Proc. A. Razmadze Math. Inst. 159(2012), 134–137

G. Berikelashvili

ON THE FOURTH-ORDER ACCURATE DIFFERENCE SCHEME FOR POISSON'S EQUATION WITH NONLOCAL BOUNDARY CONDITION

Let $\Omega = \{(x_1, x_2) :\ 0 < x_k < 1,\ k = 1, 2\}$ be a square with the boundary Γ, and $\Gamma_1 = \{(1, x_2) :\ 0 < x_2 < 1\}$, $\Gamma_0 = \Gamma \backslash \Gamma_1$.

We consider a nonlocal boundary value problem with Bitsadze-Samarskii condition and with Dirichlet conditions on a part of the boundary for Poisson's equation

$$\Delta u = f(x),\ x \in \Omega,$$
$$u(x) = 0,\ x \in \Gamma_0, \quad u(1, x_2) = \alpha u(\xi, x_2), \quad 0 < x_2 < 1, \tag{1}$$

where $\xi, \alpha \in (0, 1)$ are fixed numbers.

Consider the following grid domains in $\bar{\Omega}$: $\bar{\omega}_\alpha = \{x_\alpha = i_\alpha h :\ i_\alpha = 0, 1, \dots, n,\ h = 1/n\}$, $\omega_\alpha = \bar{\omega}_\alpha \cap (0, 1)$, $\omega_\alpha^+ = \bar{\omega}_\alpha \cap (0, 1]$, $\alpha = 1, 2$, $\omega = \omega_1 \times \omega_2$, $\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2$, $\gamma = \bar{\omega} \cap \bar{\Omega}$, $\gamma_0 = \Gamma_0 \cap \bar{\omega}$.

For grid functions and difference ratios, we use the notation

$$v_{x_i} = (v^{(+1_i)} - v)/h, \quad v_{\bar{x}_i} = (v - v^{(-1_i)})/h,$$

where $v^{(\pm 1_1)}(x) = v(x_1 \pm h, x_2)$, $v^{(\pm 1_2)}(x) = v(x_1, x_2 \pm h)$. For simplicity let us assume, that the index written below grid functions corresponds to the first coordinate: $y_i = y(ih, x_2)$.

Let us introduce the weighted inner product and the norm

$$(y, v)_r = \sum_{\omega} h^2 r y v, \quad \|y\|_r = (y, y)_r^{1/2},$$
$$\|y\|^2_{W_2^1(\omega, r)} = \|y_{\bar{x}_1}\|_r^2 + \|y_{\bar{x}_2}\|_r^2, \quad r = 1 - x_1.$$

Assume that the inner product and the norm containing the index ρ have the analogously meaning.

Let

$$\xi = (k + \theta) h, \quad 0 \le \theta < 1,$$

where k is positive integer, $2 \le k < n - 2$.

2010 *Mathematics Subject Classification.* 65N06, 35J25.

Key words and phrases. Bitsadze-Samarskii condition, difference scheme, fourth-order accuracy.

We approximate problem (1) by the difference scheme

$$y_{\bar{x}_1x_1} + y_{\bar{x}_2x_2} + \frac{h^2}{6} y_{\bar{x}_1x_1\bar{x}_2x_2} = \varphi(x), \quad x \in \omega, \tag{2}$$

$$y = 0, \quad x \in \gamma_0, \quad y(1, x_2) = \alpha Y(x_2), \quad x_2 \in \omega_2, \tag{3}$$

where φ is some average of the right-hand side of equation (1) and

$$Y(x_2) : = \frac{(1+\theta)(2-\theta)}{2}\big((1-\theta)y_k + \theta y_{k+1}\big) - \\ -\frac{\theta(1-\theta)}{6}\big((1+\theta)y_{k+2} + (2-\theta)y_{k-1}\big). \tag{4}$$

Significant moment in obtaining the main result is the selection of the weight function and establishment of estimate for the $(y_{\bar{x}_1x_1}, y)_\rho$.

Lemma 1. *For any $\theta \in [0, 1]$ the estimate*

$$Y^2 \le \frac{45}{32}\big((1-\theta)y_k^2 + \theta y_{k+1}^2\big) + \frac{5}{96}\big((1+\theta)y_{k+2}^2 + (2-\theta)y_{k-1}^2\big) \tag{5}$$

is valid.

Proof. Let $A : = (1-\theta)y_k + \theta y_{k+1}$ and $B : = (1+\theta)y_{k+2} + (2-\theta)y_{k-1}$. Then

$$|Y| \le \frac{9}{8}|A| + \frac{1}{24}|B|.$$

Consequently,

$$Y^2 \le \frac{90}{64}A^2 + \frac{10}{24^2}B^2. \tag{6}$$

Moreover, from equations

$$(1-\theta)a^2 + \theta b^2 - \big((1-\theta)a + \theta b\big)^2 = (1-\theta)\theta(a-b)^2,$$

$$3(1+\theta)a^2 + 3(2-\theta)b^2 - \big((1+\theta)a + (2-\theta)b\big)^2 = (1+\theta)(2-\theta)(a-b)^2$$

it follows that

$$\big((1-\theta)a + \theta b\big)^2 \le (1-\theta)a^2 + \theta b^2,$$

$$\big((1+\theta)a + (2-\theta)b\big)^2 \le 3(1+\theta)a^2 + 3(2-\theta)b^2.$$

Thus from (6) we get

$$Y^2 \le \frac{90}{64}\big((1-\theta)y_k^2 + \theta y_{k+1}^2\big) + \frac{10}{24^2}\big(3(1+\theta)y_{k+2}^2 + 3(2-\theta)y_{k-1}^2\big),$$

which proves the validity of Lemma 1. □

Let us pass now to the construction of a weighted function. Let

$$\beta_i = \begin{cases} (2-\theta)ih + \theta + 1 - 3\xi, & i \le k-1, \\ (1+\theta)\big(1-(k+2)h\big), & i = k, k+1, \\ (\theta+1)(1-ih), & i \ge k+2, \end{cases}$$

$$\gamma_i = \begin{cases} (1-\sigma)ih + m - \xi, & i \le k, \\ \sigma(1-ih), & i \ge k+1, \end{cases}$$

where σ is undefined yet parameter. It can be verified that

$$\beta_{\bar{x}_1 x_1,i} = -\frac{2-\theta}{h}\,\delta_{i,k-1} - \frac{1+\theta}{h}\,\delta_{i,k+2},$$

$$\gamma_{\bar{x}_1 x_1,i} = -\frac{1-\theta}{h}\,\delta_{i,k} - \frac{\theta}{h}\,\delta_{i,k+1},$$

where $\delta_{\cdot,\cdot}$ is the Kronecker symbol.

Choose

$$\rho_i = c_1\beta_i + c_2\gamma_i, \quad c_1 = \frac{5}{96}, \quad c_2 = \frac{45}{32}. \tag{7}$$

Then the identity

$$h\rho_{\bar{x}_1 x_1,i} = -c_1\big((2-\theta)\delta_{i,k-1} + (1+\theta)\delta_{i,k+2}\big) - c_2\big((1-\theta)\delta_{i,k} + \theta\delta_{i,k+1}\big) \tag{8}$$

holds.

Lemma 2. *Let*

$$\frac{25}{16}\,\xi < c_1(1+\theta) + c_2\sigma \le \frac{1}{\alpha^2}.$$

Then for every mesh function $y(x)$ satisfying the conditions (3)*, the estimate*

$$-(y_{\bar{x}_1 x_1}, y)_\rho \ge c\|y_{\bar{x}_1}\|^2_{\bar{\rho}}$$

is valid.

Proof. Let us first show that in the conditions of the lemma the inequality

$$\rho_{n-1}y^2(1,x_2) + h^2\sum_{\omega_1}\rho_{\bar{x}_1 x_1,i}y^2 \le 0. \tag{9}$$

is valid.

Indeed, according to (8), we have

$$h\sum_{\omega_1}\rho_{\bar{x}_1 x_1,i}y^2 = -c_1\big((2-\theta)y^2_{k-1} + (1+\theta)y^2_{k+2}\big) - c_2\big((1-\theta)y^2_k + \theta y^2_{k+1}\big).$$

Taking also into account Lemma 1, we can see that for the fulfilment of (9) it suffices that $\rho_{n-1}\alpha^2 \le h$, that is,

$$(c_1(1+\theta) + c_2\sigma)\alpha^2 \le 1.$$

Let now

$$\frac{25}{16}\,\xi < c_1(1+\theta) + c_2\sigma.$$

In such a case, ρ_0 turns out to be positive, and since $\rho_{\bar{x}_1 x_1} \le 0$, we have $\rho_i > 0,\ i = 0, 1, 2, \dots, n-1$.

Using summation by parts, we obtain

$$-\sum_{\omega_1} h\rho y_{\bar{x}_1 x_1} y = \sum_{\omega_1^+} h\bar{\rho} y_{\bar{x}_1}^2 -$$

$$-\frac{1}{2}\Big(\rho_{n-1} y^2(1,x_2) + h^2 \sum_{\omega_1} y^2 \rho_{\bar{x}_1 x_1}\Big),\ \bar{\rho}_i = 0.5(\rho_i + \rho_{i-1}),$$

which together with (9) proves Lemma 2. □

It is proved that auxiliary weight function $\rho(x_1)$ is equivalent to $r(x_1)$. Using procedure proposed in [1], we obtain the following

Theorem 1. *The finite difference scheme* (2), (3) *is uniquely solvable and its convergence rate is determined by the estimate*

$$\|y-u\|_{W_2^1(\omega,\rho)} \le ch^4 \|u\|_{W_2^5(\Omega)}.$$

References

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Author's addresses:

A. Razmadze Mathemetical Institute
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186
Georgia

Department of Mathematics of Georgian Technical University
77, M.Kostava Str., Tbilisi 0193
Georgia