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FINITE DIFFERENCE SOLUTION OF A MRLW EQUATION

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Abstract. We consider an initial boundary-value problem for the modified regularized long-wave (MRLW) equation. A three level conservative difference schemes are studied. The obtained algebraic equations are linear with respect to the values of a desired function for each new level. The proof of the convergence does not require any restriction on mesh steps. A numerical method for selecting artificial boundary conditions is given. It is proved that the finite difference scheme converges with the rate $O(\tau^2 + h^2)$ when the exact solution belongs to the Sobolev space W_2^3.

რეზიუმე. მოდიფიცირებული რეგულარიზებული გრძელი ტალღის განტოლებისათვის დასმული საწყის-სასაზღვრო ამოცანისთვის შესწავლილია სამშრიანი კონსერვატული სხვაობიანი სქემები. მიღებული ალგებრული განტოლებები წრფივად შეიცავენ საძიებელი ფუნქციის მნიშვნელობებს ყოველ ახალ შრეზე. შემოთავაზებულია ხელოვნური სასაზღვრო პირობების მოცემის რიცხვითი მეთოდი. დამტკიცებულია სასრულ სხვაობიანი სქემის კრებადობა $O(\tau^2 + h^2)$ სიჩქარით, როცა ზუსტი ამონახსნი მიეკუთვნება W_2^3 სობოლევის სივრცეს. კრებადობის დამტკიცება არ საჭიროებს რაიმე შეზღუდვას ბადის ბიჯებზე.

1. Introduction

The regularized long-wave (RLW) or Benjamin-Bona-Mahony (BBM) equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \gamma u \frac{\partial u}{\partial x} - \mu \frac{\partial^3 u}{\partial x^2 \partial t} = 0,$$

where γ and μ are positive constants, was first put forward as a model for small-amplitude long waves on water surface in channel by Peregrine [1,2], and first analyzed by Benjamin, et al. [3]. This equation describes phenomena with weak nonlinearity and dispersion waves, including, e.g. ion-acoustic and magneto-hydrodynamic waves in plasma.

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The RLW equation is a special case of the generalized regularized long-wave (GRLW) equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\gamma u^p\frac{\partial u}{\partial x}-\mu\frac{\partial^3 u}{\partial x^2\partial t}=0,$$

where p is a positive integer.

We consider another special case of the GRLW equation, called the modified regularized long-wave (MRLW) equation, when $p=2$:

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\gamma u^2\frac{\partial u}{\partial x}-\mu\frac{\partial^3 u}{\partial x^2\partial t}=0. \tag{1.1}$$

Physical boundary conditions require that $u\to 0$ for $x\to\pm\infty$.

While numerically solving a problem initially formulated on an unbounded domain, one typically truncates this domain, which necessitates setting of the *artificial boundary conditions* (ABCs) at the newly formed external boundary. In an ideal case, the ABCs would be specified so that the solution on the truncated domain coincides with the corresponding fragment of the original infinite-domain solution.

For the numerical solution of the equation (1.1) via initial condition $u(x,0)=u_0(x)$ artificial boundaries can be selected at some points $x=a,\ x=b,\ a<b$ and in the domain $Q_T:=(a,b)\times(0,T)$ the initial boundary-value problem with the conditions

$$u(a,t)=u(b,t)=0,\ t\in[0,T),\quad u(x,0)=u_0(x),\ x\in[a,b] \tag{1.2}$$

can be considered.

We mean that $u\in W_2^3(Q_T)$. It is easy to show that in this case $u\in C^1(\overline{Q}_T)$. Let

$$\|u\|_C:=\max_{(x,t)\in\overline{Q}_T}|u(x,t)|,\quad \delta:=\max\big(\|u\|_C;\ \|\partial u/\partial x\|_C;\ \|\partial u/\partial t\|_C\big).$$

The numerical solution of the MRLW equation has been the subject of many papers (see, e.g., [4]-[14]).

In [8], a conservative three level difference scheme is presented for the GRLW equation. But in the equality of the discrete conservation law the initial energy E^0 depends explicitly not only on the initial data. Convergence with the rate $O(h^2+\tau^2)$ is proved under the condition that the exact solution belongs to $C^{(4,3)}$. The stability is proved for a sufficiently small mesh step. A three level difference scheme with a local truncation error $O(h^2+\tau^2)$ is considered also in [9]. In these papers no way is offered for calculation of the unknown function at time level $t=\tau$.

In this paper a three level one-parameter family of difference schemes is constructed on 9-point template. Two level schemes are used to find the values of the unknown functions on the first level. It is proved that the finite difference scheme converges with the rate $O(\tau^2+h^2)$, when the exact solution belongs to the Sobolev space $W_2^3(Q_T)$. The Steklov averaging operators are used for error estimation.

In the present paper, a numerical method for selection of artificial boundary conditions are given, since the Dirichlet boundary conditions are zeros only within certain accuracy. A certain value of a free parameter of the scheme is given also, which along with artificial boundary conditions ensures a good accuracy of an approximate solution.

2. Difference Scheme and its Solvability

The finite domain $[a,b]\times[0,T]$ in a plane is divided into rectangle grids by the points $(x_i,t_j)=(a+ih,j\tau)$, $i=0,1,...,n,\ j=0,1,2,\dots,J$, where $h=(b-a)/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively.

Let $u_i^j:=u(x_i,t_j)$, $U_i^j\sim u(x_i,t_j)$,

$$(U_i^j)_x:=\frac{U_{i+1}^j-U_i^j}{h},\ (U_i^j)_{\bar x}:=\frac{U_i^j-U_{i-1}^j}{h},\ (U_i^j)_{\overset{\circ}{x}}:=\frac{1}{2}((U_i^j)_x+(U_i^j)_{\bar x}),$$

$$(U_i^j)_t:=\frac{U_i^{j+1}-U_i^j}{\tau},\ (U_i^j)_{\bar t}:=\frac{U_i^j-U_i^{j-1}}{\tau},\ (U_i^j)_{\overset{\circ}{t}}:=\frac{1}{2}((U_i^j)_t+(U_i^j)_{\bar t}).$$

The discrete inner product and discrete norms for any $U^j,\ V^j$ are defined by

$$(U^j,V^j):=\sum_{i=1}^{n-1}hU_i^jV_i^j,\quad (U^j,V^j]:=\sum_{i=1}^{n}hU_i^jV_i^j,\quad \|U^j\|^2:=(U^j,U^j),$$

$$\|U^j]|^2:=(U^j,U^j],\quad \|U^j\|_{W_2^1}=\left(\|U^j\|^2+\|U_{\bar x}^j]|^2\right)^{1/2},\quad \|U^j\|_\infty=\max_{0\le i\le n}|U_i^j|.$$

We approximate the problem (1.1), (1.2) with the help of the difference scheme:

$$\mathcal{L}U_i^j:=(U_i^j)_{\overset{\circ}{t}}+\frac{1}{2}(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}+$$
$$+\frac{\gamma}{8}\Lambda U_i^j-\mu(U_i^j)_{\bar x x\overset{\circ}{t}}=0,\ i=\overline{1,n-1},\ j=\overline{1,J-1}, \tag{2.1}$$

$$\mathcal{L}U_i^0:=(U_i^0)_t+\frac{1}{2}(U_i^1+U_i^0)_{\overset{\circ}{x}}+$$
$$+\frac{\gamma}{8}\Lambda U_i^0-\mu(U_i^0)_{\bar x xt}=0,\ i=\overline{1,n-1},\ j=0, \tag{2.2}$$

$$U_0^j=U_n^j=0,\ j=\overline{0,J},\quad U_i^0=u_0(x_i),\ i=\overline{0,n}. \tag{2.3}$$

where

$$\Lambda U_i^j:=(U_i^j)^2(U_i^{j+1}+U_i^{j-1})_{\overset{\circ}{x}}+\left((U_i^j)^2(U_i^{j+1}+U_i^{j-1})\right)_{\overset{\circ}{x}},\ j=\overline{1,J-1},$$
$$\Lambda U_i^0:=(U_i^0)^2(U_i^1+U_i^0)_{\overset{\circ}{x}}+\left((U_i^0)^2(U_i^1+U_i^0)\right)_{\overset{\circ}{x}}.$$

Theorem 2.1. *The finite difference scheme* (2.1)–(2.3) *is conservative in the sense*

$$E^j := \|U^j\|^2 + \mu\|U^j_{\bar{x}}]|^2 = \|u_0\|^2 + \mu\|u_{0,\bar{x}}]|^2 := E^0,\ j = 1, 2, \dots. \tag{2.4}$$

Proof. It is easy to check the validity of the following equalities:

$$(U^j_{\overset{\circ}{t}}, U^{j+1} + U^{j-1}) = \frac{1}{2\tau}(\|U^{j+1}\|^2 - \|U^{j-1}\|^2),$$

$$(U^0_t, U^1 + U^0) = \frac{1}{\tau}(\|U^1\|^2 - \|U^0\|^2),$$

$$-(U^j_{\bar{x}x\overset{\circ}{t}}, U^{j+1} + U^{j-1}) = \frac{1}{2\tau}(\|U^{j+1}_{\bar{x}}]|^2 - \|U^{j-1}_{\bar{x}}]|^2),$$

$$-(U^0_{\bar{x}xt}, U^1 + U^0) = \frac{1}{\tau}(\|U^1_{\bar{x}}]|^2 - \|U^0_{\bar{x}}]|^2),$$

$$((U^{j+1} + U^{j-1})_{\overset{\circ}{x}}, U^{j+1} + U^{j-1}) = 0, \quad ((U^1 + U^0)_{\overset{\circ}{x}}, U^1 + U^0) = 0,$$

$$(\Lambda U^j, U^{j+1} + U^{j-1}) = 0, \quad (\Lambda U^0, U^1 + U^0) = 0. \tag{2.5}$$

Multiplying eq. (2.1) by $(U^{j+1} + U^{j-1})$ and eq. (2.2) by $(U^1 + U^0)$ and summing over i, we respectively obtain:

$$\|U^{j+1}\|^2 + \mu\|U^{j+1}_{\bar{x}}]|^2 = \|U^{j-1}\|^2 + \mu\|U^{j-1}_{\bar{x}}]|^2,\ j = 1, 2, \dots,$$

and

$$\|U^1\|^2 + \mu\|U^1_{\bar{x}}]|^2 = \|U^0\|^2 + \mu\|U^0_{\bar{x}}]|^2.$$

From these equalities follows

$$\|U^j\|^2 + \mu\|U^j_{\bar{x}}]|^2 = \|U^0\|^2 + \mu\|U^0_{\bar{x}}]|^2,\ j = 1, 2, \dots, \tag{2.6}$$

which proves (2.4). □

On the basis of Theorem 2.1 we conclude that the problem (2.1)–(2.3) is uniquely solvable.

Theorem 2.2. *For solution of difference scheme* (2.1)–(2.3) *the estimates*

$$\|U^j\|_\infty \le c\|u_0'\|_{L_2(a,b)}, \quad \|U^j_{\bar{x}}]| \le c\|u_0'\|_{L_2(a,b)}$$

are valid, where the constant $c > 0$ *is independent of* h *and* τ.

Proof. According to (2.6),

$$\|U^j\|^2 + \mu\|U^j_{\bar{x}}]|^2 = \|u_0\|^2 + \mu\|u_{0\bar{x}}]|^2,\ j = 1, 2, \dots. \tag{2.7}$$

Since

$$|(u_{0\bar{x}})_i|^2 = \Big(\frac{1}{h}\int\limits_{x_{i-1}}^{x_i} u_0'(x)\,dx\Big)^2 \le \frac{1}{h}\int\limits_{x_{i-1}}^{x_i} (u_0')^2 dx,$$

the equality holds

$$\|u_{0\bar{x}}]|^2 = \sum_{i=1}^{n} h\Big((u_{0\bar{x}})_i\Big)^2 \le \int_a^b (u_0')^2 dx$$

holds. Taking into account the relation $\|u_0\|^2 \le (b-a)^2/8\,\|u_{0\bar{x}}]|^2$, we find from (2.7) that

$$\|U^j\|^2 + \mu\|U^j_{\bar{x}}]|^2 \le c\|u_0'\|^2_{L_2(a,b)}.$$

From this follows the second inequality of the theorem. Noticing now that $\|U^j\|_\infty \le 0,5\sqrt{b-a}\|U^j_{\bar{x}}]|$, we obtain the first inequality of the theorem.

Thus Theorem 2.2 is proved. □

3. A Priori Estimate of a Discretization Error

Let $Z := U - u$, where u is the exact solution of problem (1.1), (1.2) and U is the solution of the finite difference scheme (2.1)–(2.3). Substituting $U = Z + u$ into (2.1)–(2.3), we obtain the following problem for the error Z:

$$\begin{aligned}(Z_i^j)_{\overset{\circ}{t}} + \frac{1}{2}(Z_i^{j+1} + Z_i^{j-1})_{\overset{\circ}{x}} - \mu(Z_i^j)_{\bar{x}x\overset{\circ}{t}} &= \\ = -\frac{\gamma}{8}(\Lambda U_i^j - \Lambda u_i^j) - \mathcal{L}u_i^j,\ j = 1, 2, \dots& \qquad (3.1)\end{aligned}$$

$$(Z_i^0)_t + \frac{1}{2}(Z_i^1 + Z_i^0)_{\overset{\circ}{x}} - \mu(Z_i^0)_{\bar{x}xt} = -\frac{\gamma}{8}(\Lambda U_i^0 - \Lambda u_i^0) - \mathcal{L}u_i^0, \qquad (3.2)$$

$$Z_i^0 = 0, \quad i = 0, 1, \dots, n, \quad Z_0^j = Z_n^j = 0, \quad j = 0, 1, \dots, J. \qquad (3.3)$$

Denote

$$B^j := \|Z^j\|^2 + \|Z^{j-1}\|^2 + \mu\|Z^j_{\bar{x}}]|^2 + \mu\|Z^{j-1}_{\bar{x}}]|^2,\ j = 1, 2, \dots. \qquad (3.4)$$

Via $Z^0 = 0$ we have $B^1 = \|Z^1\|^2 + \mu\|Z^1_{\bar{x}}]|^2$.

Lemma 3.1. *For the solution of the problem* (3.2), (3.3) *the identity*

$$B^1 = -(\tau\mathcal{L}u^0, Z^1). \qquad (3.5)$$

is valid.

Proof. Taking the inner product of (3.2) with $(Z^1 + Z^0)$, we obtain:

$$\begin{aligned}\frac{1}{\tau}(\|Z^1\|^2 - \|Z^0\|^2) + \frac{\mu}{\tau}(\|Z^1_{\bar{x}}]|^2 - \|Z^0_{\bar{x}}]|^2) = -\frac{\gamma}{8}(\Lambda U^0 - \Lambda u^0, Z^1 + Z^0)-\\ -(\mathcal{L}u^0, Z^1 + Z^0). \qquad (3.6)\end{aligned}$$

It is easy to verify that

$$(\Lambda U^0 - \Lambda u^0, Z^1 + Z^0) = -(\Lambda u^0, U^1 + U^0) - (\Lambda U^0, u^1 + u^0). \qquad (3.7)$$

Applying summation by parts, we get

$$(\Lambda u^0, U^1 + U^0) = ((u^0)^2(u^1 + u^0)_{\overset{\circ}{x}}, U^1 + U^0) - ((u^0)^2(u^1 + u^0), (U^1 + U^0)_{\overset{\circ}{x}}),$$

$$(\Lambda U^0, u^1+u^0) = ((U^0)^2(U^1+U^0)_{\overset{\circ}{x}}, u^1+u^0) - ((U^0)^2(U^1+U^0), (u^1+u^0)_{\overset{\circ}{x}})$$

and via $U^0 = u^0$ (3.7) yields

$$(\Lambda U^0 - \Lambda u^0, Z^1 + Z^0) = 0. \tag{3.8}$$

According to $Z^0 = 0$ and eq. (3.8), the validity of Lemma 3.1 is obtained from (3.6). □

Lemma 3.2. *For the solution of the problem* (3.1)–(3.2) *the estimate*

$$B^{j+1} \le B^1 + \frac{\gamma\tau}{4}\sum_{k=1}^{j}|(\Lambda U^k - \Lambda u^k, Z^{k+1} + Z^{k-1})| +$$

$$+2\tau\sum_{k=1}^{j}|(\mathcal{L}u^k, Z^{k+1} + Z^{k-1})|,\ j = 1, 2, \dots, \tag{3.9}$$

is valid, where B^j *is defined by equality* (3.4).

Proof. Computing the inner product of (3.1) with $(Z^{j+1}+Z^{j-1})$, we obtain:

$$\frac{1}{2\tau}(\|Z^{j+1}\|^2 - \|Z^{j-1}\|^2) + \frac{\mu}{2\tau}(\|Z_{\bar{x}}^{j+1}]|^2 - \|Z_{\bar{x}}^{j-1}]|^2) =$$

$$= -\frac{\gamma}{8}(\Lambda U^j - \Lambda u^j, Z^{j+1} + Z^{j-1}) - (\mathcal{L}u^j, Z^{j+1} + Z^{j-1}),\ j = 1, 2, \dots \tag{3.10}$$

Taking into account notations (3.4), we get from (3.10)

$$B^{j+1} \le B^j + \frac{\gamma\tau}{4}|(\Lambda U^j - \Lambda u^j, Z^{j+1} + Z^{j-1})| +$$

$$+2\tau|(\mathcal{L}u^j, Z^{j+1} + Z^{j-1})|,\ j = 1, 2, \dots.$$

by which Lemma 3.2 is proved. □

Now we intend to estimate the terms from the right-hand side of the inequality (3.9).

Lemma 3.3. *The inequalities*

$$|(\mathcal{L}u^k, Z^{k+1} + Z^{k-1})| \le \frac{1}{4T}(\|Z^{k+1}\|^2 + \|Z^{k-1}\|^2) + 2T\|\mathcal{L}u^k\|^2 \tag{3.11}$$

$$|(\Lambda U^k - \Lambda u^k, Z^{k+1} + Z^{k-1})| \le 2T\gamma\delta^4(2 + 1/\mu)\|Z^k\|^2 +$$

$$+\frac{2}{T\gamma}(\|Z^{k+1}\|^2 + \|Z^{k-1}\|^2) + \frac{4\mu}{T\gamma}(\|Z_{\bar{x}}^{k+1}]|^2 + \|Z_{\bar{x}}^{k-1}]|^2) \tag{3.12}$$

are valid.

Proof. We have

$$|(\mathcal{L}u^k, Z^{k+1}+Z^{k-1})| \le \|\mathcal{L}u^k\|(\|Z^{k+1}\|+\|Z^{k-1}\|) \le$$

$$\le \varepsilon(\|Z^{k+1}\|^2+\|Z^{k-1}\|^2)+(1/(2\varepsilon))\|\mathcal{L}u^k\|^2.$$

Choosing here $\varepsilon = 1/(4T)$, we obtain eq. (3.11).

It is not difficult to verify that

$$\Lambda U^k - \Lambda u^k = \Lambda Z^k + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4, \tag{3.13}$$

where

$$\mathcal{R}_1 := 2Z^k u^k (Z^{k+1}+Z^{k-1})_{\overset{\circ}{x}} + 2\big(Z^k u^k (Z^{k+1}+Z^{k-1})\big)_{\overset{\circ}{x}},$$

$$\mathcal{R}_2 := (u^k)^2 (Z^{k+1}+Z^{k-1})_{\overset{\circ}{x}} + \big((u^k)^2 (Z^{k+1}+Z^{k-1})\big)_{\overset{\circ}{x}},$$

$$\mathcal{R}_3 := (Z^k)^2 (u^{k+1}+u^{k-1})_{\overset{\circ}{x}} + 2Z^k u^k (u^{k+1}+u^{k-1})_{\overset{\circ}{x}},$$

$$\mathcal{R}_4 := \big((Z^k)^2 (u^{k+1}+u^{k-1})\big)_{\overset{\circ}{x}} + 2\big(Z^k u^k (u^{k+1}+u^{k-1})\big)_{\overset{\circ}{x}}.$$

The following identities are true:

$$(\mathcal{R}_1, Z^{k+1}+Z^{k-1}) = 0, \quad (\mathcal{R}_2, Z^{k+1}+Z^{k-1}) = 0.$$

Moreover,

$$|(\mathcal{R}_3, Z^{k+1}+Z^{k-1})| = |\big(Z^k(Z^{k+1}+Z^{k-1}), (u^{k+1}+u^{k-1})_{\overset{\circ}{x}}(U^k+u^k)\big)| \le$$

$$\le 4\delta^2\|Z^k\|(\|Z^{k+1}\|+\|Z^{k-1}\|) \le 4T\gamma\delta^4\|Z^k\|^2 + \frac{2}{T\gamma}(\|Z^{k+1}\|^2+\|Z^{k-1}\|^2);$$

$$|(\mathcal{R}_4, Z^{k+1}+Z^{k-1})| = |\big(Z^k(Z^{k+1}+Z^{k-1})_{\overset{\circ}{x}}, (u^{k+1}+u^{k-1})(U^k+u^k)\big)| \le$$

$$\le 4\delta^2\|Z^k\|(\|Z^{k+1}_{\bar{x}}]|+\|Z^{k-1}_{\bar{x}}]|) \le \frac{2T\gamma\delta^4}{\mu}\|Z^k\|^2 + \frac{4\mu}{T\gamma}(\|Z^{k+1}_{\bar{x}}]|^2+\|Z^{k-1}_{\bar{x}}]|^2).$$

Via these inequalities, from (3.13) follows (3.12), since $(\Lambda Z^k, Z^{k+1} + Z^{k-1}) = 0$. □

Theorem 3.4. *For the solution of the problem* (3.1)–(3.3) *the estimates*

$$\|Z^1\|^2 + \mu\|Z^1_{\bar{x}}]|^2 \le \|\tau\mathcal{L}u^0\|^2, \tag{3.14}$$

$$\|Z^{j+1}\|^2 + \mu\|Z^{j+1}_{\bar{x}}]|^2 \le$$

$$\le 2\exp(c_1 T)(\|\tau\mathcal{L}u^0\|^2 + 4T\tau\sum_{k=1}^{j}\|\mathcal{L}u^k\|^2), \quad j = 1, 2, \dots, J-1, \tag{3.15}$$

hold, where $c_1 := (4/T) + T\delta^4\gamma^2(2 + 1/\mu)$.

Proof. On the basis of (3.11) and (3.12), we find from (3.9) that

$$B^{j+1} \le B^1 + \frac{\gamma\tau}{4}\sum_{k=1}^{j}\Bigg(\frac{2}{T\gamma}(\|Z^{k+1}\|^2 + \|Z^{k-1}\|^2) +$$

$$+2T\gamma\delta^4(2+1/\mu)\|Z^k\|^2 + \frac{4\mu}{T\gamma}(\|Z_{\bar{x}}^{k+1}]|^2 + \|Z_{\bar{x}}^{k-1}]|^2\Bigg) +$$

$$+2\tau\sum_{k=1}^{j}\Bigg(\frac{1}{4T}(\|Z^{k+1}\|^2 + \|Z^{k-1}\|^2) + 2T\|\mathcal{L}u^k\|^2\Bigg).$$

Thus

$$B^{j+1} \le B^1 + \frac{\tau}{T}\big(\|Z^{j+1}\|^2 + \mu\|Z_{\bar{x}}^{j+1}]|^2\big) +$$

$$+\Bigg(\frac{2}{T} + \frac{T\gamma^2\delta^4(2+1/\mu)}{2}\Bigg)\tau\sum_{k=1}^{j}\Bigg(\|Z^k\|^2 + \mu\|Z_{\bar{x}}^{k}]|^2\Bigg) + 4T\tau\sum_{k=1}^{j}\|\mathcal{L}u^k\|^2.$$

Taking into account that $\tau/T \le 0.5$, we get

$$B^{j+1} \le 2B^1 + c_1\tau\sum_{k=1}^{j}B^k + 8\tau T\sum_{k=1}^{j}\|\mathcal{L}u^k\|^2. \tag{3.16}$$

Let the inequalities

$$B^{j+1} \le 2B^1 + c\tau\sum_{k=1}^{j}B^k + b\tau\sum_{k=1}^{j}f^k, \quad j = 1, 2, \dots$$

be valid, where b, c, τ, B^k, f^k are nonnegative values. Then

$$B^{j+1} \le 2(1+c\tau)^{j-1}B^1 + c\tau(1+c\tau)^{j-1}B^1 + b\tau\sum_{k=1}^{j}(1+c\tau)^{j-k}f^k,$$

and therefore

$$B^{j+1} \le 2(1+c\tau)^{j}B^1 + b\tau(1+c\tau)^{j}\sum_{k=1}^{j}f^k.$$

If $j = 1, 2, \dots, J, \quad J := T/\tau$, then

$$(1+c\tau)^j < (1+c\tau)^{T/\tau} = (1+c\tau)^{(Tc)/(c\tau)} < e^{cT}.$$

Thus

$$B^{j+1} \le e^{cT}\Bigg(2B^1 + b\tau\sum_{k=1}^{j}f^k\Bigg).$$

Hence (3.16) yields

$$B^{j+1} \le e^{c_1T}\Bigg(2B^1 + 8T\tau\sum_{k=1}^{j}\|\mathcal{L}u^k\|^2\Bigg). \tag{3.17}$$

According to Lemma 3.1, we have

$$B^1 \leq 0.5\|Z^1\|^2 + 0.5\|\tau\mathcal{L}u^0\|^2 \leq 0.5B^1 + 0.5\|\tau\mathcal{L}u^0\|^2,$$

i.e. (3.14) is true. On the basis of the above inequality the estimate (3.15) follows from (3.17).

□

4. Estimation of Truncation Errors

In order to estimate the error, we will use Steklov averaging operators:

$$(\hat{\mathcal{P}}u)_i := \frac{1}{h}\int_{x_i}^{x_{i+1}} u(x,t)\,dx, \quad (\check{\mathcal{P}}u)_i := \frac{1}{h}\int_{x_{i-1}}^{x_i} u(x,t)\,dx,$$

$$(\mathring{\mathcal{P}}u)_i := 0.5(\hat{\mathcal{P}}u + \check{\mathcal{P}}u)_i, \quad (\mathcal{P}u)_i := \frac{1}{h^2}\int_{x_{i-1}}^{x_{i+1}} (h - |x_i - x|)u(x,t)\,dx,$$

$$(\hat{\mathcal{S}}u)^j := \frac{1}{\tau}\int_{t_j}^{t_{j+1}} u(x,t)\,dt, \quad (\check{\mathcal{S}}u)^j := \frac{1}{\tau}\int_{t_{j-1}}^{t_j} u(x,t)\,dt,$$

$$(\mathring{\mathcal{S}}u)^j := 0.5(\hat{\mathcal{S}}u + \check{\mathcal{S}}u)^j, \quad (\mathcal{S}u)^j := \frac{1}{\tau^2}\int_{t_{j-1}}^{t_{j+1}} (\tau - |t - t_j|)u(x,t)\,dt.$$

Let $\mathcal{I}u := u$.

For the averaging operators the following is true at the mesh points (x_i, t_j):

$$\mathcal{I} - \mathcal{P} = \frac{1}{6h}\hat{\mathcal{P}}(x - x_{i+1})^3\frac{\partial^2}{\partial x^2} - \frac{1}{6h}\check{\mathcal{P}}(x - x_{i-1})^3\frac{\partial^2}{\partial x^2}, \tag{4.1}$$

$$\mathcal{I} - \mathring{\mathcal{S}} = 0.5\check{\mathcal{S}}(t - t_{j-1})\frac{\partial}{\partial t} + 0.5\hat{\mathcal{S}}(t - t_{j+1})\frac{\partial}{\partial t}, \tag{4.2}$$

$$\begin{aligned}\mathring{\mathcal{P}} - \mathcal{P} = {} & \frac{1}{12h}\check{\mathcal{P}}(x - x_{i-1})^2(x_i + x_{i+1} - 2x)\frac{\partial^2}{\partial x^2} + \\ & + \frac{1}{12h}\hat{\mathcal{P}}(x - x_{i+1})^2(2x - x_i - x_{i-1})\frac{\partial^2}{\partial x^2}.\end{aligned} \tag{4.3}$$

Represent the truncation error in a convenient form.

Lemma 4.1. *If u is a solution to the problem* (1.1), (1.2), *then*

$$\mathcal{L}u = \psi_{(1)}\Big(\frac{\partial u}{\partial t}\Big) + \psi_{(2)}\Big(\frac{\partial u}{\partial x}\Big) + \frac{\gamma}{8}\psi_{(3)}(u) + \gamma\psi_{(2)}\Big((u)^2\frac{\partial u}{\partial x}\Big), \quad t > 0, \tag{4.4}$$

$$\mathcal{L}u = \Phi_{(1)}\Big(\frac{\partial u}{\partial t}\Big) + \Phi_{(2)}\Big(\frac{\partial u}{\partial x}\Big) + \frac{\gamma}{8}\Phi_{(3)}(u) + \gamma\Phi_{(2)}\Big((u)^2\frac{\partial u}{\partial x}\Big), \quad t = 0, \tag{4.5}$$

where

$$\psi_{(1)}(u) := \overset{\circ}{\mathcal{S}} (\mathcal{I} - \mathcal{P})u, \tag{4.6}$$

$$\psi_{(2)}(u) := 0.5 \overset{\circ}{\mathcal{P}} (\hat{u} + \check{u}) - \mathcal{P} \overset{\circ}{\mathcal{S}} u, \tag{4.7}$$

$$\psi_{(3)}(u) := \tau^2 (u)^2 u_{\overset{\circ}{x}\bar{t}t} + \tau^2 \big((u)^2 u_{\bar{t}t}\big)_{\overset{\circ}{x}} -$$

$$-h^2 u_{\overset{\circ}{x}} (u)^2_{\bar{x}x} - (4\tau^2/3)(u)^3_{\overset{\circ}{x}\bar{t}t} + \frac{4h^2}{3} (u_{\overset{\circ}{x}})^3, \tag{4.8}$$

$$\Phi_{(1)}(u) := \hat{\mathcal{S}}(\mathcal{I} - \mathcal{P})u, \tag{4.9}$$

$$\Phi_{(2)}(u) := 0.5 \overset{\circ}{\mathcal{P}} (u^1 + u^0) - \mathcal{P}\hat{\mathcal{S}}u, \tag{4.10}$$

$$\Phi_{(3)}(u) := \tau (u)^2 u_{\overset{\circ}{x}t} + \tau((u)^2 u_t)_{\overset{\circ}{x}} - h^2 u_{\overset{\circ}{x}} (u)^2_{\bar{x}x} - \frac{4\tau}{3} (u)^3_{\overset{\circ}{x}t} + \frac{4h^2}{3} (u_{\overset{\circ}{x}})^3. \tag{4.11}$$

Proof. For the sake of simplicity, we use the notation: $\hat{u} := u^{j+1}$, $\check{u} := u^{j-1}$, $u := u^j$. If we note that $\hat{u} + \check{u} = 2u + \tau^2 u_{\bar{t}t}$, then for $t > 0$ the expression Λu will have the following form:

$$\Lambda u = 2(u)^2 u_{\overset{\circ}{x}} + \tau^2 (u)^2 u_{\overset{\circ}{x}\bar{t}t} + 2(u)^3_{\overset{\circ}{x}} + \tau^2 ((u)^2 u_{\bar{t}t})_{\overset{\circ}{x}}.$$

Since

$$2(u)^2 u_{\overset{\circ}{x}} = \frac{2}{3} (u)^3_{\overset{\circ}{x}} - h^2 u_{\overset{\circ}{x}} (u)^2_{\bar{x}x} + \frac{4h^2}{3} \big(u_{\overset{\circ}{x}}\big)^3,$$

we have

$$\Lambda u = \frac{8}{3} \big((u)^3\big)_{\overset{\circ}{x}} + \tau^2 (u)^2 u_{\overset{\circ}{x}\bar{t}t} + \tau^2 \big((u)^2 u_{\bar{t}t}\big)_{\overset{\circ}{x}} + \frac{4h^2}{3} (u_{\overset{\circ}{x}})^3 - h^2 u_{\overset{\circ}{x}} (u)^2_{\bar{x}x}.$$

If here we change

$$(u)^3_{\overset{\circ}{x}} = \overset{\circ}{\mathcal{P}} \big(\frac{\partial (u)^3}{\partial x}\big) = 0.5 \overset{\circ}{\mathcal{P}} \frac{\partial}{\partial x} \Big((\hat{u})^3 + (\check{u})^3\Big) - 0.5 \tau^2 (u)^3_{\overset{\circ}{x}\bar{t}t},$$

then

$$\Lambda u = 4 \overset{\circ}{\mathcal{P}} \big((\hat{u})^2 \frac{\partial \hat{u}}{\partial x} + (\check{u})^2 \frac{\partial \check{u}}{\partial x}\big) + \psi_{(3)}(u). \tag{4.12}$$

Taking into account the differential equation (1.1), we get

$$\mu u_{\bar{x}x\overset{\circ}{t}} = \mu \mathcal{P} \overset{\circ}{\mathcal{S}} \frac{\partial^3 u}{\partial x^2 \partial t} = \mathcal{P} \overset{\circ}{\mathcal{S}} \big(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \gamma u^2 \frac{\partial u}{\partial x}\big).$$

Applying this equality and eq. (4.12), we obtain

$$\mathcal{L}u = u_{\overset{\circ}{t}} + \frac{1}{2} (\hat{u} + \check{u})_{\overset{\circ}{x}} + \frac{\gamma}{8} \Lambda u - \mu u_{\bar{x}x\overset{\circ}{t}} =$$

$$= \overset{\circ}{\mathcal{S}} \frac{\partial u}{\partial t} + \frac{1}{2} \overset{\circ}{\mathcal{P}} \frac{\partial}{\partial x} (\hat{u} + \check{u}) + \frac{\gamma}{2} \overset{\circ}{\mathcal{P}} \big((\hat{u})^2 \frac{\partial \hat{u}}{\partial x} + (\check{u})^2 \frac{\partial \check{u}}{\partial x}\big) + \frac{\gamma}{8} \psi_{(3)}(u) -$$

$$- \mathcal{P} \overset{\circ}{\mathcal{S}} \big(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \gamma (u)^2 \frac{\partial u}{\partial x}\big).$$

which proves the validity of eq. (4.4).

For $t = 0$, the expression Λu will be of the form

$$\Lambda u = \frac{8}{3}(u)^3_{\overset{\circ}{x}} - h^2 u_{\overset{\circ}{x}}(u)^2_{\bar{x}x} + \tau(u)^2 u_{\overset{\circ}{x}t} + \tau\big((u)^2 u_t\big)_{\overset{\circ}{x}} + \frac{4h^2}{3}(u_{\overset{\circ}{x}})^3$$

or, analogously to the previous case,

$$\Lambda u = 4\,\overset{\circ}{\mathcal{P}}\,\Big((u)^2\frac{\partial(u)^2}{\partial x} + (\hat{u})^2\frac{\partial(\hat{u})^2}{\partial x}\Big) + \Phi_{(3)}(u), \quad t = 0. \tag{4.13}$$

Taking into account the differential equation (1.1), we get

$$\mu u_{\bar{x}xt} = \mathcal{P}\hat{\mathcal{S}}\Big(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \gamma(u)^2\frac{\partial u}{\partial x}\Big).$$

Applying this equality and eq. (4.13), we obtain eq.(4.5). □

Now estimate the terms in the right-hand sides of (4.4), (4.5).

We denote

$$e_{i0} = \{(x,t)|\ |x - x_i| \le h,\ 0 \le t \le \tau\},$$

$$e_{ij} = \{(x,t)|\ |x - x_i| \le h,\ |t - t_j| \le \tau\},\quad j = 1, 2, \dots.$$

Lemma 4.2. *For $\Phi_{(\alpha)}$ defined from (4.9)–(4.11), the following estimates are valid:*

$$\|\Phi_{(1)}(u)\|^2 \le c\frac{h^4}{\tau}\int_a^b\int_0^\tau \Big|\frac{\partial^2 u}{\partial x^2}\Big|^2\,dxdt, \tag{4.14}$$

$$\|\Phi_{(2)}(u)\|^2 \le c\frac{h^4+\tau^4}{\tau}\int_a^b\int_0^\tau \Big(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial t^2}\Big|^2\Big)\,dxdt, \tag{4.15}$$

$$\|\Phi_{(3)}(u)\|^2 \le c\tau^2\|u\|^2_{C^1} + c\frac{h^4+\tau^4}{\tau^2}\times$$
$$\times\int_{Q_T}\Big(\Big|\frac{\partial^3 u}{\partial x\partial t^2}\Big|^2 + \Big|\frac{\partial^3 u}{\partial x^2\partial t}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial t^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x\partial t}\Big|^2\Big)\,dxdt, \tag{4.16}$$

where the constant $c > 0$ does not depend on mesh steps.

Proof. We have

$$\Phi_{(1)}(u) = \frac{1}{6\tau h^2}\int_0^\tau\Big[\int_{x_i}^{x_{i+1}}(x - x_{i+1})^3\frac{\partial^2 u}{\partial x^2}\,dx - \int_{x_{i-1}}^{x_i}(x - x_{i-1})^3\frac{\partial^2 u}{\partial x^2}\,dx\Big]dt.$$

Therefore

$$|\Phi_{(1)}(u)|^2 \le \frac{h^3}{18\tau}\int_0^\tau\int_{x_{i-1}}^{x_{i+1}}\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2\,dxdt,$$

and we get (4.14):

$$\|\Phi_{(1)}(u)\|^2 = \sum_i h|\Phi_{(1)}(u)|^2 \le \frac{h^4}{18\tau}\int\limits_a^b\int\limits_0^\tau \left|\frac{\partial^2 u}{\partial x^2}\right|^2 dxdt.$$

The identity

$$0.5(u^1+u^0) = \hat{\mathcal{S}}u - \frac{1}{2\tau}\int\limits_0^\tau t(t-\tau)\frac{\partial^2 u}{\partial t^2}\,dt \tag{4.17}$$

holds, therefore

$$\Phi_{(2)}(u) = (\overset{\circ}{\mathcal{P}} - \mathcal{P})\hat{\mathcal{S}}u - \frac{1}{2\tau}\overset{\circ}{\mathcal{P}}\int\limits_0^\tau t(t-\tau)\frac{\partial^2 u}{\partial t^2}\,dt,$$

and

$$|\Phi_{(2)}(u)| \le \frac{h}{12\tau}\int\limits_{x_{i-1}}^{x_{i+1}}\int\limits_0^\tau \left|\frac{\partial^2 u}{\partial x^2}\right| dxdt + \frac{\tau}{16h}\int\limits_{x_{i-1}}^{x_{i+1}}\int\limits_0^\tau \left|\frac{\partial^2 u}{\partial t^2}\right| dxdt$$

which proves (4.15).

In order to estimate the terms of $\Phi_{(3)}$ we will use the inequality

$$\left|\int\limits_0^\tau v\,dt\right| \le \frac{\tau}{T}\int\limits_0^T |v|\,dt + \tau\int\limits_0^T \left|\frac{\partial v}{\partial t}\right| dt,$$

which follows from

$$T\int\limits_0^\tau v(x,t)\,dt = \tau\int\limits_0^T v(x,t)\,dt + \int\limits_0^\tau dt\int\limits_0^T dt'\int\limits_{t'}^t \frac{\partial v(x,t'')}{\partial t}\,dt''.$$

We have

$$|u_{\overset{\circ}{x}t}| \le \frac{1}{2\tau h}\int\limits_{x_{i-1}}^{x_{i+1}}\left|\int\limits_0^\tau \frac{\partial^2 u}{\partial x\partial t}\,dxdt\right|,$$

from which

$$|u_{\overset{\circ}{x}t}| \le \frac{1}{2Th}\int\limits_{x_{i-1}}^{x_{i+1}}\int\limits_0^T \left|\frac{\partial^2 u}{\partial x\partial t}\right| dxdt + \frac{1}{2h}\int\limits_{x_{i-1}}^{x_{i+1}}\int\limits_0^T \left|\frac{\partial^3 u}{\partial x\partial t^2}\right| dxdt$$

and

$$\|u_{\overset{\circ}{x}t}\| \le \frac{1}{\sqrt{2T}}\left\|\frac{\partial^2 u}{\partial x\partial t}\right\|_{L_2(Q_T)} + \sqrt{\frac{T}{2}}\left\|\frac{\partial^3 u}{\partial x\partial t^2}\right\|_{L_2(Q_T)}.$$

Analogously,

$$\|(u)^3_{\overset{\circ}{x}t}\| \le \frac{1}{\sqrt{2T}}\Big\|\frac{\partial^2 (u)^3}{\partial x\partial t}\Big\|_{L_2(Q_T)} + \sqrt{\frac{T}{2}}\Big\|\frac{\partial^3 (u)^3}{\partial x\partial t^2}\Big\|_{L_2(Q_T)}.$$

Note that

$$\big((u)^2 u_t\big)_{\overset{\circ}{x}} = (u_{i+1}+u_{i-1})u_{\overset{\circ}{x},i}u_{t,i+1} + (u_{i-1})^2 u_{t\overset{\circ}{x},i},$$

therefore

$$\|\big((u)^2 u_t\big)_{\overset{\circ}{x}}\| \le 2\delta^3 + \delta^2\|u_{t\overset{\circ}{x}}\|.$$

It is easily seen that

$$|(u)^2_{\bar{x}x}| = \big|(u)^2_{x,i} - (u)^2_{\bar{x},i}\big|/h \le 2\delta^2/h, \quad |(u_{\overset{\circ}{x}})^3| \le \delta^3.$$

Applying these inequalities to the estimation of $\Phi_{(3)}$, we get (4.16). □

Lemma 4.3. *For $\psi_{(\alpha)}$ defined from equalities (4.6)–(4.8), the estimates*

$$\|\psi_{(1)}(u^j)\|^2 \le c\frac{h^4}{\tau}\int_a^b\int_{t_{j-1}}^{t_{j+1}} \Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 dxdt, \tag{4.18}$$

$$\|\psi_{(2)}(u^j)\|^2 \le c\frac{h^4+\tau^4}{\tau}\int_a^b\int_{t_{j-1}}^{t_{j+1}} \Big(\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial t^2}\Big|^2\Big) dxdt, \tag{4.19}$$

$$\|\psi_{(3)}(u^j)\|^2 \le c\frac{h^4+\tau^4}{\tau}\times$$

$$\times\int_a^b\int_{t_{j-1}}^{t_{j+1}} \Big(\Big|\frac{\partial^3 u}{\partial x\partial t^2}\Big|^2 + \Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 + \Big|\frac{\partial^3 u}{\partial x^2\partial t}\Big|^2 + \Big|\frac{\partial^2 u}{\partial t^2}\Big|^2 + \Big|\frac{\partial u}{\partial x}\Big|^2\Big) dxdt, \tag{4.20}$$

are valid, where the constant $c > 0$ does not depend on mesh steps.

Proof. Via identity (4.1) we have

$$|\psi_{(1)}(u)| = |\overset{\circ}{\mathcal{S}}(\mathcal{P}-\mathcal{I})u| \le \frac{h}{12\tau}\int_{e_{ij}}\Big|\frac{\partial^2 u}{\partial x^2}\Big| dxdt \tag{4.21}$$

and therefore

$$\|\psi_{(1)}(u^j)\|^2 = \sum_{i=1}^{n-1} h|\psi_{(1)}(u_i^j)|^2 \le \frac{h^4}{36\tau}\sum_{i=1}^{n-1}\int_{e_{ij}}\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 dxdt \le$$

$$\le \frac{h^4}{18\tau}\int_a^b\int_{t_{j-1}}^{t_{j+1}}\Big|\frac{\partial^2 u}{\partial x^2}\Big|^2 dxdt,$$

which proves (4.18).

Now we rewrite $\psi_{(2)}(u)$ in the form

$$\psi_{(2)}(u) = (\overset{\circ}{\mathcal{P}} - \mathcal{P})\, \overset{\circ}{\mathcal{S}}\, u - 0.5\, \overset{\circ}{\mathcal{P}} \overset{\circ}{\mathcal{S}} \Big((t - t_{j-1})(t - t_{j+1}) \frac{\partial^2 u}{\partial t^2} \Big). \tag{4.22}$$

If we apply the result

$$|(\overset{\circ}{\mathcal{P}} - \mathcal{P})\, \overset{\circ}{\mathcal{S}}\, u| \leq \frac{h}{24\tau} \int\limits_{e_{ij}} \Big| \frac{\partial^2 u}{\partial x^2} \Big|\, dxdt, \tag{4.23}$$

obtained from identity (4.3), then we find from (4.22) that

$$|\psi_{(2)}(u)| \leq \frac{\tau}{8h} \int\limits_{e_{ij}} \Big| \frac{\partial^2 u}{\partial t^2} \Big|\, dxdt + \frac{h}{24\tau} \int\limits_{e_{ij}} \Big| \frac{\partial^2 u}{\partial x^2} \Big|\, dxdt.$$

Therefore

$$\|\psi_{(2)}(u^j)\|^2 = \sum_{i=1}^{n-1} h |\psi_{(2)}(u_i^j)|^2 \leq$$

$$\leq \sum_{i=1}^{n-1} \Big(\frac{\tau^3}{8} \int\limits_{e_{ij}} \big| \frac{\partial^2 u}{\partial t^2} \big|^2\, dxdt + \frac{h^4}{72\tau} \int\limits_{e_{ij}} \big| \frac{\partial^2 u}{\partial x^2} \big|^2\, dxdt \Big),$$

which proves (4.19).

And finally,

$$|u_{\overset{\circ}{x}\bar{t}t}| = \Big| \overset{\circ}{\mathcal{P}}\, \mathcal{S}\big(\frac{\partial^3 u}{\partial x \partial t^2} \big) \Big| \leq \frac{1}{2\tau h} \int\limits_{e_{ij}} \big| \frac{\partial^3 u}{\partial x \partial t^2} \big|\, dxdt,$$

$$\big| \big((u)^2 u_{\bar{t}t} \big)_{\overset{\circ}{x}} \big| = \Big| \overset{\circ}{\mathcal{P}}\, \frac{\partial}{\partial x} \Big((u)^2 \mathcal{S} \frac{\partial^2 u}{\partial t^2} \Big) \Big| \leq$$

$$\leq \frac{\delta^2}{\tau h} \int\limits_{e_{ij}} \big| \frac{\partial^2 u}{\partial t^2} \big|\, dxdt + \frac{\delta^2}{2\tau h} \int\limits_{e_{ij}} \big| \frac{\partial^3 u}{\partial x \partial t^2} \big|\, dxdt.$$

Note that

$$(u)^2_{\bar{x}x,i} = (u_{i+1} + u_{i-1}) u_{\bar{x}x,i} + 2 u_{\bar{x},i} u_{x,i}.$$

Therefore

$$|(u)^2_{\bar{x}x}| \leq \frac{\delta}{\tau h} \int\limits_{e_{ij}} \big| \frac{\partial^2 u}{\partial x^2} \big|\, dxdt + \frac{\delta}{h} \int\limits_{e_{ij}} \big| \frac{\partial^3 u}{\partial x^2 \partial t} \big|\, dxdt,$$

$$|(u)^3_{\overset{\circ}{x}\bar{t}t}| = \Big| \overset{\circ}{\mathcal{P}}\, \mathcal{S}\big(\frac{\partial^3 (u)^3}{\partial x \partial t^2} \big) \Big| \leq 12\delta^3 +$$

$$+ \frac{\delta^2}{2\tau h} \int\limits_{e_{ij}} \Big(6 \big| \frac{\partial^2 u}{\partial t^2} \big| + 12 \big| \frac{\partial^2 u}{\partial x \partial t} \big| + 3 \big| \frac{\partial^3 u}{\partial x \partial t^2} \big| \Big)\, dxdt,$$

$$|(u_{\overset{\circ}{x}})^3| \leq \delta^3.$$

Applying these inequalities to the estimation of $\Psi_{(3)}$, we get (4.20). □

Theorem 4.4. *Let the exact solution of the initial-boundary value problem* (1.1), (1.2) *belong to* $W_2^3(Q_T)$. *Then the discretization error of the finite difference scheme* (2.1)–(2.3) *is determined by the estimate*

$$\|Z^j\|_{W_2^1} \le c(\tau^2+h^2)\|u\|_{W_2^3(Q_T)}, \tag{4.24}$$

where c *denotes the positive constant, independent of* h *and* τ.

Proof. According to Lemmas 4.1 and 4.2,

$$\|\mathcal{L}u^j\|^2 \le c\frac{h^4+\tau^4}{\tau}\int_a^b\int_{t_{j-1}}^{t_{j+1}}\Big(\big|\frac{\partial^2 u}{\partial x^2}\big|^2+\big|\frac{\partial^2 u}{\partial x\partial t}\big|^2+$$

$$+\big|\frac{\partial^2 u}{\partial t^2}\big|^2+\big|\frac{\partial^3 u}{\partial x^3}\big|^2+\big|\frac{\partial^3 u}{\partial x^2\partial t}\big|^3+\big|\frac{\partial^3 u}{\partial x\partial t^2}\big|^2\Big)\,dxdt, \quad j=1,2,\dots$$

Via Lemmas 4.1 and 4.3, we get

$$\|\mathcal{L}u^0\|^2 \le c\frac{h^4+\tau^4}{\tau^2}\int_{Q_T}\Big(\big|\frac{\partial^2 u}{\partial x^2}\big|^2+\big|\frac{\partial^2 u}{\partial x\partial t}\big|^2+\big|\frac{\partial^2 u}{\partial t^2}\big|^2+$$

$$+\big|\frac{\partial^3 u}{\partial x^3}\big|^2+\big|\frac{\partial^3 u}{\partial x^2\partial t}\big|^3+\big|\frac{\partial^3 u}{\partial x\partial t^2}\big|^2\Big)\,dxdt.$$

Therefore Theorem 4.4 follows from Theorem 3.4. □

Remark. The obtained results will be valid even if we use the difference scheme

$$\tilde{\mathcal{L}}U_i^j := \mathcal{L}U_i^j + \sigma(U_i^j)_{\bar{x}x\mathring{t}} = 0, \quad j\ge 1, \tag{4.25}$$

$$\tilde{\mathcal{L}}U_i^0 := \mathcal{L}U_i^0 + \sigma(U_i^0)_{\bar{x}xt} = 0 \tag{4.26}$$

instead of (2.1), (2.2), where the parameter $\sigma = O(\tau^2+h^2)$ must be chosen so that $\mu-\sigma>0$.

5. Numerical Experiments

It is well-known (see, e.g. [9])that the MRLW equation (1.1) has three independent invariants (conservation laws) given by

$$I_1=\int_{-\infty}^{\infty} u\,dx, \quad I_2=\int_{-\infty}^{\infty}\Big(u^2+\mu\big(\frac{\partial u}{\partial x}\big)^2\Big)\,dx, \quad I_3=\int_{-\infty}^{\infty}\Big(u^4-\frac{6\mu}{\gamma}\big(\frac{\partial u}{\partial x}\big)^2\Big)\,dx$$

which correspond, respectively, to mass, momentum, and energy. These invariants help us to test the numerical schemes.

In numerical examples given below we mean that $u\to 0$ as $x\to\pm\infty$. The boundary values are only approximately equal to zero, and they vary when t increases. Therefore in numerical calculations it is advisable to use artificial boundary conditions selected according to certain reasons. Namely, in the role of an artificial boundary condition for $t > 0$ and $x = a,\ x = b$

Table 1.
Invariants and error norms for $c = 0.1$, $h = 0.1$, $\tau = 0.1$, $x_0 = 40$, over $[0; 80]$.

Time	Method	I_1	I_2	I_3	L_2-error	L_∞-error
$t = 0$		8.070 838	4.100 554	0.868 352		
$t = 1$	Present	8.070 835	4.100 541	0.868 429	1.318E-5	3.110E-5
	[9]	8.070 83	4.100 50	0.868 661	3.354E-5	6.130E-5
$t = 2$	Present	8.070 824	4.100 541	0.868 429	2.675E-5	6.249E-5
	[9]	8.070 82	4.100 50	0.868 659	6.749E-5	1.200E-4
$t = 3$	Present	8.070 804	4.100 542	0.868 429	4.012E-5	9.458E-5
	[9]	8.070 81	4.100 50	0.868 659	1.024E-4	1.805E-4
$t = 4$	Present	8.070 774	4.100 542	0.868 429	5.313E-5	1.276E-4
	[9]	8.070 78	4.100 50	0.868 659	1.314E-4	2.372E-4
$t = 5$	Present	8.070 731	4.100 542	0.868 429	6.576E-5	1.614E-4
	[9]	8.070 74	4.100 50	0.868 659	1.526E-4	2.860E-4
$t = 6$	Present	8.070 671	4.100 541	0.868 429	7.810E-5	1.958E-4
	[9]	8.070 68	4.100 50	0.868 660	1.703E-4	3.254E-4
$t = 7$	Present	8.070 588	4.100 541	0.868 429	9.426E-5	2.307E-4
	[9]	8.070 59	4.100 50	0.868 660	1.860E-4	3.701E-4
$t = 8$	Present	8.070 473	4.100 541	0.868 429	1.097E-4	2.657E-4
	[9]	8.070 47	4.100 50	0.868 660	1.989E-4	4.086E-4
$t = 9$	Present	8.070 314	4.100 541	0.868 429	1.245E-4	3.008E-4
	[9]	8.070 29	4.100 50	0.868 660	2.100E-4	4.451E-4
$t = 10$	Present	8.070 091	4.100 541	0.868 429	1.387E-4	3.358E-4
	[9]	8.070 04	4.100 50	0.868 659	2.200E-4	4.799E-4

We can use approximation of equality $\partial u/\partial t + \partial u/\partial x = 0$:

$$U_0^{j+1} = \big(1 + \frac{\tau}{h}\big)U_0^j - \frac{\tau}{h}U_1^j, \quad U_n^{j+1} = \big(1 - \frac{\tau}{h}\big)U_n^j + \frac{\tau}{h}U_{n-1}^j,$$

which corresponds to a negligible alteration in the directions x and t.

We consider the MRLW equation (1.1) with $\gamma = \mu = 1$ and the initial data

$$u(x,0) = \sqrt{6c}\, sech\,\big[k\big(x - x_0\big)\big], \quad k := \sqrt{\frac{c}{c+1}}.$$

Note that the exact solution of this problem is

$$u(x,t) = \sqrt{6c}\, sech\,\big[k\big(x - (c+1)t - x_0\big)\big].$$

For numerical experiments in (4.25), (4.26) we take a free parameter

$$\sigma = 0.34\tau^2 + 0.002\tau h + 0.18h^2.$$

Table 2.

Invariants and error norms for $c = 0.03$, $h = 0.2$, $\tau = 0.025$, $x_0 = 0$, over $[-40; 60]$.

Time	Method	I_1	I_2	I_3	L_2-error	L_∞-error
$t = 0$		7.804 305	2.129 885	0.130 251		
$t = 1$	Present	7.805 14	2.129 883	0.130 268	2.300E-6	2.737E-6
	[9]	7.805 13	2.129 88	0.130 307	5.219E-5	6.892E-5
$t = 2$	Present	7.805 84	2.129 883	0.130 268	6.260E-6	6.793E-6
	[9]	7.805 92	2.129 88	0.130 309	9.491E-5	1.386E-4
$t = 3$	Present	7.806 41	2.129 884	0.130 268	1.238E-5	1.3117E-5
	[9]	7.806 60	2.129 88	0.130 311	1.287E-4	2.101E-4
$t = 4$	Present	7.806 89	2.129 884	0.130 268	2.115E-5	2.271E-5
	[9]	7.807 20	2.129 88	0.130 312	1.543E-4	2.824E-4
$t = 5$	Present	7.807 28	2.129 884	0.130 268	3.303E-5	3.662E-5
	[9]	7.807 71	2.129 88	0.130 313	1.723E-4	3.553E-4
$t = 6$	Present	7.807 60	2.129 884	0.130 268	4.840E-5	5.589E-5
	[9]	7.808 16	2.129 88	0.130 314	1.836E-4	4.280E-4
$t = 7$	Present	7.807 87	2.129 884	0.130 268	6.758E-5	8.151E-5
	[9]	7.808 54	2.129 88	0.130 314	1.893E-4	4.992E-4
$t = 8$	Present	7.808 09	2.129 884	0.130 268	9.076E-5	1.144E-4
	[9]	7.808 86	2.129 88	0.130 315	1.933E-4	5.684E-4
$t = 9$	Present	7.808 27	2.129 884	0.130 268	1.180E-4	1.556E-4
	[9]	7.809 12	2.129 88	0.130 315	1.970E-4	6.349E-4
$t = 10$	Present	7.808 42	2.129 885	0.130 268	1.494E-4	2.056E-4
	[9]	7.809 32	2.129 88	0.130 315	1.995E-4	6.983E-4

Accuracy of the method at any time $t = j\tau$ is measured by the

$$L_2 := \|u^j - U^j\|, \quad L_\infty := \|u^j - U^j\|_\infty$$

error norms.

The conservation properties of the proposed method are examined by calculating the invariants over a region $[a, b]$.

Values of the three invariants as well as L_2 and L_∞-error norms are computed and recorded in Tables 1-3 for different values of time t.

Numerical results have shown the good conservation and accuracy properties of the proposed scheme.

Table 3.

Invariants and error norms for $c = 0.1$, $h = 0.01$, $\tau = 0.01$, $x_0 = 40$, over $[0; 80]$.

Time	I_1	I_2	I_3	L_2-error	L_∞-error
$t = 0$	8.070 838	4.100 554	0.868 352		
$t = 1$	8.070 835	4.100 554	0.868 353	3.585E-7	4.382E-7
$t = 2$	8.070 826	4.100 554	0.868 353	9.637E-7	1.056E-6
$t = 3$	8.070 809	4.100 554	0.868 353	1.894E-6	2.021E-6
$t = 4$	8.070 784	4.100 554	0.868 353	3.229E-6	3.489E-6
$t = 5$	8.070 747	4.100 554	0.868 353	5.044E-6	5.620E-6
$t = 6$	8.070 694	4.100 554	0.868 353	7.406E-6	8.578E-6
$t = 7$	8.070 619	4.100 554	0.868 353	1.037E-5	1.252E-5
$t = 8$	8.070 513	4.100 554	0.868 353	1.399E-5	1.763E-5
$t = 9$	8.070 364	4.100 554	0.868 353	1.829E-5	2.403E-5
$t = 10$	8.070 154	4.100 554	0.868 353	2.330E-5	3.190E-5

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