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ON A THREE LEVEL DIFFERENCE SCHEME FOR THE REGULARIZED LONG WAVE EQUATION

Abstract. We consider an initial boundary-value problem for the Regularized Long Wave equation. A three level conservative difference scheme is studied. On the first level a two level scheme is used to find the values of the unknown functions which ensures the expression of the initial energies only by the initial data. The obtained algebraic equations are linear with respect to the values of the unknown function for each new level. The use of the Gronwall lemma does not require any restriction on mesh steps. It is proved that the finite difference scheme converges with the rate $O(\tau^2 + h^2)$ when the exact solution belongs to the Sobolev space W_2^3.

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1. Introduction

We consider the Regularized Long Wave (RLW) equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^2 \partial t} = 0, \ \ (x,t) \in Q_T. \tag{1.1}$$

The RLW equation was first put forward by Peregrine [1] as a model for small-amplitude long waves on the water surface in a channel and later by Benjamen et al. [2]. This equation describes phenomena with weak nonlinearity and dispersion waves, including, e.g., ion-acoustic and magneto hydrodynamic waves in plasma.

In the domain $Q_T := (0, a) \times (0, T)$, for the equation (1.1) we consider the initial boundary-value problem with the following conditions

$$u(0, t) = u(a, t) = 0, \ t \in [0, T), \quad u(x, 0) = u_0(x), \ x \in [0, a]. \qquad (1.2)$$

The numerical solution of the RLW equation has been the subject of many papers. The first finite-difference scheme is given by Peregrine [1]. The schemes offered in [3], [4] are not conservative. The scheme in [5] is conservative but the passage from one level to another requires iterations.

In [6] a three level difference scheme is presented for the problem (1.1), (1.2). The scheme is conservative but in the equality of the discrete conservation law the initial energy E^0 depends explicitly not only on initial data. Besides, the scheme for the first level is nonlinear with respect to the values of the unknown function. Convergence with the rate $O(h^2 + \tau^2)$ is proved under condition that the exact solution belongs to $C^{4,3}$.

A three level scheme is considered in [7] and convergence with the rate $O(h^2 + \tau^2)$ is shown when the exact solution belongs to C^5. Stability is proved for a sufficiently small mesh step. No way is offered for calculation of the unknown function on the first level. The values of the unknown function on the first level are involved in the expression of the initial energy.

In this paper the three level scheme has the same form as in [6], [7], but the scheme for the first level linearly contains the values of the unknown function. It is proved that the finite difference scheme converges with the rate $O(\tau^2 + h^2)$ when the exact solution belongs to the Sobolev space $W_2^3(Q_T)$. The Steklov averaging operators are used for error estimation.

The paper is organized as follows. In the next section we present the statement of the problem and main results. Then, in Section 3, we propose auxiliary statements which are used in the proof of Theorem 2.2. In Section 4 we give the results needed in the proof of Theorem 2.3.

2. Statement of the Problem and Main Results

Let Q_T be a rectangle where the problem (1.1), (1.2) is to be solved. We assume that $u \in W_2^3(Q_T)$. It is easy to show that in that case $u \in C^1(Q_T)$ and $u_0 \in C^1(0, a)$.

For convenience we introduce the following notation:

$$x_i = ih, \ t_j = j\tau, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, J,$$

where $h = a/n$ and $\tau = T/J$ denote the spatial and temporal mesh sizes, respectively. Let $u_i^j := u(x_i, t_j)$, $U_i^j \sim u(x_i, t_j)$,

$$(U_i^j)_x := \frac{U_{i+1}^j - U_i^j}{h}, \ (U_i^j)_{\bar{x}} := \frac{U_i^j - U_{i-1}^j}{h}, \ (U_i^j)_{\overset{\circ}{x}} := \frac{1}{2}\left((U_i^j)_x + (U_i^j)_{\bar{x}}\right),$$

$$(U_i^j)_t := \frac{U_i^{j+1} - U_i^j}{\tau}, \ (U_i^j)_{\bar{t}} := \frac{U_i^j - U_i^{j-1}}{\tau}, \ (U_i^j)_{\overset{\circ}{t}} := \frac{1}{2}\left((U_i^j)_t + (U_i^j)_{\bar{t}}\right),$$

$$(U^j, V^j) := \sum_{i=1}^{n-1} h U_i^j V_i^j, \quad (U^j, V^j] := \sum_{i=1}^{n} h U_i^j V_i^j,$$

$$\|U^j\|^2 := (U^j, U^j), \quad \|U^j]|^2 := (U^j, U^j], \quad \|U^j\|_\infty = \max_{1 \le i \le n-1} |U_i^j|.$$

We approximate the problem $(1.1), (1.2)$ with the help of the difference scheme:

$$\mathcal{L} U_i^j := (U_i^j)_{\overset{\circ}{t}} + \frac{1}{2} (U_i^{j+1} + U_i^{j-1})_{\overset{\circ}{x}} +$$

$$+ \frac{1}{6} (\Lambda U)_i^j - (U_i^j)_{\overline{x}x\overset{\circ}{t}} = 0, \quad i = \overline{1, n-1}, \;\; j = \overline{1, J-1}, \quad (2.1)$$

$$\mathcal{L} U_i^0 := (U_i^0)_t + \frac{1}{2} (U_i^1 + U_i^0)_{\overset{\circ}{x}} +$$

$$+ \frac{1}{6} (\Lambda U)_i^0 - (U_i^0)_{\overline{x}xt} = 0, \quad i = \overline{1, n-1}, \quad (2.2)$$

$$U_0^j = U_n^j = 0, \;\; j = \overline{0, J}, \quad U_i^0 = u_0(x_i), \;\; i = \overline{0, n}, \quad (2.3)$$

where

$$(\Lambda U)_i^j := U_i^j (U_i^{j+1} + U_i^{j-1})_{\overset{\circ}{x}} + (U_i^j (U_i^{j+1} + U_i^{j-1}))_{\overset{\circ}{x}}, \;\; j = \overline{1, J-1},$$

$$(\Lambda U)_i^0 := U_i^0 (U_i^1 + U_i^0)_{\overset{\circ}{x}} + (U_i^0 (U_i^1 + U_i^0))_{\overset{\circ}{x}}.$$

It is well known (see, e.g., [8]) that the problem $(1.1), (1.2)$ possesses an invariant corresponding to the conservation of energy which can be expressed in the form

$$E(t) := \int_0^a \left(u^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right) dx = E(0).$$

The same property is kept for the difference scheme.

Theorem 2.1. *The finite difference scheme* (2.1)–(2.3) *possesses the following invariant*

$$E^j := \|U^j\|^2 + \|U_{\overline{x}}^j]|^2 = \|u_0\|^2 + \|u_{0,\overline{x}}]|^2 = E^0, \;\; j = 1, 2, \dots . \quad (2.4)$$

Proof. It is easy to check the validity of the following equalities:

$$(U_{\overset{\circ}{t}}^j, U^{j+1} + U^{j-1}) = \frac{1}{2\tau} \left(\|U^{j+1}\|^2 - \|U^{j-1}\|^2 \right),$$

$$-(U_{\overline{x}x\overset{\circ}{t}}^j, U^{j+1} + U^{j-1}) = \frac{1}{2\tau} \left(\|U_{\overline{x}}^{j+1}]|^2 - \|U_{\overline{x}}^{j-1}]|^2 \right),$$

$$((\Lambda U)^j, U^{j+1} + U^{j-1}) = 0, \quad ((U^{j+1} + U^{j-1})_{\overset{\circ}{x}}, U^{j+1} + U^{j-1}) = 0,$$

$$(U_t^0, U^1 + U^0) =$$

$$= \frac{1}{\tau} \left(\|U^1\|^2 - \|U^0\|^2 \right) - (U_{\overline{x}xt}^0, U^1 + U^0) = \frac{1}{\tau} \left(\|U_{\overline{x}}^1]|^2 - \|U_{\overline{x}}^0]|^2 \right),$$

$$((\Lambda U)^0, U^1 + U^0) = 0, \quad ((U^1 + U^0)_{\overset{\circ}{x}}, U^1 + U^0) = 0.$$

Multiplying (2.1) by $(U^{j+1} + U^{j-1})$ and (2.2) by $(U^1 + U^0)$ and summing over i, we obtain respectively:

$$\|U^{j+1}\|^2 + \|U_{\overline{x}}^{j+1}]\|^2 = \|U^{j-1}\|^2 + \|U_{\overline{x}}^{j-1}]\|^2, \;\; j = 1, 2, \ldots,$$

and

$$\|U^1\|^2 + \|U_{\overline{x}}^1]\|^2 = \|U^0\|^2 + \|U_{\overline{x}}^0]\|^2.$$

From these equalities it follows

$$\|U^j\|^2 + \|U_{\overline{x}}^j]\|^2 = \|U^0\|^2 + \|U_{\overline{x}}^0]\|^2, \;\; j = 1, 2, \ldots, \tag{2.5}$$

which proves (2.4). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Let $Z := U - u$, where u is the exact solution of the problem $(1.1), (1.2)$ and U is the solution of the finite difference scheme (2.1)–(2.3). Substituting $U = Z + u$ into (2.1)–(2.3), we obtain the following problem for the error Z:

$$(Z_i^j)_{\overset{\circ}{t}} + \frac{1}{2}(Z_i^{j+1} + Z_i^{j-1})_{\overset{\circ}{x}} - (Z_i^j)_{\overline{x}x\overset{\circ}{t}} =$$

$$= -\frac{1}{6}\left((\Lambda U)_i^j - (\Lambda u)_i^j\right) - \mathcal{L}u_i^j, \;\; j = 1, 2, \ldots, \tag{2.6}$$

$$(Z_i^0)_t + \frac{1}{2}\left(Z_i^1 + Z_i^0\right)_{\overset{\circ}{x}} - (Z_i^0)_{\overline{x}xt} = -\frac{1}{6}\left((\Lambda U)_i^0 - (\Lambda u)_i^0\right) - \mathcal{L}u_i^0, \tag{2.7}$$

$$Z_i^0 = 0, \;\; i = 0, 1, \ldots, n, \;\;\; Z_0^j = Z_n^j = 0, \;\; j = 0, 1, \ldots, J. \tag{2.8}$$

Theorem 2.2. *For the solution of the problem* (2.6)–(2.8) *the following estimates hold:*

$$\|Z^1\|^2 + \|Z_{\overline{x}}^1]\|^2 \leq \|\tau\mathcal{L}u^0\|^2, \tag{2.9}$$

$$\|Z^{j+1}\|^2 + \|Z_{\overline{x}}^{j+1}]\|^2 \leq$$

$$\leq 2\exp(c_1 T)\Big(\|\tau\mathcal{L}u^0\|^2 + 4T\tau\sum_{k=1}^{j}\|\mathcal{L}u^k\|^2\Big), \;\; j = 1, 2, \ldots, J-1, \tag{2.10}$$

where $c_1 := (4/T) + (9Tc_*^2)$.

Theorem 2.3. *Let the exact solution of the initial-boundary value problem* $(1.1), (1.2)$ *belong to* $W_2^3(Q_T)$. *Then the discretization error of the finite difference scheme* (2.1)–(2.3) *is determined by the estimate*

$$\|Z^j\|^2 + \|Z_{\overline{x}}^j]\|^2 \leq c_2(\tau^2 + h^2)^2\|u\|_{W_2^3(Q_T)}^2,$$

where c_2 *denotes a positive constant independent of* h *and* τ.

3. Auxiliary Statements and Proof of Theorem 2.2

Lemma 3.1. *For the solution of the difference scheme* (2.1)–(2.3), *the following estimates*

$$\|U^j\|_\infty^2 \leq (a/4)\|U_{\overline{x}}^j]\|^2, \;\;\; \|U^j\| \leq c\|u_0'\|_{L_2(0,a)}$$

are valid.

Denote

$$B^j := \|Z^j\|^2 + \|Z^{j-1}\|^2 + \|Z_{\overline{x}}^j]\|^2 + \|Z_{\overline{x}}^{j-1}]\|^2, \quad j = 1, 2, \dots . \qquad (3.1)$$

Lemma 3.2. *For the solution of the problem* (2.7), (2.8), *the following identity is valid*

$$B^1 := \|Z^1\|^2 + \|Z_{\overline{x}}^1]\|^2 = -(\tau \mathcal{L} U^0, Z^1).$$

Lemma 3.3. *For the solution of the problem* (2.6)–(2.8), *the estimate*

$$B^{j+1} \leq B^1 + \frac{\tau}{3} \sum_{k=1}^{j} \left| \left((\Lambda U)^k - (\Lambda u)^k, Z^{k+1} + Z^{k-1} \right) \right| +$$

$$+ 2\tau \sum_{k=1}^{j} \left| \left(\mathcal{L} u^k, Z^{k+1} + Z^{k-1} \right) \right|, \quad j = 1, 2, \dots, \qquad (3.2)$$

is valid, where B^j is defined by the equality (3.1).

Now we intend to estimate the terms in the right-hand side of the inequality (3.2).

Lemma 3.4. *The following inequalities*

$$\left| (\mathcal{L} u^k, Z^{k+1} + Z^{k-1}) \right| \leq \frac{1}{4T} \left(\|Z^{k+1}\|^2 + \|Z^{k-1}\|^2 \right) + 2T \|\mathcal{L} u^k\|^2, \qquad (3.3)$$

$$\left((\Lambda U)^k - (\Lambda u)^k, Z^{k+1} + Z^{k-1} \right) \leq \frac{3}{2T} \left(\|Z^{k+1}\|^2 + \|Z^{k-1}\|^2 \right) +$$

$$+ \frac{4Tc_*^2}{3} \|Z_{\overline{x}}^k]\|^2 + \frac{3}{T} \left(\|Z_{\overline{x}}^{k+1}]\|^2 + \|Z_{\overline{x}}^{k-1}]\|^2 \right) + \frac{8Tc_*^2}{3} \|Z^k\|^2 \qquad (3.4)$$

are valid, where

$$c_* := \max_{k} \|u^k\|_C.$$

Proof of Theorem 2.2. On the basis of (3.3) and (3.4), we get from (3.2):

$$B^{j+1} \leq B^1 + \frac{\tau}{3} \sum_{k=1}^{j} \left(\frac{3}{2T} \left(\|Z^{k+1}\|^2 + \|Z^{k-1}\|^2 \right) + \right.$$

$$+ \frac{4Tc_*^2}{3} \|Z_{\overline{x}}^k]\|^2 + \frac{3}{T} \left(\|Z_{\overline{x}}^{k+1}]\|^2 + \|Z_{\overline{x}}^{k-1}]\|^2 \right) + \frac{8Tc_*^2}{3} \|Z^k\|^2 \Big) +$$

$$+ 2\tau \sum_{k=1}^{j} \left(\frac{1}{4T} \left(\|Z^{k+1}\|^2 + \|Z^{k-1}\|^2 \right) + 2T \|\mathcal{L} u^k\|^2 \right).$$

Thus

$$B^{j+1} \leq B^1 + \frac{\tau}{T} \left(\|Z^{j+1}\|^2 + \|Z_{\overline{x}}^{j+1}\|^2 \right) +$$

$$+ \left(\frac{2\tau}{T} + \frac{8\tau T c_*^2}{9} \right) \sum_{k=1}^{j} \left((\|Z^k\|^2 + \|Z_{\overline{x}}^k]\|^2) \right) + 4\tau T \sum_{k=1}^{j} \|\mathcal{L} u^k\|^2.$$

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Taking into account that $\tau/T \le 0.5$, we get

$$B^{j+1} \le 2B^1 + c\tau \sum_{k=1}^{j} B^k + 8\tau T \sum_{k=1}^{j} \|\mathcal{L}u^k\|^2, \quad c := (4/T) + (16Tc_*^2)/9. \quad (3.5)$$

Let the inequalities

$$B_{j+1} \le 2B_1 + c\tau \sum_{k=1}^{j} B_k + b\tau \sum_{k=1}^{j} f_k, \quad j = 1, 2, \ldots,$$

be valid, where b, c, τ, B_k, f_k are non-negative numbers. Then

$$B_{j+1} \le 2(1 + c\tau)^{j-1} B_1 + c\tau(1 + c\tau)^{j-1} B_1 + b\tau \sum_{k=1}^{j} (1 + c\tau)^{j-k} f_k,$$

and therefore

$$B_{j+1} \le 2(1 + c\tau)^j B_1 + b\tau(1 + c\tau)^j \sum_{k=1}^{j} f_k.$$

If $j = 1, 2, \ldots, J$, $J := T/\tau$, then

$$(1 + c\tau)^j < (1 + c\tau)^{T/\tau} = (1 + c\tau)^{(Tc)/(c\tau)} < e^{cT}.$$

Thus

$$B_{j+1} \le e^{cT}\Big(2B_1 + b\tau \sum_{k=1}^{j} f_k\Big).$$

So (3.5) yields

$$B^{j+1} \le e^{cT}\Big(2B^1 + 8T\tau \sum_{k=1}^{j} \|\mathcal{L}u^k\|^2\Big). \quad (3.6)$$

According to Lemma 3.2, we have

$$B^1 \le 0.5\|Z^1\|^2 + 0.5\|\tau\mathcal{L}u^0\|^2 \le 0.5B^1 + 0.5\|\tau\mathcal{L}u^0\|^2,$$

i.e., (2.9) is true. On the basis of this inequality, the estimate (2.10) follows from (3.6).

Theorem 2.2 is proved. $\square$

4. ESTIMATION OF TRUNCATION ERRORS AND PROOF OF THEOREM 2.3

In order to estimate the error, we will use the Steklov averaging operators:

$$(\widehat{\mathcal{P}}u)_i := \frac{1}{h} \int_{x_i}^{x_{i+1}} u(x,t)\,dx, \quad (\check{\mathcal{P}}u)_i := \frac{1}{h} \int_{x_{i-1}}^{x_i} u(x,t)\,dx,$$

$$(\overset{\circ}{\mathcal{P}}u)_i := 0.5(\widehat{\mathcal{P}}u + \check{\mathcal{P}}u)_i, \quad (\mathcal{P}u)_i := \frac{1}{h^2} \int_{x_{i-1}}^{x_{i+1}} (h - |x_i - x|)u(x,t)\,dx,$$

$$(\widehat{\mathcal{S}}u)^j := \frac{1}{\tau} \int\limits_{t_j}^{t_{j+1}} u(x,t)\, dt, \quad (\check{\mathcal{S}}u)^j := \frac{1}{\tau} \int\limits_{t_{j-1}}^{t_j} u(x,t)\, dt,$$

$$(\overset{\circ}{\mathcal{S}}u)^j := 0.5(\widehat{\mathcal{S}}u + \check{\mathcal{S}}u)^j, \quad (\mathcal{S}u)^j := \frac{1}{\tau^2} \int\limits_{t_{j-1}}^{t_{j+1}} (\tau - |t-t_j|)u(x,t)\, dt.$$

Represent the approximation error in a convenient form.

Lemma 4.1. *If u is a solution to the problem $(1.1), (1.2)$, then*

$$\mathcal{L}u = \psi_{(1)}\Big(\frac{\partial u}{\partial t}\Big) + \psi_{(2)}\Big(\frac{\partial u}{\partial x}\Big) + \frac{1}{6}\,\psi_{(3)}(u) + \psi_{(2)}\Big(u\frac{\partial u}{\partial x}\Big), \quad t > 0, \qquad (4.1)$$

$$\mathcal{L}u = \Phi_{(1)}\Big(\frac{\partial u}{\partial t}\Big) + \Phi_{(2)}\Big(\frac{\partial u}{\partial x}\Big) + \frac{1}{6}\,\Phi_{(3)}(u) + \Phi_{(2)}\Big(u\frac{\partial u}{\partial x}\Big), \quad t = 0, \qquad (4.2)$$

where

$$\psi_{(1)}(u) := \overset{\circ}{\mathcal{S}}\,(\mathcal{I} - \mathcal{P})u, \quad \mathcal{I}u := u, \tag{4.3}$$

$$\psi_{(2)}(u) := 0.5\,\overset{\circ}{\mathcal{P}}\,(\widehat{u} + \check{u}) - \mathcal{P}\,\overset{\circ}{\mathcal{S}}\,u, \tag{4.4}$$

$$\psi_{(3)}(u) := \tau^2 u u_{\overset{\circ}{x}\bar{t}t} + \tau^2 (u u_{\bar{t}t})_{\overset{\circ}{x}} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}x} - (3\tau^2/2)(u^2)_{\overset{\circ}{x}\bar{t}t}, \tag{4.5}$$

$$\Phi_{(1)}(u) := \widehat{\mathcal{S}}(\mathcal{I} - \mathcal{P})u, \tag{4.6}$$

$$\Phi_{(2)}(u) := 0.5\,\overset{\circ}{\mathcal{P}}\,(u^1 + u^0) - \mathcal{P}\widehat{\mathcal{S}}u, \tag{4.7}$$

$$\Phi_{(3)}(u) := \tau u u_{\overset{\circ}{x}t} + \tau(u u_t)_{\overset{\circ}{x}} - h^2 u_{\overset{\circ}{x}} u_{\overline{x}x} - (3\tau/2)(u)^2_{\overset{\circ}{x}t}. \tag{4.8}$$

Now estimate the terms in the right-hand sides of (4.1), (4.2).

When estimating the truncation error, we will assume that the solution to the problem (1.1), (1.2) $u \in W_2^3(Q_T)$. In this case, on the basis of embedding theorems we conclude that $u \in C^1(\overline{Q}_T)$. We denote

$$\delta := \max_{(x,t)\in \overline{Q}_T} \Big(|u| + \Big|\frac{\partial u}{\partial x}\Big| + \Big|\frac{\partial u}{\partial t}\Big|\Big).$$

Also,

$$e_{i0} = \big\{(x,t)|\ |x - x_i| \le h,\ 0 \le t \le \tau\big\},$$

$$e_{ij} = \big\{(x,t)|\ |x - x_i| \le h,\ |t - t_j| \le \tau\big\}, \quad j = 1, 2, \dots.$$

Lemma 4.2. *For $\psi_{(\alpha)}$ defined from the equalities (4.3)–(4.5), the following estimates*

$$\|\psi_{(1)}(u^j)\|^2 \le c\,\frac{h^4}{\tau} \int\limits_0^a \int\limits_{t_{j-1}}^{t_{j+1}} \Big|\frac{\partial^2 u}{\partial x^2}\Big|^2\, dxdt,$$

$$\|\psi_{(2)}(u^j)\|^2 \le c\,\frac{h^4 + \tau^4}{\tau} \int\limits_0^a \int\limits_{t_{j-1}}^{t_{j+1}} \Big(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial t^2}\Big|^2\Big)\, dxdt,$$

$$\|\psi_{(3)}(u^j)\|^2 \le c\,\frac{h^4+\tau^4}{\tau}\int\limits_0^a\int\limits_{t_{j-1}}^{t_{j+1}}\left(\Big|\frac{\partial^3 u}{\partial x\partial t^2}\Big|^2+\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2+\Big|\frac{\partial^2 u}{\partial x\partial t}\Big|^2+\Big|\frac{\partial^2 u}{\partial t^2}\Big|^2\right)dxdt$$

are valid, where the constant $c > 0$ does not depend on the mesh steps.

Lemma 4.3. *For $\Phi_{(\alpha)}$ defined from (4.6)–(4.8), the following estimates are valid*

$$\|\Phi_{(1)}(u)\|^2 \le c\,\frac{h^4}{\tau}\int\limits_0^a\int\limits_0^\tau \Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 dxdt,$$

$$\|\Phi_{(2)}(u)\|^2 \le c\,\frac{h^4+\tau^4}{\tau}\int\limits_0^a\int\limits_0^\tau \left(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2+\Big|\frac{\partial^2 u}{\partial t^2}\Big|^2\right)dxdt,$$

$$\|\Phi_{(3)}(u)\|^2 \le c\tau^2\|u\|_{C^1}^2 + c\,\frac{h^4+\tau^4}{\tau^2}\times$$

$$\times\int\limits_{Q_T}\left(\Big|\frac{\partial^3 u}{\partial x\partial t^2}\Big|^2+\Big|\frac{\partial^3 u}{\partial x^2\partial t}\Big|^2+\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2+\Big|\frac{\partial^2 u}{\partial t^2}\Big|^2+\Big|\frac{\partial^2 u}{\partial x\partial t}\Big|^2\right)dxdt,$$

where the constant $c > 0$ does not depend on the mesh steps.

Proof of Theorem 2.3. According to Lemmas 4.1 and 4.2,

$$\|\mathcal{L}u^j\|^2 \le c\,\frac{h^4+\tau^4}{\tau}\int\limits_0^a\int\limits_{t_{j-1}}^{t_{j+1}}\left(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2+\Big|\frac{\partial^2 u}{\partial x\partial t}\Big|^2+\right.$$

$$\left.+\Big|\frac{\partial^2 u}{\partial t^2}\Big|^2+\Big|\frac{\partial^3 u}{\partial x^3}\Big|^2+\Big|\frac{\partial^3 u}{\partial x^2\partial t}\Big|^3+\Big|\frac{\partial^3 u}{\partial x\partial t^2}\Big|^2\right)dxdt,\ \ j=1,2,\dots.$$

Via Lemmas 4.1 and 4.3 we get

$$\|\mathcal{L}u^0\|^2 \le c\,\frac{h^4+\tau^4}{\tau^2}\times$$

$$\times\int\limits_{Q_T}\left(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2+\Big|\frac{\partial^2 u}{\partial x\partial t}\Big|^2+\Big|\frac{\partial^2 u}{\partial t^2}\Big|^2+\Big|\frac{\partial^3 u}{\partial x^3}\Big|^2+\Big|\frac{\partial^3 u}{\partial x^2\partial t}\Big|^3+\Big|\frac{\partial^3 u}{\partial x\partial t^2}\Big|^2\right)dxdt.$$

Therefore Theorem 2.3 follows from Theorem 2.2. $\qquad\square$

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