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Mathematics

The Cauchy-Nicoletti Multipoint Boundary Value Problem for Systems of Linear Generalized Differential Equations with Singularities

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ABSTRACT. The Cauchy-Nicoletti multipoint boundary value problem

$$dx(t) = dA(t)\cdot x(t) + df(t) \quad for \quad t \in [a,b],$$

$$x_i(t_i+) = 0, \quad x_i(t_i-) = 0, \ (i = 1,\dots,n),$$

is considered, where $x_1,\dots,x_n$ are the components of the desired solution *x*, $-\infty < a < t_i \le t_{i+1} < b < \infty$, $f = (f_l)_{l=1}^n : [a,b] \to R^n$ is a vector-function the components of which are functions with bounded variations, and $A = (a_{il})_{i,l=1}^n : [a,b] \to R^{n\times n}$ is a matrix-function such that the functions $a_{i1},\dots,a_{il}$ have bounded variations on every interval from [*a*, *b*] which do not include the point t_i for every $i \in \{1,\dots,n\}$.

The sufficient conditions are established for the unique solvability of this problem in the case when the considered system is singular, i. e., the components of the matrix-function *A* do not have bounded variation on the interval [*a*, *b*]. *© 2012 Bull. Georg. Natl. Acad. Sci.*

Key words: *systems of linear generalized ordinary differential equations, singularity, the Lebesgue-Stiltjes integral, a multipoint boundary value problem.*

1. Statement of the Problem and Basic Notation

In the paper for the system of linear singular generalized ordinary differential equations

$$dx(t) = dA(t)\cdot x(t) + df(t) \quad for \quad t \in [a,b], \tag{1}$$

we consider the Cauchy-Nicoletti multipoint boundary value problem

$$x_i(t_i+) = 0, \quad x_i(t_i-) = 0, \ (i = 1,\dots,n), \tag{2}$$

where $-\infty < a < t_i \le t_{i+1} < b < \infty$, $x_1,\dots,x_n$ are the components of the desired solution *x*,

$f = (f_l)_{l=1}^{n} : [a,b] \to R^n$ is a vector-function the components of which are functions with bounded variations on the interval $[a,b]$, and $A = (a_{il})_{i,l=1}^{n} : [a,b] \to R^{n\times n}$ is a matrix-function the components $a_{i1},....,a_{il}$ of which have bounded variations on every closed interval contained in $[a,t_i[\cup]t_i,b]$ for every $i \in \{1,...,n\}$. We have investigated the question of the unique solvability of the problem (1), (2) in the singular case, i.e., in the case when the components of the matrix-functions A may have unbounded variation on the closed interval $[a,b]$.

We give a general theorem for solvability of the problem (1), (2). On the basis of this theorem we have obtained effective criteria for the solvability of this problem.

Analogous and related questions are investigated in [1-7] (see also the references therein) for the singular boundary value problems for linear and nonlinear systems of ordinary differential equations, and in [8-14] for regular and singular multipoint boundary value problems for systems of linear and nonlinear generalized differential equations. As to multipoint singular boundary value problems for generalized differential systems, they have not been sufficiently studied yet, and, despite some results [13, 14], their theory is far from completion even in the linear case. Therefore, the problem considered in the paper is actual.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view [8-18] and the references therein).

Throughout the paper the following notation and definitions will be used. $R =]-\infty,\infty[$, $[a,b]$, $]a,b[$ and $[a,b[$, $]a,b]$ $(a,b \in R)$ are, respectively, closed, open and semi-open intervals. $R^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{i,l=1})^{n,m}$ with the norm

$$\| X \| = \sum_{i,l=1}^{n,m} | x_{il} |;$$

$R_{+}^{n\times m} = \{X = (x_{il})_{i,l=1}^{n,m} : x_{il} \ge 0 \ \ (i = 1,...,n;\ l = 1,...,m)\}$; $|X| = (|x_{il}|)_{i,l=1}^{n,m}$; $O_{n\times m}$ $(or\, O)$ is the zero $n\times m$ – matrix.

If $X = (x_{il})_{i,l=1}^{n} \in R^{n\times n}$, then X^{-1}, $\det(X)$ and $r(X)$ are, respectively, the matrix inverse to X the determinant of X and the spectral radius of X; I_n is the identity $n\times n$ – matrix.

$R^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $R_{+}^{n} = R_{+}^{n\times 1}$.

$V_c^d(X)$, where $a < c < d < b$, is the total variation of the matrix-function $X = (x_{il})_{i,l=1}^{n} : [a,b] \to R^{n\times n}$, i.e., the sum of total variations of the latter's components x_{il} $(i = 1,...,n;\ l = 1,...,m)$; if $d < c$, then $V_c^d(X) = -V_d^c(X)$; $V(X)(t) = (v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(c_0) = 0$, $v(x_{il})(t) = V_{c_0}^{t}(x_{il})$ for $a < t < b$, $c_0 = (a+b)/2$; $X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X :]a,b[\to R^{n\times m}$ at the point $t \in]a,b[$ (we will assume $X(t) = X(a+)$ for $t \le a$ and $X(t) = X(b-)$ for $t \ge b$ if necessary); $d_1X(t) = x(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$BV([a,b], R^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to R^{n\times m}$ (i.e., such that $V_a^b(X) < \infty$;

$BV_{loc}(]a,b[, R^{n\times m})$ is the set of all matrix-functions $X :]a,b[\to R^{n\times m}$ such that $V_a^b(X) < \infty$ for every $a < c < d < b$;

If I is an arbitrary interval from R and $t_1,\dots,t_n \in I$, then $BV_{loc}(I,t_1,\dots,t_n;R^{n\times m})$ is the set of all matrix-functions $X : I \to R^{n\times m}$ the restrictions of which on every closed interval $[c,d] \subset I \setminus \{t_1,\dots,t_n\}$ belongs to $BV([a,b],R^{n\times m})$.

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If a function $\alpha \in BV([a,b],R)$ has no more than a finite number of points of discontinuity, and $m \in \{1,2\}$, then by $D_{\alpha m} = \{t_{\alpha m1},\dots,t_{\alpha mn_{\alpha m}}\}$ $(t_{\alpha m1} < \dots < t_{\alpha mn_{\alpha m}})$ we denote the set of all points $t \in [a,b]$ for which $d_m\alpha(t) = 0$; moreover, we put $\mu_{\alpha m} = \max\{d_m\alpha(t) : t \in D_{\alpha m}\}$.

If $\beta \in BV([a,b],R)$, then

$$\nu_{\alpha m\beta j} = max\left\{ d_j\beta(t_{\alpha ml}) + \sum_{t_{\alpha ml+1-m}<\tau<t_{\alpha ml+2-m}} d_j\beta(\tau) : l = 1,\dots,n_{\alpha m} \right\} \ (j,m = 1,2),$$

where $t_{\alpha 20} = a-1$, $t_{\alpha 2n_{\alpha 1}+1} = b+1$.

$s_j : BV([a,b],R) \to BV([a,b],R)$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(b) = 0,$$

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau),\ s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \quad for\ a < t \le b$$

and

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \quad for\ a \le t \le b.$$

If $g : [a,b] \to R$ is a nondecreasing function, and $a \le s \le t \le b$, then

$$\int_s^t x(\tau)dg(\tau) = \int_{]s,t[} x(\tau)ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau\le t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)ds_0(g)(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_0(g))$ corresponding to the function $s_0(g)$; moreover, we assume $\int_t^s x(\tau)dg(\tau) = -\int_s^t x(\tau)dg(\tau)$ and $\int_s^s x(\tau)dg(\tau) = 0$;

$L([a,b],R;g)$ is the space of all functions $x : [a,b] \to R$, measurable and integrable with respect to the measure $\mu(g)$ with the norm

$$\| X \|_{L,g} = \int_a^b | x(t) | dg(t).$$

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)dg(\tau) = \int_s^t x(\tau)dg_1(\tau) - \int_s^t x(\tau)dg_2(\tau) \quad for\ s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset R^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to D$ such that $x_{kj}\in L([a,b],R;g_{ik})(i=1,\dots,l;k=1,\dots,n;j=1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau)=\left(\sum_{k=1}^{n}\int_s^t x_{kj}(\tau)dg_{ik}(\tau)\right)_{i,j=1}^{l,n} \quad for\ \ a\le s\le t\le b,$$

$$S_j(G)(t)\equiv(s_j(g_{ik})(t))_{i,k=1}^{l,n}\ (j=0,1,2).$$

If $G(t)\equiv G_1(t)-G_2(t)$, where $G_1(t)$ and $G_2(t)$ are nondecreasing matrix-functions, then

$$\int_s^t dG(\tau)\cdot X(\tau)=\int_s^t dG_1(\tau)\cdot X(\tau)-\int_s^t dG_2(\tau)\cdot X(\tau)\ for\ s\le t,$$

$$S_k(G)=S_k(G_1)-S_k(G_2)\ (k=0,1,2),$$

$$L([a,b],D;G)=L([a,b],D;G_1)\cap L([a,b],D;G_2).$$

The inequalities between the matrices are understood component-wise.

A vector-function $x\in BV_{loc}([a,b],t_1,\dots,t_n;R^n)$ is said to be a solution of the system (1) if

$$x(t)=x(s)+\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ for\ s<t,\ [s,t]\subset[a,b]\setminus\{t_1,\dots,t_n\}.$$

By a solution of the problem (1), (2) we mean the solution $x=(x_i)_{i=1}^{n}$ of the system (1) such that the one-sided limits $x_i(t_i-),\ x_i(t_i+)\ (i=1,\dots,n)$ exist and the equalities (2) are valid.

A vector-function $x\in BV_{loc}([a,b],t_1,\dots,t_n;R^n)$ is said to be a solution of the system of generalized differential inequalities $dx(t)\le dB(t)\cdot x(t)+df(t)\ (\ge)\quad for\quad t\in[a,b]$, if

$$x(t)\le x(s)+\int_s^t dB(\tau)\cdot x(\tau)+f(t)-f(s)\ \ for\, s<t,\ \ [s,t]\subset[a,b]\setminus\{t_1,\dots,t_n\}.$$

Without loss of generality we assume that $A(a)=O_{n\times n},\ f(0)=0_n$. Let, moreover,

$$\det(I_n+(-1)^j dA(t))\ne 0\ \ for\ \ t\in[a,b]\setminus\{t_1,\dots,t_n\}\ (j=1,2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding system (1) (see [18, Theorem III.1.4]).

If $s\in]a,b[$ and $\alpha\in BV_{loc}(]a,b[,R)$ are such that

$$1+(-1)^j d_j\alpha(t)\ne 0\ \ for\ t\in]a,b[\ (j=1,2),$$

then by $\gamma_\alpha(.,s)$ we denote the solution of the Cauchy problem $d\gamma(t)=\gamma(t)d\alpha(t),\ \gamma(s)=1$.

It is known (see [15], [16]) that this problem has a unique solution and it is given by

$$\gamma_\alpha(t,s)=\begin{cases}\exp\big(s_0(\alpha)(t)-s_0(\alpha)(s)\big)\prod\limits_{s<\tau\le t}\big(1-d_1\alpha(\tau)\big)^{-1}\prod\limits_{s\le\tau\le t}\big(1+d_2\alpha(\tau)\big) & for\ t>s,\\ \exp\big(s_0(\alpha)(t)-s_0(\alpha)(s)\big)\prod\limits_{s<\tau\le t}\big(1-d_1\alpha(\tau)\big)\prod\limits_{s\le\tau\le t}\big(1+d_2\alpha(\tau)\big)^{-1} & for t<s,\\ 1 & for\ t=s.\end{cases}$$

Definition 1. We say that a matrix-function $C=\left(c_{il}\right)_{i,l=1}^{n}\in BV([a,b],R^{n\times m})$ belongs to the set $\mathrm{U}([a,b],t_1,\dots,t_n)$ if the functions $c_{il}\ (i\ne l;\ i,l=1,\dots,n)$ are nondecreasing on $[a,b]$ and the system

$$\operatorname{sgn}(t-t_i)\cdot dx_i(t)\le\sum_{i=1}^{n}x_l(t)dc_{il}(t)\ for\ t\in[a,b]\ (i=1,\dots,n)$$

has no nontrivial, nonnegative solution satisfying the condition (2).

A similar definition of set $U([a,b],t_1,\dots,t_n)$ has been introduced by I. Kiguradze for ordinary differential equations (see [4]).

We note that the problem (1),(2) under the condition $a\le t_i\le t_{i+1}\le b\ (i=1,\dots,n)$ is reduced to the case given above. Indeed, if $t_i=a\ (t_i=b)$ for some $i\in\{1,\dots,n\}$, then setting $A(t)\equiv A(a)$ and $f(t)\equiv f(a)$ for $t\le a\ (A(t)\equiv A(b)$ and $f(t)\equiv f(b)$ for $t\ge b)$, we can consider the problem on every interval $[a_0,b_0]$, where $a_0<a<b<b_0$. Moreover, without loss of generality we assume that $a<t_i-1/k<t_i+1/k<b$ for every natural *k*.

2. Formulation of Main Results

Theorem 1. Let the vector-function $f=(f_l)_{l=1}^n$ belong to $BV([a,b],R^n)$, and the matrix-function $A=\left(a_{il}\right)_{i,l=1}^n\in BV_{loc}([a,b],t_1,\dots,t_n;R^{n\times n})$ be such that the conditions

$$\left(s_0\left(a_{ii}\right)(t)-s_0\left(a_{ii}\right)(s)\right)\operatorname{sgn}\left(t-t_i\right)\le s_0\left(c_{ii}-\alpha_i\right)(t)-s_0\left(c_{ii}-\alpha_i\right)(s)$$
$$for\ a\le s<t<t_i\ or\ t_i<s<t\le b\ (i=1,\dots,n),\tag{3}$$

$$(-1)^j\left(\left|1+(-1)^j d_j a_{ii}(t)\right|-1\right)\operatorname{sgn}\left(t-t_i\right)\le d_j\left(c_{ii}(t)-\alpha_i(t)\right)\ for\ t\in[a,t_i[\cup]t_i,b]\ (j=1,2;i=1,\dots,n),\tag{4}$$

$$\left|s_0\left(a_{il}\right)(t)-s_0\left(a_{il}\right)(s)\right|\le s_0\left(c_{il}\right)(t)-s_0\left(c_{il}\right)(s)\ for\ a\le s<t<t_i\ or\ t_i<s<t\le b\ (i\ne l;i,l=1,\dots,n)\tag{5}$$

and

$$\left|d_j a_{il}(t)\right|\le d_j c_{il}(t)\ for\ t\in[a,t_i[\cup]t_i,b]\ (j=1,2;i\ne l;i,l=1,\dots,n)\tag{6}$$

are fulfilled, where

$$C=\left(c_{il}\right)_{i,l=1}^n\in \mathrm{U}([a,b],t_1,\dots,t_n),\quad \alpha_i:[a,t_i[\cup]t_i,b]\to R\ (l=1,\dots,n)$$

are functions, nondecreasing on every interval $[a,t_i[$ and $]t_i,b]$, having one-sided limits $\alpha_i(t_i-)$ and $\alpha_i(t_i-)$ and satisfying the conditions

$$\lim_{t\to t_i+}d_1\alpha_i(t)<1\ (i=1,\dots,n_0),\ \lim_{t\to t_i-}d_2\alpha_i(t)<1\ (i=n_0+1,\dots,n)\tag{7}$$

and

$$\begin{aligned}&\lim_{t\to t_i+}\sup\left\{\gamma_{\alpha_i}(t,t_i+1/k):k=1,2,\dots\right\}=0\ (i=1,\dots,n),\\&\lim_{t\to t_i-}\sup\left\{\gamma_{\alpha_i}(t,t_i-1/k):k=1,2,\dots\right\}=0\ (i=1,\dots,n).\end{aligned}\tag{8}$$

Then the problem (1), (2) has one and only one solution.

Corollary 1. Let the vector-function $f=(f_l)_{l=1}^n$ belong to $BV([a,b],R^n)$, and the matrix-function $A=\left(a_{il}\right)_{i,l=1}^n \in BV_{loc}([a,b],t_1,...,t_n;R^{n\times n})$ be such that the conditions

$$\left(s_0\left(a_{ii}\right)(t)-s_0\left(a_{ii}\right)(s)\right)\operatorname{sgn}\left(t-t_i\right)\le-\left(s_0\left(\alpha_i\right)(t)-s_0\left(\alpha_i\right)(s)\right)+\int_s^t h_{ii}(\tau)ds_0\left(\beta_i\right)(\tau)$$
$$\text{for } a\le s<t<t_i \text{ or } t_i<s<t\le b \ (i=1,...,n),$$

$$(-1)^j\left(\left|1+(-1)^j d_j a_{ii}(t)\right|-1\right)\operatorname{sgn}\left(t-t_i\right)\le h_{ii}(t)d_j\beta_i(t)-d_j\alpha_i(t)$$
$$\text{for } \ t\in[a,t_i[\cup]t_i,b](j=1,2;i=1,...,n),$$

$$\left|s_0\left(a_{il}\right)(t)-s_0\left(a_{il}\right)(s)\right|\le\int_s^t h_{il}(\tau)ds_0\left(\beta_l\right)(\tau) \text{ for } a\le s<t<t_i \text{ or } t_i<s<t\le b(i\ne l;i,l=1,...,n)$$

$$\left|d_j a_{il}(t)\right|\le h_{il}(t)d_j\beta_l(t) \text{ for } t\in[a,t_i[\cup]t_i,b](j=1,2;i\ne l;i,l=1,...,n)$$

are fulfilled, where $\alpha_i:[a,t_i[\cup]t_i,b]\to R$ $(i=1,...,n)$ are functions, nondecreasing on every interval $[a,t_i[$ and $]t_i,b]$, having one-sided limits $\alpha_i(t_i-)$ and $\alpha_i(t_i+)$ and satisfying the conditions (3), (4); β_l $(l=1,...,n)$ are functions nondecreasing on $[a,b]$ and having not more than a finite numder of points of discontinuity; $h_{ii}\in L^\mu\left(\left[a,b\right],R;\beta_i\right)$, $h_{il}\in L^\mu\left(\left[a,b\right],R_+;\beta_i\right)$ $(i\ne l;i,l=1,...,n)$, $1\le\mu\le\infty$. Let, moreover,

$$r(H)<1,$$

where $3n\times 3n$ -matrix $H=\left(H_{j+1m+1}\right)_{j,m=0}^2$ is defined by

$$H_{j+1m+1}=\left(\lambda_{kmij}\left\|h_{ik}\right\|_{\mu,s_m(\beta_i)}\right)_{i,k=1}^n \quad (j,m=0,1,2),$$

$$\xi_{ij}=\left(s_j\left(\beta_i\right)(b)-s_j\left(\beta_i\right)(a)\right)^{\frac{1}{\nu}} \ (j=0,1,2;i=1,...,n);$$

$$\lambda_{k0i0}=\begin{cases}\left(\dfrac{4}{\pi^2}\right)^{\frac{1}{\nu}}\xi_{k0}{}^2 & \text{if } \ s_0\left(\beta_i\right)(t)\equiv s_0\left(\beta_k\right)(t),\\ \xi_{ko}\xi_{i0} & \text{if } \ s_0\left(\beta_i\right)(t)\ne s_0\left(\beta_k\right)(t)(i,k=1,...,n);\end{cases}$$

$$\lambda_{kmij}=\xi_{km}\xi_{ij} \text{ if } m^2+j^2>0, mj=0 \ (j,m=0,1,2;\ k=1,...,n),$$

$$\lambda_{kmij}=\left(\frac{1}{4}\mu_{\alpha_k m}\nu_{\alpha_k m\alpha_i j}\sin^{-2}\frac{\pi}{4n_{\alpha_k m}+2}\right)^{\frac{1}{\nu}} \ (j,m=1,2;k=1,...,n),$$

and $\dfrac{1}{\mu}+\dfrac{2}{\nu}=1$. Then the problem (1), (2) has one and only one solution.

Remark 1. In Corollary 1, $3n\times 3n$ -matrix H can be replaced by the $n\times n$ -matrix

$$\left(\max\left\{\sum_{j=0}^2\lambda_{kmij}\left\|h_{ik}\right\|_{\mu,s_m(\alpha_k)}:m=0,1,2\right\}\right)_{i,k=1}^n.$$

By Remark 1, Corollary 1 has the following form for $h_{il}(t)\equiv h_{il}=\text{const}(i,l=1,\dots,n)$, $\alpha_i(t)\equiv\alpha(t)\,(i=1,\dots,n)$, $\beta_i(t)\equiv\beta(t)\ (i=1,\dots,n)$ and $\mu=\infty$.

Corollary 2. Let the vector-function $f=(f_l)_{l=1}^n$ belong to $BV([a,b],R^n)$, and the matrix-function $A=(a_{il})_{i,l=1}^n\in BV_{loc}([a,b],t_1,\dots,t_n;R^{n\times n})$ be such that the conditions

$$\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}(t-t_i)\le h_{ii}\big(s_0(\beta)(t)-s_0(\beta)(s)\big)-\big(s_0(\alpha)(t)-s_0(\alpha)(s)\big)$$
$$\text{for } a\le s<t<t_i \text{ or } t_i<s<t\le b\ (i=1,\dots,n),$$

$$(-1)^j\Big(\big|1+(-1)^j d_j a_{ii}(t)\big|-1\Big)\operatorname{sgn}(t-t_i)\le h_{ii}d_j\beta(t)-d_j\alpha(t)\,\text{for } t\in[a,t_i[\cup]t_i,b]\,(j=1,2;i=1,\dots,n),$$

$$\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\le h_{ii}\big(s_0(\beta)(t)-s_0(\beta)(s)\big)\,\text{for } a\le s<t<t_i \text{ or } t_i<s<t\le b(i\ne l;i,l=1,\dots,n),$$

$$\big|d_j a_{il}(t)\big|\le h_{il}d_j\beta_l(t)\,\text{for } t\in[a,t_i[\cup]t_i,b]\ (j=1,2;i\ne l;i,l=1,\dots,n)$$

are fulfilled, where $\alpha_i:[a,t_i[\cup]t_i,b]\to R\ (i=1,\dots,n)$ are functions, nondecreasing on every interval $[a,t_i[$ and $]t_i,b]$, having one-sided limits $\alpha_i(t_i-)$ and $\alpha_i(t_i+)$ and satisfying the conditions

$$\lim_{t\to t_i+}d_1\alpha(t)<1\ (i=1,\dots,n_0),\ \lim_{t\to t_i-}d_2\alpha(t)<1\ \ (i=1,\dots,n)$$

and

$$\lim_{t\to t_i+}\sup\{\gamma_\alpha(t,t_i+1/k):k=1,2,\dots\}=0\ \ (i=1,\dots,n),$$
$$\lim_{t\to t_i-}\sup\{\gamma_\alpha(t,t_i-1/k):k=1,2,\dots\}=0\ (i=1,\dots,n);$$

β is a function nondecreasing on $[a,b]$ and having not more than a finite number of points of discontinuity; $h_{ii}\in R$, $h_{il}\in R\ (i\ne l;\ i,l=1,\dots,n)$. Let, moreover,

$$\rho_0 r(H)<1,$$

where $H=(h_{ik})_{i,k=1}^n$,

$$\rho_0=\max\left\{\sum_{j=0}^{2}\lambda_{mj}:m=0,1,2\right\},\ \lambda_{00}=\frac{2}{\pi}\big(s_0(\beta)(b)-s_0(\beta)(a)\big),$$

$$\lambda_{0j}=\lambda_{0j}=\big(s_0(\beta)(b)-s_0(\beta)(a)\big)^{\frac12}\cdot\big(s_j(\beta)(b)-s_j(\beta)(a)\big)^{\frac12}\ (j=1,2),$$

$$\lambda_{mj}=\left(\frac14\mu_{\alpha m}\nu_{\alpha m\alpha j}\sin^{-2}\frac{\pi}{4n_{\alpha m}+2}\right)^{\frac{1}{\nu}}\ \ (m,j=1,2).$$

Then the problem (1), (2) has one and only one solution.

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მათემატიკა

კოში-ნიკოლეტის მრავალწერტილოვანი სასაზღვრო ამოცანა წრფივ განზოგადებულ დიფერენციალურ განტოლებათა სისტემებისათვის სინგულარობებით

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განხილულია კოში-ნიკოლეტის მრავალწერტილოვანი სასაზღვრო ამოცანა

$$dx(t) = dA(t) \cdot x(t) + df(t) \quad for \quad t \in [a,b],$$

$$x_i(t_i+) = 0, \quad x_i(t_i-) = 0, \ (i = 1,\dots,n),$$

სადაც $x_1,\dots,x_n$ **საძიებელი** x **ამონახსნის კომპონენტებია,** $-\infty < a < t_i \le t_{i+1} < b < \infty$, $f = (f_l)_{l=1}^n : [a,b] \to R^n$ **არის ვექტორული ფუნქცია, რომლის კომპონენტები სასრული ვარიაციის მქონე ფუნქციებია, ხოლო მატრიცული ფუნქცია** $A = (a_{il})_{i,l=1}^n : [a,b] \to R^{n \times n}$ **ისეთია, რომ ყოველი** $i \in \{1,\dots,n\}$ **-თვის** $a_{i1},\dots,a_{il}$ **ფუნქციებს გააჩნია სასრული ვარიაციები [*a*, *b*]-ში შემავალ ნებისმიერ შუალედზე, რომელიც არ შეიცავს** t_i **წერტილს.**

დადგენილია აღნიშნული სასაზღვრო ამოცანის ცალსახად ამოხსნადობისათვის საკმარისი პირობები იმ შემთხვევაში, როცა განსახილველი დიფერენციალური სისტემა სინგულარულია, ანუ როცა A **მატრიცული ფუნქციის კომპონენტებს არ გააჩნია სასრული ვარიაციები [*a*, *b*].**

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