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ON THE WELL-POSEDNESS OF ANTIPERIODIC PROBLEM FOR SYSTEMS OF LINEAR GENERALIZED DIFFERENTIAL EQUATIONS

Abstract. The question of well-posedness of antiperiodic boundary value problem for systems of linear generalized differential equations is considered. The necessary and sufficient as well as the effective sufficient conditions are found for the well-posedness of the problem.

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We consider the question of well-posedness of the ω-antiperiodic problem for linear generalized ordinary differential equations of the form

$$dx(t) = dA(t) \cdot x(t) + df(t) \ \text{ for } \ t \in \mathbb{R}, \tag{1}$$

$$x(t + \omega) = -x(t) \ \text{ for } \ t \in \mathbb{R}, \tag{2}$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R} \to \mathbb{R}^n$ are, respectively, the matrix- and vector-functions with bounded variation components on the every closed interval $[a, b]$ from $\mathbb{R}$, and ω is a fixed positive number.

Let the system (1) have a unique ω-antiperiodic solution x^0.

Along with the system (1), consider a sequence of systems

$$dx(t) = dA_k(t) \cdot x(t) + df_k(t) \ \ (k = 1, 2, \dots) \tag{1_k}$$

where $A_k : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $f_k : \mathbb{R} \to \mathbb{R}^n$ are, respectively, the matrix- and vector-functions with bounded variation components on every closed interval $[a, b]$ from $\mathbb{R}$.

In the present paper, the necessary and sufficient conditions are given for a sequence of ω-antiperiodic problems $(1_k), (2)$ $(k = 1, 2, \dots)$ to have a unique solution x_k for a sufficiently large k and

$$\lim_{k\to+\infty} x_k(t) = x^0(t) \ \text{ uniformly on } \ \mathbb{R}. \tag{3}$$

The analogous questions for the linear general boundary value problems are investigated in [2, 6, 10, 11, 19] (see also the references therein) for linear generalized differential systems, in [3–5, 14] (see also the references therein) for nonlinear generalized differential systems and equations, and in [1, 9, 12, 13, 16] (see also the references therein) for ordinary differential and impulsive systems.

The problem on the solvability of the ω-antiperiodic boundary value problem (1), (2) can be found in [8].

As to the well-posedness question concerning of the antiperiodic problem, it is sufficiently far from by completeness. Thus the problem considered in the present paper is actual.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see [3,7,14,15,17,18] and the references therein).

The theory of generalized ordinary differential equations has been introduced by J. Kurzweil [14,15] in connection with the investigation of the well-posed problem for the Cauchy problem for ordinary differential equations.

In the paper, the use will be made of the following notation and definitions:

$\mathbb{R} =]-\infty, +\infty[$ is the real axis;

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\|=\max_{j=1,\dots,m}\sum_{i=1}^{n}|x_{ij}|;$$

$O_{n\times m}$ (or O) is the zero $n\times m$ matrix; I_n is the identity $n\times n$-matrix.

$\mathbb{R}^n=\mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^{n}$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such. The inequalities between the real matrices are understood componentwise.

If $X:[a,b]\to\mathbb{R}^{n\times m}$ is a matrix-function, then $\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of its components x_{ij} $(i=1,\dots,n;\ j=1,\dots,m)$; $V(X)(t)=(V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a)=0$, $V(x_{ij})(t)=\bigvee\limits_a^t(x_{ij})$ for $a<t\le b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(a-)=X(a),\ X(b+)=X(b))$; $d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., $\bigvee\limits_a^b(X)<\infty$) with the norm $\|X\|_s=\sup\{\|X(t)\|:\ t\in[a,b]\}$.

$\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ from $\mathbb{R}$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

$\mathrm{BV}_\omega^+(\mathbb{R},\mathbb{R}^{n\times m})$ and $\mathrm{BV}_\omega^-(\mathbb{R},\mathbb{R}^{n\times m})$ are the sets of all matrix-functions $G:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on $[0,\omega]$ belong to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$, and there exist a constant matrix $C\in\mathbb{R}^{n\times m}$ such that, respectively,

$$G(t+\omega)=G(t)+C \ \text{ and } \ G(t+\omega)=-G(t)+C \ \text{ for } \ t\in\mathbb{R}.$$

$s_c,s_j:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ $(j=1,2)$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(a)=0,$$

$$s_1(x)(t)=\sum_{a<\tau\le t}d_1x(\tau) \ \text{ and } \ s_2(x)(t)=\sum_{a\le\tau<t}d_2x(\tau) \ \text{ for } \ a<t\le b,$$

and

$$s_c(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t) \ \text{ for } \ t\in[a,b].$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\le s<t\le b$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)+\sum_{s<\tau\le t}x(\tau)d_1g(\tau)+\sum_{s\le\tau<t}x(\tau)d_2g(\tau),$$

where $\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_c(g))$ corresponding to the function $s_c(g)$.

If $a=b$, then we assume

$$\int_a^b x(t)\,dg(t)=0,$$

and if $a>b$, then we assume

$$\int_a^b x(t)\,dg(t)=-\int_b^a x(t)\,dg(t).$$

Thus $\int_a^b x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [14–19]).

If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_s^t x(\tau)\,dg_1(\tau)-\int_s^t x(\tau)\,dg_2(\tau) \ \text{ for } \ s\le t.$$

If $G=(g_{ik})_{i,k=1}^{l,n}\in \mathrm{BV}([a,b],\mathbb{R}^{l\times n})$ and $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to\mathbb{R}^{n\times m}$, then

$$S_c(G)(t)\equiv\big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n},\ \ S_j(G)(t)\equiv\big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n}\ \ (j=1,2)$$

and

$$\int_a^b dG(\tau)\cdot X(\tau)=\bigg(\sum_{k=1}^n\int_a^b x_{kj}(\tau)\,dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m}.$$

We introduce the operators. If $X\in \mathrm{BV}_{loc}(\mathbb{R},;\mathbb{R}^{n\times n})$ and $Y\in \mathrm{BV}_{loc}(\mathbb{R},;\mathbb{R}^{n\times m})$, then

$$\mathcal{B}(X,Y)(t)=X(t)Y(t)-X(0)Y(0)-\int_0^t dX(\tau)\cdot Y(\tau);$$

if, in addition, $\det(X(t))\neq 0$ for $t\in\mathbb{R}$, then

$$\mathcal{I}(X,Y)(t)=\int_0^t d\big(X(\tau)+\mathcal{B}(X,Y)(\tau)\big)\cdot X^{-1}(\tau);$$

and if, moreover, $\det(I_n+(-1)^j d_jX(t))\neq 0$ for $t\in\mathbb{R}$ $(j=1,2)$, then

$$\begin{aligned}
&\mathcal{A}(X,Y)(0)=O_{n\times m},\\
&\mathcal{A}(X,Y)(t)=Y(t)-Y(0)+\sum_{0<\tau\le t}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)\\
&\qquad\qquad-\sum_{0\le\tau<t}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \ \text{for}\ \ t>0,\\
&\mathcal{A}(X,Y)(t)=-\mathcal{A}(X,Y)(t)\ \ \text{for}\ \ t<0.
\end{aligned}$$

We say that the matrix-function $X\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ satisfies the Lappo–Danilevskiĭ condition if the matrices $S_c(X)(t)$, $S_1(X)(t)$ and $S_2(X)(t)$ are pairwise permutable for every $t\in[a,b]$, and there exists $t_0\in[a,b]$ such that

$$\int_{t_0}^t S_c(X)(\tau)\,dS_c(X)(\tau)=\int_{t_0}^t dS_c(X)(\tau)\cdot S_c(X)(\tau)\ \ \text{for}\ \ t\in[a,b].$$

A vector-function $\mathrm{BV}_{loc}(\mathbb{R},\mathbb{R}^{n\times m})$ is said to be a solution of the system (1) if

$$x(t)-x(s)=\int_s^t dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \ \text{for}\ \ s<t;\ \ s,t\in\mathbb{R}.$$

We assume that

$$A,A_k\in \mathrm{BV}_\omega^+(\mathbb{R},\mathbb{R}^{n\times n})\ \ \text{and}\ \ f,f_k\in \mathrm{BV}_\omega^-(\mathbb{R},\mathbb{R}^n)\ \ (k=1,2,\dots),$$

i.e.,

$$A(t+\omega)=A(t)+C,\quad A_k(t+\omega)=A_k(t)+C_k\ \ \text{for}\ \ t\in\mathbb{R}\ \ (k=1,2,\dots)$$

and

$$f(t+\omega)=-f(t)+c,\quad f_k(t+\omega)=-f_k(t)+c_k\ \ \text{for}\ \ t\in\mathbb{R}\ \ (k=1,2,\dots),$$

where $C,C_k\in\mathbb{R}^{n\times n}$ $(k=1,2,\dots)$ and $c,c_k\in\mathbb{R}^n$ $(k=1,2,\dots)$ are, respectively, some constant matrix and vector. In addition, without loss of generality, we assume that

$$A(0)=A_k(0)=O_{n\times n},\ \ f(0)=f_k(0)=0\ \ (k=1,2,\dots)$$

(the last condition is assumed for every generalized linear systems, as well). Moreover, we assume

$$\det\big(I_n+(-1)^jd_jA(t)\big)\neq 0\ \ \text{for}\ \ t\in\mathbb{R}\ \ (j=1,2).$$

Alongside with the system (1), we consider the corresponding homogeneous system

$$dx(t)=dA(t)\cdot x(t). \tag{4_0}$$

Moreover, along with the problem (2), we consider the problem

$$x(0) = -x(\omega). \tag{5}$$

If the matrix-function A satisfies the Lappo–Danilevskiĭ's condition, then the fundamental matrix Y, $Y(0) = I_n$, of the system (4_0) is defined by

$$Y(t) \equiv \exp(S_0(A)(t)) \prod_{0\leq\tau<t} (I_n + d_2A(\tau)) \prod_{0<\tau\leq t} (I_n - d_1A(\tau))^{-1} \ \text{ for } \ t \in [0,\omega].$$

Definition 1. We say that a sequence (A_k, f_k) $(k = 1, 2, \dots)$ belongs to the set $\mathcal{S}(A, f)$ if the ω-antiperiodic problem $(1_k), (2)$ has a unique solution x_k for any sufficiently large k, and the condition (3) holds.

Proposition 1. *The following statements are valid:*

(a) *if x is a solution of the system* (1), *then the vector-function $y(t) = -x(t + \omega)$ $(t \in \mathbb{R})$ will be a solution of the system* (1), *as well;*

(b) *the problem* $(1), (2)$ *is solvable if and only if the system* (1) *on the closed interval* $[0, \omega]$ *has a solution satisfying the boundary condition* (5). *Moreover, the set of restrictions of solutions of the problem* $(1), (2)$ *on* $[0, \omega]$ *coincides with the set of solutions of the problem* $(1), (5)$.

Theorem 1. *The inclusion*

$$\big((A_k, f_k)\big)_{k=1}^{+\infty} \in \mathcal{S}(A, f) \tag{6}$$

is valid if and only if there exists a sequence of matrix-functions $H, H_k \in \mathrm{BV}([0,\omega], \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ *such that*

$$\lim_{k\to+\infty} \sup \bigvee_a^b (H_k + \mathcal{B}(H_k, A_k)) < +\infty, \tag{7}$$

$$\inf\Big\{\big|\det(H(t))\big| : \ t \in [0, \omega]\Big\} > 0, \tag{8}$$

and the conditions

$$\lim_{k\to+\infty} H_k(t) = H(t), \tag{9}$$

$$\lim_{k\to+\infty} \mathcal{B}(H_k, A_k)(t) = \mathcal{B}(H, A)(t), \tag{10}$$

$$\lim_{k\to+\infty} \mathcal{B}(H_k, f_k)(t) = \mathcal{B}(H, f)(t)$$

are fulfilled uniformly on $[0, \omega]$.

Theorem 2. *Let* $A_* \in \mathrm{BV}([0, \omega], \mathbb{R}^{n\times n})$, $f_* \in \mathrm{BV}([0, \omega], \mathbb{R}^n)$ *be such that*

$$\det\big(I_n + (-1)^j d_jA_*(t)\big) \neq 0 \ \ \textit{for} \ \ t \in [0, \omega] \ \ (j = 1, 2) \tag{11}$$

and the system

$$dx(t) = dA_*(t) \cdot x(t) + df_*(t) \tag{12}$$

have a unique ω-antiperiodic solution x_*. Let, moreover, there exist sequences of matrix- and vector-functions $H_k \in \mathrm{BV}([0, \omega], \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ *and* $h_k \in \mathrm{BV}([0, \omega], \mathbb{R}^n)$ $(k = 1, 2 \dots)$, *respectively, such that* $h_k(0) = -h_k(\omega)$ $(k = 1, 2, \dots)$,

$$\inf\Big\{\big|\det(H_k(t))\big| : \ t \in [0, \omega]\Big\} > 0 \ \ (k = 1, 2, \dots), \tag{13}$$

and

$$\lim_{k\to+\infty} \sup \bigvee_a^b A_{*k} < +\infty, \tag{14}$$

and the conditions

$$\lim_{k\to+\infty} A_{*k}(t) = A_*(t), \tag{15}$$

$$\lim_{k\to+\infty} f_{*k}(t) = f_*(t)$$

are fulfilled uniformly on $[0,\omega]$, *where*

$$A_{*k}(t) \equiv \mathcal{I}_k(H_k, A_k)(t) \ \ (k=1,2,\dots),$$

$$f_{*k}(t) \equiv h_k(t) - h_k(0) + \mathcal{B}_k(H_k, f_k)(t) - \int\limits_0^t dA_{*k}(\tau)\cdot h_k(t) \ \ (k=1,2,\dots).$$

Then the system (1_k) *has a unique* ω*-antiperiodic solution* x_k *for any sufficiently large* k, *and*

$$\lim_{k\to+\infty} \|H_k x_k + h_k - x_*\|_s = 0.$$

Corollary 1. *Let the conditions* (7) *and* (8) *hold, and let the conditions* (9), (10) *and*

$$\lim_{k\to+\infty}\bigg(\mathcal{B}(H_k, f_k-\varphi_k)(t) + \int\limits_0^t d\mathcal{B}(H_k, A_k)(s)\cdot\varphi_k(s)\bigg) = \mathcal{B}(H,f)(t)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H, H_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$. *Then the system* (1_k) *has a unique* ω*-antiperiodic solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to+\infty} \|x_k - \varphi_k - x_*\|_s = 0.$$

Corollary 2. *Let the conditions* (7) *and* (8) *hold, and let the conditions* (9),

$$\lim_{k\to+\infty}\int\limits_0^t H_k(s)\,dA_k(s) = \int\limits_0^t H(s)\,dA(s), \quad \lim_{k\to+\infty}\int\limits_0^t H_k(s)\,df_k(s) = \int\limits_0^t H(s)\,df(s),$$

$$\lim_{k\to+\infty} d_jA_k(t) = d_jA(t) \ \ (j=1,2), \ \ \textit{and} \ \ \lim_{k\to+\infty} d_jf_k(t) = d_jf(t) \ \ (j=1,2)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H, H_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$. *Let, moreover, either*

$$\lim_{k\to+\infty}\sup\sum_{a\le t\le b}\big(\|d_jA_k(t)\| + \|d_jf_k(t)\|\big) < +\infty \ \ (j=1,2)$$

or

$$\lim_{k\to+\infty}\sup\sum_{a\le t\le b}\|d_jH_k(t)\| < +\infty \ \ (j=1,2). \tag{16}$$

Then the inclusion (6) *holds.*

Corollary 3. *Let the conditions* (7) *and* (8) *hold, and let the conditions* (9),

$$\lim_{k\to+\infty} A_k(t) = A(t), \tag{17}$$

$$\lim_{k\to+\infty} f_k(t) = f(t)), \tag{18}$$

$$\lim_{k\to+\infty}\int\limits_0^t d\big(H^{-1}(s)H_k(s)\big)\cdot A_k(s) = A_*(t),$$

$$\lim_{k\to+\infty}\int\limits_0^t d\big(H^{-1}(s)H_k(s)\big)\cdot f_k(s) = f_*(t)$$

be fulfilled uniformly on $[0,\omega]$, *where* $H, H_k, A_* \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, *and* $f_* \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$. *Let, moreover, the system*

$$dx(t) = d\big(A(t)-A_*(t)\big)\cdot x(t) + d\big(f(t)-f_*(t)\big)$$

have a unique ω*-antiperiodic solution. Then*

$$\big((A_k, f_k)\big)_{k=1}^{+\infty} \in \mathcal{S}(A-A_*, f-f_*).$$

Corollary 4. *Let there exist a natural number m and matrix-functions $B_j \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(j = 0,\dots,m-1$ such that*

$$\lim_{k\to+\infty} \sup \bigvee_a^b (A_{km}) < +\infty,$$

and the conditions

$$\lim_{k\to+\infty} \big(A_{km}(t) - A_{km}(0)\big) = A(t),$$
$$\lim_{k\to+\infty} \big(f_{km}(t) - f_{km}(0)\big) = f(t)$$

be fulfilled uniformly on $[0,\omega]$, where

$$H_{k0}(t) \equiv I_n, \quad H_{k\,j+1\,0}(t) \equiv \prod_{j+1}^{1} \big(I_n - A_{kl}(t) + A_{kl}(0) + B_l(t) - B_l(0)\big),$$
$$A_{k\,j+1} \equiv H_{kj}(t) + \mathcal{B}(H_{kj}, A_k)(t), \quad f_{k\,j+1} \equiv \mathcal{B}(H_{kj}, f_k)(t).$$

Then the inclusion (6) holds.

If $m = 1$, then Corollary 4 has the following form

Corollary 5. *Let*

$$\lim_{k\to+\infty} \sup \bigvee_a^b (A_k) < +\infty$$

and the conditions (17) and (18) be fulfilled uniformly on $[0,\omega]$. Then the inclusion (6) holds.

Theorem 1′. *Let $A_* \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$, $f_* \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$ be such that the condition (11) hold and the system (12) has a unique ω-antiperiodic solution x_*. Let, moreover, there exist sequences of matrix- and vector-functions $H_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$ and $B, B_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k = 1,2,\dots)$, and a sequence of vector-functions $h_k \in \mathrm{BV}([0,\omega],\mathbb{R}^n)$ $(k = 1,2\dots)$, respectively, such that $h_k(0) = -h_k(\omega)$ $(k = 1,2,\dots)$, the conditions (13),*

$$\lim_{k\to+\infty} \sup \bigvee_a^b (A_{*k} - B_k) < +\infty, \tag{19}$$

$$\det\big(I_n + (-1)^j d_j B(t)\big) \neq 0, \ \ \det\big(I_n + (-1)^j d_j B_k(t)\big) \neq 0 \ \text{ for } t \in [0,\omega] \ \ (j=1,2; \ \ k=0,1,\dots) \tag{20}$$

hold, and the conditions

$$\lim_{k\to+\infty} Z_k(t) = Z(t), \tag{21}$$
$$\lim_{k\to+\infty} \mathcal{B}\big(Z_k^{-1}, A_{*k}(t)\big) = \mathcal{B}(Z^{-1}, A_*(t)), \tag{22}$$
$$\lim_{k\to+\infty} \mathcal{B}\big(Z_k^{-1}, f_{*k}(t)\big) = \mathcal{B}(Z^{-1}, f_*(t)) \tag{23}$$

are fulfilled uniformly on $[0,\omega]$, where A_{*k} and f_{*k} are the matrix- and vector-functions appearing in Theorem 2, and Z_k (Z) is the fundamental matrix of the system

$$dx(t) = dB_k(t)\cdot x(t) \ \ \big(dx(t) = dB(t)\cdot x(t)\big) \tag{24}$$

under the condition

$$Z_k(0) = I_n \ \ (Z(0) = I_m) \ \ (k = 1,2,\dots). \tag{25}$$

Then the conclusion of Theorem 2 is true.

Below, everywhere, just as in the above theorem, it will be assumed that Z_k (Z) is the fundamental matrix of the system (24) under the condition (25) for every $k \in \{1,2,\dots\}$, as well.

Corollary 6. *Let the conditions (8), (19),*

$$\lim_{k\to+\infty} \sup \sum_{0\le t\le\omega} \|d_j B_k(t)\| < +\infty \ \ (j = 1,2) \tag{26}$$

and

$$\det\big(I_n+(-1)^j d_jB(t)\big)\neq 0 \ \ \textit{for} \ \ t\in[0,\omega] \ \ (j=1,2; \ \ k=0,1,\dots) \tag{27}$$

hold and let the conditions (9),

$$\lim_{k\to+\infty} B_k(t)=B(t), \tag{28}$$

$$\lim_{k\to+\infty}\int_0^t Z_k^{-1}(s)\,d\mathcal{A}(B_k,A_{*k})(s)=\int_0^t Z^{-1}(s)\,d\mathcal{A}(B,A_*)(s) \tag{29}$$

and

$$\lim_{k\to+\infty}\int_0^t Z_k^{-1}(s)\,d\mathcal{A}(B_k,f_{*k})(s)=\int_0^t Z^{-1}(s)\,d\mathcal{A}(B,f_*)(s) \tag{30}$$

be fulfilled uniformly on $[0,\omega]$, *where* H, $H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, *and* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *satisfy the Lappo–Danilevskiĭ condition;* $A_{*k}(t)\equiv\mathcal{I}(H_k,A_k)(t)$ $(k=1,2,\dots)$,

$$f_{*k}(t)\equiv -H_k(t)\varphi_k(t)+H_k(0)\varphi_k(0)+\mathcal{B}(H_k,f_k)(t)+\int_0^t dA_{*k}(s)\cdot H_k(s)\varphi_k(s),$$
$$\varphi_k\in\mathrm{BV}([0,\omega],\mathbb{R}^n) \ \ (k=1,2,\dots),$$

and A_* *and* f_* *are the matrix- and vector-functions appearing in Theorem* $1'$. *Then the conclusion of Corollary* 1 *is true.*

In the Lappo–Danilevskiĭ case, for every $k\in\{1,2,\dots\}$, we have

$$Z_k(t)\equiv\exp(S_0(B_k)(t))\prod_{0\le\tau<t}\big(I_n+d_2B_k(\tau)\big)\prod_{0<\tau\le t}\big(I_n-d_1B_k(\tau)\big)^{-1}.$$

Corollary 7. *Let the conditions* (8), (19) *hold and let the conditions* (9), (15), (27) *and*

$$\lim_{k\to+\infty}\int_0^t \exp(-B_k(s))\,df_{*k}(s)=\int_0^t \exp(-B(s))\,df_*(s)$$

be fulfilled uniformly on $[0,\omega]$, *where* H, $H_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, *and* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *are the continuous matrix-functions satisfying the Lappo–Danilevskiĭ condition; and* A_*, A_{*k} *and* f_*, f_{*k}, φ_k $(k=1,2,\dots)$ *are, respectively, matrix- and vector-functions appearing in Corollary* 6. *Then the conclusion of Corollary* 1 *is true.*

Corollary 8. *Let there exist a sequence of matrix-functions* H *and* H_k $(k=0,1,\dots)$ *from* $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ *such that the matrix-functions* $S_c(A)$ *and* $S_c(A_{*k})$ $(k=1,2,\dots)$ *satisfy the Lappo–Danilevskiĭ condition and the conditions* (8) *and*

$$\lim_{k\to+\infty}\sup\sum_{0\le t\le\omega}\|d_jA_{*k}(t)\|<+\infty \ \ (j=1,2)$$

hold, let the conditions (9),

$$\lim_{k\to+\infty}S_c(A_{*k})(t)=S_c(A_*)(t), \quad \lim_{k\to+\infty}S_j(A_{*k})=S_j(A_*)(t) \ \ (j=1,2)$$

and

$$\lim_{k\to+\infty}\int_0^t \exp\big(-S_c(A_{*k})(s)\big)\,df_{*k})(s)=\int_0^t \exp\big(-S_c(A_{*k})(s)\big)\,df_*)(s)$$

be fulfilled uniformly on $[0,\omega]$, *where* A_*, A_{*k} *and* f_*, f_{*k}, φ_k $(k=1,2,\dots)$ *are, respectively, the matrix-and vector-functions appearing in Corollary* 6. *Then the conclusion of Corollary* 1 *is true.*

Theorem 2′. *The inclusion* (6) *is valid if and only if there exist the sequences of matrix-functions* H, H_k *and* B, $B_k \in \mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ *such that the conditions* (8), (20) *and*

$$\lim_{k\to+\infty}\sup\bigvee_a^b(\mathcal{I}(H_k,A_k)-B_k)<+\infty$$

hold, and the conditions (9), (21),

$$\lim_{k\to+\infty}\mathcal{B}\big(Z_k^{-1},\mathcal{I}(H_k,A_k)\big)(t)=\mathcal{B}\big(Z^{-1},\mathcal{I}(H,A)\big)(t)$$

and

$$\lim_{k\to+\infty}\mathcal{B}\big(Z_k^{-1},\mathcal{I}(H_k,f_k)\big)(t)=\mathcal{B}\big(Z^{-1},\mathcal{I}(H,f)\big)(t)$$

are fulfilled uniformly on $[0,\omega]$.

Corollary 9. *Let the conditions* (20) *and*

$$\lim_{k\to+\infty}\sup\bigvee_a^b(A_k-B_k)<+\infty \tag{31}$$

hold and the conditions (21),

$$\lim_{k\to+\infty}\mathcal{B}(Z_k^{-1},A_k)(t)=\mathcal{B}(Z^{-1},A)(t) \tag{32}$$

and

$$\lim_{k\to+\infty}\mathcal{B}(Z_k^{-1},f_k)(t)=\mathcal{B}(Z^{-1},f)(t) \tag{33}$$

be fulfilled uniformly on $[0,\omega]$, *where* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$. *Then the inclusion* (6) *holds.*

Corollary 10. *Let the conditions* (26), (27) *and* (31) *hold and the conditions* (29),

$$\lim_{k\to+\infty}\int_0^t Z_k^{-1}(s)\,d\mathcal{A}(B_k,A_k)(s)=\int_0^t Z^{-1}(s)\,d\mathcal{A}(B,A)(s)$$

and

$$\lim_{k\to+\infty}\int_0^t Z_k^{-1}(s)\,d\mathcal{A}(B_k,f_k)(s)=\int_0^t Z^{-1}(s)\,d\mathcal{A}(B,f)(s)$$

be fulfilled uniformly on $[0,\omega]$, *where* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *satisfy the Lappo–Danilevskiĭ condition. Then the inclusion* (6) *holds.*

Corollary 11. *Let the condition* (31) *hold and the conditions* (17), (29) *and*

$$\lim_{k\to+\infty}\int_0^t \exp(-B_k(s))\,df_k(s)=\int_0^t \exp(-B(s))\,df(s)$$

be fulfilled uniformly on $[0,\omega]$, *where* B *and* $B_k\in\mathrm{BV}([0,\omega],\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ *are the continuous matrix-function satisfying the Lappo–Danilevskiĭ condition. Then the inclusion* (6) *holds.*

Corollary 12. *Let the matrix-functions* $S_c(A)$ *and* $S_c(A_k)$ $(k=0,1,\dots)$, $A(t)\equiv A_0(t)$, *satisfy the Lappo–Danilevskiĭ condition and the condition*

$$\lim_{k\to+\infty}\sup\sum_{0\le t\le\omega}\|d_jA_k(t)\|<+\infty\ \ (j=1,2)$$

hold. Let, moreover, the conditions

$$\lim_{k\to+\infty}S_c(A_k)(t)=S_c(A)(t),\quad \lim_{k\to+\infty}S_j(A_k)=S_j(A)(t)\ \ (j=1,2)$$

and

$$\lim_{k\to+\infty}\int_0^t \exp\big(-S_c(A_k)(s)\big)\,df_k(s)=\int_0^t \exp\big(-S_c(A)(s)\big)\,df(s)$$

be fulfilled uniformly on $[0,\omega]$*. Then the inclusion* (6) *holds.*

Remark 1. The condition (8) is equivalent to the condition

$$\det\big(H(t-)\cdot H(t+)\big)\neq 0 \ \text{ for } \ t\in[0,\omega].$$

Remark 2. Let $A_*(t)\equiv\mathcal{I}(H,A)(t)$ and (9) be fulfilled uniformly on $[0,\omega]$. Then the condition (14) holds and (15) is fulfilled uniformly on $[0,\omega]$ if and only if the condition (7) holds and (10) is fulfilled uniformly on $[0,\omega]$, respectively.

Remark 3. Without loss of generality we can assume that $H(t)\equiv I_n$ in Theorems 1 and 1′ and in the above corollaries.

Remark 4. In designations of Theorem 1′:

(a) if (19) holds and the conditions (21),

$$\lim_{k\to+\infty}\int\limits_0^t Z_k^{-1}(s)\,d\big(A_{*k}(s)-B_k(s)\big)=\int\limits_0^t Z_k^{-1}(s)\,d\big(A_*(s)-B(s)\big) \tag{34}$$

and

$$\lim_{k\to+\infty} d_j\big(A_{*k}(t)-B_k(t)\big)=d_j\big(A_*(t)-B(t)\big) \ \ (j=1,2) \tag{35}$$

are fulfilled uniformly on $[0,\omega]$, then (22) is fulfilled uniformly on $[0,\omega]$, as well. On the other hand, if the condition (19) holds and the conditions (21) and

$$\lim_{k\to+\infty}\big(A_{*k}(t)-B_k(t)\big)=A_*(t)-B(t)$$

are fulfilled uniformly on $[0,\omega]$, then the conditions (34) and (35) are fulfilled uniformly on $[0,\omega]$, as well;

(b) if

$$\lim_{k\to+\infty}\sup\sum_{0\le t\le\omega}\|d_jf_{*k}(t)\|<+\infty \ \ (j=1,2)$$

and the conditions (21),

$$\lim_{k\to+\infty}\int\limits_0^t Z_k^{-1}(s)\,df_{*k}(s)=\int\limits_0^t Z_k^{-1}(s)\,df_*(s) \tag{36}$$

and

$$\lim_{k\to+\infty} d_jf_{*k}(t)=d_jf_*(t) \ \ (j=1,2) \tag{37}$$

are fulfilled uniformly on $[0,\omega]$, then the condition (24) is fulfilled uniformly on $[0,\omega]$, as well;

(c) if $B(t)\equiv A_*(t)$ and $B_k(t)\equiv A_{*k}(t)$ $(k=1,2,\dots)$, then (19) vanishes and (22) follows from (21).

Remark 5. In designations of Corollary 6:

(a) if (19) holds and (15) and (28) are fulfilled uniformly on $[0,\omega]$, then (29) is fulfilled uniformly on $[0,\omega]$, as well;

(b) if (26) and (27) holds and (28), (36) and (37) are fulfilled uniformly on $[0,\omega]$, then (30) is fulfilled uniformly on $[0,\omega]$, as well.

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