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ON A WEAK SOLUTION OF ONE NONLOCAL BOUNDARY-VALUE PROBLEM

Nonlocal boundary value problems are a very interesting generalization of classical boundary value problems (e.g., see [1, 2]); at the same time, they naturally arise in the mathematical modeling of contaminant propagation in watercourses and atmosphere, and in other problems of ecology, physics, and engineering when it is impossible to determine the boundary values of the unknown function.

Our goal is to study the existence of weak solutions of nonlocal boundary value problem for the Poisson equation. Analogous question is investigated in [3].

Let $\Omega = \{(x_1, x_2):\ 0 < x_k < 1,\ k = 1,2\}$ be a square with a boundary Γ, and let $\Gamma_* = \{(x_1, 1),\ (1,\ x_2):\ 0 \le x_k \le 1,\ k = 1,2\}$.

Consider the nonlocal boundary-value problem

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = -f(x), \quad x \in \Omega, \tag{1}$$

$$u\big|_{x_k=1} = 0, \quad 0 \le x_{3-k} \le 1, \quad k = 1,2, \tag{2}$$

$$l_k(u) := \int\limits_0^1 \beta(x_k) u(x)\, dx_k = 0, \quad 0 \le x_{3-k} \le 1, \quad k = 1,2, \tag{3}$$

where $\beta(t) = \varepsilon t^{\varepsilon - 1}$, $\varepsilon \in (0;1)$.

We introduce weighted Hardy operators, associated to conditions (3):

$$H_1 v = \frac{1}{\rho(x_1)} \int\limits_0^{x_1} \beta(t) v(t, x_2)\, dt, \quad H_2 v = \frac{1}{\rho(x_2)} \int\limits_0^{x_2} \beta(t) v(x_1, t)\, dt,$$

where $\rho(t) = \int_0^t \beta(\xi)\, d\xi = t^\varepsilon$.

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By $L_2(\Omega,\rho)$ we denote the weighted Lebesgue space of all real-valued functions $u(x)$ on Ω with the inner product and the norm

$$(u,v)_\rho = \int\limits_\Omega \rho(x_1)\rho(x_2)u(x)v(x)\,dx, \quad \|u\|_\rho = (u,u)_\rho^{1/2}.$$

The weighted Sobolev space $W_2^1(\Omega,\rho)$ is usually defined as a linear set of all functions $u(x) \in L_2(\Omega,\rho)$, whose derivatives $\partial u/\partial x_k,\ k=1,2$ (in the generalized sense) belong to $L_2(\Omega,\rho)$. It is a normed linear space if equipped with the norm

$$\|u\|_{W_2^1(\Omega,\rho)} = \Big(\|u\|_\rho^2 + |u|^2_{W_2^1(\Omega,\rho)}\Big)^{1/2}, \quad |u|^2_{W_2^1(\Omega,\rho)} = \left|\left|\frac{\partial u}{\partial x_1}\right|\right|_\rho^2 + \left|\left|\frac{\partial u}{\partial x_1}\right|\right|_\rho^2.$$

It is well-known (see, e.g., [5, p. 10], [6, Theorem 3.1]) that $W_2^1(\Omega,\rho)$ is a Banach space and $C^\infty(\bar\Omega)$ is dense in $W_2^1(\Omega,\rho)$ and in$L_2(\Omega,\rho)$. As an immediate consequence, we can define the space $W_2^1(\Omega,\rho)$ as a closure of $C^\infty(\bar\Omega)$ with respect to the norm $\|\cdot\|_{W_2^1(\Omega,\rho)}$, and these both definitions are equivalent.

Define the subspace of space $W_2^1(\Omega,\rho)$ which can be obtained by closing the set

$$\overset{*}{C^\infty}(\bar\Omega) = \big\{u \in C^\infty(\bar\Omega) : \operatorname{supp} u \cap \Gamma_* = \emptyset,\ \ l_k(u)=0,\ 0 < x_{3-k} < 1,\ k=1,2\big\}$$

with the norm $\|\cdot\|_{W_2^1(\Omega,\rho)}$. Denote it by $\overset{*}{W_2^1}(\Omega,\rho)$.

Define the identity minus Hardy operators:

$$G_1 = I - H_1,\ G_2 = I - H_2,\ Iv := v.$$

Let the right-hand side $f(x)$ in equation (1) be a linear continuous functional on $\overset{*}{W_2^1}(\Omega,\rho)$ which can be represented as

$$f = f_0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, \quad f_k(x) \in L_2(\Omega,\rho),\ \ k=0,1,2. \tag{4}$$

We say that the function $u \in \overset{*}{W_2^1}(\Omega,\rho)$ is a weak solution of problem (1)–(3), if the relation

$$a(u,v) = \langle f,v\rangle, \quad \forall v \in \overset{*}{W_2^1}(\Omega,\rho) \tag{5}$$

holds, where

$$a(u,v) = \left(\frac{\partial u}{\partial x_1}, G_2\frac{\partial u}{\partial x_1}\right)_\rho + \left(\frac{\partial u}{\partial x_2}, G_1\frac{\partial u}{\partial x_2}\right)_\rho, \tag{6}$$

$$\langle f,v\rangle = (f_0, G_1G_2)_\rho - \left(f_1, G_2\frac{\partial v}{\partial x_1}\right)_\rho - \left(f_2, G_1\frac{\partial v}{\partial x_2}\right)_\rho. \tag{7}$$

Equality (5) formally follows from $(\Delta u + f, G_1G_2v)_\rho = 0$ by integration by parts, taking into account that

$$\left(\frac{\partial v}{\partial x_k}, G_1G_2u\right) = -\left(v, G_{3-k}\frac{\partial u}{\partial x_k}\right), \quad k = 1, 2.$$

To prove the existence of the unique solution of problem (5) (weak solution of problem (1)–(3)) we will apply the Lax-Milgram lemma [4].

Lemma 1. *If the function $v \in L_2(\Omega, \rho)$ satisfies the condition $l_k(v) = 0$, $k = 1, 2$, then*

$$(v, G_kv)_\rho \geq \|v\|_\rho^2, \tag{8}$$

$$\|G_kv\|_\rho \leq \frac{1+\varepsilon}{1-\varepsilon}\|v\|_\rho, \tag{9}$$

$$\|G_1G_2v\|_\rho \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \|v\|_\rho. \tag{10}$$

Proof. By the definition of the operator G_k

$$(v, G_kv)_\rho = \|v\|_\rho^2 - (v, H_kv)_\rho. \tag{11}$$

Further,

$$\|H_kv\|_\rho^2 = \frac{-2\varepsilon}{1-\varepsilon}(v, H_kv)_\rho. \tag{12}$$

Indeed, in the case $k = 1$

$$\|H_1v\|_\rho^2 = \frac{1}{1-\varepsilon}\int_0^1 \rho(x_2)\left(\int_0^1 \left(\int_0^{x_1} \beta(t)v(t, x_2)\,dt\right)^2 dx_1^{1-\varepsilon}\right)dx_2 =$$

$$= \frac{1}{1-\varepsilon}\int_0^1 \rho(x_2)\left(0 - 2\int_0^1 x_1^{1-\varepsilon}\int_0^{x_1} \beta(t)v(t, x_2)\,dt\,\beta(x_1)v(x)\,dx_1\right)dx_2 =$$

$$= \frac{-2\varepsilon}{1-\epsilon}(v, H_1v)_\rho.$$

Using (12), from (11) we obtain

$$(v, G_kv)_\rho = \|v\|_\rho^2 + \frac{1-\varepsilon}{2\varepsilon}\|H_kv\|_\rho^2 \tag{13}$$

which proves (8).

From (12) it is easy to get

$$\|H_kv\|_\rho \leq \frac{2\varepsilon}{1-\varepsilon}\|v\|_\rho, \tag{14}$$

therefore

$$\|G_kv\|_\rho \leq \|v\|_\rho + \|H_kv\|_\rho,$$

which together with (14) implies (9).

(10) is a consequence of (9), taking into account that $l_1(G_2v)$=0. □

Lemma 2. *Let* $u \in \overset{*}{W}{}_2^1(\Omega,\rho)$. *Then*

$$|u|_{W_2^1(\Omega,\rho)} \le \|u\|_{W_2^1(\Omega,\rho)} \le c_1 |u|_{W_2^1(\Omega,\rho)}, \quad c_1 = \sqrt{5}.$$

Proof. The left inequality of the lemma is obvious.

Integration by parts results

$$\|v\|_\rho^2 = \frac{1}{\varepsilon+1}\int_0^1 \rho(x_2)\Big(\int_0^1 v^2(x)\,dx_1^{\varepsilon+1}\Big)dx_2 =$$
$$= \frac{-2}{\varepsilon+1}\int_\Omega \rho(x_1)\rho(x_2)x_1 v(x)\frac{\partial v}{\partial x_1}dx.$$

From this

$$\|v\|_\rho \le \frac{2}{\varepsilon+1}\left\|\frac{\partial v}{\partial x_1}\right\| \le \frac{2}{\varepsilon+1}|v|_{W_2^1(\Omega,\rho)},$$

and the second inequality of the lemma follows from the definition of the norm $\|\cdot\|_{W_2^1(\Omega,\rho)}$. □

Application of lemmas 1,2 and condition (3), (6) gives the continuity

$$a(u,v)| \le c\|u\|_{W_2^1(\Omega,\rho)}\|v\|_{W_2^1(\Omega,\rho)},\ c>0, \quad \forall u,v \in \overset{*}{W}{}_2^1(\Omega,\rho)$$

and W_2^1-ellipticity

$$a(u,u) \ge c\|u\|^2_{W_2^1(\Omega,\rho)},\ c>0, \quad \forall u \in \overset{*}{W}{}_2^1(\Omega,\rho)$$

of the bilinear form $a(u,v)$.

By applying lemmas 1,2 from (7) we obtain the continuity of linear form $\langle f,v\rangle$:

$$|\langle f,v\rangle| \le c\|v\|_{W_2^1(\Omega,\rho)}, c>0, \quad \forall v \in \overset{*}{W}{}_2^1(\Omega,\rho).$$

Thus, all conditions of the Lax-Milgram lemma are fulfilled. Therefore, the following theorem is true.

Theorem 1. *The problem* (1)–(3) *has unique weak solution from* $\overset{*}{W}{}_2^1(\Omega,\rho)$.

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