G_h y)_\omega \geq \|y\|_\rho^2$, $(y, G_h y)_{\omega_1 \times \omega_2^+} \geq \|y]|_\rho^2$ are valid.*

Proof. We can show that

$$-\sum_{i=1}^{N-1} h y(ih, x_2) P y(ih, x_2) = \frac{1}{2\beta_1} \Big(\frac{h}{2}\,\beta_0^+ y(0, x_2)\Big)^2 + J_3, \tag{20.16}$$

where $J_3 = 0$ for $N = 2$ and $J_3 = \frac{1}{2} \sum\limits_{i=2}^{N-1} \big(\frac{1}{\beta_i} - \frac{1}{\beta_{i-1}}\big) \big(Py(ih, x_2) - \frac{h}{2}\,\beta_i y(ih, x_2)\big)^2$ for $N > 2$.

If we observe that $J_3 \geq 0$ by $(1/\beta_i) - (1/\beta_{i-1}) > 0$, and also $\beta_0^+ > \beta_1$, then from (20.16) it follows that Lemma 20.3 is valid. □

Lemma 20.4. *For any function $y \in H$ the estimate*

$$(L_h y, G_h y)_\omega \geq c_5 \|y\|_1^2, \;\; c_5 = \frac{\nu}{4}\,, \tag{20.17}$$

is valid.

Proof. The identities

$$\sum_{\omega_1} h v_{x_1} G_h y = -\sum_{\omega_1^+} h\rho^- v y_{\overline{x}_1}, \quad \sum_{\omega_1} h v_{\overline{x}_1} G_h y = -\sum_{\omega_1^-} h\rho^+ v y_{x_1},$$

where v is an arbitrary mesh function, are valid. Therefore

$$-(y_{\overline{x}_1 x_1}, G_h y)_\omega = \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\rho^-(y_{\overline{x}_1})^2 + \frac{1}{2}\sum_{\omega_1^-\times\omega_2} h^2\rho^+(y_{x_1})^2, \tag{20.18}$$

$$-(y_{\overset{\circ}{x}_1\overset{\circ}{x}_2}, G_h y)_\omega = \frac{1}{2}\sum_{\omega_1^-\times\omega_2} h^2\rho^+ y_{x_1} y_{\overset{\circ}{x}_2} + \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\rho^- y_{\overline{x}_1} y_{\overset{\circ}{x}_2}. \tag{20.19}$$

Moreover, using Lemma 20.3, we obtain

$$-(y_{\overline{x}_2 x_2}, G_h y)_\omega \ge \|y_{\overline{x}_2}\|^2_{(2)}. \tag{20.20}$$

Let $\widehat{\rho} = \rho + \frac{h}{2}\,\beta^+ - \frac{h}{4}\,\beta_0^+$, $\check{\rho} = \rho - \frac{h}{2}\,\beta^- + \frac{h}{4}\,\beta_0^+$. Then $\overline{\rho} = \frac{1}{2}\,(\widehat{\rho} + \check{\rho})$, $\widehat{\rho}_0 = \frac{h}{4}\,\beta_0^+$ and after some transformations, (20.18) yields

$$-(y_{\overline{x}_1 x_1}, G_h y)_\omega = \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\check{\rho}(y_{\overline{x}_1})^2 + \frac{1}{2}\sum_{\omega_1^-\times\omega_2} h^2\widehat{\rho}(y_{x_1})^2, \tag{20.21}$$

from (20.19) it follows

$$-(y_{\overset{\circ}{x}_1\overset{\circ}{x}_2}, G_h y)_\omega = \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\check{\rho} y_{\overline{x}_1} y_{\overset{\circ}{x}_2} + \frac{1}{2}\sum_{\omega_1^-\times\omega_2^-} h^2\widehat{\rho} y_{x_1} y_{\overset{\circ}{x}_2}, \tag{20.22}$$

from (20.20) we have

$$-(y_{\overline{x}_2 x_2}, G_h y)_\omega \ge \frac{1}{2}\sum_{\omega_1^-\times\omega_2^+} h^2\widehat{\rho}(y_{\overline{x}_2})^2 + \frac{1}{2}\sum_{\omega_1\times\omega_2^-} h^2\check{\rho}(y_{x_2})^2. \tag{20.23}$$

Taking into account (20.21), (20.22) and (20.23), from (20.15) we obtain

$$4(L_h y, G_h y)_\omega \ge \sum_{\omega_1^+\times\omega_2^-} h^2\check{\rho}F(y_{\overline{x}_1}, y_{x_2}) + \sum_{\omega_1^+\times\omega_2^+} h^2\check{\rho}F(y_{\overline{x}_1}, y_{\overline{x}_2}) +$$

$$+ \sum_{\omega_1^-\times\omega_2^-} h^2\widehat{\rho}F(y_{x_1}, y_{x_2}) + \sum_{\omega_1^-\times\omega_2^+} h^2\widehat{\rho}F(y_{x_1}, y_{\overline{x}_2}) + a_0(y, G_h y)_\omega, \tag{20.24}$$

where $F(t_1,t_2) = a_{11}t_1^2 + 2a_{12}t_1t_2 + a_{22}t_2^2$. If we notice that

$$\check{\rho} = \frac{1}{h}\int\limits_{x_1-h}^{x_1} \rho(t)\,dt + \frac{1}{2h}\int\limits_0^h \rho(t)\,dt > 0, \quad x_1 \in \omega_1^+,$$

$$\widehat{\rho} = \frac{1}{h}\int\limits_{x_1}^{x_1+h} \rho(t)\,dt - \frac{1}{2h}\int\limits_0^h \rho(t)\,dt > 0, \quad x_1 \in \omega_1^-,$$

then by the ellipticity conditions (20.3), from (20.24) follows the estimate $(L_h y, G_h y)_\omega \ge \nu_1 \|\nabla y\|^2$, which together with $\|y\| \le 2\|\nabla y\|$ proves the lemma. □

Thus if $\varphi = 0$, $x \in \omega$, then $y = 0$, and hence the difference scheme (20.15) is uniquely solvable.

Lemma 20.5. *If the mesh function y defined on $\overline{\omega}$ satisfies the conditions $\ell_h(y) = 0$, $y(1, x_2) = 0$, $x_2 \in \omega_2$, then*

$$\Big|\sum_{\omega_1} hvG_h y\Big| \le c\Big(\sum_{\omega_1} h\overline{\rho}v^2\Big)^{1/2}\Big(\sum_{\omega_1} h\overline{\rho}y^2\Big)^{1/2},$$

where v is an arbitrary mesh function.

Proof. From the definition of the operator G_h it follows that

$$\Big|\sum_{\omega_1} hvG_h y\Big| \le \Big(\sum_{\omega_1} h\overline{\rho}v^2\Big)^{1/2}\Big(\Big(\sum_{\omega_1} h\overline{\rho}y^2\Big)^{1/2} + J_4(y)\Big), \tag{20.25}$$

where $J_4^2(y) = \sum\limits_{\omega_1} h(\overline{\rho})^{-1}(Py)^2$.

Denote $2(\widetilde{P}y)_i = \sum\limits_{k=0}^{i} h\beta_k y(kh, x_2)$, $\sigma_i = \sum\limits_{k=1}^{i} \frac{h}{\overline{\rho}_k}$, $\sigma_0 = 0$. Then $(\widetilde{P}y)_i + (\widetilde{P}y)_{i-1} = (Py)_i$, $(\widetilde{P}y)_i - (\widetilde{P}y)_{i-1} = \frac{h\beta_i}{2}\, y(ih, x_2)$, $(\widetilde{P}v)_{N-1} = 0$, $\sigma_i - \sigma_{i-1} = \frac{h}{\overline{\rho}_i}$, and we have

$$\begin{aligned} J_4^2(y) &\le 2\sum_{i=1}^{N-1} (\sigma_i - \sigma_{i-1})\big((\widetilde{P}y)_i^2 + (\widetilde{P}y)_{i-1}^2\big) = \\ &= -2\sum_{i=1}^{N-1} (\sigma_i + \sigma_{i-1})\big((\widetilde{P}y)_i^2 - (\widetilde{P}y)_{i-1}^2\big) = \\ &= -\sum_{i=1}^{N-1} (\sigma_i + \sigma_{i-1})h\beta_i y(ih, x_2)(Py)_i. \end{aligned} \tag{20.26}$$

Let us show that

$$(\sigma_i + \sigma_{i-1})\beta_i \le c. \tag{20.27}$$

Indeed, for $i=1$, $\sigma_1 + \sigma_0 = 0.5(1+\varepsilon)(2h)^{1-\varepsilon}$, $\beta_1 = h^{\varepsilon-1}(\varepsilon+1)^{-1}(2^{\varepsilon+1} - 2)$ and $(\sigma_1 + \sigma_0)\beta_1 < 1$.

For $i > 1$, we have $\overline{\rho}_k = \frac{2}{h}\int\limits_{(k-1)h}^{(k+1)h} \rho(t)\,dt > t_{k-1}^{\varepsilon}$, $t_k = kh$. By the Lagrange theorem, $t_{k-1}^{1-\varepsilon} - t_{k-2}^{1-\varepsilon} = (1-\varepsilon)h\xi^{-\varepsilon} > (1-\varepsilon)ht_{k-1}^{-\varepsilon}$, $t_{k-2} < \xi < t_{k-1}$.

Therefore $(\overline{\rho}_k)^{-1} < (t_{k-1})^{-\varepsilon} < \frac{t_{k-1}^{1-\varepsilon} - t_{k-2}^{1-\varepsilon}}{(1-\varepsilon)h}$, $k = 2, 3, \dots$, $\sum\limits_{k=2}^{i} h(\overline{\rho}_k)^{-1} < \frac{t_{i-1}^{1-\varepsilon}}{1-\varepsilon}$, and since $h(\overline{\rho}_1)^{-1} = \frac{1+\varepsilon}{2}(2h)^{1-\varepsilon} < \frac{t_{i-1}^{1-\varepsilon}}{1-\varepsilon}$, therefore $\sum\limits_{k=1}^{i} h(\overline{\rho}_k)^{-1} < \frac{2t_{i-1}^{1-\varepsilon}}{1-\varepsilon}$, $\sigma_i + \sigma_{i-1} \le 2\sigma_i < \frac{4t_{i-1}^{1-\varepsilon}}{1-\varepsilon}$, $i = 2, 3, \dots$. Moreover, $\beta_i < \varepsilon t_{i-1}^{\varepsilon-1}$, $i =$

$2, 3, \ldots$, and hence $(\sigma_i + \sigma_{i-1}) < \frac{4\varepsilon}{1-\varepsilon}$, $i = 2, 3, \ldots$. The inequality (20.27) is proved.

Using (20.27), from (20.26) we obtain

$$J_4^2(y) \le c\sum_{\omega_1} h|yPy| \le c\Big(\sum_{\omega_1} h\overline{\rho}y^2\Big)^{1/2} J_4(y), \quad \text{i.e.} \quad J_4(y) \le c\Big(\sum_{\omega_1} h\overline{\rho}y^2\Big)^{1/2},$$

which together with (20.25) completes the proof of Lemma 20.5. □

To study the problem on the convergence and accuracy of the difference scheme (20.15), we consider the error $z = y - u$, where y is a solution of the difference scheme, and $u = u(x)$ is a solution of the differential problem. For z we obtain the problem

$$L_h z = \psi, \;\; x \in \omega, \;\; z = 0, \;\; x \in \gamma_*, \;\; \ell_h(z) = \chi(x_2), \;\; x_2 \in \omega_2, \qquad (20.28)$$

where $\psi = a_{11}\eta_{11\overline{x}_1 x_1} + a_{12}\eta_{12x_1x_2} + a_{22}\eta_{22\overline{x}_2x_2} + a_0\eta_0$, $\eta_0 = T_1T_2u - u$, $\eta_{\alpha\alpha} = u - T_{3-\alpha}u$, $\alpha = 1, 2$, $\eta_{12} = \frac{1}{2}\left(u + u^{(-1_1)} + u^{(-1_2)} + u^{(-1_1,-1_2)}\right) - 2S_1^- S_2^- u(x)$, $\chi = \ell(u) - \ell_h(u)$.

Noticing that

$$\ell_h(u) = \sum_{\omega_1^+} \int\limits_{x_1-h}^{x_1} \beta(t)\Big(\frac{x_1 - t}{h}\, u(x_1 - h, x_2) + \frac{t - x_1 + h}{h}\, u(x_1, x_2)\Big)\, dt,$$

we can represented χ in the form

$$\chi = \sum_{\omega_1^+} \eta, \quad \eta = \int\limits_{x_1-h}^{x_1} \beta(t)\, \frac{t - x_1}{h} \int\limits_{x_1-h}^{t} (\xi - x_1 + h) \frac{\partial^2 u(\xi, x_2)}{\partial \xi^2}\, d\xi\, dt +$$
$$+ \int\limits_{x_1-h}^{x_1} \beta(t) \frac{t - x_1 + h}{h} \int\limits_{t}^{x_1} (\xi - x_1) \frac{\partial^2 u(\xi, x_2)}{\partial \xi^2}\, d\xi\, dt.$$

Obviously, $\chi = 0$ for $u(x) = 1 - x_1$. Consequently, $\ell_h(1 - x_1) = \ell(1 - x_1) = 1/(1 + \varepsilon)$, and substitution

$$z(x) = \widetilde{z}(x) + \frac{1 - x_1}{1 + \varepsilon}\, \chi(x_2) \qquad (20.29)$$

reduces the problem (20.28) (in which the nonlocal condition is inhomogeneous) to the problem with the homogeneous conditions

$$L_h \widetilde{z} = \widetilde{\psi}, \;\; x \in \omega, \;\; \widetilde{z} \in H, \qquad (20.30)$$

where $\widetilde{\psi} = \psi + 2a_{12}\big(\frac{1-x_1}{1+\varepsilon}\, \chi\big)_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} + a_{22}\big(\frac{1-x_1}{1+\varepsilon}\chi\big)_{\overline{x}_2 x_2} - a_0\, \frac{1-x_1}{1+\varepsilon}\, \chi$.

Using Lemmas 20.4 and 20.5, for the problem (20.30) we obtain the a priori estimate

$$\|\widetilde{z}\|_1 \le c\big(\|\eta_{11\overline{x}_1}\|_{\omega_1^+ \times \omega_2} + \|\eta_{12x_2}\|_{\omega_1^+ \times \omega_2} +$$
$$+ \|\eta_{22\overline{x}_2}\|_{\omega_1 \times \omega_2^+} + \|\eta_0\|_\omega + \|\chi\|_* + \|\chi_{\overline{x}_2}|]_*\big). \qquad (20.31)$$

For the error of the method we obtain from (20.29) the estimate $\|z\|_1 \le \|\widetilde{z}\|_1 + c\big(\|\chi\|_* + \|\chi_{\overline{x}_2}|]_*\big)$, which together with (20.31) yields

$$\|z\|_1 \le c\big(\|\eta_{11\overline{x}_1}\|_{\omega_1^+\times\omega_2} + \|\eta_{12x_2}\|_{\omega_1^+\times\omega_2} + \\ + \|\eta_{22\overline{x}_2}\|_{\omega_1\times\omega_2^+} + \|\eta_0\|_\omega + \|\chi\|_* + \|\chi_{\overline{x}_2}|]_*\big). \quad (20.32)$$

To obtain the estimate of the convergence rate of the difference scheme (20.15) it is sufficient to estimate the corresponding norms of χ and the components η_{11}, η_{12}, η_{22}, η_0 of the approximation error in the right-hand side of (20.32).

Let us estimate $\chi_{\overline{x}_2}$. Note that the linear with respect to $u(x)$ functional $\eta_{\overline{x}_2}$ is bounded in $W_p^m(\Omega)$, $pm > 1$, and vanishes on the polynomials of second order. Consequently, using the Bramble–Hilbert lemma, for the above functional we obtain the estimate $|\eta_{\overline{x}_2}| \le ch^{m-1-2/p} \int\limits_{x_1-h}^{x_1} \beta(t)\,dt\,|u|_{W_p^m(e)}$, $pm > 1$, $m \in (1,3]$, $e = (x_1-h, x_1)\times(x_2-h, x_2)$. Thus

$$|\eta_{\overline{x}_2}| \le ch^{m-1-1/p}\bigg(\int\limits_{x_1-h}^{x_1} t^{\frac{(\varepsilon-1)p}{p-1}}\,dt\bigg)^{\frac{p-1}{p}} |u|_{W_p^m(e)},$$

$$|\chi_{\overline{x}_2}| \le ch^{m-1-1/p}\sum_{\omega_1}\bigg(\int\limits_{x_1-h}^{x_1} t^{\frac{(\varepsilon-1)p}{p-1}}\,dt\bigg)^{\frac{p-1}{p}} |u|_{W_p^m(e)}.$$

Applying Hölder's inequality, we obtain

$$|\chi_{\overline{x}_2}| \le ch^{m-1-1/p}\bigg(\int\limits_0^1 t^{\frac{(\varepsilon-1)p}{p-1}}\,dt\bigg)^{\frac{p-1}{p}} |u|_{W_p^m(\overline{e})}, \quad \overline{e} = (0,1)\times(x_2-h, x_2).$$

Obviously, $\int\limits_0^1 t^{\frac{(\varepsilon-1)p}{p-1}}\,dt = \frac{p-1}{\varepsilon p-1}$, $p > \frac{1}{\varepsilon}$. Therefore, choosing $p = 2$ for $\varepsilon \in (0.5, 1)$, and $p > 1/\varepsilon$ for $\varepsilon \in (0, 0.5]$, we have $|\chi_{\overline{x}_2}| \le ch^{m-1-1/p}|u|_{W_p^m(\overline{e})}$, $\|\chi_{\overline{x}_2}|]_*^2 \le ch^{2m-2-2/p}\sum\limits_{\omega_2^+}|u|^2_{W_p^m(\overline{e})}$. Taking here into account the inequality $\sum\limits_{\omega_2^+}|u|^2_{W_p^m(\overline{e})} \le ch^{-1+2/p}|u|^2_{W_p^m(\overline{e})}$ we finally have $\|\chi_{\overline{x}_2}|]_* \le ch^{m-1}|u|_{W_p^m(\Omega)}$.

An analogous estimate is obtained for $\|\chi\|_*$. Taking also into account the well-known estimates for η_{11}, η_{12}, η_{22}, η_0, from (20.32) we prove the theorem on the convergence.

Theorem 20.2. *The finitely-difference scheme* (20.15) *converges, and for its convergence rate the estimate* (20.1) *is valid.*

Bibliography

1. V. B. Andreev, The stability of difference schemes for fourth order elliptic equations in a rectangle with respect to boundary conditions of the first kind. (Russian) *Vychisl. Metody i Programmirovanie*, 1977, No. 27, 116–165.
2. V. V. Badagadze, On the calculation of freely supported orthotropic plates by the grid method. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **24**(1960), 265–272.
3. V. V. Badagadze, On the numerical solution of a second-order equation of elliptic type. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **30**(1963), No. 6, 689–696.
4. N. S. Bakhvalov, Convergence of a relaxation method under natural constraints on an elliptic operator. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **6**(1966), 861–883.
5. I. G. Belukhina, Difference schemes for the solution of certain statistical problems in elasticity theory. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **8**(1968), 808–823.
6. G. K. Berikelashvili, An approach to the difference method of solving the problem of bending of orthotropic plates. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **96**(1979), No. 2, 281–284.
7. G. K. Berikelashvili and G. I. Sulkhanishvili, On the convergence of thr difference schemes for elliptic equations with variable coefficnets and solutions from Sobolev spaces. (Russian) *Gruz. NIINTI*, 1985, No. 190-G, p. 16.
8. G. K. Berikelashvili and G. I. Sulkhanishvili, The order of convergence of difference schemes for elliptic systems with solutions in Sobolev spaces. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **87**(1987), 21–28.
9. G. K. Berikelashvili, On the question of error estimates of the Richardson extrapolation method for strongly elliptic systems. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **118** (1985), No. 1, 45–48.
10. G. K. Berikelashvili, The order of convergence of difference schemes for an elliptic equation with mixed boundary conditions. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **118**(1985), No. 2, 285–288.
11. G. K. Berikelashvili, On a difference sheme of high accuracy for an elliptic equation with mixed derivative. (Russian) *Gruz NIINTI*, 1986, No. 250-G, p. 8.
12. G. K. Berikelashvili, Convergence of some difference schemes for elliptic equations with variable coefficients. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **90**(1988), 16–24.
13. G. K. Berikelashvili, On the convergence in W_2^2 of the difference solution of the Dirichlet problem. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **30**(1990), No. 3, 470–474; English transl.: *U.S.S.R. Comput. Math. and Math. Phys.* **30**(1990), No. 2, 89–92.
14. G. K. Berikelashvili, On the convergence of Richardson's extrapolation method for elliptic equations. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **96**(1991), 3–14.
15. G. K. Berikelashvili and M. G. Mirianashvili, On the convergence of the difference solution of the first biharmonic boundary value problem. *Numerical methods* (*Miskolc*, 1990), 133–143, *Colloq. Math. Soc. Janos Bolyai,* 59, *North–Holland, Amsterdam*, 1991.

16. G. Berikelashvili and M. Chkhartishvili, On the convergence of the difference schemes on W_2^1 for one mixed boundary value problem of theory of elasticity. *Bull. Georgian Acad. Sci.* **153**(1996), No. 2, 195–198.
17. G. K. Berikelashvili, On the convergence of the difference solution of the third boundary value problem in elasticity theory. (Russian) *Zh. Vychisl. Mat. Mat. Fiz.* **38**(1998), No. 2, 310–314; English transl.: *Comput. Math. Math. Phys.* **38**(1998), No. 2, 300–304.
18. G. Berikelashvili, The difference schemes of high order accuracy for elliptic equations with lower derivatives. *Proc. A. Razmadze Math. Inst.* **117**(1998), 1–6.
19. G. Berikelashvili, On the rate of convergence of the difference solution of the bending problem of the orthotropic plates. *Proc. A. Razmadze Math. Inst.* **118**(1998), 21–32.
20. G. K. Berikelashvili, On the definition of a nonlocal trace of a function. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.* **13**(1998), No. 2–3, 2–5.
21. G. K. Berikelashvili, On the rate of convergence of a difference solution of the first boundary value problem for a fourth-order elliptic equation. (Russian) *Differ. Uravn.* **35**(1999), No. 7, 958–963; English transl.: *Differential Equations* **35**(1999), No. 7, 967–973.
22. G. Berikelashvili, The convergence in W_2^1 of the difference solution to the third boundary value problem of elasticity theory. *Rep. Enlarged Sess. Semin. I. Vekua Inst. Appl. Math.* **14**(1999), No. 3, 24–26.
23. G. Berikelashvili, On the solvability of a nonlocal boundary value problem in the weighted Sobolev spaces. *Proc. A. Razmadze Math. Inst.* **119**(1999), 3–11.
24. G. Berikelashvili, Finite difference schemes for elliptic equations with mixed boundary conditions. *Proc. A. Razmadze Math. Inst.* **122**(2000), 21–31.
25. G. Berikelashvili, Finite difference schemes for some mixed boundary value problems. *Proc. A. Razmadze Math. Inst.* **127**(2001), 77–87.
26. G. Berikelashvili, On the convergence of finite-difference scheme for a nonlocal elliptic boundary value problem. *Publ. Inst. Math.* (*Beograd*) (*N.S.*) **70(84)**(2001), 69–78.
27. G. K. Berikelashvili, On the convergence of difference schemes for the third boundary value problem in the theory of elasticity. (Russian) *Zh. Vychisl. Mat. Mat. Fiz.* **41**(2001), No. 8, 1242–1249; English transl.: *Comput. Math. Math. Phys.* **41**(2001), No. 8, 1182–1189.
28. G. Berikelashvili, On a nonlocal boundary-value problem for two-dimensional elliptic equation. *Comput. Methods Appl. Math.* **3**(2003), No. 1, 35–44 (electronic).
29. G. K. Berikelashvili, On the rate of convergence of the difference solution of a nonlocal boundary value problem for a second-order elliptic equation. (Russian) *Differ. Uravn.* **39**(2003), No. 7, 896–903, 1004–1005; English transl.: *Differ. Equ.* **39**(2003), No. 7, 945–953.
30. M. Sh. Birman, N. Ya. Vilenkin, E. A. Gorin, P. P. Zabreĭko, I. S. Iokhvidov, M. Ĭ. Kadets', A. G. ostyuchenko, M. A. Krasnosel'skiĭ, S. G. Kreĭn, B. S. Mityagin, Yu. I. Petunin, Ya. B. Rutitskiĭ, E. M. Semenov, V. I. Sobolev, V. Ya. Stetsenko, L. D. Faddeev, and E. S. Tsitlanadze, Functional analysis. (Russian) *Nauka, Moscow*, 1972.
31. A. V. Bicadze, and A. A. Samarskiĭ, Some elementary generalizations of linear elliptic boundary value problems. (Russian) *Dokl. Akad. Nauk SSSR* **185**(1969), 739–740.
32. V. Bogunovic, Calculation on the difference method for free revolving layered rectangular plates. *Z. Angew. Math. Mech.* **52**(1972), 125–127.
33. J. H. Bramble and S. R. Hilbert, Bounds for a class of linear functionals with applications to Hermite interpolation. *Numer. Math.* **16**(1970/1971), 362–369.
34. J. R. Cannon, A Cauchy problem for the heat equation. *Ann. Mat. Pura Appl. (4)* **66**(1964), 155–165.

35. M. D. Chkhartishvili and G. K. Berikelashvili, On the convergence in W_2^1 of the difference solution for an elliptic equation with mixed boundary conditions. (Russian) *Soobshch. Akad. Nauk Gruzii* **148**(1993), No. 2, 180–184.
36. P. G. Ciarlet, The finite element method for elliptic problems. (Russian) *Mir, Moscow*, 1980.
37. E. G. D'yakonov, Difference methods of solving the boundary value problems. Vyp. 1. (Russian) *Izd. MGU, Moscow*, 1971.
38. T. Dupont and R. Scott, Polynomial approximation of functions in Sobolev spaces. *Math. Comp.* **34**(1980), No. 150, 441–463.
39. A. V. Dzhishkariani, Similar difference operators. (Russian) Boundary properties of analytic functions, singular integral equations and some questions of harmonic analysis. *Trudy Tbiliss. Mat. Inst. Razmadze* **65**(1980), 38–50.
40. I. P. Gavrilyuk, R. D. Lazarov, V. L. Makarov, and S. P. Pirnazarov, Estimates for the rate of convergence of difference schemes for fourth-order equations of elliptic type. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **23**(1983), No. 2, 355–365.
41. I. P. Gavrilyuk, V. G. Prikazchikov, and A. N. Khimich, The accuracy of a difference boundary value problem for a fourth-order elliptic operator with mixed boundary conditions. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **26**(1986), No. 12, 1821–1830.
42. K. Godev and R. Lazarov, On the convergence of the difference scheme for the second boundary problem for the biharmonic equation with solution from W_p^k. *Mathematical models in physics and chemistry and numerical methods of their realization* (*Visegrad*, 1982), 130–141, *Teubner-Texte Math.*, 61, *Teubner, Leipzig*, 1984.
43. D. G. Gordeziani, A certain method of solving the Bicadze-Samarskiĭboundary value problem. (Russian) *Gamoqeneb. Math. Inst. Sem. Mohsen. Anotacie* **2**(1970), 39–41.
44. D. G. Gordeziani, Methods for solving a class of nonlocal boundary value problems. (Russian) *Tbilis. Gos. Univ., Inst. Prikl. Mat., Tbilisi*, 1981.
45. D. Gordeziani and G. Avalishvili, Investigation of the nonlocal initial boundary value problems for some hyperbolic equations. *Hiroshima Math. J.* **31**(2001), No. 3, 345–366.
46. M. M. Gupta, Numerical solution of a second biharmonic boundary value problem. *Nordisk Tidskr. Informationsbehandling* (*BIT*) **13**(1973), 160–164.
47. M. M. Gupta, R. P. Manohar, and J. W. Stephenson, A fourth order, cost effective and stable finite difference scheme for the convection-diffusion equation. *Numerical properties and methodologies in heat transfer* (*College Park, Md.*, 1981), 201–209, *Ser. Comput. Methods Mech. Thermal Sci., Hemisphere, Washington, D.C.-London*, 1983.
48. G. H. Hardy and J. E. Littlewood, Inequalities. *Gosudarstv. Izdat. Inostr. Lit., Moscow*, 1948.
49. V. A. Il'in and E. I. Moiseev, A two-dimensional nonlocal boundary value problem for the Poisson operator in the differential and the difference interpretation. (Russian) *Mat. Model.* **2**(1990), No. 8, 139–156.
50. L. D. Ivanovich, B. S. Jovanović, and E. E. Shili, Convergence of difference schemes for the biharmonic equation. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **26**(1986), No. 5, 776–779.
51. B. S. Jovanović, E. E. Süli, and L. D. Ivanović, On finite difference schemes of high order accuracy for elliptic equations with mixed derivatives. *Mat. Vesnik* **38**(1986), No. 2, 131–136.
52. B. S. Jovanović, Approximation of generalized solutions by means of finite differences. (Russian) *Arch. Math.* (*Brno*) **23**(1987), No. 1, 9–14.
53. B. S. Jovanović, L. D. Ivanović, and E. E. Suli, Convergence of finite-difference schemes for elliptic equations with variable coefficients. *IMA J. Numer. Anal.* **7**(1987), No. 3, 301–305.

54. B. Jovanović, Sur la convergence des schemas aux differences finies pour des equations elliptiques du quatrieme ordre avec des solutions irregulieres. *Publ. Inst. Math.* (*Beograd*) (*N.S.*) **46(60)**(1989), 214–222.
55. B. S. Jovanović, Finite-difference approximations of elliptic equations with nonsmooth coefficients. *Numerical methods and applications* (*Sofia*, 1989), 207–211, *Publ. House Bulgar. Acad. Sci., Sofia*, 1989.
56. A. Kufner and A.-M. Sändig, Some applications of weighted Sobolev spaces. With German, French and Russian summaries. *BSB B. G. Teubner Verlagsgesellschaft, Leipzig*, 1987.
57. A. Kufner, O. John, and S. Fučik, Function spaces. *Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. Noordhoff International Publishing, Leyden; Academia, Prague*, 1977.
58. O. A. Ladyzhenskaya and N. N. Uraltseva, Linear and quasilinear equations of elliptic type (Russian) *Nauka, Moscow*, 1973.
59. R. D. Lazarov, On the convergence of difference solutions to generalized solutions of a biharmonic equation in a rectangle. (Russian) *Differentsial'nye Uravneniya* **17**(1981), No. 7, 1295–1303.
60. V. L. Makarov and V. M. Kalinin, Consistent estimates for the rate of convergence of difference schemes in L_2-norm for the third boundary value problem of elasticity theory. (Russian) *Differentsial'nye Uravneniya* **22**(1986), No. 7, 1265–1268.
61. G. I. Marčuk and V. V. Šaĭdurov, Increasing the accuracy of solutions of difference schemes. (Russian) *Nauka, Moscow*, 1979.
62. Š. E. Mikeladze, New finite difference equations for the computation of rectangular plates, freely supported along the boundaries. *Soobshch. Akad. Nauk Gruzin. SSR* **4**(1943), 737–744.
63. Ju. I. Mokin, Estimations of the L_p-norms of mesh functions in the limiting cases. (Russian) *Differencial'nye Uravneniya* **11**(1975), No. 9, 1652–1663.
64. J. Nitsche and J. C. C. Nitsche, Error estimates for the numerical solution of elliptic differential equations. *Arch. Rational Mech. Anal.* **5**(1960), 293–306.
65. A. Nekvinda and L. Pick, A note on the Dirichlet problem for the elliptic linear operator in Sobolev spaces with weight d_M^ε. *Comment. Math. Univ. Carolin.* **29**(1988), No. 1, 63–71.
66. L. A. Oganesjan and L. A. Ruhovec, Variation-difference methods for solving elliptic equations. (Russian) *Akad. Nauk Armyan. SSR, Erevan*, 1979.
67. J. Peetre, On the differentiability of the solutions of quasilinear partial differential equations. *Trans. Amer. Math. Soc.* **104**(1962), 476–482.
68. V. G. Prikazchikov and G. K. Berikelashvili, Schemes of increased accuracy for an elliptic equation with a mixed derivative. (Russian) *Differentsial'nye Uravneniya* **25**(1989), No. 9, 1622–1624.
69. A. A. Samarskiĭ, Theory of difference schemes. (Russian) *Nauka, Moscow*, 1977.
70. A. A. Samarskiĭ and V. B. Andreev, Difference methods for elliptic equations. (Russian) *Nauka, Moscow*, 1976.
71. A. A. Samarskiĭ, R. D. Lazarov, and V. L. Makarov, Difference schemes for differential equations with generalized solutions. (Russian) *Vyssh. Schola, Moscow*, 1987.
72. M. P. Sapagovas, Solution of a nonlinear ordinary differential equation with an integral condition. (Russian) *Litovsk. Mat. Sb.* **24**(1984), No. 1, 155–166.
73. M. P. Sapagovas, A difference scheme for two-dimensional elliptic problems with an integral condition. (Russian) *Litovsk. Mat. Sb.* **23**(1983), No. 3, 155–159.
74. A. L. Skubachevskiĭ, On the spectrum of some nonlocal elliptic boundary value problems. (Russian) *Mat. Sb.* (*N.S.*) **117(159)**(1982), No. 4, 548–558.
75. Kh. Trieble, Interpolation theory, functional spaces, differential operators. (Russian) *Mir, Moscow*, 1980.

76. S. A. Voĭtsekhovskiĭ, On the convergence of difference solutions to generalized solutions of the Dirichlet problem for a second-order elliptic equation. (Russian) *Vychisl. Prikl. Mat.* (*Kiev*), 1985, No. 57, 21–26; English transl.: *J. Soviet Math.* **58**(1992), No. 3, 208–212.
77. S. A. Voĭtsekhovskiĭ and V. M. Kalinin, An estimate for the rate of convergence of difference schemes for the first boundary value problem in elasticity theory in the anisotropic case. (Russian) *Zh. Vychisl. Mat. i Mat. Fiz.* **29**(1989), No. 7, 1088–1092, 1104; English transl.: *U.S.S.R. Comput. Math. and Math. Phys.* **29**(1989), No. 4, 87–91.
78. S. A. Voĭtsekhovskiĭ, V. M. Kalinin, and V. L. Makarov, An estimate for the rate of convergence of difference schemes for a system of equations of the equilibrium of an inhomogeneous anisotropic elastic solid under conditions of rigid fixing. (Russian) *Vychisl. Prikl. Mat.* (*Kiev*), 1987, No. 62, 14–19; English transl.: *J. Soviet Math.* **63**(1993), No. 5, 517–521.
79. S. A. Voĭtsekhovskiĭ and V. L. Makarov, An estimate for the rate of convergence of difference schemes for elliptic equations with discontinuous coefficients on a regular grid. (Russian) *Vychisl. Prikl. Mat.* (*Kiev*), 1985, No. 55, 21–26; English transl.: *J. Soviet Math.* **58**(1992), No. 1, 17–21.
80. S. A. Voĭcehovskiĭ, V. L. Makarov, and V. G. Prikazčikov, Error of the method of nets in an eigenvalue problem for fourth-order ordinary differential equations. (Russian) *Vychisl. Prikl. Mat.* (*Kiev*), 1979, No. 38, 138–145.
81. S. A. Voĭtsekhovskiĭ and V. N. Novichenko, Estimates for the rate of convergence of difference schemes of improved accuracy for elliptic equations with a mixed derivative. (Russian) *Model. Mekh.* **2**(1988), No. 5, 34–38.
82. E. A. Volkov, On methods of refinement using higher-order differences and h^2-extrapolations. (Russian) *Dokl. Akad. Nauk SSSR* **150**(1963), 455–458.
83. V. Vaĭnel't, R. D. Lazarov, and V. L. Makarov, Convergence of difference schemes for elliptic equations with mixed derivatives and generalized solutions. (Russian) *Differentsial'nye Uravneniya* **19**(1983), No. 7, 1140–1145.

(Received 24.05.2005)

Author's address:

A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: bergi@rmi.acnet.ge