ON THE OPIAL TYPE CRITERION FOR THE WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Received February 3, 2016
Communicated by Alexandr Lomtatidze

Dedicated to the 90th birthday anniversary of Professor Jaroslav Kurzweil

Abstract. The Cauchy problem for the system of linear generalized ordinary differential equations in the J. Kurzweil sense $\mathrm{d}x(t) = \mathrm{d}A_0(t) \cdot x(t) + \mathrm{d}f_0(t)$, $x(t_0) = c_0$ $(t \in I)$ with a unique solution x_0 is considered. Necessary and sufficient conditions are obtained for a sequence of the Cauchy problems $\mathrm{d}x(t) = \mathrm{d}A_k(t) \cdot x(t) + \mathrm{d}f_k(t)$, $x(t_k) = c_k$ $(k = 1, 2, \ldots)$ to have a unique solution x_k for any sufficiently large k such that $x_k(t) \to x_0(t)$ uniformly on I. Presented results are analogous to the sufficient conditions due to Z. Opial for linear ordinary differential systems. Moreover, efficient sufficient conditions for the problem of well-posedness are given.

Keywords: linear system of generalized ordinary differential equations in the Kurzweil sense; Cauchy problem; well-posedness; Opial type necessary condition; Opial type sufficient condition; efficient sufficient condition

MSC 2010: 34A12, 34A30, 34K06

1. Statement of the problem and basic notation

Let $A_0 \in \mathrm{BV}_{\mathrm{loc}}(I; \mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}_{\mathrm{loc}}(I; \mathbb{R}^n)$ and $t_0 \in I$, where $I \subset \mathbb{R}$ is an arbitrary interval non-degenerated to a point. Let x_0 be a unique solution of the Cauchy problem

$$\mathrm{d}x(t) = \mathrm{d}A_0(t) \cdot x(t) + \mathrm{d}f_0(t) \quad \text{for } t \in I, \tag{1.1}$$

$$x(t_0) = c_0, \tag{1.2}$$

where $c_0 \in \mathbb{R}^n$ is a constant vector.

DOI: 10.21136/MB.2016.15

Along with the Cauchy problem (1.1), (1.2), we consider the sequence of Cauchy problems

$$\mathrm{d}x(t) = \mathrm{d}A_k(t) \cdot x(t) + \mathrm{d}f_k(t), \tag{1.1$_k$}$$

$$x(t_k) = c_k \tag{1.2$_k$}$$

($k = 1, 2, \ldots$), where $A_k \in \mathrm{BV}_{\mathrm{loc}}(I; \mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}_{\mathrm{loc}}(I; \mathbb{R}^n)$, $t_k \in I$ and $c_k \in \mathbb{R}^n$ ($k = 1, 2, \ldots$).

In the paper we establish necessary and sufficient and efficient sufficient conditions for the Cauchy problem (1.1_k), (1.2_k) to have a unique solution x_k for every sufficiently large k such that

$$\lim_{k\to\infty} x_k(t) = x_0(t) \quad \text{uniformly on } I. \tag{1.3}$$

The obtained necessary and sufficient criterion has the Opial type form considered in [20] for the case of ordinary differential equations and it differs from analogous ones given in [6], [3], [8], [23] for linear generalized differential systems. The Opial type sufficient condition for the well-posedness is obtained in [19] for linear generalized differential equations in the Banach space.

Some well-posedness problems for the linear and nonlinear boundary value problems for generalized differential equations are studied in [9], [4], [10], [17], [16], [15], [19], [23] (see also the references therein).

Analogous questions for the Cauchy problem and linear and nonlinear boundary value problems for systems of ordinary differential equations are studied in [7], [13], [12], [14], [18], [20] (see also references therein).

The idea of the theory of generalized ordinary differential equations belongs to Kurzweil (see [17], [16], [15]). In [17] he investigated the well-posedness question for the Cauchy problem for linear ordinary differential systems and constructed an example of a sequence of problems whose sequence of solutions (absolutely continuous) converges to the discontinuity function (it is evident that the convergence is not uniform). Kurzweil constructed some types of integral and differential equations (so-called generalized ordinary differential equations) such that the above mentioned discontinuous "limit" function is a solution of some generalized equation. Moreover, from the theorem on the well-posedness (in the pointwise sense) of the Cauchy problem for generalized differential equations the above convergence process follows as a particular case.

To a considerable extent, the interest in the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate linear ordinary differential, impulsive and difference equations from a unified point of view; in particular, these equations of various types can be rewritten in form (1.1). Moreover, the convergence conditions for difference schemes corresponding to systems of ordinary differential and impulsive equations can be obtained from the results on the well-posedness (in the uniform sense) of the corresponding problems for systems of generalized ordinary differential equations (see [5], [2], [1], [11], [21], [22] and the references therein).

In the paper, the use will be made of the following notation and definitions.

- ▷ $\mathbb{R} =]-\infty, \infty[$; $[a, b]$ and $]a, b[$ are, respectively, closed and open intervals.
- ▷ I is an arbitrary, non-degenerated to a point, finite or infinite interval from $\mathbb{R}$, and $\xi \in I$ is a fixed point.
- ▷ $\mathbb{R}^{n\times m}$ is the space of all real $(n \times m)$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

- ▷ $O_{n\times m}$ is the zero $(n \times m)$-matrix.
- ▷ $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all column n-vectors $x = (x_i)_{i=1}^{n}$; o_n is the zero n-vector.
- ▷ $\mathbb{R}^{n\times n}$ is the space of all real quadratic $(n \times n)$-matrices $X = (x_{ij})_{i,j=1}^{n}$.
- ▷ I_n is the identity $(n \times n)$-matrix; $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$; δ_{ij} is the Kronecker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, \dots)$.
- ▷ If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X; $\operatorname{diag} X = \operatorname{diag}(x_{11}, \dots, x_{nn})$ is the diagonal matrix corresponding to X.
- ▷ A matrix-function is said to be continuous, integrable, nondecreasing, etc., if such is each of its component.
- ▷ $\bigvee\limits_a^b(X)$ is the sum of total variations of the components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$ of the matrix-function $X\colon [a, b] \to \mathbb{R}^{n\times m}$, $\bigvee\limits_b^a(X) = -\bigvee\limits_a^b(X)$.
- ▷ $\bigvee\limits_I(X) = \lim\limits_{a\to\alpha+,b\to\beta-} \bigvee\limits_a^b(X)$, where $\alpha = \inf I$ and $\beta = \sup I$.
- ▷ $V(X)(t) = (V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(t) = \bigvee\limits_{t_0}^{t}(x_{ij})$ for $t \in I$ $(i = 1, \dots, n;\ j = 1, \dots, m)$.
- ▷ $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t $(X(\alpha-) = X(\alpha)$ if $\alpha \in I$ and $X(\beta+) = X(\beta)$ if $\beta \in I)$.
- ▷ $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.
- ▷ $\mathrm{BV}(I; \mathbb{R}^{n\times m})$ is the normed space of all bounded variation matrix-functions $X\colon I \to \mathbb{R}^{n\times m}$ (i.e. such that $\bigvee\limits_I(X) < \infty$) with the norm $\|X\|_s = \sup\{\|X(t)\|\colon t \in I\}$.

- $\triangleright$ $\mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times m})$ is the set of all $X\colon I\to\mathbb{R}^{n\times m}$ for which the restriction to $[a,b]$ belongs to $\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ for every closed interval $[a,b]$ from I.
- $\triangleright$ $s_c, s_j\colon \mathrm{BV}_{\mathrm{loc}}(I,\mathbb{R})\to\mathrm{BV}_{\mathrm{loc}}(I,\mathbb{R})$ $(j=1,2)$ are the operators defined, respectively, by $s_c(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)$ for $t\in I$; $s_1(x)(\xi)=s_2(x)(\xi)=0$,

$$s_1(x)(t)-s_1(x)(s)=\sum_{s<\tau\leqslant t} d_1x(\tau)$$

and

$$s_2(x)(t)-s_2(x)(s)=\sum_{s\leqslant\tau<t} d_2x(\tau)\quad\text{if } s<t.$$

- $\triangleright$ If $g\colon I\to\mathbb{R}$ is a nondecreasing function, $x\colon I\to\mathbb{R}$ and $s<t$, $s,t\in I$, then

$$\int_s^t x(\tau)\,\mathrm{d}g(\tau)=\int_{]s,t[} x(\tau)\,\mathrm{d}s_c(g)(\tau)+\sum_{s<\tau\leqslant t}x(\tau)d_1g(\tau)+\sum_{s\leqslant\tau<t}x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,\mathrm{d}s_c(g)(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$. We assume

$$\int_s^t x(t)\,\mathrm{d}g(t)=0\quad\text{if } s=t,\quad\text{and}\quad\int_s^t x(t)\,\mathrm{d}g(t)=-\int_t^s x(t)\,\mathrm{d}g(t)\quad\text{if } t>s.$$

Thus the integral considered is the Kurzweil-Stielties one (see [17], [16], [21], [22]).
- $\triangleright$ If $g(t)\equiv g_1(t)-g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,\mathrm{d}g(\tau)=\int_s^t x(\tau)\,\mathrm{d}g_1(\tau)-\int_s^t x(\tau)\,\mathrm{d}g_2(\tau)\quad\text{for } s\leqslant t.$$

- $\triangleright$ If $G=(g_{ik})_{i,k=1}^{l,n}\in\mathrm{BV}(I;\mathbb{R}^{l\times n})$, $X=(x_{kj})_{k,j=1}^{n,m}\in\mathrm{BV}(I;\mathbb{R}^{n\times m})$ and $Y,Z\in\mathrm{BV}(I;\mathbb{R}^{n\times n})$, and $Z(t)$ is nonsingular for $t\in I$, then

$$S_c(G)(t)=(s_c(g_{ik})(t))_{i,k=1}^{l,n},\ S_j(G)(t)=(s_j(g_{ik})(t))_{i,k=1}^{l,n}\quad\text{for } t\in I\ (j=1,2),$$
$$\int_s^t \mathrm{d}G(\tau)\cdot X(\tau)=\left(\sum_{k=1}^n\int_s^t x_{kj}(\tau)\,\mathrm{d}g_{ik}(\tau)\right)_{i,j=1}^{l,m}\quad\text{for } s,t\in I,$$
$$\mathcal{B}(G,X)(t)=G(t)X(t)-G(\xi)X(\xi)-\int_\xi^t \mathrm{d}G(\tau)\cdot X(\tau)\quad\text{for } t\in I,$$
$$\mathcal{I}(Y,Z)(t)=\int_\xi^t \mathrm{d}(Y(\tau)+\mathcal{B}(Y,Z)(\tau))\cdot Z^{-1}(\tau)\quad\text{for } t\in I.$$

▷ If $X \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times n})$, $\det(I_n + (-1)^j d_j X(t)) \neq 0$ for $t \in I$ $(j = 1,2)$, and $Y \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times m})$, then

$$
\begin{aligned}
(1.4)\quad & \mathcal{A}(X,Y)(\xi) = O_{n\times m}, \\
& \mathcal{A}(X,Y)(t) - \mathcal{A}(X,Y)(s) = Y(t) - Y(s) \\
& \qquad + \sum_{s<\tau\leqslant t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1} d_1Y(\tau) \\
& \qquad - \sum_{s\leqslant\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1} d_2Y(\tau) \quad \text{if } s<t.
\end{aligned}
$$

▷ We say that a matrix-function $X \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ satisfies the Lappo-Danilevskiĭ condition if the matrices $S_c(X)(t)$, $S_1(X)(t)$ and $S_2(X)(t)$ are pairwise permutable and

$$\int_s^t S_c(X)(\tau)\,\mathrm{d}S_c(X)(\tau) = \int_s^t \mathrm{d}S_c(X)(\tau)\cdot S_c(X)(\tau) \quad \text{for } s,t\in[a,b].$$

For $f,g \in \mathrm{BV}([a,b];\mathbb{R})$ and $t\in[a,b]$, the use will be made of the following formulas:

$$(1.5)\quad \int_a^b f(t)\,\mathrm{d}g(t) = \int_a^b f(t)\,\mathrm{d}g(t-) + f(b)d_1g(b) = \int_a^b f(t)\,\mathrm{d}g(t+) + f(a)d_2g(a),$$

$$
\begin{aligned}
(1.6)\quad & \int_a^b f(t)\,\mathrm{d}g(t) + \int_a^b f(t)\,\mathrm{d}g(t) = f(b)g(b) - f(a)g(a) + \sum_{a<t\leqslant b} d_1f(t)\cdot d_1g(t) \\
& \qquad - \sum_{a\leqslant t<b} d_2f(t)\cdot d_2g(t) \qquad \text{(integration-by-parts formula)},
\end{aligned}
$$

$$
\begin{aligned}
(1.7)\quad & \int_a^b h(t)\,\mathrm{d}(f(t)g(t)) = \int_a^b h(t)f(t)\,\mathrm{d}g(t) + \int_a^b h(t)g(t)\,\mathrm{d}f(t) \\
& \qquad - \sum_{a<t\leqslant b} h(t)d_1f(t)\cdot d_1g(t) + \sum_{a\leqslant t<b} h(t)d_2f(t)\cdot d_2g(t) \\
& \qquad \text{(general integration-by-parts formula)},
\end{aligned}
$$

$$(1.8)\quad \int_a^b f(t)\,\mathrm{d}s_1(g)(t) = \sum_{a<t\leqslant b} f(t)d_1g(t),\ \int_a^b f(t)\,\mathrm{d}s_2(g)(t) = \sum_{a\leqslant t<b} f(t)d_2g(t),$$

$$(1.9)\quad \int_a^b f(t)\,\mathrm{d}\Big(\int_a^t g(s)\,\mathrm{d}h(s)\Big) = \int_a^b f(t)g(t)\,\mathrm{d}h(t),$$

$$(1.10)\quad d_j\Big(\int_a^t f(s)\,\mathrm{d}g(s)\Big) = f(t)d_jg(t) \quad \text{for } j=1,2.$$

▷ If Y ($Y(a) = I_n, a \in I$) is the fundamental matrix of system (1.1), then (see e.g. [22], Proposition I.2.15)

$$Y^{-1}(t) = I + A(a) - Y^{-1}(t)A(t) + \int_a^t \mathrm{d}Y^{-1}(\tau) \cdot A(\tau) \quad \text{for } t \in I, \tag{1.11}$$

$$d_j X^{-1}(t) = -X^{-1}(t) d_j A(t) \cdot (I_n + (-1)^j d_j A(t))^{-1} \quad \text{for } t \in I \ (j = 1, 2). \tag{1.12}$$

The proofs of formulas (1.5), (1.6), (1.8) and (1.9) can be found e.g. in [22]. As to formula (1.7), it can be easily shown using the integration-by-parts formula (1.6).

If $t \in I$, then we denote $I_t = I \setminus \{t\}$. Moreover, we use the notation

$$\|x\|_{kj} = \sup\{\|x(t)\| \colon t \in I_{kj}\} \quad \text{for } x \in \mathrm{BV}_{\mathrm{loc}}(I; \mathbb{R}^n) \ (j = 1, 2; \ k = 0, 1, \ldots),$$

where $I_{kj} = \{t \in I \colon (-1)^j (t - t_k) > 0\}$ $(j = 1, 2; k = 0, 1, \ldots)$.

We will assume that $A_k = (a_{kil})_{i,l=1}^n$ and $f_k = (f_{kil})_{i,l=1}^n$ $(k = 0, 1, \ldots)$, and, without loss of generality, either $t_k < t_0$ $(k = 1, 2, \ldots)$, or $t_k = t_0$ $(k = 1, 2, \ldots)$, or $t_k > t_0$ $(k = 1, 2, \ldots)$.

Along with systems (1.1) and (1.1_k) $(k = 1, 2, \ldots)$, we consider the corresponding homogeneous systems

$$\mathrm{d}x(t) = \mathrm{d}A_0(t) \cdot x(t) \tag{1.1_0}$$

and

$$\mathrm{d}x(t) = \mathrm{d}A_k(t) \cdot x(t). \tag{1.1_{k0}}$$

2. Formulation of the main results

Definition 2.1. We say that a sequence $(A_k, f_k; t_k)$ $(k = 1, 2, \ldots)$ belongs to the set $\mathcal{S}(A_0, f_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k = 1, 2, \ldots)$ such that

$$\lim_{k \to \infty} c_k = c_0, \tag{2.1}$$

problem (1.1_k), (1.2_k) has a unique solution x_k for any sufficiently large k and condition (1.3) holds.

We also consider the case when

$$\lim_{k \to \infty} c_{kj} = c_{0j} \quad \text{if } j \in \{1, 2\} \text{ is such that } (-1)^j (t_k - t_0) \geqslant 0 \ (k = 0, 1, \ldots), \tag{2.1_j}$$

where

$$(2.2)\qquad c_{kj} = c_k + (-1)^j(d_j A_k(t_k)c_k + d_j f_k(t_k)) \quad (j = 1, 2;\ k = 0, 1, \dots).$$

Note that if

$$\lim_{k\to\infty} d_j A_k(t_k) = d_j A_0(t_0)$$

for some $j \in \{1, 2\}$, then condition (2.1_j) follows from (2.1).

Theorem 2.1. *Let $A_0 \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I; \mathbb{R}^n)$ and $t_k \in I$ $(k = 0, 1, \dots)$ be such that*

$$(2.3)\qquad \det(I_n + (-1)^j d_j A_0(t)) \neq 0 \quad \textit{for } t \in I,\ (-1)^j(t - t_0) < 0,\ \textit{and also}$$
$$\textit{for } t = t_0 \textit{ if } j \in \{1, 2\} \textit{ is such that } (-1)^j(t_k - t_0) > 0 \textit{ for every } k \in \{1, 2, \dots\},$$

and

$$(2.4)\qquad \lim_{k\to\infty} t_k = t_0.$$

Then

$$(2.5)\qquad ((A_k, f_k; t_k))_{k=1}^{\infty} \in \mathcal{S}(A_0, f_0; t_0)$$

if and only if there exists a sequence of matrix-functions $H_k \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$ such that

$$(2.6)\qquad \inf\{|\det(H_0(t))|\colon\ t \in I\} > 0,$$

and the conditions

$$(2.7)\qquad \lim_{k\to\infty} H_k(t) = H_0(t),$$

$$(2.8)\quad \lim_{k\to\infty}\left\{\|\mathcal{I}(H_k, A_k)(\tau)|_{t_k}^{t} - \mathcal{I}(H_0, A_0)(\tau)|_{t_0}^{t}\|\left(1 + \left|\bigvee_{t_k}^{t}(\mathcal{I}(H_k, A_k))\right|\right)\right\} = 0$$

and

$$(2.9)\quad \lim_{k\to\infty}\left\{\|\mathcal{B}(H_k, f_k)(\tau)|_{t_k}^{t} - \mathcal{B}(H_0, f_0)(\tau)|_{t_0}^{t}\|\left(1 + \left|\bigvee_{t_k}^{t}(\mathcal{I}(H_k, A_k))\right|\right)\right\} = 0$$

hold uniformly on I.

Theorem 2.2. *Let $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k = 0,1,\dots)$ be such that conditions* (2.1), (2.1$_j$), (2.3) *and* (2.4) *hold, and the conditions*

$$\lim_{k\to\infty} \sup_{t\in I, t\neq t_k} \left\{ \|A_{kj}(t) - A_{0j}(t)\| \left(1 + \left|\bigvee_{t_k}^{t}(A_k)\right|\right)\right\} = 0 \tag{2.10}$$

and

$$\lim_{k\to\infty} \sup_{t\in I, t\neq t_k} \left\{ \|f_{kj}(t) - f_{0j}(t)\| \left(1 + \left|\bigvee_{t_k}^{t}(A_k)\right|\right)\right\} = 0 \tag{2.11}$$

are fulfilled for some $j \in \{1,2\}$, where c_{kj} $(k = 0,1,\dots)$ are defined by (2.2),

$$A_{kj}(t) \equiv (-1)^j (A_k(t) - A_k(t_k)) - d_j A_k(t_k) \quad (j = 1,2;\ k = 0,1,\dots)$$

and

$$f_{kj}(t) \equiv (-1)^j (f_k(t) - f_k(t_k)) - d_j f_k(t_k) \quad (j = 1,2; k = 0,1,\dots).$$

Then the Cauchy problem (1.1$_k$), (1.2$_k$) *has a unique solution x_k for any sufficiently large k and*

$$\lim_{k\to\infty} \sup_{t\in I, t\neq t_k} \{\|x_k(t) - x_0(t)\|\} = 0. \tag{2.12}$$

R e m a r k 2.1. In Theorem 2.2, it is evident that the sequence x_k $(k = 1,2,\dots)$ converges to x_0 uniformly on the set $\{t \in I, t \leqslant t_0\}$ if $t_k > t_0$ $(k = 1,2,\dots)$, and on the set $\{t \in I, t \geqslant t_0\}$ if $t_k < t_0$ $(k = 1,2,\dots)$. Moreover, in Theorem 2.2, if conditions (2.10) and (2.11) hold uniformly on the set I instead of the sets I_{t_k} $(k = 1,2,\dots)$, then these conditions are equivalent, respectively, to the conditions

$$\lim_{k\to\infty} \left\{ \|(A_k(t) - A_k(t_k)) - (A_0(t) - A_0(t_0))\| \left(1 + \left|\bigvee_{t_k}^{t}(A_k)\right|\right)\right\} = 0 \tag{2.13}$$

and

$$\lim_{k\to\infty} \left\{ \|(f_k(t) - f_k(t_k)) - (f_0(t) - f_0(t_0))\| \left(1 + \left|\bigvee_{t_k}^{t}(A_k)\right|\right)\right\} = 0 \tag{2.14}$$

uniformly on I, since (2.10) and (2.11) imply that

$$\lim_{k\to\infty} d_j A_k(t) = d_j A_0(t) \quad \text{and} \quad \lim_{k\to\infty} d_j f_k(t) = d_j f_0(t)$$

uniformly on I for every $j \in \{1,2\}$. In addition, by (2.2), conditions (2.1$_j$) $(j = 1,2)$ immediately follow from the last equalities. Thus, in this case, condition (1.3) holds.

Theorem 2.3. *Let $A_0^*, A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$; $f_0^*, f_k \in \mathrm{BV}(I;\mathbb{R}^n)$; $c_0^*, c_k \in \mathbb{R}^n$ and $t_0, t_k \in I$ $(k=1,2,\dots)$ be such that condition* (2.4) *holds,*

$$\det(I_n + (-1)^j d_j A_0^*(t)) \neq 0 \quad \text{for } t \in I,\ (-1)^j(t-t_0) < 0, \text{ and also}$$
$$\text{for } t = t_0 \text{ if } j \in \{1,2\} \text{ is such that } (-1)^j(t_k - t_0) > 0 \text{ for every } k \in \{1,2,\dots\}, \tag{2.15}$$

the Cauchy problem

$$\mathrm{d}x(t) = \mathrm{d}A_0^*(t)\cdot x(t) + \mathrm{d}f_0^*(t), \tag{2.16}$$
$$x(t_0) = c_0^* \tag{2.17}$$

has a unique solution x_0^* and there exist sequences $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $h_k \in \mathrm{BV}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ such that the conditions

$$\inf\{|\det(H_k(t)|\colon\ t \in I_{t_k}\} > 0 \text{ for every sufficiently large } k, \tag{2.18}$$
$$\lim_{k\to\infty} c_k^* = c_0^*, \quad \lim_{k\to\infty} c_{kj}^* = c_{0j}^*, \tag{2.19}$$
$$\lim_{k\to\infty} \sup_{t\in I, t\neq t_k} \left\{ \|A_{kj}^*(t) - A_{0j}^*(t)\| \left(1 + \left|\bigvee_{t_k}^{t}(A_k^*)\right|\right)\right\} = 0 \tag{2.20}$$

and

$$\lim_{k\to\infty} \sup_{t\in I, t\neq t_k} \left\{ \|f_{kj}^*(t) - f_{0j}^*\| \left(1 + \left|\bigvee_{t_k}^{t}(A_k^*)\right|\right)\right\} = 0 \tag{2.21}$$

hold for some $j \in \{1,2\}$, where

$$A_{kj}^*(t) = (-1)^j(A_k^*(t) - A_k^*(t_k)) - d_j A_k^*(t_k)$$

and

$$f_{kj}^*(t) = (-1)^j(f_k^*(t) - f_k^*(t_k)) - d_j f_k^*(t_k) \quad \text{for } t \in I \ (j=1,2; k=0,1,\dots),$$
$$A_k^*(t) = \mathcal{I}(H_k, A_k)(t)$$

and

$$f_k^*(t) = h_k(t) - h_k(t_k) + \mathcal{B}(H_k, f_k)(t) - \mathcal{B}(H_k, f_k)(t_k)$$
$$- \int_{t_k}^{t} \mathrm{d}A_k^*(s)\cdot h_k(s) \quad \text{for } t \in I \ (k=1,2,\dots);$$
$$c_k^* = H_k(t_k)c_k + h_k(t_k) \quad (k=1,2,\dots),$$
$$c_{kj}^* = c_k^* + (-1)^j(d_j A_k^*(t_k)c_k^* + d_j f_k^*(t_k)) \quad (j=1,2;\ k=0,1,\dots).$$

Then problem (1.1_k), (1.2_k) *has a unique solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to\infty} \sup_{t\in I, t\neq t_k} \{\|H_k(t)x_k(t) + h_k(t) - x_0^*(t)\|\} = 0. \tag{2.22}$$

Remark 2.2. In Theorem 2.3, the vector-function $x_k^*(t) = H_k(t)x_k(t) + h_k(t)$ for every sufficiently large k is a solution of the problem

$$\mathrm{d}x(t) = \mathrm{d}A_k^*(t) \cdot x(t) + \mathrm{d}f_k^*(t), \tag{2.16_k}$$

$$x(t_k) = c_k^*. \tag{2.17_k}$$

Below we consider, mainly, the well-posedness question on the whole interval I. For the last case, in view of Remark 2.1 conditions (2.20) and (2.21) have, respectively, the form

$$\lim_{k\to\infty} \left\{ \|(A_k^*(t) - A_k^*(t_k)) - (A_0^*(t) - A_0^*(t_0))\| \left(1 + \left|\bigvee_{t_k}^{t}(A_k^*)\right|\right)\right\} = 0 \tag{2.23}$$

and

$$\lim_{k\to\infty} \left\{ \|(f_k^*(t) - f_k^*(t_k)) - (f_0^*(t) - f_0^*(t_0))\| \left(1 + \left|\bigvee_{t_k}^{t}(A_k^*)\right|\right)\right\} = 0 \tag{2.24}$$

uniformly on I.

Corollary 2.1. *Let* $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ *and* $t_k \in I$ $(k = 0, 1, \ldots)$ *be such that conditions* (2.3), (2.4), (2.6) *and*

$$\lim_{k\to\infty} (c_k - \varphi_k(t_k)) = c_0 \tag{2.25}$$

hold, and conditions (2.7), (2.8) *and*

$$\lim_{k\to\infty} \left\{ \left\| \mathcal{B}(H_k, f_k - \varphi_k)(\tau)\Big|_{t_k}^{t} - \mathcal{B}(H_0, f_0)(\tau)\Big|_{t_0}^{t} + \int_{t_k}^{t} \mathrm{d}\mathcal{I}(H_k, A_k)(\tau) \cdot \varphi_k(\tau) \right\| \left(1 + \left|\bigvee_{t_k}^{t}(\mathcal{I}(H_k, A_k))\right|\right)\right\} = 0 \tag{2.26}$$

are fulfilled uniformly on I, *where* $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ *and* $\varphi_k \in \mathrm{BV}(I;\mathbb{R}^n)$ $(k = 0, 1, \ldots)$. *Then problem* (1.1_k), (1.2_k) *has a unique solution* x_k *for any sufficiently large* k *and*

$$\lim_{k\to\infty} (x_k(t) - \varphi_k(t)) = x_0(t) \text{ uniformly on } I. \tag{2.27}$$

Below, we give some sufficient conditions guaranteeing inclusion (2.5). To this end we establish a theorem different from Theorem 2.1 concerning the necessary and sufficient conditions for inclusion (2.5) as well, and the corresponding propositions.

Theorem 2.1′. *Let* $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$ *and* $t_k \in I$ $(k = 0,1,\ldots)$ *be such that conditions* (2.3) *and* (2.4) *hold. Then inclusion* (2.5) *holds if and only if there exists a sequence of matrix-functions* $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k = 0,1,\ldots)$ *such that conditions* (2.6) *and*

$$\limsup_{k\to\infty} \bigvee_I (H_k + \mathcal{B}(H_k, A_k)) < \infty \tag{2.28}$$

hold, and conditions (2.7),

$$\lim_{k\to\infty} (\mathcal{B}(H_k, A_k)(t) - \mathcal{B}(H_k, A_k)(t_k)) = \mathcal{B}(H_0, A_0)(t) - \mathcal{B}(H_0, A_0)(t_0) \tag{2.29}$$

and

$$\lim_{k\to\infty} (\mathcal{B}(H_k, f_k)(t) - \mathcal{B}(H_k, f_k)(t_k)) = \mathcal{B}(H_0, f_0)(t) - \mathcal{B}(H_0, f_0)(t_0) \tag{2.30}$$

are fulfilled uniformly on I.

Remark 2.3. Due to (2.6), (2.7), there exists a positive number r such that

$$\sup\left\{\left|\bigvee_{t_0}^{t}(\mathcal{I}(H_k, A_k))\colon\ t \in I\right|\right\} \leqslant r \bigvee_I (H_k + \mathcal{B}(H_k, A_k)) \quad (k = 0,1,\ldots).$$

Further, in view of Lemma 3.3 (see below), by conditions (2.28) and (2.29) we get

$$\lim_{k\to\infty} (\mathcal{I}(H_k, A_k)(t) - \mathcal{I}(H_k, A_k)(t_k)) = \mathcal{I}(H_0, A_0)(t) - \mathcal{I}(H_0, A_0)(t_0)$$

uniformly on I. Therefore, thanks to this, (2.28) and (2.30), conditions (2.8) and (2.9) are fulfilled uniformly on I

Theorem 2.2′. *Let* $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ *and* $t_k \in I$ $(k = 0,1,\ldots)$ *be such that conditions* (2.1), (2.3), (2.4) *and*

$$\limsup_{k\to\infty} \bigvee_I (A_k) < \infty \tag{2.31}$$

hold, and the conditions

$$\lim_{k\to\infty} (A_k(t) - A_k(t_k)) = A_0(t) - A_0(t_0) \tag{2.32}$$

and

$$\lim_{k\to\infty}(f_k(t)-f_k(t_k))=f_0(t)-f_0(t_0) \tag{2.33}$$

are fulfilled uniformly on I. Then the Cauchy problem (1.1_k), (1.2_k) *has a unique solution x_k for any sufficiently large k and condition* (1.3) *holds.*

Theorem 2.3′. *Let $A_0^*, A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0^*, f_k \in \mathrm{BV}(I;\mathbb{R}^n)$, $c_0^*, c_k \in \mathbb{R}^n$ and $t_0, t_k \in I$ $(k=1,2,\dots)$ be such that conditions* (2.4) *and* (2.15) *hold, the Cauchy problem* (2.16), (2.17) *has a unique solution x_0^* and there exist sequences $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$ and $h_k \in \mathrm{BV}(I;\mathbb{R}^n)$ $(k=1,2,\dots)$ such that conditions* (2.18),

$$\lim_{k\to\infty}(H_k(t_k)c_k+h_k(t_k))=c_0^* \tag{2.34}$$

and

$$\lim_{k\to\infty}\sup\bigvee_I(A_k^*)<\infty \tag{2.35}$$

hold, and the conditions

$$\lim_{k\to\infty}(A_k^*(t)-A_k^*(t_k))=A_0^*(t)-A_0^*(t_0) \tag{2.36}$$

and

$$\lim_{k\to\infty}(f_k^*(t)-f_k^*(t_k))=f_0^*(t)-f_0^*(t_0) \tag{2.37}$$

are fulfilled uniformly on I, where the matrix- and vector-functions A_k^* and f_k^* $(k=1,2,\dots)$ are defined as in Theorem 2.3. *Then problem* (1.1_k), (1.2_k) *has a unique solution x_k for any sufficiently large k and condition* (2.22) *holds uniformly on I.*

Corollary 2.1′. *Let $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k=0,1,\dots)$ be such that conditions* (2.3), (2.4), (2.6), (2.25) *and* (2.28) *hold, and conditions* (2.7), (2.29) *and*

$$\lim_{k\to\infty}\left(\mathcal{B}(H_k,f_k-\varphi_k)(\tau)\big|_{t_k}^{t}+\int_{t_k}^{t}\mathrm{d}\mathcal{B}(H_k,A_k)(\tau)\cdot\varphi_k(\tau)\right)=\mathcal{B}(H_0,f_0)(\tau)\big|_{t_0}^{t} \tag{2.38}$$

are fulfilled uniformly on I, where $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ and $\varphi_k \in \mathrm{BV}(I;\mathbb{R}^n)$ $(k=0,1,\dots)$. Then problem (1.1_k), (1.2_k) *has a unique solution x_k for any sufficiently large k and condition* (2.27) *holds.*

Corollary 2.2. *Let $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$ and $t_k \in I$ $(k=0,1,\dots)$ be such that conditions* (2.3), (2.4), (2.6) *and* (2.28) *hold, and conditions* (2.7),

$$\lim_{k\to\infty}\int_{t_k}^{t} H_k(s)\,\mathrm{d}A_k(s) = \int_{t_0}^{t} H_0(s)\,\mathrm{d}A_0(s), \tag{2.39}$$

$$\lim_{k\to\infty}\int_{t_k}^{t} H_k(s)\,\mathrm{d}f_k(s) = \int_{t_0}^{t} H_0(s)\,\mathrm{d}f_0(s), \tag{2.40}$$

$$\lim_{k\to\infty} d_j A_k(t) = d_j A_0(t) \quad (j=1,2) \tag{2.41}$$

and

$$\lim_{k\to\infty} d_j f_k(t) = d_j f_0(t) \quad (j=1,2) \tag{2.42}$$

are fulfilled uniformly on I, where $H_k \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$. Let, moreover, either

$$\lim_{k\to\infty}\sup_{t\in I}\{\|d_j A_k(t)\| + \|d_j f_k(t)\|\} < \infty \quad (j=1,2) \tag{2.43}$$

or

$$\lim_{k\to\infty}\sup_{t\in I}\{\|d_j H_k(t)\|\} < \infty \quad (j=1,2). \tag{2.44}$$

Then inclusion (2.5) *holds.*

Corollary 2.3. *Let $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$ and $t_k \in I$ $(k=0,1,\dots)$ be such that conditions* (2.3), (2.4), (2.6) *and* (2.28) *hold, and conditions* (2.7), (2.32), (2.33),

$$\lim_{k\to\infty}\int_{t_k}^{t} \mathrm{d}H_k(s)\cdot A_k(s) = A^*(t) - A^*(t_0) \tag{2.45}$$

and

$$\lim_{k\to\infty}\int_{t_k}^{t} \mathrm{d}H_k(s)\cdot f_k(s) = f^*(t) - f^*(t_0) \tag{2.46}$$

are fulfilled uniformly on I, where $H_0(t) = I_n$, $H_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=1,2,\dots)$, $A^* \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f^* \in \mathrm{BV}(I;\mathbb{R}^n)$. Let, moreover, problem (2.16), (1.2), *where $A_0^*(t) = A_0(t) - A^*(t)$ and $f_0^*(t) = f_0(t) - f^*(t)$, have a unique solution x_0^*. Then*

$$((A_k, f_k; t_k))_{k=1}^{\infty} \in \mathcal{S}(A_0 - A^*, f_0 - f^*; t_0).$$

Corollary 2.4. *Let $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$ and $t_k \in I$ $(k=0,1,\dots)$ be such that conditions* (2.3) *and* (2.4) *hold and there exist a natural number m and matrix-functions $B_l \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(l=1,\dots,m-1)$ such that*

$$\lim_{k\to\infty} \sup \bigvee_I (A_{km}) < \infty, \tag{2.47}$$

and the conditions

$$\lim_{k\to\infty} H_{km-1}(t) = I_n, \tag{2.48}$$

$$\lim_{k\to\infty} (A_{km}(t) - A_{km}(t_k)) = A_0(t) - A_0(t_0), \tag{2.49}$$

$$\lim_{k\to\infty} (f_{km}(t) - f_{km}(t_k)) = f_0(t) - f_0(t_0) \tag{2.50}$$

are fulfilled uniformly on I, where

$$H_{k0}(t) = I_n,\ \ H_{kj+1}(t) = \big(I_n - (A_{kl}(t) - A_{kl}(t_k)) + (B_l(t) - B_l(t_k))\big)H_{kj}(t),$$
$$A_{kj+1}(t) = H_{kj}(t) + \mathcal{B}(H_{kj}, A_k)(t)$$

and

$$f_{kj+1}(t) = \mathcal{B}(H_{kj}, f_k)(t) \quad \text{for } t \in I \ (j=0,\dots,m-1).$$

Then inclusion (2.5) *holds.*

If $m=1$, then Corollary 2.1 coincides with Theorem 2.2′.

Corollary 2.5. *Let $A_0 \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_0 \in \mathrm{BV}(I;\mathbb{R}^n)$ and $t_k \in I$ $(k=0,1,\dots)$ be such that conditions* (2.3) *and* (2.4) *hold. Then inclusion* (2.5) *holds if and only if there exist matrix-functions $B_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\dots)$ such that*

$$\lim_{k\to\infty} \sup \bigvee_I (A_k - B_k) < \infty \tag{2.51}$$

and

$$\det(I_n + (-1)^j d_j B_k(t)) \neq 0 \quad \text{for } t \in I \ (j=1,2;\ k=0,1,\dots), \tag{2.52}$$

and the conditions

$$\lim_{k\to\infty} Z_k^{-1}(t) = Z_0^{-1}(t), \tag{2.53}$$

$$\lim_{k\to\infty} (\mathcal{B}(Z_k^{-1}, A_k)(t) - \mathcal{B}(Z_k^{-1}, A_k)(t_k)) = \mathcal{B}(Z_0^{-1}, A_0)(t) - \mathcal{B}(Z_0^{-1}, A_0)(t_0) \tag{2.54}$$

and

$$\lim_{k\to\infty} (\mathcal{B}(Z_k^{-1}, f_k)(t) - \mathcal{B}(Z_k^{-1}, f_k)(t_k)) = \mathcal{B}(Z_0^{-1}, f_0)(t) - \mathcal{B}(Z_0^{-1}, f_0)(t_0) \tag{2.55}$$

are fulfilled uniformly on I, where Z_k $(Z_k(t_k) = I_n)$ is a fundamental matrix of the homogeneous system $\mathrm{d}x(t) = \mathrm{d}B_k(t)\cdot x(t)$ *for every* $k \in \{0, 1 \ldots\}$.

Corollary 2.6. *Let $A_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I;\mathbb{R}^n)$ and $t_k \in I$ $(k = 0, 1, \ldots)$ be such that conditions* (2.3) *and* (2.4) *hold and there exist matrix-functions $B_k \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$ $(k = 0, 1, \ldots)$, satisfying the Lappo-Danilevskiĭ condition, such that conditions* (2.51) *and*

$$\det(I_n + (-1)^j d_j B_0(t)) \neq 0 \quad \text{for } t \in I \ (j = 1, 2) \tag{2.56}$$

hold, and the conditions

$$\lim_{k\to\infty} (B_k(t) - B_k(t_k)) = B_0(t) - B_0(t_0), \tag{2.57}$$

$$\lim_{k\to\infty} \int_{t_k}^{t} Z_k^{-1}(\tau)\,\mathrm{d}\mathcal{A}(B_k, A_k)(\tau) = \int_{t_0}^{t} Z_0^{-1}(\tau)\,\mathrm{d}\mathcal{A}(B_0, A_0)(\tau) \tag{2.58}$$

and

$$\lim_{k\to\infty} \int_{t_k}^{t} Z_k^{-1}(\tau)\,\mathrm{d}\mathcal{A}(B_k, f_k)(\tau) = \int_{t_0}^{t} Z_0^{-1}(\tau)\,\mathrm{d}\mathcal{A}(B_0, f_0)(\tau) \tag{2.59}$$

are fulfilled uniformly on I, where $\mathcal{A}$ is the operator defined by (1.4), *and Z_k $(Z_k(t_k) = I_n)$ is a fundamental matrix of the homogeneous system given in Corollary* 2.5 *for every $k \in \{0, 1 \ldots\}$. Then inclusion* (2.5) *holds.*

R e m a r k 2.4. In Corollaries 2.5 and 2.6, if we assume that the matrix functions B_k $(k = 0, 1, \ldots)$ are continuous, then conditions (2.52) and (2.56) are valid obviously. Moreover, due to the integration-by-parts formula and definitions of operators $\mathcal{B}$ and $\mathcal{A}$, each of conditions (2.54) and (2.58) has the form

$$\lim_{k\to\infty} \int_{t_k}^{t} Z_k^{-1}(\tau)\,\mathrm{d}A_k(\tau) = \int_{t_0}^{t} Z_0^{-1}(\tau)\,\mathrm{d}A_0(\tau), \tag{2.60}$$

and each of conditions (2.55) and (2.59) has the form

$$\lim_{k\to\infty} \int_{t_k}^{t} Z_k^{-1}(\tau)\,\mathrm{d}f_k(\tau) = \int_{t_0}^{t} Z_0^{-1}(\tau)\,\mathrm{d}f_0(\tau). \tag{2.61}$$

R e m a r k 2.5. If a matrix-function $B \in \mathrm{BV}(I;\mathbb{R}^{n\times n})$, satisfying the Lappo-Danilevskiĭ condition, and $s \in I$ are such that $\det(I_n + (-1)^j d_j B(t)) \neq 0$ for $t \in I$, $(-1)^j(t - s) < 0$ $(j = 1, 2)$, then the fundamental matrix Z $(Z(s) = I_n)$ of the homogeneous system $\mathrm{d}x(t) = \mathrm{d}B(t) \cdot x(t)$ has the form (see [11])

$$(2.62)\qquad Z(t) = \begin{cases} \exp(S_0(B)(t) - S_0(B)(s)) \\ \qquad \times \prod\limits_{s<\tau\leqslant t} (1 - d_1 B(\tau))^{-1} \prod\limits_{s\leqslant\tau<t} (1 + d_2 B(\tau)) & \text{for } t > s, \\ \exp(S_0(B)(t) - S_0(B)(s)) \\ \qquad \times \prod\limits_{t<\tau\leqslant s} (1 - d_1 B(\tau)) \prod\limits_{t\leqslant\tau<s} (1 + d_2 B(\tau))^{-1} & \text{for } t < s, \\ I_n & \text{for } t = s. \end{cases}$$

Corollary 2.7. *Let $A_k \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I; \mathbb{R}^n)$ and $t_k \in I$ $(k = 0, 1, \dots)$ be such that conditions* (2.3) *and* (2.4) *hold, the matrix-functions $S_c(A_k)$ $(k = 0, 1, \dots)$ satisfy the Lappo-Danilevskiĭ condition and*

$$(2.63)\qquad \lim_{k\to\infty} \sup \sum_{t\in I} \|d_j A_k(t)\| < \infty \quad (j = 1, 2),$$

and the conditions

$$(2.64)\qquad \lim_{k\to\infty} (S_c(A_k)(t) - S_c(A_k)(t_k)) = S_c(A_0)(t) - S_c(A_0)(t_0),$$

$$(2.65)\qquad \lim_{k\to\infty} d_j A_k(t) = d_j A_0(t) \quad (j = 1, 2)$$

and

$$(2.66)\qquad \lim_{k\to\infty} \int_{t_k}^{t} \exp(-S_c(A_k)(\tau) + S_c(A_k)(t_k))\, \mathrm{d}f_k(\tau) = \int_{t_0}^{t} \exp(-S_c(A_0)(\tau) + S_c(A_k)(t_0))\, \mathrm{d}f_0(\tau)$$

are fulfilled uniformly on I. Then inclusion (2.5) *holds.*

Corollary 2.8. *Let $A_k \in \mathrm{BV}(I; \mathbb{R}^{n\times n})$, $f_k \in \mathrm{BV}(I; \mathbb{R}^n)$ and $t_k \in I$ $(k = 0, 1, \dots)$ be such that conditions* (2.3), (2.4),

$$\lim_{k\to\infty} \sup \sum_{i,l=1; i\neq l}^{n} \bigvee_I (a_{kil}) < \infty$$

and

$$1 + (-1)^j d_j a_{0ii}(t)) \neq 0 \quad \text{for } t \in I \ (j = 1, 2;\ i = 1, \dots, n)$$

hold, and the conditions

$$\lim_{k\to\infty}(a_{kii}(t)-a_{kii}(t_k))=a_{0ii}(t)-a_{0ii}(t_0)\quad (i=1,\dots,n),$$

$$\lim_{k\to\infty}\int_{t_k}^{t} z_{kii}^{-1}(\tau)\,\mathrm{d}\mathcal{A}(a_{kii},a_{kil})(\tau)=\int_{t_0}^{t} z_{0ii}^{-1}(\tau)\,\mathrm{d}\mathcal{A}(a_{0ii},a_{0il})(\tau)\quad (i\neq l;\ i,l=1,\dots,n)$$

and

$$\lim_{k\to\infty}\int_{t_k}^{t} z_{kii}^{-1}(\tau)\,\mathrm{d}\mathcal{A}(a_{kii},f_{ki})(\tau)=\int_{t_0}^{t} z_{0ii}^{-1}(\tau)\,\mathrm{d}\mathcal{A}(a_{0ii},f_{0i})(\tau)\quad (i=1,\dots,n)$$

are fulfilled uniformly on I, where $\mathcal{A}$ is the operator defined by (1.4), *and z_{kii}, defined according to* (2.62), *is a solution of the Cauchy problem* $\mathrm{d}z(t)=z(t)\,\mathrm{d}a_{kii}(t)$, $z(t_k)=1$ *for $i\in\{1,\dots,n\}$ and every sufficiently large k. Then inclusion* (2.5) *holds.*

Remark 2.6. In Theorems 2.1′–2.3′ and Corollaries 2.1′, 2.2–2.8, we can assume $H_0(t)=I_n$, without loss of generality. In this case, it is evident that

$$\mathcal{I}(H_0,Y)(t)-\mathcal{I}(H_0,Y)(s)=Y(t)-Y(s)\quad \text{for } Y\in\mathrm{BV}(I;\mathbb{R}^{n\times n}) \text{ and } t,s\in I.$$

Remark 2.7. The following example shows that if condition (2.63) is violated, then the statement of Corollary 2.7 is not true in general.

Example. Let $I=[0,1]$, $A_0(t)=0$, $f_0(t)=f_k(t)=0$, $t_k=t_0=0$, $c_k=c_0=1$,

$$A_k(t)=\begin{cases} k^{-1} & \text{for } t\in\bigcup\limits_{i=1}^{2k^2}\,]t_{2i-1k},t_{2ik}],\\ 0 & \text{for } t\notin\bigcup\limits_{i=1}^{2k^2}\,]t_{2i-1k},t_{2ik}],\end{cases}$$

where $t_{ik}=(2k^2+1)^{-1}i$ $(i=0,\dots,2k^2)$ for every natural k. Then all the conditions of Corollary 2.7 are fulfilled except of (2.63). It is evident that $x_0(t)\equiv 1$. On the other hand, the Cauchy problem (1.1_k), (1.2_k) has a unique solution x_k and, in addition, $x_k(1)=\left(1-1/k^2\right)^{k^2}$. Therefore, condition (1.3) is not valid since

$$\lim_{k\to\infty}x_k(1)=\exp(-1)\neq x_0(1).$$

Lemma 3.1. *Let $a \in I$ be a fixed point. Then:*
(a) *if $X \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times m})$, $Y \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{m\times l})$ and $Z \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{l\times k})$, then*

$$\mathcal{B}(X,\mathcal{B}(Y,Z))(t) = \mathcal{B}(XY,Z)(t) \quad \text{for } t \in I,$$
$$\mathcal{B}\Big(X,\int_a^{\cdot} \mathrm{d}Y(s)\cdot Z(s)\Big)(t) = \int_a^t \mathrm{d}\mathcal{B}(X,Y)(s)\cdot Z(s) \quad \text{for } t \in I;$$

(b) *if $X \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times n})$, $Y \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times n})$ and $Z \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times n})$, then*

$$\mathcal{I}(X,\mathcal{I}(Y,Z))(t) = \mathcal{I}(XY,Z)(t) \quad \text{for } t \in I.$$

Lemma 3.2. *Let $h \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^n)$, and let $H \in \mathrm{BV}_{\mathrm{loc}}(I;\mathbb{R}^{n\times n})$ be a nonsingular matrix-function. Then the mapping $x \to y = Hx + h$ establishes a one-to-one correspondence between the solutions x and y of systems $\mathrm{d}x(t) = \mathrm{d}A(t)\cdot x(t) + \mathrm{d}f(t)$ and $\mathrm{d}y(t) = \mathrm{d}A_*(t)\cdot y(t) + \mathrm{d}f_*(t)$, respectively, where*

$$A_*(t) = \mathcal{I}(H,A)(t), \quad f_*(t) = h(t) - h(a) + \mathcal{B}(H,f)(t) - \int_a^t \mathrm{d}A_*(s)\cdot h_k(s) \quad \text{for } t \in I,$$

and $a \in I$ is a fixed point. Besides,

$$I_n + (-1)^j d_j A_*(t) \equiv (H(t) + (-1)^j d_j H(t))\cdot(I_n + (-1)^j d_j A(t))H^{-1}(t) \quad (j=1,2).$$

Lemma 3.3. *Let $\alpha_k, \beta_k \in \mathrm{BV}(I;\mathbb{R})$ $(k = 0,1,\dots)$ be such that*

$$\lim_{k\to\infty} \|\beta_k - \beta_0\|_s = 0 \quad \text{and} \quad \lim_{k\to\infty} \sup \bigvee_I(\alpha_k) < \infty,$$

and let the condition

$$\lim_{k\to\infty} (\alpha_k(t) - \alpha_k(a)) = \alpha_0(t) - \alpha_0(a)$$

be fulfilled uniformly on I, where $a \in I$ is a fixed point. Then

$$\lim_{k\to\infty} \int_a^t \beta_k(\tau)\,\mathrm{d}\alpha_k(\tau) = \int_a^t \beta_0(\tau)\,\mathrm{d}\alpha_0(\tau) \quad \text{uniformly on } I.$$

Lemma 3.4. *Let*

$$\det(I_n + (-1)^j d_j A_0(t)) \neq 0 \quad \textit{for } t \in I \ (j = 1, 2)$$

and let

$$\lim_{k\to\infty} Y_k(t) = Y_0(t) \quad \textit{uniformly on } I, \tag{3.1}$$

where Y_0 and Y_k ($k = 1, 2, \ldots$) are fundamental matrices of homogeneous systems (1.1_0) *and* (1.1_{k0}) $(k = 1, 2, \ldots)$*, respectively. Then*

$$\inf\{|\det(Y_0(t))| \colon\ t \in I\} > 0, \tag{3.2}$$

$$\inf\{|\det(Y_0^{-1}(t))| \colon\ t \in I\} > 0, \tag{3.3}$$

and

$$\lim_{k\to\infty} Y_k^{-1}(t) = Y_0^{-1}(t) \quad \textit{uniformly on } I. \tag{3.4}$$

We omit the proofs of the above lemmas. One can find the proofs of Lemmas 3.1 and 3.2 in [3], and Lemmas 3.3 and 3.4 in [6].

The conclusion of the next lemma is often used implicitly in various papers (e.g. [7], [22]). We give the proof from those papers.

Lemma 3.5. *Let sequences of matrix-functions $B_k \in \mathrm{BV}_{\mathrm{loc}}(I; \mathbb{R}^{n\times n})$ and points $t_k \in I$ ($k = 0, 1, \ldots$) be such that conditions* (2.4),

$$\det(I_n + (-1)^j d_j B_0(t)) \neq 0 \quad \textit{for } t \in I, \quad (-1)^j (t - t_0) < 0 \ (j = 1, 2) \tag{3.5}$$

and

$$\lim_{k\to\infty} \sup\{\|d_j B_k(t) - d_j B_0(t)\| \colon\ t \in I,\ (-1)^j (t - t_k) < 0\} = 0 \ (j = 1, 2) \tag{3.6}$$

hold. Then

$$\det(I_n + (-1)^j d_j B_k(t)) \neq 0 \quad \textit{for } t \in I,\ (-1)^j (t - t_k) < 0 \ (j = 1, 2) \tag{3.7}$$

and there exists a positive number r_0 such that

$$\|(I_n + (-1)^j d_j B_0(t))^{-1}\| \leqslant r_0 \quad \textit{for } t \in I,\ (-1)^j (t - t_0) < 0 \tag{3.8}$$

and

$$\|(I_n + (-1)^j d_j B_k(t))^{-1}\| \leqslant r_0 \quad \textit{for } t \in I,\ (-1)^j (t - t_k) < 0 \ (j = 1, 2)$$

for every sufficiently large k.

P r o o f. Since $\bigvee\limits_I B_0 < \infty$, the series $\sum\limits_{t\in I} \|d_j B_0(t)\|$ $(j=1,2)$ converge. Thus for any $j\in\{1,2\}$ the inequality $\|d_j B_0(t)\| \geqslant 1/2$ may hold only for a finite number of points $t_{j1},\dots,t_{jm_j}$ from I. Therefore,

$$\|d_j B_0(t)\| < \frac{1}{2} \quad \text{for } t\in I,\ t\neq t_{ji}\ (i=1,\dots,m_j). \tag{3.9}$$

First consider the case when $j=2$ and $t_k \geqslant t_0$ for every sufficiently large k. We can assume that $t_{2i} \geqslant t_k$ $(i=1,\dots,m_2)$ for every sufficiently large k.

It follows from (3.5), (3.6) and (3.9) that $\det(I_n + d_2 B_k(t_{2i})) \neq 0$ $(i=1,\dots,m_2)$ and $\|d_j B_k(t)\| < 1/2$ for $t\in I_{t_k}$, $t\neq t_{2i}$ $(i=1,\dots,m_2)$ for every sufficiently large k. The latter inequalities imply that the matrices $I_n + d_2 B_k(t)$ $(j=1,2)$ are invertible for $t\in I_{t_0}$, $t\neq t_{ji}$ $(i=1,\dots,m_j)$, too. From this, it is evident that condition (3.7) is fulfilled and there exists a positive number r_0 for which estimates (3.8) hold. Analogously we can prove this estimate for the other cases. □

4. Proof of the main results

P r o o f of Theorem 2.2. In virtue of (2.10),

$$\lim_{k\to\infty} \sup_{t\in I, t\neq t_k} \{\|d_j A_k(t) - d_j A_0(t)\|\} = 0 \quad (j=1,2). \tag{4.1}$$

So, according to Lemma 3.5 there exists a positive number r_0 such that

$$\det(I_n + (-1)^j d_j A_k(t)) \neq 0 \quad \text{for } t\in I,\ (-1)^j(t-t_k)<0\ (j=1,2)$$

and

$$\|(I_n + (-1)^j d_j A_0(t))^{-1}\| \leqslant r_0 \quad \text{for } t\in I,\ (-1)^j(t-t_k)<0\ (j=1,2) \tag{4.2}$$

for every sufficiently large k. Hence, there exists a natural k_0 such that problem (1.1_k), (1.2_k) has a unique solution x_k for every $k\geqslant k_0$.

Let $z_k(t) = x_k(t) - x_0(t)$ for $k\in\{k_0,k_0+1,\dots\}$. First, consider the case when $t_k > t_0$ $(k=k_0,k_0+1,\dots)$. Below, for this case, we assume that $k\geqslant k_0$.

Let ε be an arbitrary small positive number. It is not difficult to check that

$$z_k(t) = z_k(t_k+\varepsilon) + \int_{t_k+\varepsilon}^t \mathrm{d}A_0(s)\cdot z_k(s) + \int_{t_k+\varepsilon}^t \mathrm{d}\bar{A}_{k2}(s)\cdot x_k(s) + \bar{f}_{k2}(t) - \bar{f}_{k2}(t_k+\varepsilon)$$

for $t\geqslant t_k+\varepsilon$, where $\bar{A}_{kj}(t) = A_{kj}(t) - A_{0j}(t)$, $\bar{f}_{kj}(t) = f_{kj}(t) - f_{0j}(t)$ $(j=1,2)$. Thanks to (1.10) and the definition of a solution of system (1.1_k), we get

$$d_j x_k(t) = d_j A_k(t)\cdot x_k(t) + d_j f(t) \quad \text{for } t\in I\ (j=1,2). \tag{4.3}$$

Using the integration-by-parts formula (1.6), equalities (4.3), the general integration-by-parts formula (1.7) and equality (1.9) we conclude

$$
\begin{aligned}
&\int_{t_k+\varepsilon}^{t} \mathrm{d}\bar{A}_{k2}(s)\cdot x_k(s)\\
&\qquad = \bar{A}_{k2}(t)\cdot x_k(t) - \bar{A}_{k2}(t_k+\varepsilon)\cdot x_k(t_k+\varepsilon) - \int_{t_k+\varepsilon}^{t} \bar{A}_{k2}(s)\,\mathrm{d}x_k(s)\\
&\qquad\quad + \sum_{t_k+\varepsilon<s\leqslant t} d_1\bar{A}_{k2}(s)\cdot d_1x_k(s) - \sum_{t_k+\varepsilon\leqslant s<t} d_2\bar{A}_{k2}(s)\cdot d_2x_k(s)\\
&\qquad = \bar{A}_{k2}(t)\cdot x_k(t) - \bar{A}_{k2}(t_k+\varepsilon)\cdot x_k(t_k+\varepsilon)\\
&\qquad\quad - \int_{t_k+\varepsilon}^{t} \bar{A}_{k2}(s)(\,\mathrm{d}A_k(s)\cdot x_k(s) + \,\mathrm{d}f_k(s))\\
&\qquad\quad + \sum_{t_k+\varepsilon<s\leqslant t} d_1\bar{A}_{k2}(s)\cdot (d_1A_k(s)\cdot x_k(s) + d_1f_k(s))\\
&\qquad\quad - \sum_{t_k+\varepsilon\leqslant s<t} d_2\bar{A}_{k2}(s)\cdot (d_2A_k(s)\cdot x_k(s) + d_2f_k(s)) \quad \text{for } t\geqslant t_k+\varepsilon.
\end{aligned}
$$

Therefore,

$$
z_k(t) = z_k(t_k+\varepsilon) + \mathcal{J}_{k2}(t,t_k+\varepsilon) + \mathcal{Q}_{k2}(t,t_k+\varepsilon) + \int_{t_k+\varepsilon}^{t} \mathrm{d}A_0(s)\cdot z_k(s) \tag{4.4}
$$

for $t\geqslant t_k+\varepsilon$, where

$$
\begin{aligned}
\mathcal{J}_{kj}(t,\tau) &= \bar{A}_{kj}(t)\cdot x_k(t) - \bar{A}_{kj}(\tau)\cdot x_k(\tau) - \int_{\tau}^{t} \bar{A}_{kj}(s)\,\mathrm{d}A_k(s)\cdot x_k(s)\\
&\quad + \sum_{s\in]\tau,t]} d_1\bar{A}_{kj}(s)\cdot d_1A_k(s)\cdot x_k(s)\\
&\quad - \sum_{s\in[\tau,t[} d_2\bar{A}_{kj}(s)\cdot d_2A_k(s)\cdot d_2x_k(s) \quad \text{for } \tau<t \ (j=1,2),\\
\mathcal{J}_{kj}(t,t) &\equiv O_{n\times n} \ (j=1,2), \quad \mathcal{J}_{kj}(t,\tau) = -\mathcal{J}_{kj}(\tau,t) \text{ for } t<\tau \ (j=1,2);\\
\mathcal{Q}_{kj}(t,\tau) &\equiv \bar{f}_{kj}(t) - \bar{f}_{kj}(\tau) - \mathcal{B}(\bar{A}_{kj},f_k)(t) + \mathcal{B}(\bar{A}_{kj},f_k)(\tau) \ (j=1,2).
\end{aligned}
$$

Let B_0 be a matrix-function defined by $B_0(t_k+\varepsilon) = A_0(t_k+\varepsilon)$ and $B_0(s) = A_0(s-)$ for $s>t_k+\varepsilon$. Obviously,

$$
d_2B_0(t_k+\varepsilon) = d_2A_0(t_k+\varepsilon) \quad \text{and} \quad d_1(B_0(s)-A_0(s)) = -d_1A_0(s) \quad \text{for } s>t_k+\varepsilon.
$$

Hence, according to (1.5),

$$
\int_{t_k+\varepsilon}^{t} \mathrm{d}A_0(s)\cdot z_k(s) = \int_{t_k+\varepsilon}^{t} \mathrm{d}B_0(s)\cdot z_k(s) + d_1A_0(t)\cdot z_k(t) \quad \text{for } t>t_k+\varepsilon.
$$

Consequently, thanks to (2.3), it follows from (4.4) that

$$z_k(t) = (I_n - d_1 A_0(t))^{-1}\biggl(z_k(t_k+\varepsilon) + \mathcal{J}_{k2}(t, t_k+\varepsilon) + \mathcal{Q}_{k2}(t, t_k+\varepsilon) \\ + \int_{t_k+\varepsilon}^{t} \mathrm{d}B_0(s)\cdot z_k(s)\biggr) \quad \text{for } t > t_k+\varepsilon.$$

Let $r_1 = r_0 + 1$. Due to (4.1) and estimate (4.2), without loss of generality we get

$$\|z_k(t)\| \leqslant r_1\biggl(\|z_k(t_k+\varepsilon)\| + \|\mathcal{J}_{k2}(t, t_k+\varepsilon)\| + \|\mathcal{Q}_{k2}(t, t_k+\varepsilon)\| \\ + \int_{t_k+\varepsilon}^{t} \|z_k(\tau)\|\,\mathrm{d}\|V(B_0)(\tau)\|\biggr) \quad \text{for } t \geqslant t_k+\varepsilon. \tag{4.5}$$

Let

$$\alpha_k = \sup_{t\in I, t\neq t_k}\{\|\bar{A}_{k2}(t)\|\}, \quad \beta_k = \sup_{t\in I, t\neq t_k}\{\|\bar{f}_{k2}(t)\|\}, \quad \gamma_k = \sup_{t\in I, t\neq t_k}\biggl\{\biggl|\bigvee_{]t_k,t[}(A_k)\biggr|\biggr\}.$$

Then by (2.10) and (2.11) we have

$$\lim_{k\to\infty}\alpha_k(1+\gamma_k) = \lim_{k\to\infty}\beta_k(1+\gamma_k) = 0. \tag{4.6}$$

It is evident that

$$\|\mathcal{J}_{k2}(t, t_k+\varepsilon)\| \leqslant 2\alpha_k\|x_k\|_{k2} + \alpha_k\gamma_k\|x_k\|_{k2} \\ + 2\alpha_k\|x_k\|_{k2}\biggl(\sum_{t_k+\varepsilon<s\leqslant t}\|d_1A_k(s)\| + \sum_{t_k+\varepsilon\leqslant s<t}\|d_2A_k(s)\|\biggr)$$

and, therefore,

$$\|\mathcal{J}_{k2}(t, t_k+\varepsilon)\| \leqslant \varepsilon_k\|x_k\|_{k2} \quad \text{for } t \geqslant t_k+\varepsilon, \tag{4.7}$$

where $\varepsilon_k = \alpha_k(2+3\gamma_k)$ $(k = 1, 2, \ldots)$. Moreover, if we take into account the fact that the operator $\mathcal{B}$ is linear with respect to each of its variables and equals zero if the second variable is a constant function, then we obtain

$$\|\mathcal{B}(\bar{A}_{k2}, f_k)(t) - \mathcal{B}(\bar{A}_{k2}, f_k)(t_k+\varepsilon)\| \\ \leqslant \|\mathcal{B}(\bar{A}_{k2}, \bar{f}_{k2})(t) - \mathcal{B}(\bar{A}_{k2}, \bar{f}_{k2})(t_k+\varepsilon)\| \\ + \|\mathcal{B}(\bar{A}_{k2}, f_0)(t) - \mathcal{B}(\bar{A}_{k2}, f_0)(t_k+\varepsilon)\| \quad \text{for } t \geqslant t_k+\varepsilon.$$

Let $r_2 = \bigvee_I(A_0)$ and $r_3 = \bigvee_I(f_0)$. By the definition of the operator $\mathcal{B}$, we have

$$\|\mathcal{B}(\bar{A}_{k2}, \bar{f}_{k2})(t) - \mathcal{B}(\bar{A}_{k2}, \bar{f}_{k2})(t_k+\varepsilon)\| \leqslant 2\alpha_k\beta_k + \beta_k(\gamma_k + r_2) \quad \text{for } t \geqslant t_k + \varepsilon.$$

Moreover, using the integration-by-parts formula we find

$$\begin{aligned}&\|\mathcal{B}(\bar{A}_{k2}, f_0)(t) - \mathcal{B}(\bar{A}_{k2}, f_0)(t_k+\varepsilon)\|\\ &\quad\leqslant \alpha_k \bigvee_{t_k+\varepsilon}^{t}(f_0) + 2\alpha_k\Bigg(\sum_{t_k+\varepsilon<s\leqslant t}\|d_1 f_0(s)\| + \sum_{t_k+\varepsilon\leqslant s<t}\|d_2 f_0(s)\|\Bigg) \quad \text{for } t \geqslant t_k+\varepsilon\end{aligned}$$

and, consequently,

$$\|\mathcal{B}(\bar{A}_{k2}, f_0)(t) - \mathcal{B}(\bar{A}_{k2}, f_0)(t_k+\varepsilon)\| \leqslant 3\alpha_k r_3 \quad \text{for } t \geqslant t_k+\varepsilon.$$

Further,

$$\|\mathcal{Q}_{k2}(t, t_k+\varepsilon)\| \leqslant \delta_k \quad \text{for } t \geqslant t_k+\varepsilon, \tag{4.8}$$

where $\delta_k = \beta_k(2 + 2\alpha_k + \gamma_k + r_2) + 3\alpha_k r_3$. From (4.5), by (4.7) and (4.8) we get

$$\|z_k(t)\| \leqslant r_1\Bigg(\|z_k(t_k+\varepsilon)\| + \varepsilon_k\|x_k\|_{k2} + \delta_k + \int_{t_k+\varepsilon}^{t}\|z_k(\tau)\|\,\mathrm{d}\|V(B_0)(\tau)\|\Bigg) \tag{4.9}$$

for $t \geqslant t_k+\varepsilon$. So, according to the Gronwall inequality (see [22], Theorem I.4.30)

$$\begin{aligned}\|z_k(t)\| &\leqslant r_1(\|z_k(t_k+\varepsilon)\| + \varepsilon_k\|x_k\|_{k2} + \delta_k)\exp(r_1\|V(B_0)(t) - V(B_0)(t_k)\|)\\ &\leqslant r_1\big(\|z_k(t_k+\varepsilon)\| + \varepsilon_k\|x_k\|_{k2} + \delta_k\big)\exp(r_1 r_2) \quad \text{for } t \geqslant t_k+\varepsilon.\end{aligned}$$

Now, passing to the limit in the last inequality for $\varepsilon \to 0$, we conclude

$$\|z_k\|_{k2} \leqslant r_1\big(\|z_k(t_k+)\| + \varepsilon_k\|x_k\|_{k2} + \delta_k\big)\exp(r_1 r_2). \tag{4.10}$$

In virtue of (4.6), we find

$$\lim_{k\to\infty}\varepsilon_k = 0. \tag{4.11}$$

Therefore, there exists a natural $k_1 \geqslant k_0$ such that

$$r_1\exp(r_1 r_2)\varepsilon_k < \frac{1}{2} \quad (k \geqslant k_1).$$

Due to this inequality, (4.10) implies

$$\|x_k\|_{k2} \leqslant \|x_0\|_{k2} + r_1(\|z_k(t_k+)\| + \varepsilon_k\|x_k\|_{k2} + \delta_k)\exp(r_1r_2) \quad (k \geqslant k_1),$$

which, due to (2.1_2) yields that the sequence $\|x_k\|_{k2}$ $(k = k_1, k_1+1, \ldots)$ is bounded.

In view of conditions (2.10) and (2.11) we have

$$\lim_{k\to\infty} \delta_k = 0. \tag{4.12}$$

Moreover, using (2.1_2) we get

$$\begin{aligned}
\lim_{k\to\infty} z_k(t_k+) &= \lim_{k\to\infty} (x_k(t_k+) - x_0(t_k+)) = \lim_{k\to\infty} (x_k(t_k+) - x_0(t_0+)) \\
&= \lim_{k\to\infty} \big([(I_n + d_2A(t_k))x_k(t_k) + d_2 f_k(t_k)] \\
&\quad - [(I_n + d_2A(t_0))x_0(t_0) + d_2 f_0(t_0)]\big) \\
&= \lim_{k\to\infty} (c_{k2} - c_{02}) = 0.
\end{aligned}$$

Therefore, by this, (4.11) and (4.12), it follows from (4.10) that

$$\lim_{k\to\infty} \|z_k\|_{k2} = 0.$$

Analogously to (4.4), we can show that

$$z_k(t) = z_k(t_k-\varepsilon) - \mathcal{J}_{k2}(t_k-\varepsilon, t) - \mathcal{Q}_{k2}(t_k-\varepsilon, t) - \int_t^{t_k-\varepsilon} \mathrm{d}A_0(s)\cdot z_k(s) \tag{4.13}$$

for $t \leqslant t_k - \varepsilon$. Let now the matrix-function B_0 be defined by $B_0(t_k-\varepsilon) = A_0(t_k-\varepsilon)$ and $B_0(s) = A_0(s+)$ for $s < t_k - \varepsilon$. It is evident that

$$d_1B_0(t_k-\varepsilon) = d_1A_0(t_k-\varepsilon) \quad \text{and} \quad d_2(B_0(s) - A_0(s)) = -d_2A_0(s) \quad \text{for } s < t_k - \varepsilon.$$

Hence, according to (1.5),

$$\int_t^{t_k-\varepsilon} \mathrm{d}A_0(s)\cdot z_k(s) = \int_t^{t_k-\varepsilon} \mathrm{d}B_0(s)\cdot z_k(s) + d_2A_0(t)\cdot z_k(t) \quad \text{for } t < t_k - \varepsilon.$$

Using these equalities, from (4.13) we obtain

$$\begin{aligned}
z_k(t) = (I_n + d_2A_0(t))^{-1}\bigg(& z_k(t_k-\varepsilon) - \mathcal{J}_{k2}(t_k-\varepsilon, t) - \mathcal{Q}_{k2}(t_k-\varepsilon, t) \\
& - \int_t^{t_k-\varepsilon} \mathrm{d}A_0(s)\cdot z_k(s)\bigg) \quad \text{for } t < t_k - \varepsilon.
\end{aligned}$$

From this, analogously as above, we have

$$\|z_k\|_{k1} \leqslant r_1(\|z_k(t_k-)\| + \varepsilon_k\|x_k\|_{k1} + \delta_k)\exp(r_1 r_2) \tag{4.14}$$

and, in addition, the sequence $\|x_k\|_{k2}$ $(k = k_1, k_1+1, \ldots)$ is bounded.

Thanks to (2.10) and (2.11),

$$\lim_{k\to\infty} \big(\|d_1A(t_k) + d_2A(t_k)\| + \|d_1f_k(t_k) + d_2f_k(t_k)\|\big) = 0.$$

Using this and (2.1_2), we conclude

$$\begin{aligned}
\lim_{k\to\infty} z_k(t_k-) &= \lim_{k\to\infty}(x_k(t_k-) - x_0(t_k-)) = \lim_{k\to\infty}(x_k(t_k-) - x_0(t_0+)) \\
&= \lim_{k\to\infty}\big([(I_n - d_1A(t_k))x_k(t_k) - d_1f_k(t_k)] \\
&\qquad\qquad - [(I_n + d_2A(t_0))x_0(t_0) + d_2f_0(t_0)]\big) \\
&= \lim_{k\to\infty}\big([(I_n + d_2A(t_k))c_k + d_2f_k(t_k)] \\
&\qquad\qquad - [(I_n + d_2A(t_0))x_0(t_0) + d_2f_0(t_0)]\big) \\
&\quad - \lim_{k\to\infty}\big(d_1A(t_k) + d_2A(t_k))c_k - (d_1f_k(t_k) + d_2f_k(t_k))\big) \\
&= \lim_{k\to\infty}(c_{k2} - c_{02}) = 0.
\end{aligned}$$

Therefore, due to (4.14), taking into account (4.11) and (4.12), we find

$$\lim_{k\to\infty}\|z_k\|_{k1} = 0.$$

So, condition (2.12) holds for $t_k > t_0$ $(k = 1, 2, \ldots)$.

In a similar way, we can prove the theorem for the cases when $t_k < t_0$ $(k = 1, 2, \ldots)$ or $t_k = t_0$ $(k = 1, 2, \ldots)$, as well. □

P r o o f of Theorem 2.3. In view of condition (2.15), analogously to the proof of Theorem 2.2, we can show that the Cauchy problem (2.17_k), (2.18_k) has a unique solution x_k^* for every sufficiently large k. Moreover, according to Lemma 2.2, the mapping $x \to H_kx + h_k$ establishes a one-to-one correspondence between the solutions of problem (1.1_k), (1.2_k) and the solutions of problem (2.17_k), (2.18_k) for every natural k. So, problem (1.1_k), (1.2_k) has a unique solution x_k, and $x_k^*(t) = H_k(t)x_k(t) + h_k(t)$ for every sufficiently large k.

Conditions (2.15), (2.18)–(2.21) guarantee the fulfillment of the conditions of Theorem 2.2 for the Cauchy problem (2.16), (2.17) and the sequence of the Cauchy problems (2.17_k), (2.18_k) for every sufficiently large k. Hence, by Theorem 2.2,

$$\lim_{k\to\infty}\sup_{t\in I, t\neq t_k}\{\|x_k^*(t) - x_0^*(t)\|\} = 0.$$

Thus condition (2.22) holds. □

P r o o f of Corollary 2.1. Let us verify the conditions of Theorem 2.3. From (2.6), (2.7) it follows that condition (2.18) holds, as well as the condition

$$\lim_{k\to\infty} H_k^{-1}(t) = H_0^{-1}(t) \quad \text{uniformly on } I. \tag{4.15}$$

Put $h_k(t) \equiv -H_k(t)\varphi_k(t)$ $(k = 1, 2, \ldots)$. In view of (2.4) and (2.7), we get

$$\lim_{k\to\infty} H_k(t_k) = Q_0,$$

where $Q_0 = H_0(t_0-)$ if $t_k < t_0$, $Q_0 = H_0(t_0)$ if $t_k = t_0$, and $Q_0 = H_0(t_0+)$ if $t_k > t_0$ for every sufficiently large k. By this and (2.25), (2.19) is fulfilled for $c_0^* = Q_0(t_0)c_0$.

Further, by (2.8) and (2.9), conditions (2.23) and (2.24) hold uniformly on I, where

$$\begin{aligned}
&A_k^*(t) = \mathcal{I}(H_k, A_k)(t) - \mathcal{I}(H_k, A_k)(t_k) \quad \text{for } t \in I \ (k = 0, 1, \ldots);\\
&f_0^*(t) = \mathcal{B}(H_0, f_0)(t) - \mathcal{B}(H_0, f_0)(t_0)
\end{aligned}$$

and

$$\begin{aligned}
f_k^*(t) = {} & \mathcal{B}(H_k, f_k - \varphi_k)(t) - \mathcal{B}(H_k, f_k - \varphi)(t_k)\\
& + \int_{t_k}^{t} \mathrm{d}\mathcal{I}(H_k, A_k)(s) \cdot \varphi_k(s) \quad \text{for } t \in I \ (k = 1, 2, \ldots).
\end{aligned}$$

Taking into account Lemma 3.2, it is not difficult to see that problem (2.16), (2.17) has a unique solution $x_0^*(t) \equiv H_0(t)x_0(t)$.

Thanks to Theorem 2.3 and Remark 2.1, we have

$$\lim_{k\to\infty} (H_k(t)x_k(t) - H_k(t)\varphi_k(t)) = x_0^*(t)$$

uniformly on I. Hence, by virtue of conditions (2.7) and (4.15), condition (2.27) is valid. □

P r o o f of Theorem 2.1. The sufficiency follows from Corollary 2.1 if we assume $\varphi_k(t) = 0$ $(k = 1, 2, \ldots)$.

Let us show the necessity. Let $c_k \in \mathbb{R}^n$ $(k = 0, 1, \ldots)$ be an arbitrary sequence of constant vectors satisfying (2.1) and let $e_j = (\delta_{ij})_{i=1}^n$ $(j = 1, \ldots, n)$.

In view of (2.3), we can assume without loss of generality that problem (1.1_k), (1.2_k) has a unique solution x_k for every natural k.

For any $k \in \{0, 1, \ldots\}$ and $j \in \{1, \ldots, n\}$, let $y_{kj}(t) = x_k(t) - x_{kj}(t)$, where x_{kj} is the unique solution of system (1.1_k) under the Cauchy condition $x(t_k) = c_k - e_j$. Moreover, let $Y_k(t)$ be the matrix-function whose columns are $y_{k1}(t), \ldots, y_{kn}(t)$.

It can be easily shown that Y_0 and Y_k $(k = 1, 2, \ldots)$ satisfy, respectively, homogeneous systems (1.1_0) and (1.1_{k0}) $(k = 1, 2, \ldots)$, and

$$y_{kj}(t_k) = e_j \quad (j = 1, \ldots, n;\ k = 0, 1, \ldots). \tag{4.16}$$

If

$$\sum_{j=1}^{n} \alpha_j y_{kj}(t) \equiv o_n$$

for some natural k and $\alpha_j \in \mathbb{R}$ $(j = 1, \ldots, n)$; then using (4.16) we have

$$\sum_{j=1}^{n} \alpha_j e_j = o_n$$

and, therefore, $\alpha_1 = \ldots = \alpha_n = 0$, i.e. Y_0 and Y_k $(k = 1, 2, \ldots)$ are the fundamental matrices, respectively, of the homogeneous systems (1.1_0) and (1.1_{k0}) $(k = 1, 2, \ldots)$. Thanks to Corollary 2.1 and Lemma 3.4, conditions (3.1)–(3.4) hold uniformly on I.

We can assume without loss of generality that $Y_k(t_k) = I_n$ $(k = 0, 1, \ldots)$. We put $H_k(t) \equiv Y_k^{-1}(t)$ $(k = 0, 1, \ldots)$ and verify conditions (2.6)–(2.9).

Conditions (2.6) and (2.7) coincide with (3.3) and (3.4), respectively.

According to equality (1.11), we have

$$H_k(t) + \mathcal{B}(H_k, A_k)(t) \equiv I_n \quad (k = 0, 1, \ldots). \tag{4.17}$$

Thus condition (2.8) is evident, since by the definition of the operator $\mathcal{I}$ we find

$$\mathcal{I}(H_k, A_k)(t) \equiv O_{n\times n} \quad (k = 0, 1, \ldots). \tag{4.18}$$

On the other hand, in view of (4.17) and equalities $H_k(t_k) = I_n$ $(k = 0, 1, \ldots)$, according to Lemma 3.1 and the definition of a solution of system (1.1_k), we have

$$\begin{aligned}
\mathcal{B}(H_k, f_k)(\tau)\big|_{t_k}^{t} &= \mathcal{B}(H_k, x_k - y_k)(\tau)\big|_{t_k}^{t} = \mathcal{B}(H_k, x_k)(\tau)\big|_{t_k}^{t} - \mathcal{B}(H_k, y_k)(\tau)\big|_{t_k}^{t} \\
&= \mathcal{B}(H_k, x_k)(\tau)\big|_{t_k}^{t} - \int_{t_k}^{t} \mathrm{d}\mathcal{B}(H_k, A_k)(s) \cdot x_k(s) \\
&= H_k(t)x_k(t) - H_k(t_k)x_k(t_k) \\
&\quad - \int_{t_k}^{t} \mathrm{d}H_k(s) \cdot x_k(s) - \int_{t_k}^{t} \mathrm{d}(I_n - H_k(s)) \cdot x_k(s) \\
&= H_k(t)x_k(t) - x_k(t_k) \quad \text{for } t \in I \ (k = 0, 1, \ldots),
\end{aligned}$$

where $y_k(t) = \int_{t_k}^t \mathrm{d}A_k(s) \cdot x_k(s)$ $(k = 0, 1, \ldots)$. Hence,

$$\begin{aligned}(4.19)\qquad & \mathcal{B}(H_k, f_k)(\tau)|_{t_k}^t - \mathcal{B}(H_0, f_0)(\tau)|_{t_0}^t \\ &= (H_k(t)x_k(t) - H_0(t)x_0(t)) - (x_k(t_k) - x_0(t_0)) \\ &\qquad \text{for } t \in I \ (k = 1, 2, \ldots).\end{aligned}$$

Due to the necessity conditions of the theorem condition (1.3) holds. This, (2.1), (4.18) and (4.19) imply that condition (2.9) holds uniformly on I. □

Proof of Theorem 2.2′. Thanks to conditions (2.31), (2.32) and (2.33), conditions (2.10) and (2.11) hold. The theorem follows from Theorem 2.2 and Remark 2.1. □

Proof of Theorem 2.3′. Condition (2.34) is equivalent to condition (2.19). Moreover, due to (2.35), (2.36) and (2.37), conditions (2.23) and (2.24) hold. Therefore, the theorem follows from Theorem 2.3 and the remark analogous to Remark 2.1. □

Proof of Corollary 2.1′. Let us verify the conditions of Theorem 2.3′. The validity of conditions (2.18), (2.34) and (4.15) can be shown in a way similar to the proof of Corollary 2.1. In addition, by (4.15) there exists a positive number r such that

$$\|H_k^{-1}(t)\| \leqslant r \quad \text{for } t \in I \ (k = 0, 1, \ldots).$$

Using Lemma 3.1, from this estimate, (2.7), (2.28), (2.29), (2.38) and (4.15) we find that condition (2.35) holds, and conditions (2.36) and (2.37) are fulfilled uniformly on I, where

$$\begin{aligned} & h_k(t) = -H_k(t)\varphi_k(t), \ A_k^*(t) = \mathcal{I}(H_k, A_k)(\tau)|_{t_k}^t \quad \text{for } t \in I \ (k = 0, 1, \ldots); \\ & f_0^*(t) = \mathcal{B}(H_0, f_0)(\tau)|_{t_0}^t, \ f_k^*(t) = \mathcal{B}(H_k, f_k - \varphi_k)(\tau)|_{t_k}^t + \int_{t_k}^t \mathrm{d}\mathcal{B}(H_k, A_k)(s) \cdot \varphi_k(s) \\ & \qquad \text{for } t \in I \ (k = 1, 2, \ldots). \end{aligned}$$

Further, the rest of the proof coincides with the proof of Corollary 2.1. □

Proof of Theorem 2.1′. Sufficiency follows from Corollary 2.1′ if we assume $\varphi_k(t) = o_n$ $(k = 1, 2, \ldots)$. The proof of necessity is the same as in the proof of Theorem 2.1. We only note that by condition (2.7) and equality (4.17), condition (2.28) is valid, and condition (2.29) is fulfilled uniformly on I. Moreover, according to Remark 2.3, it is evident that sufficiency immediately follows from Theorem 2.1. □

P r o o f of Corollary 2.2. By (2.41), (2.42) and (2.43) (or (2.44)), the conditions

$$\lim_{k\to\infty}\sum_{s\leqslant t;s,t\in I}(d_1H_k(s)\cdot d_1A_k(s)-d_1H_0(s)\cdot d_1A_0(s))=O_{n\times n},$$
$$\lim_{k\to\infty}\sum_{s\leqslant t;s,t\in I}(d_1H_k(s)\cdot d_1f_k(s)-d_1H_0(s)\cdot d_1f_0(s))=o_n,$$
$$\lim_{k\to\infty}\sum_{s\leqslant t;s,t\in I}(d_2H_k(s)\cdot d_2A_k(s)-d_2H_0(s)\cdot d_2A_0(s))=O_{n\times n}$$

and

$$\lim_{k\to\infty}\sum_{s\leqslant t;\, s,t\in I}(d_2H_k(s)\cdot d_2f_k(s)-d_2H_0(s)\cdot d_2f_0(s))=o_n$$

are fulfilled uniformly on I. From this, the integration-by-parts formula, (2.39) and (2.40) we obtain that conditions (2.29) and (2.30) are fulfilled uniformly on I. Therefore, the corollary follows from Theorem 2.1′. □

P r o o f of Corollary 2.3. Using (2.7), (2.32) and (2.45) we conclude that $d_jA_0^*(t)\equiv O_{n\times n}$ $(j=1,2)$. Hence, in view of (2.3) we have

$$\det(I_n+(-1)^jd_jA_0^*(t))\neq 0 \quad \text{for } t\in I,\ (-1)^j(t-t_0)<0 \quad \text{and also}$$
$$\text{for } t=t_0 \text{ if } j\in\{1,2\} \text{ is such that } (-1)^j(t_k-t_0)>0 \text{ for every } k\in\{1,2,\dots\}.$$

On the other hand, (2.7), (2.32), (2.33), (2.45) and (2.46) yield that the conditions

$$\lim_{k\to\infty}(\mathcal{B}(H_k,A_k)(t)-\mathcal{B}(H_k,A_k)(t_k))=\mathcal{B}(I_n,A_0^*)(t)-\mathcal{B}(I_n,A_0^*)(t_0)$$

and

$$\lim_{k\to\infty}(\mathcal{B}(H_k,f_k)(t)-\mathcal{B}(H_k,f_k)(t_k))=\mathcal{B}(I_n,f_0^*)(t)-\mathcal{B}(I_n,f_0^*)(t_0)$$

hold uniformly on I. Thus, Corollary 2.3 is a direct consequence of Theorem 2.1′. □

P r o o f of Corollary 2.4. Let

$$C_{kl}(t)=I_n-(A_{kl}(t)-A_{kl}(t_k))+(B_l(t)-B_l(t_k)) \quad (l=1,\dots,m;\ k=1,2,\dots).$$

Thanks to (2.48), without loss of generality we can assume that the matrix-functions H_{kl} $(l=1,\dots,m)$ and C_{kl} $(l=1,\dots,m)$ are nonsingular for every natural k. Using now Lemma 3.1, we find that

$$\mathcal{B}(C_{kj}, \mathcal{B}(H_{kj-1}, A_k))(\tau)|_{t_k}^{t} \equiv \mathcal{B}(H_{kj}, A_k)(\tau))|_{t_k}^{t},$$
$$\mathcal{B}(C_{kj}, \mathcal{B}(H_{kj-1}, f_k))(\tau)|_{t_k}^{t} \equiv \mathcal{B}(H_{kj}, f_k)(\tau)|_{t_k}^{t}$$

and

$$\mathcal{I}(C_{kj}, \mathcal{I}(H_{kj-1}, A_k))(\tau)|_{t_k}^{t} \equiv \mathcal{I}(H_{kj}, A_k)(\tau)|_{t_k}^{t} \quad (j = 1, \dots, m;\ k = 1, 2, \dots).$$

In addition, by conditions (2.47)–(2.50), conditions (2.6) and (2.28) hold, and conditions (2.7), (2.29) and (2.30) are fulfilled uniformly on I, where $H_0(t) = I_n$ and $H_k(t) = H_{km-1}(t)$ $(k = 1, 2, \dots)$. The corollary follows from Theorem 2.1′. □

P r o o f of Corollary 2.5. Let us show sufficiency. Let $H_k(t) = Z_k^{-1}(t)$ $(k = 0, 1, \dots)$ in Theorem 2.1′. Thanks to (2.53), there exists a positive number r such that $\|Z_k^{-1}(t)\| \leqslant r$ for $t \in I$ $(k = 0, 1, \dots)$. Using this estimate, by (1.11), the definition of the operator $\mathcal{B}$ and the integration-by-parts formula, we have

$$\begin{aligned}
&\|Z_k^{-1}(t) + \mathcal{B}(Z_k^{-1}, A_k)(t) - Z_k^{-1}(s) - \mathcal{B}(Z_k^{-1}, A_k)(s)\| \\
&\qquad = \|\mathcal{B}(Z_k^{-1}, A_k - B_k)(t) - \mathcal{B}(Z_k^{-1}, A_k - B_k)(s)\| \\
&\qquad = \Big\| \int_s^t Z_k^{-1}(\tau)\,\mathrm{d}(A_k(\tau) - B_k(\tau)) - \sum_{s<\tau\leqslant t} d_1 Z_k^{-1}(\tau) \cdot d_1(A_k(\tau) - B_k(\tau)) \\
&\qquad\qquad + \sum_{s\leqslant\tau<t} d_2 Z_k^{-1}(\tau) \cdot d_2(A_k(\tau) - B_k(\tau)) \Big\| \\
&\qquad \leqslant r \bigvee_s^t (A_k - B_k) + 2r \sum_{s<\tau\leqslant t} \|d_1(A_k(\tau) - B_k(\tau))\| \\
&\qquad\qquad + 2r \sum_{s\leqslant\tau<t} \|d_2(A_k(\tau) - B_k(\tau))\| \\
&\qquad \leqslant 5r \bigvee_s^t (A_k - B_k) \quad \text{for } s < t\ (k = 0, 1, \dots).
\end{aligned}$$

Consequently,

$$\bigvee_I (H_k + \mathcal{B}(H_k, A_k)) \leqslant 5r \bigvee_I (A_k - B_k) \quad (k = 0, 1, \dots)$$

and due to (2.51) estimate (2.28) holds. Conditions (2.29) and (2.30) coincide with conditions (2.54) and (2.55), respectively. Sufficiency follows from Theorem 2.1′.

Let us show necessity. Let $B_k(t) = A_k(t)$ $(k = 0, 1, \ldots)$. Then $Z_k(t) \equiv Y_k(t)$ $(k = 0, 1, \ldots)$, where Y_0 and Y_k $(k = 1, 2, \ldots)$ are fundamental matrices, respectively, of systems (1.1_0) and (1.1_{k0}). Analogously, as in the proof of Theorem 2.1, conditions (2.53) and (4.19) are valid. In addition, condition (2.54) coincides with condition (2.29), and condition (2.55) follows from condition (4.19). □

P r o o f of Corollary 2.6. Due to conditions (2.56) and (2.57), without loss of generality, we can assume that condition (2.52) holds for every natural k. Condition (2.53) follows from condition (2.57) by representation (2.62).

Let us verify condition (2.54). Using the integration-by-parts formula we find

$$\begin{aligned}\mathcal{B}(Z_k^{-1}, A_k)(t) - \mathcal{B}(Z_k^{-1}, A_k)(s) \\ &= \int_s^t Z_k^{-1}(\tau)\,\mathrm{d}A_k(\tau) - \sum_{s<\tau\leqslant t} d_1 Z_k^{-1}(\tau)\cdot d_1 A_k(\tau) \\ &\quad + \sum_{s\leqslant\tau<t} d_2 Z_k^{-1}(\tau)\cdot d_2 A_k(\tau) \quad \text{for } s<t \ (k=0,1,\ldots).\end{aligned}$$

In addition, in virtue of equalities (1.12), we have

$$d_j Z_k^{-1}(t) \equiv -Z_k^{-1}(t) d_j B_k(t)\cdot (I_n + (-1)^j d_j B_k(t))^{-1} \quad (j=1,2;\ k=0,1,\ldots).$$

Consequently, due to (1.4), we get

$$\mathcal{B}(Z_k^{-1}, A_k)(t) - \mathcal{B}(Z_k^{-1}, A_k)(s) = \int_s^t Z_k^{-1}(\tau)\,\mathrm{d}\mathcal{A}(B_k, A_k)(\tau)\ (k=0,1,\ldots)$$

for $s < t$. In the same way we establish the last equalities for the case when $t < s$.

Analogously, we check the equalities

$$\begin{aligned}\mathcal{B}(Z_k^{-1}, f_k)(t) - \mathcal{B}(Z_k^{-1}, f_k)(s) \\ &= \int_s^t Z_k^{-1}(\tau)\,\mathrm{d}\mathcal{A}(B_k, f_k)(\tau) \quad \text{for } s,t\in I\ (k=0,1,\ldots).\end{aligned}$$

Therefore, equalities (2.54) and (2.55) coincide, respectively, with equalities (2.58) and (2.59). The corollary follows from Corollary 2.5. □

P r o o f of Corollary 2.7. The corollary follows from Corollary 2.6 if we assume $B_k(t) = S_c(A_k)(t)$ $(k = 0, 1, \ldots)$. In addition, we note that condition (2.54) has the form (2.63), equality (2.57) is equivalent to conditions (2.64) and (2.65), and by virtue of (2.62) condition (2.58) coincides with (2.66). □

P r o o f of Corollary 2.8. The corollary follows from Corollary 2.6 if we assume that $B_k(t) = \operatorname{diag}(A_k(t))$ $(k = 0, 1, \ldots)$. □

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