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Original article

On the Opial type criterion for the well-posedness of the Cauchy problem for linear systems of ordinary differential equations

Malkhaz Ashordia*

A. Razmadze Mathematical Institute I, Javakhishvili Tbilisi State University, 6 Tamarashvili st., Tbilisi 0177, Georgia
Sukhumi State University, 12 Politkovskaia st., Tbilisi 0186, Georgia

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Abstract

There are obtained necessary and sufficient conditions for the well-posedness of the Cauchy problem for the systems of linear ordinary differential equations, analogous to the sufficient condition by Z. Opial for the problem one. Moreover, there are given the efficient sufficient conditions for the problem one.

Keywords: Linear systems of ordinary differential equations; The Cauchy problem; Well-posedness; The Opial type condition; Necessary and sufficient conditions; Efficient sufficient conditions

1. Statement of the problem and basic notation

Let $P_0 \in L_{loc}(I, \mathbb{R}^{n\times n})$, $q_0 \in L_{loc}(I, \mathbb{R}^n)$ and $t_0 \in I$, where I is an arbitrary interval from $\mathbb{R}$ non-degenerated in the point. Let x_0 be a unique solution of the Cauchy problem

$$\frac{dx}{dt} = \mathcal{P}_0(t)\,x + q_0(t), \tag{1.1}$$

$$x(t_0) = c_0, \tag{1.2}$$

where $c_0 \in \mathbb{R}^n$ is a constant vector.

Consider sequences of matrix- and vector-functions $P_k \in L_{loc}(I, \mathbb{R}^{n\times n})$ $(k = 1, 2, \ldots)$ and $q_k \in L_{loc}(I, \mathbb{R}^n)$ $(k = 1, 2, \ldots)$, respectively; sequence of points t_k $(k = 1, 2, \ldots)$ and sequence of constant vectors $c_k \in \mathbb{R}^n$ $(k = 1, 2, \ldots)$.

* Correspondence to: A. Razmadze Mathematical Institute I, Javakhishvili Tbilisi State University, 6 Tamarashvili st., Tbilisi 0177, Georgia.
E-mail address: ashord@rmi.ge.
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In [1–8] (see, also the references therein), the sufficient conditions are given such that a sequence of unique solutions x_k $(k = 1, 2, \dots)$ of the Cauchy problems

$$\frac{dx}{dt} = \mathcal{P}_k(t)\,x + q_k(t), \tag{1.1$_k$}$$

$$x(t_k) = c_k \tag{1.2$_k$}$$

$(k = 1, 2, \dots)$ satisfy the condition

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \quad \text{uniformly on } I. \tag{1.3}$$

In the present paper necessary and sufficient conditions are established for the sequence of the Cauchy problems (1.1$_k$), (1.2$_k$) $(k = 1, 2, \dots)$ to have the above-mentioned property. The obtained criterion are based on the concept by Z. Opial, concerning to the sufficient condition considered in [8], and it differs from analogous one given in [1].

The Opial type sufficient conditions are investigated in [5] for the well-posedness problem of the Cauchy problem for linear functional-differential equations.

In the paper the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$; $[a, b]$ and $]a, b[(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

I is an arbitrary, non-degenerated in the point, finite or infinite interval from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$O_{n\times m}$ is the zero $n \times m$-matrix.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; o_n is the zero n-vector.

$\mathbb{R}^{n\times n}$ is the space of all real quadratic $n \times n$-matrices $X = (x_{ij})_{i,j=1}^n$;

I_n is the identity $n \times n$-matrix; $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \dots, \lambda_n$; δ_{ij} is the Kronecker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j\,(i, j = 1, \dots)$;

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X; $\operatorname{diag}X = \operatorname{diag}(x_{11}, \dots, x_{nn})$ is the diagonal matrix corresponding to X.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

We say that the matrix-function $X \in L_{loc}(I, \mathbb{R}^{n\times n})$ satisfies the Lappo-Danilevskiĭ condition if for every $\tau \in I$ the following condition holds

$$X(t)\int_{\tau}^{t} X(\tau)\,d\tau = \int_{\tau}^{t} X(\tau)d\tau \cdot X(t) \quad \text{for a. a. } t \in I.$$

$\bigvee\limits_a^b(X)$ is the sum total variation of the components $x_{ij}\,(i = 1, \dots, n;\ j = 1, \dots, m)$ of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$; $\bigvee\limits_b^a(X) = -\bigvee\limits_a^b(X)$;

$\bigvee\limits_I(X) = \lim_{a\to\alpha+, b\to\beta-} \bigvee\limits_a^b(X)$, where $\alpha = \inf I$ and $\beta = \sup I$.

$C(I; \mathbb{R}^{m\times n})$ is a space of continuous and bounded matrix-functions $X : I \to \mathbb{R}^{m\times n}$ with the norm

$$\|X\|_c = \sup\{\|X(t)\| : t \in I\};$$

$C(I; D)$, where $D \subset \mathbb{R}^{m\times n}$, is the set of continuous and bounded matrix-functions $X : I \to D$;

$C_{loc}(I; D)$ is the set of continuous matrix-functions $X : I \to D$;

$\widetilde{C}(I; D)$ is the set of absolutely continuous matrix-functions $X : I \to D$;

$\widetilde{C}_{loc}(I; D)$ is the set of matrix-functions $X : I \to D$ which are absolutely continuous on the every closed interval $[a, b]$ from I.

$L(I; D)$, where $D \subset \mathbb{R}^{m\times n}$, is the set of matrix-functions $X : I \to D$ whose components are Lebesgue-integrable;

$L_{loc}(I; D)$ is the set of matrix-functions $X : I \to D$ whose components are Lebesgue-integrable on the every closed interval $[a, b]$ from I.

We introduce the operators. If $G \in L(I; \mathbb{R}^{l\times n})$, $X \in L(I; \mathbb{R}^{n\times m})$, $Y \in L(I; \mathbb{R}^{n\times n})$, and $H \in \widetilde{C}(I; \mathbb{R}^{n\times n})$ is nonsingular, then

$$\mathcal{B}_c(G, X)(t) = \int_\alpha^t G(\tau)\, X(\tau) d\tau \quad \text{for } t \in I,$$

$$\mathcal{I}_c(H, Y)(t) = \int_\alpha^t (H'(\tau) + H(\tau)\, Y(\tau))\, H^{-1}(\tau) d\tau \quad \text{for } t \in I.$$

The vector-function $x : I \to \mathbb{R}^n$ is said to be a solution of the system (1.1) if it belongs to $\widetilde{C}_{loc}(I; \mathbb{R}^n)$ and satisfies the equality $x'(t) = \mathcal{P}_0(t)x(t) + q_0(t)$ at almost all $t \in I$.

Under a solution of the Cauchy problem (1.1), (1.2) we understand a solution of system (1.1) satisfying condition (1.2).

We will assume that $P_k = (p_{kil})_{i,l=1}^n$ and $q_k = (q_{kl})_{l=1}^n (k = 0, 1, \dots)$.

Along with systems (1.1) and (1.1_k) we consider the corresponding homogeneous systems

$$\frac{dx}{dt} = P_0(t)\, x \tag{1.1_0}$$

and

$$\frac{dx}{dt} = P_k(t)\, x \tag{1.1_{k0}}$$

$(k = 1, 2, \dots)$.

2. Formulation of the main results

Definition 2.1. We say that the sequence $(P_k, q_k; t_k)$ $(k = 1, 2, \dots)$ belongs to the set $\mathcal{S}(P_0, q_0; t_0)$ if for every $c_0 \in \mathbb{R}^n$ and a sequence $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ satisfying the condition

$$\lim_{k\to+\infty} c_k = c_0, \tag{2.1}$$

condition (1.3) holds, where x_k is the unique solution of problem (1.1_k), (1.2_k) for every natural k.

Theorem 2.1. *Let $P_0 \in L(I, \mathbb{R}^{n\times n})$, $q_0 \in L(I, \mathbb{R}^n)$ and $t_k \in I (k = 0, 1, \dots)$ be such that*

$$\lim_{k\to+\infty} t_k = t_0. \tag{2.2}$$

Then

$$((P_k, q_k; t_k))_{k=1}^{+\infty} \in \mathcal{S}(P_0, q_0; t_0) \tag{2.3}$$

if and only if there exists a sequence of matrix-functions $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$ such that

$$\inf\Big\{|\det(H_0(t))| : t \in I\Big\} > 0, \tag{2.4}$$

and the conditions

$$\lim_{k\to+\infty} H_k(t) = H_0(t), \tag{2.5}$$

$$\lim_{k\to+\infty} \left\{ \left\| \mathcal{I}_c(H_k, P_k)(\tau)\Big|_{t_k}^t - \mathcal{I}_c(H_0, P_0)(\tau)\Big|_{t_0}^t \right\| \times \left(1 + \left| \mathop{\mathrm{V}}_{t_k}^{t}(\mathcal{I}_c(H_k, P_k)) \right| \right) \right\} = 0 \tag{2.6}$$

and

$$\lim_{k\to+\infty} \left\{ \left\| \mathcal{B}_c(H_k, q_k)(\tau)\Big|_{t_k}^t - \mathcal{B}_c(H_0, q_0)(\tau)\Big|_{t_0}^t \right\| \times \left(1 + \left| \mathop{\mathrm{V}}_{t_k}^{t}(\mathcal{I}_c(H_k, P_k)) \right| \right) \right\} = 0 \tag{2.7}$$

hold uniformly on I.

Theorem 2.2. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k = 0, 1, \dots)$ be such that conditions (2.1) and (2.2) hold, and the conditions*

$$\lim_{k\to+\infty}\left\{\left\|\int_{t_k}^{t} P_k(\tau)d\tau - \int_{t_0}^{t} P_0(\tau)d\tau\right\|\left(1+\left|\int_{t_k}^{t}\|P_k(\tau)\|d\tau\right|\right)\right\} = 0 \tag{2.8}$$

and

$$\lim_{k\to+\infty}\left\{\left\|\int_{t_k}^{t} q_k(\tau)d\tau - \int_{t_0}^{t} q_0(\tau)d\tau\right\|\left(1+\left|\int_{t_k}^{t}\|P_k(\tau)\|d\tau\right|\right)\right\} = 0 \tag{2.9}$$

are fulfilled uniformly on I. Then condition (1.3) holds.

Theorem 2.3. *Let x_0^* be a unique solution of the Cauchy problem*

$$\frac{dx}{dt} = \mathcal{P}_0^*(t)\,x + q_0^*(t), \tag{2.10}$$

$$x(t_0) = c_0^*, \tag{2.11}$$

where $P_0^* \in L(I, \mathbb{R}^{n\times n})$, $q_0^* \in L(I, \mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, $t_0 \in I$. Let, moreover, $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I\,(k = 1, 2, \dots)$ be such that conditions (2.2),

$$\inf\{|\det(H_k(t))| : t \in I_{t_k}\} > 0 \quad \text{for every sufficiently large } k, \tag{2.12}$$

and

$$\lim_{k\to+\infty} c_k^* = c_0^* \tag{2.13}$$

hold, and conditions (2.6) and

$$\lim_{k\to+\infty}\left\{\left\|\int_{t_k}^{t} q_k^*(\tau)d\tau - \int_{t_0}^{t} q_0^*(\tau)d\tau\right\|\left(1+\left|\bigvee_{t_k}^{t}(\mathcal{I}_c(H_k, P_k))\right|\right)\right\} = 0 \tag{2.14}$$

are fulfilled uniformly on I, where $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})$, $h_k \in \widetilde{C}(I; \mathbb{R}^n)$ $(k = 1, 2, \dots)$,

$$q_k^*(t) = H_k(t)\,q_k(t) + h_k'(t) - (H_k'(t) + H_k(t)\,P_k(t))\,H_k^{-1}(t)\,h_k(t) \quad \text{for } t \in I\ (k = 1, 2, \dots)$$

and

$$c_k^* = H_k(t_k)\,c_k + h_k(t_k) \quad (k = 1, 2, \dots).$$

Then

$$\lim_{k\to+\infty}(H_k(t)\,x_k(t) + h_k(t)) = x_0^*(t) \quad \text{uniformly on } I. \tag{2.15}$$

Remark 2.1. In Theorem 2.3, the vector function $x_k^*(t) = H_k(t)\,x_k(t) + h_k(t)$ is a solution of problem

$$\frac{dx}{dt} = \mathcal{P}_k^*(t)\,x + q_k^*(t), \tag{2.10$_k$}$$

$$x(t_k) = c_k^* \tag{2.11$_k$}$$

for every natural k.

Corollary 2.1. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I\,(k = 0, 1, \dots)$ be such that conditions (2.2), (2.4) and*

$$\lim_{k\to+\infty}(c_k - \varphi_k(t_k)) = c_0 \tag{2.16}$$

hold, and conditions (2.5), (2.6) and

$$\lim_{k\to+\infty}\left\{\left\|\int_{t_k}^{t} H_k(\tau)\left(q_k(\tau)-\varphi_k'(\tau)+P_k(\tau)\,\varphi_k(\tau)\right)d\tau-\int_{t_0}^{t} H_0(\tau)\,q_0(\tau)d\tau\right\|\right.$$
$$\left.\times\left(1+\left|\mathop{\mathrm{V}}_{t_k}^{t}(\mathcal{I}_c(H_k,P_k))\right|\right)\right\}=0$$

are fulfilled uniformly on I, where $H_k \in \widetilde{C}(I;\mathbb{R}^{n\times n})$ and $\varphi_k \in \widetilde{C}(I;\mathbb{R}^{n})$ $(k=0,1,\ldots)$. Then

$$\lim_{k\to+\infty}(x_k(t)-\varphi_k(t))=x_0(t) \quad \text{uniformly on } I. \tag{2.17}$$

Below, we give some sufficient conditions guaranteeing inclusion (2.3). To this connection we give a theorem different from Theorem 2.1 concerning the necessary and sufficient condition for inclusion (2.3), as well, and corresponding propositions.

Theorem 2.1′. *Let $P_0 \in L(I,\mathbb{R}^{n\times n})$, $q_0 \in L(I,\mathbb{R}^{n})$, $t_0 \in I$, and $t_k \in I$ $(k=1,2,\ldots)$ be such that condition (2.2) hold. Then inclusion (2.3) holds if and only if there exists a sequence of matrix-functions $H_k \in \widetilde{C}(I;\mathbb{R}^{n\times n})$ $(k=0,1,\ldots)$ such that conditions (2.4) and*

$$\lim_{k\to+\infty}\sup\int_I \|H_k'(\tau)+H_k(\tau)\,P_k(\tau)\|d\tau<+\infty \tag{2.18}$$

hold, and conditions (2.5),

$$\lim_{k\to+\infty}\int_{t_k}^{t} H_k(\tau)\,P_k(\tau)d\tau=\int_{t_0}^{t} H_0(\tau)\,P_0(\tau)d\tau \tag{2.19}$$

and

$$\lim_{k\to+\infty}\int_{t_k}^{t} H_k(\tau)\,q_k(\tau)d\tau=\int_{t_0}^{t} H_0(\tau)\,q_0(\tau)d\tau \tag{2.20}$$

are fulfilled uniformly on I.

Remark 2.2. Due to (2.4), (2.5), there exists a positive number r such that

$$\sup\left\{\left|\mathop{\mathrm{V}}_{t_k}^{t}(\mathcal{I}_c(H_k,P_k))\right|: t\in I\right\}\le r\int_I \|H_k'(\tau)+H_k(\tau)\,P_k(\tau)\|d\tau \quad (k=0,1,\ldots).$$

In addition, in view of Lemma 3.2 (see below), by conditions (2.18) and (2.19) we get

$$\lim_{k\to+\infty}(\mathcal{I}_c(H_k,P_k)(t)-\mathcal{I}_c(H_k,P_k)(t_k))=\mathcal{I}_c(H_0,P_0)(t)-\mathcal{I}_c(H_0,P_0)(t_0)$$

uniformly on I. Therefore, thanks to this, (2.18) and (2.20), conditions (2.6) and (2.7) are fulfilled uniformly on I

Theorem 2.2′. *Let $P_k \in L(I,\mathbb{R}^{n\times n})$, $q_k \in L(I,\mathbb{R}^{n})$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k=0,1,\ldots)$ be such that conditions (2.1), (2.2) and*

$$\lim_{k\to+\infty}\sup\int_I \|P_k(\tau)\|d\tau<+\infty \tag{2.21}$$

hold, and the conditions

$$\lim_{k\to+\infty}\int_{t_k}^{t} P_k(\tau)d\tau=\int_{t_0}^{t} P_0(\tau)d\tau \tag{2.22}$$

and

$$\lim_{k\to+\infty}\int_{t_k}^{t} q_k(\tau)d\tau=\int_{t_0}^{t} q_0(\tau)d\tau \tag{2.23}$$

are fulfilled uniformly on I. Then condition (1.3) holds.

Theorem 2.3′. *Let x_0^* be a unique solution of the Cauchy problem (2.10), (2.11), where $P_0^* \in L(I, \mathbb{R}^{n\times n})$, $q_0^* \in L(I, \mathbb{R}^n)$, $c_0^* \in \mathbb{R}^n$, $t_0 \in I$. Let, moreover, $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k = 1, 2, \ldots)$ be such that conditions (2.2), (2.12), (2.18) and*

$$\lim_{k\to+\infty} (H_k(t_k)\, c_k + h_k(t_k)) = c_0^* \tag{2.24}$$

hold, and the conditions

$$\lim_{k\to+\infty} (\mathcal{I}_c(H_k, P_k)(t) - \mathcal{I}_c(H_k, P_k)(t_k)) = \mathcal{I}_c(H_0, P_0^*)(t) - \mathcal{I}_c(H_0, P_0^*)(t_0), \tag{2.25}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} q_k^*(\tau)d\tau = \int_{t_0}^{t} q_0^*(\tau)d\tau \tag{2.26}$$

are fulfilled uniformly on I, where $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})$, $h_k \in \widetilde{C}(I; \mathbb{R}^n)(k = 1, 2, \ldots)$, and the vector-functions $q_k^*(k = 1, 2, \ldots)$ are defined as in Theorem 2.3. Then condition (1.3) holds.

Corollary 2.1′. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k = 0, 1, \ldots)$ be such that conditions (2.2), (2.4), (2.16) and (2.18) hold, and conditions (2.5), (2.19) and*

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k(\tau) \big(q_k(\tau) - \varphi_k'(\tau) + P_k(\tau)\, \varphi_k(\tau)\big)d\tau = \int_{t_0}^{t} H_0(\tau)\, q_0(\tau)d\tau$$

are fulfilled uniformly on I, where $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})$ and $\varphi_k \in \widetilde{C}(I; \mathbb{R}^n)(k = 0, 1, \ldots)$. Then condition (2.17) holds.

Corollary 2.2. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I(k = 0, 1, \ldots)$ be such that conditions (2.2), (2.4) and (2.18) hold, and conditions (2.5), (2.22), (2.23),*

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k'(\tau) \left(\int_{t_k}^{\tau} P_k(s)ds\right) d\tau = \int_{t_0}^{t} P^*(\tau)d\tau \tag{2.27}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k'(\tau) \left(\int_{t_k}^{\tau} q_k(s)ds\right) d\tau = \int_{t_0}^{t} q^*(\tau)d\tau \tag{2.28}$$

are fulfilled uniformly on I, where $H_0(t) = I_n$, $H_k \in \widetilde{C}(I; \mathbb{R}^{n\times n})(k = 1, 2, \ldots)$, $P^* \in L(I, \mathbb{R}^{n\times n})$, $q^* \in L(I, \mathbb{R}^n)$. Then

$$((P_k, q_k; t_k))_{k=1}^{+\infty} \in \mathcal{S}(P_0 - P^*, q_0 - q^*; t_0).$$

Corollary 2.3. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I(k = 0, 1, \ldots)$ be such that condition (2.2) holds and let there exist a natural number m and matrix-functions $P_{0l} \in L(I; \mathbb{R}^{n\times n})(l = 1, \ldots, m-1)$ such that*

$$\lim_{k\to+\infty} \sup \int_I \|H_{k\,m-1}'(t) + H_{k\,m-1}(t)\, P_k(t)\|dt < +\infty, \tag{2.29}$$

and the conditions

$$\lim_{k\to+\infty} H_{k\,m-1}(t) = I_n, \tag{2.30}$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_{k\,m-1}(\tau)\, P_k(\tau)d\tau = \int_{t_0}^{t} P_0(\tau)d\tau, \tag{2.31}$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_{k\,m-1}(\tau)\, q_k(\tau)d\tau = \int_{t_0}^{t} q_0(\tau)d\tau \tag{2.32}$$

hold uniformly on I, where

$$H_{k0}(t) = I_n, \qquad H_{k\,j+1}(t) = \left(I_n - \int_{t_k}^{t} (P_{k\,j+1}(\tau) - P_{0l}(\tau))d\tau\right) H_{kj}(t),$$
$$P_{kj+1}(t) = H'_{kj}(t) + H_{kj}(t)\,P_k(t), \qquad q_{kj+1}(t) = H_{kj}(t)\,q_k(t)$$
$$\text{for } t \in I \ (j = 0, \dots, m-1;\ k = 0, 1, \dots).$$

Then inclusion (2.3) holds.

If $m = 1$, then Corollary 2.3 coincides to Theorem 2.2′.
If $m = 2$, then Corollary 2.3 has the following form.

Corollary 2.3′. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$, $c_k \in \mathbb{R}^n$ and $t_k \in I$ $(k = 0, 1, \dots)$ be such that condition (2.2) holds and let there exist a matrix-function $P_{01} \in L(I; \mathbb{R}^{n\times n})$ such that*

$$\lim_{k\to+\infty} \sup \int_I \left\| P_{01}(t) - \int_{t_k}^{t} (P_k(\tau) - P_{01}(\tau))d\tau \cdot P_k(t) \right\| dt < +\infty,$$

and the conditions

$$\lim_{k\to+\infty} \int_{t_k}^{t} P_k(\tau)d\tau = \int_{t_0}^{t} P_{01}(\tau)d\tau,$$
$$\lim_{k\to+\infty} \int_{t_k}^{t} \left((P_k(\tau) - P_{01}(\tau)) \int_{t_k}^{\tau} P_k(s)ds\right)d\tau = \int_{t_0}^{t} (P_0(\tau) - P_{01}(\tau))d\tau$$

and

$$\lim_{k\to+\infty} \left\{ \int_{t_k}^{t} q_k(\tau)d\tau + \int_{t_k}^{t} \left((P_k(\tau) - P_{01}(\tau)) \int_{t_k}^{\tau} q_k(s)ds\right)d\tau \right\} = \int_{t_0}^{t} q_0(\tau)d\tau$$

are fulfilled uniformly on I. Then inclusion (2.3) holds.

Corollary 2.4. *Let $P_0 \in L(I, \mathbb{R}^{n\times n})$, $q_0 \in L(I, \mathbb{R}^n)$, $t_0 \in I$, and $t_k \in I (k = 1, 2, \dots)$ be such that condition (2.2) holds. Then inclusion (2.3) holds if and only if there exists a sequence of matrix-functions $Q_k \in L(I; \mathbb{R}^{n\times n})(k = 0, 1, \dots)$ such that the condition*

$$\lim_{k\to+\infty} \sup \int_I \| P_k(\tau) - Q_k(\tau) \| d\tau < +\infty \tag{2.33}$$

holds, and the conditions

$$\lim_{k\to+\infty} Z_k^{-1}(t) = Z_0^{-1}(t), \tag{2.34}$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} Z_k^{-1}(\tau)\,P_k(\tau)d\tau = \int_{t_0}^{t} Z_0^{-1}(\tau)\,P_0(\tau)d\tau \tag{2.35}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} Z_k^{-1}(\tau)\,q_k(\tau)d\tau = \int_{t_0}^{t} Z_0^{-1}(\tau)\,q_0(\tau)d\tau \tag{2.36}$$

are fulfilled uniformly on I, where $Z_k(Z_k(t_k) = I_n)$ is a fundamental matrices of the homogeneous problems

$$\frac{dx}{dt} = Q_k(t)x \tag{2.37}$$

for every $k \in \{0, 1, \dots\}$.

Corollary 2.5. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I (k = 0, 1, \dots)$ be such that condition (2.2) holds and let there exist a sequence of matrix-functions $Q_k \in L(I; \mathbb{R}^{n\times n})$ $(k = 0, 1, \dots)$, satisfying the Lappo-Danilevskiĭ condition, such that condition (2.33) holds, and the conditions*

$$\lim_{k\to+\infty} \int_{t_k}^{t} Q_k(\tau)d\tau = \int_{t_0}^{t} Q_0(\tau)d\tau,$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} \exp\left(-\int_{t_k}^{\tau} Q_k(s)ds\right) P_k(\tau)d\tau = \int_{t_0}^{t} \exp\left(-\int_{t_0}^{\tau} Q_0(s)ds\right) P_0(\tau)d\tau \tag{2.38}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} \exp\left(-\int_{t_k}^{\tau} Q_k(s)ds\right) q_k(\tau)d\tau = \int_{t_0}^{t} \exp\left(-\int_{t_0}^{\tau} Q_0(s)ds\right) q_0(\tau)d\tau \tag{2.39}$$

are fulfilled uniformly on I. Then inclusion (2.3) holds.

Corollary 2.6. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I (k = 0, 1, \dots)$ be such that condition (2.2) holds, the matrix functions $P_k (k = 0, 1, \dots)$ satisfy the Lappo-Danilevskiĭ condition, and the conditions*

$$\lim_{k\to+\infty} \int_{t_k}^{t} P_k(\tau)d\tau = \int_{t_0}^{t} P_0(\tau)d\tau, \tag{2.40}$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} \exp\left(-\int_{t_k}^{\tau} P_k(s)ds\right) q_k(\tau)d\tau = \int_{t_0}^{t} \exp\left(-\int_{t_0}^{\tau} P_0(s)ds\right) q_0(\tau)d\tau \tag{2.41}$$

are fulfilled uniformly on I. Then inclusion (2.3) holds.

Corollary 2.7. *Let $P_k \in L(I, \mathbb{R}^{n\times n})$, $q_k \in L(I, \mathbb{R}^n)$ and $t_k \in I (k = 0, 1, \dots)$ be such that conditions (2.2) and*

$$\lim_{k\to+\infty} \sup \sum_{i,l=1;\, i\neq l}^{n} \int_I \|p_{kil}(\tau)\|d\tau < +\infty$$

hold, and the conditions

$$\lim_{k\to+\infty} \int_{t_k}^{t} p_{kii}(\tau)d\tau = \int_{t_0}^{t} p_{0ii}(\tau)d\tau \quad (i = 1, \dots, n)$$

$$\lim_{k\to+\infty} \int_{t_k}^{t} z_{kii}^{-1}(\tau)p_{kil}(\tau)d\tau = \int_{t_0}^{t} z_{0ii}^{-1}(\tau)p_{0il}(\tau)d\tau \quad (i \neq l;\ i, l = 1, \dots, n)$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} z_{kii}^{-1}(\tau)q_{ki}(\tau)d\tau = \int_{t_0}^{t} z_{0ii}^{-1}(\tau)q_{0i}(\tau)d\tau \quad (i = 1, \dots, n)$$

are fulfilled uniformly on I, where

$$z_{kii}(t) = \exp\left(\int_{t_k}^{t} p_{kii}(s)ds\right) \quad \text{for } t \in I \ (i = 1, \dots, n;\ k = 1, 2, \dots).$$

Then inclusion (2.3) holds.

Remark 2.3. In Theorems 2.1′–2.3′ and Corollaries 2.1′, 2.2–2.7, we can assume $H_0(t) = I_n$, without loss of generality. It is evident that

$$\mathcal{I}_c(H_0, Y)(t) - \mathcal{I}_c(H_0, Y)(s) = \int_s^t Y(\tau)d\tau \quad \text{for } Y \in L(I; \mathbb{R}^{n\times n}) \text{ and } s, t \in I,$$

in this case.

Remark 2.4. In Theorem 2.2′, condition (2.21) is essential and it cannot be removed. In connection with this we give the example from [4].

Example 2.1. Let $I = [0, 2\pi]$, $n = 1$, $c_k = c_0 = 0$, $P_0(t) = q_0(t) = 0$, $P_k(t) = k\cos^2 k^2 t$, $q_k(t) = -k\sin k^2 t$, $t_0 = t_k = 0 (k = 1, 2, \ldots)$. Then

$$x_0(t) \equiv 0, \qquad x_k(t) \equiv -k\int_0^t \exp\Big(\frac{\sin k^2 t}{k} - \frac{\sin k^2 \tau}{k}\Big)\sin k^2\tau\, d\tau \quad (k = 1, 2, \ldots)$$

and

$$\lim_{k\to+\infty} x_k(t) = x_0(t) + \frac{t}{2} \quad \text{uniformly on } [0, 2\pi].$$

It is evident that, in the case, all conditions of Theorem 2.2′ are valid except of (2.21). On the other hand, the case coordinates to Corollary 2.2 because its conditions hold and the function $x_0^*(t) = t/2$ is a solution of problem (2.10), (2.11), where $P_0^*(t) = 0$, $q_0^*(t) = t/2$, and

$$H_k(t) = \exp\Big(-\frac{\sin k^2 t}{k}\Big) \quad (k = 1, 2, \ldots).$$

Example 2.2. Let $I = [0, 2\pi]$, $n = 2$, $t_0 = t_k = 0$ $(k = 1, 2, \ldots)$,

$$c_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad c_k = \begin{pmatrix} 1 \\ 1/k \end{pmatrix} \quad (k = 1, 2, \ldots);$$
$$P_0(t) = \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix}, \qquad P_k(t) = \begin{pmatrix} k\cos k^2 t & 0 \\ -k\sin k^2 t & 0 \end{pmatrix} \quad (k = 1, 2, \ldots);$$
$$q_0(t) = q_k(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (k = 1, 2, \ldots).$$

Then

$$x_0(t) \equiv \begin{pmatrix} 1 \\ -t/2 \end{pmatrix}, \qquad x_k(t) \equiv \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \end{pmatrix} \quad (k = 1, 2, \ldots),$$

where

$$x_{1k}(t) \equiv \exp\Big(\frac{\sin k^2 t}{k}\Big), \qquad x_{2k}(t) \equiv \frac{1}{k} - k\int_0^t \exp\Big(\frac{\sin k^2\tau}{k}\Big)\sin k^2\tau\, d\tau \quad (k = 1, 2, \ldots).$$

It is not difficult to verify that condition (1.3) is fulfilled uniformly on I. Note that, in the case, condition (2.21) is not hold. But, all conditions of Theorem 2.1′ hold if we assume $H_k(t) = Y_k(t) (k = 0, 1, \ldots)$ therein, where Y_0 and $Y_k (k = 1, 2, \ldots)$, $Y_0(0) = Y_k(0) = I_2$, are is the fundamental matrix of the systems (1.1_0) and (1.1_{k0}) $(k = 1, 2, \ldots)$, respectively.

Remark 2.5. As compared with Theorem 2.1′ and Theorem 2.2′, it is not assumed, in Theorem 2.1′, that the equalities (2.22) and (2.23) hold uniformly on I. Below we will give an example of a sequence of initial value problems for which inclusion (2.3) holds but condition (2.22) is not fulfilled uniformly on I.

Example 2.3. Let $I = [0, \pi]$, $n = 2$, $t_0 = t_k = 0$ $(k = 1, 2, \ldots)$,

$$c_0 = c_k = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (k = 1, 2, \ldots);$$

$$P_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_k(t) = \begin{pmatrix} 0 & p_{k1}(t) \\ 0 & p_{k2}(t) \end{pmatrix} \quad (k = 1, 2, \ldots);$$

$$q_0(t) = q_k(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (k = 1, 2, \ldots);$$

$$p_{k1}(t) = \begin{cases} (\sqrt{k} + \sqrt[4]{k}) \sin kt & \text{for } t \in I_k, \\ \sqrt{k} \sin kt & \text{for } t \in [0, 2\pi] \setminus I_k \ (k = 1, 2, \ldots); \end{cases}$$

$$p_{k2}(t) = \begin{cases} -\alpha_k'(t)\,(1 - \alpha_k(t))^{-1} & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0, 2\pi] \setminus I_k \ (k = 1, 2, \ldots); \end{cases}$$

$$\beta_k(t) = \int_0^t (1 - \alpha_k(\tau))\, p_{k1}(\tau)\, d\tau \quad (k = 1, 2, \ldots);$$

$$\alpha_k(t) = \begin{cases} 4\pi^{-1}(\sqrt[4]{k} + 1)^{-1} \sin kt & \text{for } t \in I_k, \\ 0 & \text{for } t \in [0, 2\pi] \setminus I_k \ (k = 1, 2, \ldots); \end{cases}$$

where

$$I_k = \bigcup_{m=0}^{k-1}]2mk^{-1}\pi, (2m+1)k^{-1}\pi[\quad (k = 1, 2, \ldots).$$

Let, moreover, Y_0 and $Y_k (k = 1, 2, \ldots)$, $Y_0(0) = Y_k(0) = I_2$, be the fundamental matrix of the systems (1.1$_0$) and (1.1$_{k0}$) $(k = 1, 2, \ldots)$, respectively. It can easily be shown that

$$Y_0(t) \equiv I_2, \qquad Y_k(t) \equiv \begin{pmatrix} 1 & \beta_k(t) \\ 0 & 1 - \alpha_k(t) \end{pmatrix} \quad (k = 1, 2, \ldots)$$

and

$$\lim_{k \to +\infty} Y_k(t) = Y_0(t) \quad \text{uniformly on } [0, 2\pi],$$

since

$$\lim_{k \to +\infty} \|\alpha_k\|_c = \lim_{k \to +\infty} \|\beta_k\|_c = 0.$$

Note that

$$\lim_{k \to +\infty} \int_0^{2\pi} p_{k1}(t)\, dt = 2 \lim_{k \to +\infty} \sqrt[4]{k} = +\infty$$

and

$$\lim_{k \to +\infty} \sup \int_0^{2\pi} |p_{k2}(t)|\, dt = +\infty.$$

Therefore, condition (2.22) is not fulfilled uniformly on I.

On the other hand, if we assume that $H_0(t) = I_n$ and $H_k(t) = Y_k^{-1}(t) (k = 1, 2, \ldots)$, then all conditions of Theorem 2.1′ hold.

3. Auxiliary propositions

We will use the following simple lemma.

Lemma 3.1. *Let $h \in \widetilde{C}_{loc}(I; \mathbb{R}^n)$, and $H \in \widetilde{C}_{loc}(I; \mathbb{R}^{n \times n})$ be a nonsingular matrix-function. Then the mapping*

$$x \to y = Hx + h$$

establishes a one-to-one corresponding between the solution between the solutions x and y of systems

$$\frac{dx}{dt} = \mathcal{P}(t)\, x + q(t)$$

and

$$\frac{dy}{dt} = \mathcal{P}_*(t)\, y + q_*(t)$$

respectively, where the matrix- and vector-functions P_* *and* q_* *are defined, respectively, by*

$$P_*(t) \equiv (H'(t) + H(t)P(t))\, H^{-1}(t), \qquad q_*(t) = H(t)\, q(t) + h'(t) - P^*(t)\, h(t).$$

Lemma 3.2. *Let* $\alpha_k, \beta_k \in L(I; \mathbb{R})$ $(k = 0, 1, \dots)$ *be such that*

$$\lim_{k\to+\infty} \|\beta_k - \beta_0\|_s = 0, \qquad \lim_{k\to+\infty} \sup \int_I |\alpha_k(t)|dt < +\infty,$$

and the condition

$$\lim_{k\to+\infty} \int_a^t \alpha_k(\tau)d\tau = \int_a^t \alpha_0(\tau)d\tau$$

hold uniformly on I*, where* $a \in I$ *is some fixed point. Then*

$$\lim_{k\to+\infty} \int_a^t \beta_k(\tau)\alpha_k(\tau)d\tau = \int_a^t \beta_0(\tau)\alpha_0(\tau)d\tau$$

uniformly on I*, as well.*

The proof of the lemma one can find in [3,6].

4. Proof of the main results

Proof of Theorem 2.2. Let $z_k(t) = x_k(t) - x_0(t)$ for $t \in I\,(k = 1, 2, \dots\}$.

It is not difficult to check that

$$z_k(t) = z_k(t_k) + \int_{t_k}^t P_0(s)\, z_k(s)ds + \int_{t_k}^t \bar{P}_k(s)\, x_k(s)ds + \int_{t_k}^t \bar{q}_k(s)ds \quad \text{for } t \in I \;(k = 1, 2, \dots),$$

where

$$\bar{P}_k(t) = P_k(t) - P_0(t), \qquad \bar{q}_k(t) = q_k(t) - q_0(t) \quad (k = 1, 2, \dots).$$

Using the integration-by-parts formula we conclude

$$\int_{t_k}^t \bar{P}_k(s)\, x_k(s)ds = \int_{t_k}^t \bar{P}_k(s)ds \cdot x_k(t) - \int_{t_k}^t \left(\int_{t_k}^s \bar{P}_k(\tau)d\tau\right) x_k'(s)ds$$

$$= \int_{t_k}^t \bar{P}_k(s)ds \cdot x_k(t) - \int_{t_k}^t \left(\int_{t_k}^s \bar{P}_k(\tau)d\tau\right) (P_k(s)\, x_k(s) + q_k(s))ds \quad \text{for } t \in I \;(k = 1, 2, \dots).$$

Therefore,

$$z_k(t) = z_k(t_k) + \mathcal{J}_k(t) + \mathcal{Q}_k(t) + \int_{t_k}^t P_0(s)\, z_k(s)ds \quad \text{for } t \in I \;(k = 1, 2, \dots) \tag{4.1}$$

where

$$\mathcal{J}_k(t) = \int_{t_k}^t \bar{P}_k(s)ds \cdot x_k(t) - \int_{t_k}^t \left(\int_{t_k}^s \bar{P}_k(\tau)d\tau\right) P_k(s)\, x_k(s)ds \quad (k = 1, 2, \dots),$$

and

$$\mathcal{Q}_k(t) = \int_\tau^t \bar{q}_k(s)ds - \int_{t_k}^t \left(\int_{t_k}^s \bar{P}_k(\tau)d\tau\right) q_k(s)ds \quad (k = 1, 2, \dots).$$

Due to (4.1) we get

$$\|z_k(t)\| \le \|z_k(t_k)\| + \|\mathcal{J}_k(t)\| + \|\mathcal{Q}_k(t)\| + \left|\int_{t_k}^t \|P_0(s)\|\, \|z_k(s)\|ds\right| \quad \text{for } t \in I \;(k = 1, 2, \dots). \tag{4.2}$$

Let

$$\alpha_k = \sup_{t\in I}\left\|\int_{t_k}^{t}\bar{P}_k(s)ds\right\|,\qquad \beta_k = \sup_{t\in I}\left\|\int_{t_k}^{t}\bar{q}_k(s)ds\right\|$$

and

$$\gamma_k = \sup_{t\in I}\left|\int_{t_k}^{t}\|P_k(s)\|ds\right| \quad (k = 1, 2, \ldots).$$

Then by (2.8) and (2.9) we have

$$\lim_{k\to+\infty}\alpha_k(1+\gamma_k) = \lim_{k\to+\infty}\beta_k(1+\gamma_k) = 0. \tag{4.3}$$

It is evident that

$$\|\mathcal{J}_k(t)\| \le \varepsilon_k\|x_k\|_c \quad \text{for } t \in I \ (k = 1, 2, \ldots) \tag{4.4}$$

where $\varepsilon_k = \alpha_k(1+\gamma_k)(k = 1, 2, \ldots)$.

Further, we have

$$\left\|\int_{t_k}^{t}\left(\int_{t_k}^{s}\bar{P}_k(\tau)d\tau\right)q_0(s)ds\right\| \le r_0\alpha_k \quad \text{for } t \in I \ (k = 1, 2, \ldots)$$

and, in addition, using the integration-by-parts formulae we get

$$\left\|\int_{t_k}^{t}\left(\int_{t_k}^{s}\bar{P}_k(\tau)d\tau\right)\bar{q}_k(s)ds\right\| \le \alpha_k\beta_k + \beta_k(\gamma_k + r_1) \quad \text{for } t \in I \ (k = 1, 2, \ldots),$$

where

$$r_0 = \int_I \|q_0(t)\|dt, \qquad r_1 = \int_I \|P_0(t)\|dt.$$

Due to the last two estimates, thanks to the inequalities

$$\left\|\int_{t_k}^{t}\left(\int_{t_k}^{s}\bar{P}_k(\tau)d\tau\right)q_k(s)ds\right\| \le \left\|\int_{t_k}^{t}\left(\int_{t_k}^{s}\bar{P}_k(\tau)d\tau\right)\bar{q}_k(s)ds\right\| + \left\|\int_{t_k}^{t}\left(\int_{t_k}^{s}\bar{P}_k(\tau)d\tau\right)q_0(s)ds\right\| \quad \text{for } t \in I \ (k = 1, 2, \ldots),$$

we conclude

$$\|\mathcal{Q}_k(t)\| \le \delta_k \quad \text{for } t \in I \ (k = 1, 2, \ldots), \tag{4.5}$$

where $\delta_k = \alpha_k(\beta_k + r_0) + \beta_k(\gamma_k + r_1)$.

From (4.2), by (4.4) and (4.5) we find

$$\|z_k(t)\| \le \|z_k(t_k)\| + \varepsilon_k\|x_k\|_c + \delta_k + \left|\int_{t_k}^{t}\|P_0(s)\|\,\|z_k(s)\|ds\right| \quad \text{for } t \in I \ (k = 1, 2, \ldots).$$

Hence, according to the Gronwall inequality (see [4])

$$\|z_k\|_c \le \big(\|z_k(t_k)\| + \varepsilon_k\|x_k\|_c + \delta_k\big)\exp(r_1) \quad (k = 1, 2, \ldots). \tag{4.6}$$

In virtue of (4.3) we have

$$\lim_{k\to+\infty}\varepsilon_k = 0. \tag{4.7}$$

Therefore, there exists a natural k_0 such that

$$\varepsilon_k < \frac{1}{2}\exp(-r_1) \quad \text{for } k > k_0.$$

From this and (4.6) it follows

$$\|x_k\|_c \le \|x_0\|_c + \|z_k\|_c \le \|x_0\|_c + \big(\|z_k(t_k)\| + \varepsilon_k \|x_k\|_c + \delta_k\big) \exp(r_1) \quad (k > k_1).$$

So, the sequence $\|x_k\|_c (k = 1, 2, \dots)$ is bounded. In addition, in view of conditions (2.8) and (2.9) we have

$$\lim_{k\to+\infty} \delta_k = 0, \tag{4.8}$$

and using (2.1) we conclude

$$\lim_{k\to+\infty} z_k(t_k) = \lim_{k\to+\infty} (x_k(t_k) - x_0(t_k)) = \lim_{k\to+\infty} c_k - x_0(t_0) = 0.$$

Therefore, by this, (4.7) and (4.8), it follows from (4.6)

$$\lim_{k\to+\infty} \|z_k\|_c = 0,$$

since the sequence $\|x_k\|_c (k = 1, 2, \dots)$ is bounded. □

Proof of Theorem 2.3. According to Theorem 2.2 the mapping $x \to H_k x + h_k$ establishes a one-to-one corresponding between the solution x_k of problem (1.1_k), (1.2_k) and the solution x_k^* of the Cauchy problem (2.10_k), (2.11_k) and, in addition, $x_k^*(t) \equiv H_k(t)\, x_k(t) + h_k(t)$ for every natural k.

Conditions (2.12)–(2.14) guarantee the fulfillment of the conditions of Theorem 2.2 for the Cauchy problem (2.10), (2.11) and sequence of the Cauchy problems (2.10_k), (2.11_k) $(k = 1, 2, \dots)$. Therefore, according to Theorem 2.2

$$\lim_{k\to+\infty} x_k^*(t) = x_0^*(t) \quad \text{uniformly on } I.$$

So, condition (2.15) holds. □

Proof of Corollary 2.1. Verifying the conditions of Theorem 2.3. From (2.4) and (2.5) it follows that condition (2.12) holds, and the condition

$$\lim_{k\to+\infty} H_k^{-1}(t) = H_0^{-1}(t) \quad \text{uniformly on } I. \tag{4.9}$$

Put

$$h_k(t) = -H_k(t)\, \varphi_k(t) \quad \text{for } t \in I \ (k = 1, 2, \dots).$$

Due to (2.2) and (2.5) we get

$$\lim_{k\to+\infty} H_k(t_k) = H_0(t_0).$$

By this and (2.16) condition (2.13) is fulfilled for $c_0^* = H_0(t_0)\, c_0$.

Let $q_k^* (k = 1, 2, \dots)$ are the vector-functions given in Theorem 2.3. It is not difficult to verify that

$$q_k^*(t) \equiv q_k(t) - \varphi_k'(t) + P_k(t)\, \varphi_k(t) \quad (k = 1, 2, \dots)$$

in the case. Further, by (2.6) and (2.1) condition (2.14) holds uniformly on I for the functions q_k^* $(k = 1, 2, \dots)$ given above, $q_0^*(t) = H_0(t)\, q_0(t)$ and $c_k^* = H_k(t_k)\,(c_k - \varphi_k(t))(k = 1, 2, \dots)$. In view of Lemma 3.1, the vector-function $x_0^*(t) = H_0(t)\, x_0(t)$ is the unique solution of problem (2.10), (2.11). By Theorem 2.3 we have

$$\lim_{k\to+\infty} (H_k(t)\, x_k(t) - H_k(t)\, \varphi_k(t)) = x_0^*(t) \quad \text{uniformly on } I.$$

Therefore, by (2.5) and (4.9), condition (2.17) holds. □

Proof of Theorem 2.1. Sufficiency follows from Corollary 2.1 if we assume $\varphi_k(t) = o_n (k = 1, 2, \dots)$ therein.

Let us show necessity. Let $c_k \in \mathbb{R}^n (k = 0, 1, \dots)$ be an arbitrary sequence of constant vectors satisfying (2.1) and let $e_j = (\delta_{ij})_{i=1}^n \delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \ne j\, (i, j = 1, \dots, n)$.

Let x_k be a unique solution of problem (1.1_k), (1.2_k) for every natural k.

For any $k \in \{0, 1, \dots\}$ and $j \in \{1, \dots, n\}$ let us denote

$$y_{kj}(t) = x_k(t) - x_{kj}(t),$$

where x_{kj} is a unique solution of the system (1.1$_k$) under the Cauchy condition

$$x(t_k) = c_k - e_j.$$

Moreover, let $Y_k(t)$ be matrix-function whose columns are $y_{k1}(t), \dots, y_{kn}(t)$.

It can be easily shown that Y_0 and $Y_k (k = 1, 2, \dots)$ satisfy, respectively, of homogeneous systems (1.1$_0$) and (1.1$_{k0}$) $(k = 1, 2, \dots)$ and

$$y_{kj}(t_k) = e_j \quad (k = 0, 1, \dots) \tag{4.10}$$

for every $j \in \{1, \dots, n\}$. If for some natural k and $\alpha_j \in \mathbb{R} (j = 1, \dots, n)$

$$\sum_{j=1}^{n} \alpha_j y_{kj}(t) \equiv o_n,$$

then using (4.10) we have

$$\sum_{j=1}^{n} \alpha_j e_j = o_n$$

and, therefore,

$$\alpha_1 = \dots = \alpha_n = 0,$$

i.e., Y_0 and $Y_k (k = 1, 2, \dots)$ are the fundamental matrices, respectively, of homogeneous systems (1.1$_0$) and (1.1$_{k0}$) $(k = 1, 2, \dots)$.

Thanks to Corollary 2.1 we have

$$\lim_{k\to+\infty} Y_k(t) = Y_0(t) \quad \text{uniformly on } I$$

and, consequently,

$$\lim_{k\to+\infty} Y_k^{-1}(t) = Y_0^{-1}(t) \quad \text{uniformly on } I, \tag{4.11}$$

as well.

We may assume without loss of generality that

$$Y_k(t_k) = I_n \quad (k = 0, 1, \dots).$$

We put

$$H_k(t) = Y_k^{-1}(t) \quad \text{for } t \in I \ (k = 0, 1, \dots)$$

and verify conditions (2.4)–(2.7) of the theorem.

Condition (2.4) is evident, and condition (2.5) coincides to (4.11).

Using the equality

$$(Y_k^{-1}(t))' = -Y_k^{-1}(t)\, P_k(t) \quad \text{for } t \in I \ (k = 0, 1, \dots), \tag{4.12}$$

we show

$$\mathcal{I}_c(H_k, A_k)(t) - \mathcal{I}_c(H_k, A_k)(t_k) = \int_{t_k}^{t} \Big((Y_k^{-1}(t))' + Y_k^{-1}(t)\, P_k(t) \Big)\, d\tau = O_{n\times n} \quad \text{for } t \in I \ (k = 0, 1, \dots).$$

Thus condition (2.6) is evident.

On the other hand, using integration-by-parts formulae we find

$$\mathcal{B}_c(H_k, q_k)(t) - \mathcal{B}_c(H_k, q_k)(t_k) = \int_{t_k}^{t} Y_k^{-1}(\tau) q_k(\tau) d\tau = \int_{t_k}^{t} Y_k^{-1}(\tau) \big(x_k'(\tau) - P_k(\tau)\, x_k(\tau) \big)\, d\tau$$
$$= Y_k^{-1}(t)\, x_k(t) - Y_k^{-1}(t_k)\, x_k(t_k) = Y_k^{-1}(t)\, x_k(t) - c_k \quad \text{for } t \in I \ (k = 0, 1, \dots).$$

Hence,

$$\int_{t_k}^{t} Y_k^{-1}(\tau)\, q_k(\tau)d\tau - \int_{t_0}^{t} Y_k^{-1}(\tau)\, q_0(\tau)d\tau = (Y_k^{-1}(t)\, x_k(t) - Y_0^{-1}(t)\, x_0(t))$$
$$- (c_k - c_0) \quad \text{for } t \in I \ (k = 1, 2, \dots). \tag{4.13}$$

By this, (2.1), (4.11) and (4.13), if we take account that due to necessity of theorem condition (1.3) holds uniformly on I, we conclude that condition (2.7) holds uniformly on I, as well. □

Proof of Theorem 2.2′. It is evident that doe to conditions (2.21), (2.22) and (2.23) conditions (2.8) and (2.9) are valid. So, the theorem follows from Theorem 2.2. □

Proof of Theorem 2.3′. In the case, condition (2.24) is equivalent to condition (2.13). Moreover, due to conditions (2.18), (2.25) and (2.26) conditions (2.6) and (2.14) are fulfilled uniformly on I. So, the theorem follows from Theorem 2.3. □

Proof of Corollary 2.1′. From (2.4) and (2.5) it follows that conditions (2.12) and (4.9) are valid. By (4.9) there exists a positive number is r such that

$$\|H_k^{-1}(t)\| \le r \quad \text{for } t \in I \ (k = 0, 1, \dots).$$

Therefore, due to Remark 2.2 and (2.18) we get

$$\sup\left\{\left|\bigvee_{t_k}^{t}(\mathcal{I}_c(H_k, P_k))\right| : t \in I\right\} \le r r_0 < +\infty \quad (k = 0, 1, \dots),$$

where r_0 is the right hand of inequality (2.18). So, thanks to this, the uniform fulfillment on I of conditions (2.19) and (2.20), guarantees, respectively, the same property for conditions (2.6) and (2.7). Hence, the corollary follows from Corollary 2.1. □

Proof of Theorem 2.1′. Sufficiency follows from Corollary 2.1′ if we assume $\varphi_k(t) = o_n$ $(k = 1, 2, \dots)$ therein. The proof of the necessity is the same as in the proof of Theorem 2.1. We only note that by condition (2.5) and equality (4.12) condition (2.18) is valid, and condition (2.19) is fulfilled uniformly on I. Moreover, according to Remark 2.2, it is evident that the sufficiency immediately follows from Theorem 2.1. □

Proof of Corollary 2.2. In virtue of the integration-by-parts formula, conditions (2.5), (2.22), (2.23), (2.27) and (2.28) yield that the conditions

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k(\tau)\, P_k(\tau)d\tau = \int_{t_0}^{t} (P_0(\tau) - P^*(\tau))d\tau$$

and

$$\lim_{k\to+\infty} \int_{t_k}^{t} H_k(\tau)\, q_k(\tau)d\tau = \int_{t_0}^{t} (q_0(\tau) - q^*(\tau))d\tau$$

are fulfilled uniformly on I. Corollary 2.2 follows from Theorem 2.1′. □

Proof of Corollary 2.3. Let

$$C_{kl}(t) = I_n - \int_{t_k}^{t} (P_{kl}(\tau) - P_{0l}(\tau))d\tau \quad (l = 1, \dots, m;\ k = 1, 2, \dots).$$

Thanks to (2.30), without loss of generality we can assume that the matrix-functions H_{kl} and $C_{kl}(l = 1, \dots, m)$ are nonsingular for every natural k.

Based on the definitions of the operators $\mathcal{B}_c$ ad $\mathcal{I}_c$, it is not difficult to verify the equality

$$\mathcal{B}_c(C_{kj}, H_{kj-1}P_k)(\tau)\Big|_{t_k}^{t} \equiv \mathcal{B}_c(H_{kj}, P_k)(\tau)\Big|_{t_k}^{t},$$
$$\mathcal{B}_c(C_{kj}, H_{kj-1}f_k)(\tau)\Big|_{t_k}^{t} \equiv \mathcal{B}_c(H_{kj}, f_k)(\tau)\Big|_{t_k}^{t}$$

and

$$\mathcal{I}_c(C_{kj},(H'_{kj-1}+H_{kj-1}P_k)H^{-1}_{kj-1})(\tau)\Big|_{t_k}^{t}\equiv\mathcal{I}_c(H_{kj},P_k)(\tau)\Big|_{t_k}^{t}\quad(j=1,\dots,m;\ k=1,2,\dots).$$

In addition, by conditions (2.29)–(2.32) conditions (2.4) and (2.18) hold, and conditions (2.5) and (2.19) and (2.20) are fulfilled uniformly on I, where $H_0(t)=I_n$ and $H_k(t)=H_{km-1}(t)(k=1,2,\dots)$. So, the corollary follows from Theorem 2.1′. □

Proof of Corollary 2.4. Let us show the sufficiency. Let $H_k(t)=Z_k^{-1}(t)(k=0,1,\dots)$ in Theorem 2.1′. Thanks to (2.34), there exists a positive number r such that

$$\|Z_k^{-1}(t)\|\le r\quad\text{for } t\in I\ (k=0,1,\dots).$$

Using this estimate and the equality

$$(Z_k^{-1}(t))'=-Z_k^{-1}(t)\,Q_k(t)\quad\text{for } t\in I\ (k=0,1,\dots),$$

by the integration-by-parts formulae we have

$$\left\|Z_k^{-1}(t)-Z_k^{-1}(s)+\int_s^t Z_k^{-1}(\tau)P_k(\tau)d\tau\right\|=\left\|\int_s^t Z_k^{-1}(\tau)(P_k(\tau)-Q_k(\tau))d\tau\right\|$$
$$\le r\int_s^t\|P_k(\tau)-Q_k(\tau)\|d\tau\quad\text{for } s<t\ (k=0,1,\dots).$$

Therefore,

$$\int_I\|H_k'(\tau)+H_k(\tau)P_k(\tau)\|d\tau\le r\int_I\|P_k(\tau)-Q_k(\tau)\|d\tau\quad(k=0,1,\dots)$$

and due to (2.33) estimate (2.18) holds. Moreover, conditions (2.19) and (2.20) coincide to conditions (2.35) and (2.36), respectively. So, the sufficiently follows from Theorem 2.1′.

Let us show the necessity. Let $Q_k(t)=P_k(t)(k=0,1,\dots)$. Then $Z_k(t)\equiv Y_k(t)(k=0,1,\dots)$, where Y_0 and Y_k $(k=1,2,\dots)$ are fundamental matrices, respectively, of the homogeneous systems (1.1_0) and (1.1_{k0}). Analogously, as in the proof of Theorem 2.1, conditions (2.34) and equality (4.13) are valid. In addition, condition (2.35) coincides to condition (2.19), and condition (2.36) follows from equality (4.13). □

Proof of Corollary 2.5. The corollary immediately follows from Corollary 2.4 if we note the fundamental matrix of $Z_k(t)(Z_k(t_k)=I_n)$ of system (2.37), in the case, has the form

$$Z_k(t)\equiv\exp\left(\int_{t_k}^t Q_k(\tau)d\tau\right)\quad(k=0,1,\dots).\quad\square$$

Proof of Corollary 2.6. The corollary follows from Corollary 2.5 if we assume that therein $Q_k(t)=P_k(t)$ $(k=0,1,\dots)$ and, in addition, we note that condition (2.38) is equivalent to condition (2.40), and condition (2.39) coincides to (2.41). □

Proof of Corollary 2.7. The corollary follows from Corollary 2.4 if we assume therein that $Q_k(t)=\mathrm{diag}(P_k(t))(k=0,1,\dots)$. □

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