Proc. A. Razmadze Math. Inst. 167(2015), 93–98

M. Ashordia and G. Ekhvaia

ON THE CRITERIA OF WELL-POSED OF THE PERIODIC PROBLEM FOR LINEAR SYSTEMS OF IMPULSIVE EQUATIONS WITH FINITE AND FIXED POINTS OF IMPULSES ACTIONS

Let $P \in L([0,\omega];\mathbb{R}^{n\times n})$, $p \in L([0,\omega];\mathbb{R}^n)$, $Q_j \in \mathbb{R}^{n\times n}$ $(j=1,\dots,m)$, $q_j \in \mathbb{R}^n$ $(j=1,\dots,m)$, $0=\tau_0<\tau_1<\dots<\tau_m<\tau_{m+1}=\omega$ and ω be a fixed positive number.

Consider the linear the impulsive system

$$\frac{dx}{dt} = P(t)x + p(t), \tag{1}$$

$$x(\tau_j+) - x(\tau_j-) = Q_j x(\tau_j) + q_j \ \ (j=1,\dots,m). \tag{2}$$

For the system (1), (2) consider the ω periodic problem

$$x(0) = x(\omega).$$

Let the system (1), (2) has the unique ω periodic solution x_0.

Consider sequences of matrix- and vector-functions $P_k \in L([0,\omega];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $p_k \in L([0,\omega];\mathbb{R}^n)$ $(k=1,2,\dots)$, sequences of constant matrices $Q_{kj} \in \mathbb{R}^{n\times n}$ $(j=1,\dots,m;\ k=1,2,\dots)$ and constant vectors $q_{kj} \in \mathbb{R}^n$ $(j=1,\dots,m;\ k=1,2,\dots)$.

In this paper necessary and sufficient conditions as well as effective sufficient conditions are established for a sequence of boundary value problems

$$\frac{dx}{dt} = P_k(t)x + p_k(t), \tag{3}$$

$$x(\tau_j+) - x(\tau_j-) = Q_{kj} x(\tau_j) + q_{kj} \ \ (j=1,\dots,m), \tag{4}$$

$(k=1,2,\dots)$ to have a unique ω solution x_k for sufficiently large k and

$$\lim_{k\to\infty} x_k(t) = x_0(t) \tag{5}$$

uniformly on $[a,b]$.

Analogous questions for the general linear boundary value problems and ω-periodic problems are investigated e.g. in [1], [2], [6], [7] (see the references

---

2010 *Mathematics Subject Classification:* 34B37.

Key words and phrases. Linear impulsive systems, periodic problem, criteria of well-posedness.

therein, too) for systems of ordinary differential equations, in [3], [4] for systems of generalized ordinary differential equations, and in [5] for systems of impulsive equations.

Throughout the paper, the following notation and definitions will be used.

$\mathbb{R} =]-\infty, \infty[$. $\mathbb{R}^{n\times l}$ is the space of all real $n \times l$-matrices $X = (x_{ij})_{i,j=1}^{n,l}$ with the norm

$$\|X\| = \max_{j=1,\dots,l} \sum_{i=1}^{n} |x_{ij}|.$$

$O_{n\times l}$ is the zero $n \times l$-matrix.

$\det(X)$ is the determinant of a matrix $X \in \mathbb{R}^{n\times n}$.

I_n is the identity $n \times n$-matrix.

δ_{ij} is the Kroneker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, \dots)$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$.

$\mathrm{BVC}([0,\omega]; \tau_1, \dots, \tau_m; \mathbb{R}^{n\times l})$ is the normed space of all continuous on the intervals $[0, \tau_1]$, $]\tau_k, \tau_{k+1}]$ $(k = 1, \dots, m)$ matrix-functions of bounded variation $X : [0, \omega] \to \mathbb{R}^{n\times l}$ with the norm

$$\|X\|_s = \sup\big\{\|X(t)\| : \ t \in [0, \omega]\big\}.$$

$L([0,\omega]; \mathbb{R}^{n\times l})$ is the set of all measurable and Lebesgue integrable on $[0, \omega]$ matrix-functions.

$C([0,\omega]; \mathbb{R}^{n\times l})$ is the set of all continuous on $[0, \omega]$ matrix-functions.

$\widetilde{C}([0,\omega]; \mathbb{R}^{n\times l})$ is the set of all absolutely continuous on $[0, \omega]$ matrix-functions.

$\widetilde{C}([0,\omega] \setminus \{\tau_j\}_{j=1}^{m}; \mathbb{R}^{n\times l})$ is the set of all matrix-functions restrictions of which on every closed interval $[c, d]$ from $[0,\omega] \setminus \{\tau_j\}_{j=1}^{m}$ belong to $\widetilde{C}([0,\omega]; \mathbb{R}^{n\times l})$.

On the set $C([0,\omega]; \mathbb{R}^{n\times l}) \times \underbrace{\mathbb{R}^{n\times l} \times \cdots \times \mathbb{R}^{n\times l}}_{m} \times L([0,\omega]; \mathbb{R}^{l\times k})$ we introduce the operator

$$\mathcal{B}_0(\Phi, G_1, \dots, G_m, X)(t) \equiv \int\limits_0^t \Phi(s) X(s)\, ds + \sum_{j=0,\ \tau_j \in [0,t[}^{m} G_j \int\limits_{\tau_j}^{t} X(s)\, ds,$$

where $G_0 = O_{n\times n}$.

Under a solution of the system (1), (2) we understand a continuous from the left vector-function $x \in \widetilde{C}([0,\omega] \setminus \{\tau_j\}_{j=1}^{m}; \mathbb{R}^{n\times l}) \cap \mathrm{BVC}([0,\omega]; \tau_1, \dots, \tau_m; \mathbb{R}^n)$ satisfying the system (1) for a.e. $t \in [0, \omega]$ and the equality (2) for every $j \in \{1, \dots, n\}$.

We assume everywhere that

$$\det(I_n + Q_j) \neq 0 \quad (j = 1, \dots, m).$$

Note that this condition guarantees the unique solvability of the system (1), (2) under the Cauchy condition $x(t_0) = c_0$.

Definition 1. We say that a sequence $(P_k, p_k, \{Q_{kj}\}_{j=1}^m, \{q_{kj}\}_{j=1}^m)$ $(k = 1, 2, \dots)$ belongs to the set $S(P, p, \{Q_j\}_{j=1}^m, \{q_j\}_{j=1}^m)$ if the system (3), (4) has the unique ω-periodic solution x_k for any sufficiently large k and the condition (5) holds uniformly on $[0, \omega]$.

Theorem 1. *The include*

$$\big((P_k, p_k, \{Q_{kj}\}_{j=1}^m, \{q_{kj}\}_{j=1}^m, \ell_k)\big)_{k=1}^{\infty} \in S(P, p, \{Q_j\}_{j=1}^m, \{q_j\}_{j=1}^m, \ell) \tag{6}$$

holds if and only if there exist sequences of matrix-functions Φ, $\Phi_k \in \widetilde{C}([a, b]; \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ *and constant matrices* G_j, $G_{kj} \in \mathbb{R}^{n\times n}$, $G_0 = G_{k0} = O_{n\times n}$ $(j = 0, \dots, m;\ k = 1, 2, \dots)$ *such that*

$$\lim_{k\to\infty} \sup \sum_{j=0}^{m} \int\limits_{\tau_j}^{\tau_{j+1}} \bigg\| \Phi_k'(t) + \bigg(\Phi_k(t) + \sum_{i=0}^{j} Q_{kj}\bigg) P_k(t) \bigg\|\, dt < \infty, \tag{7}$$

$$\inf \bigg\{ \bigg| \det \bigg(\Phi(t) + \sum_{i=0}^{j} G_i\bigg) \bigg| :\ t \in]\tau_j, \tau_{j+1}] \bigg\} > 0 \ \ (j = 0, \dots, m), \tag{8}$$

$$\lim_{k\to\infty} G_{kj} = G_j \ \ (j = 1, \dots, m), \tag{9}$$

$$\lim_{k\to\infty} Q_{kj} = Q_j, \quad \lim_{k\to\infty} q_{kj} = q_j \ \ (j = 1, \dots, m), \tag{10}$$

and the conditions

$$\lim_{k\to\infty} \Phi_k(t) = \Phi(t), \tag{11}$$

$$\lim_{k\to\infty} \mathcal{B}_0(\Phi_k, G_{k1}, \dots, G_{km}, P_k)(t) = \mathcal{B}_0(\Phi, G_1, \dots, G_m, P)(t), \tag{12}$$

$$\lim_{k\to\infty} \mathcal{B}_0(\Phi_k, G_{k1}, \dots, G_{km}, p_k)(t) = \mathcal{B}_0(\Phi, G_1, \dots, G_m, p)(t) \tag{13}$$

are fulfilled uniformly on $[a, b]$.

Remark 1. The conditions (12) and (13) are fulfilled uniformly on $[a, b]$ if and only if the conditions

$$\lim_{k\to\infty} \int\limits_{\tau_j}^{t} \bigg(\Phi_k(s) + \sum_{i=0}^{j} G_{ki}\bigg) P_k(s)\, ds = \int\limits_{\tau_j}^{t} \bigg(\Phi(s) + \sum_{i=0}^{j} G_i\bigg) P(s)\, ds,$$

$$\lim_{k\to\infty} \int\limits_{\tau_j}^{t} \bigg(\Phi_k(s) + \sum_{i=0}^{j} G_{ki}\bigg) p_k(s)\, ds = \int\limits_{\tau_j}^{t} \bigg(\Phi(s) + \sum_{i=0}^{j} G_i\bigg) p(s)\, ds,$$

are fulfilled uniformly on $[\tau_j, \tau_{j+1}]$ for every $j \in \{0, \dots, m\}$.

Corollary 1. *Let the condition* (10) *hold. Let, moreover, there exist matrix-functions* Φ, $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *such that the conditions* (7) *and*

$$\inf\Big\{\big|\det\big(\Phi(t)+(1-\delta_{0j})jI_n\big)\big| :\ t\in\,]\tau_j,\tau_{j+1}]\Big\}>0 \ \ (j=0,\dots,m)$$

hold and the conditions (11),

$$\lim_{k\to\infty}\int\limits_{\tau_j}^{t}\big(\Phi_k(s)+(1-\delta_{0j})jI_n\big)P_k(s)\,ds=\int\limits_{\tau_j}^{t}\big(\Phi(s)+(1-\delta_{0j})jI_n\big)P(s)\,ds$$

and

$$\lim_{k\to\infty}\int\limits_{\tau_j}^{t}\big(\Phi_k(s)+(1-\delta_{0j})jI_n\big)p_k(s)\,ds=\int\limits_{\tau_j}^{t}\big(\Phi(s)+(1-\delta_{0j})jI_n\big)p(s)\,ds$$

be fulfilled uniformly on $[\tau_j,\tau_{j+1}]$ *for every* $j\in\{0,\dots,m\}$*. Then the condition* (6) *holds.*

Corollary 2. *Let the condition* (10) *hold. Let, moreover, there exist matrix-functions* Φ, $\Phi_k \in \widetilde{C}([a,b];\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *such that*

$$\lim_{k\to\infty}\sup\int\limits_{a}^{b}\big\|\Phi_k'(t)+\Phi_k(t)P_k(t)\big\|\,dt<\infty,\quad \inf\Big\{\big|\det(\Phi(t))\big| :\ t\in[a,b]\Big\}>0$$

and the conditions (11) *and*

$$\lim_{k\to\infty}\int\limits_{a}^{t}\Phi_k(s)P_k(s)\,ds=\int\limits_{a}^{t}\Phi(s)P(s)\,ds,$$

$$\lim_{k\to\infty}\int\limits_{a}^{t}\Phi_k(s)p_k(s)\,ds=\int\limits_{a}^{t}\Phi(s)p(s)\,ds$$

are fulfilled uniformly on $[a,b]$*. Then the condition* (6) *holds.*

Corollary 3. *Let the conditions* (9) *and* (10) *hold. Let, moreover, there exist constant matrices* G_j, $G_{kj}\in\mathbb{R}^{n\times n}$, $G_0=G_{k0}=O_{n\times n}$ $(j=0,\dots,m;$ $k=1,2,\dots)$ *such that*

$$\lim_{k\to\infty}\sup\sum_{j=0}^{m}\int\limits_{\tau_j}^{\tau_{j+1}}\left\|\left(I_n+\sum_{i=0}^{j}Q_{ki}\right)P_k(t)\right\|dt<\infty, \tag{14}$$

$$\det\left(I_n+\sum_{i=1}^{j}G_i\right)\neq 0 \ \ (j=1,\dots,m)$$

and the conditions

$$\lim_{k\to\infty}\int\limits_{\tau_j}^{t}\Big(I_n+\sum_{i=0}^{j}G_{ki}\Big)P_k(s)\,ds=\int\limits_{\tau_j}^{t}\Big(I_n+\sum_{i=0}^{j}G_i\Big)P(s)\,ds,$$

$$\lim_{k\to\infty}\int\limits_{\tau_j}^{t}\Big(I_n+\sum_{i=0}^{j}G_{ki}\Big)p_k(s)\,ds=\int\limits_{\tau_j}^{t}\Big(I_n+\sum_{i=0}^{j}G_i\Big)p(s)\,ds$$

are fulfilled uniformly on $[\tau_j,\tau_{j+1}]$ *for every* $j\in\{0,\dots,m\}$. *Then the condition* (6) *holds.*

Corollary 4. *Let the conditions* (10) *and* (14) *hold and the conditions*

$$\lim_{k\to\infty}\int\limits_{a}^{t}P_k(s)\,ds=\int\limits_{a}^{t}P(s)\,ds,\quad \lim_{k\to\infty}\int\limits_{a}^{t}p_k(s)\,ds=\int\limits_{a}^{t}p(s)\,ds \tag{15}$$

be fulfilled uniformly on $[a,b]$. *Then the condition* (6) *holds.*

Corollary 5. *Let the condition* (10)*, and* (14) *hold and the condition* (15) *be fulfilled uniformly on* $[a,b]$. *Then the condition* (6) *holds.*

Remark 2. In Theorem 1 and Corollaries 1–5 we can assume without loss of generality that $\Phi(t)\equiv I_n$ and $G_j=O_{n\times n}$ $(j=1,\dots,m)$ everywhere they appear. So that the condition (8) in Theorem 1 as well as the analogous conditions in the corollaries are valid automatically.

These results follow from analogous results for a system of so-called generalized differential equations contained in [4] because the system (1), (2) is its particular of one.

Acknowledgement

The present paper was supported by the Shota Rustaveli National Science Foundation (Grant No. FR/182/5-101/11).

References

1. M. Ashordia, On the stability of solutions of linear boundary value problems for the system of ordinary differential equations. This paper also appears in Georgian Math. J. 1 (1994), No. 2, 115–126.
2. M. Ashordia, Criteria of correctness of linear boundary value problems for systems of ordinary differential equations. *Rep. Enlarged Sess. Semin. I. Vekua Appl. Math.* **15** (2000), No. 1-3, 40–43.
3. M. Ashordia, On the correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Georgian Math. J.* **1** (1994), No. 4, 343–-351.
4. M. Ashordia, Criteria of correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Czechoslovak Math. J.* **46(121)** (1996), No. 3, 385–404.

5. M. Ashordia and G. Ekhvaia, Criteria of correctness of linear boundary value problems for systems of impulsive equations with finite and fixed points of impulses actions. *Mem. Differential Equations Math. Phys.* **37** (2006), 154–157.
6. I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Itogi Nauki i Tekhniki*, *Current problems in mathematics. Newest results*, Vol. 30. (Russian) 3–103, 204, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987; traranslation in *J. Soviet Math.* **43** (1988), No. 2, 2259–-2339.
7. I. Kiguradze, The initial value problem and boundary value problems for systems of ordinary differential equations. Vol. I. (Russian) *Metsniereba, Tbilisi*, 1997.

Authors' addresses:

M. Ashordia

A. Razmadze Mathematical Institute
Iv. Javakhishvili Tbilisi State University
6, Tamarashvili St., Tbilisi 0177, Georgia

Sukhumi University 12
Politkovskaia St., Tbilisi, 0186,
Georgia
E-mail: ashord@rmi.ge

G. Ekhvaia

Sukhumi University 9
Politkovskaia St., Tbilisi, 0186
Georgia
E-mail: goderdzi.ekhvaia@mail.ru