THIN SHELLS WITH LIPSCHITZ BOUNDARY(1)

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In [Du2] we have revised an asymptotic model of a shell (Koiter, Sanchez-Palencia, Ciarlet etc.), based on the the calculus of tangent Günter's derivatives, developed in the papers of R. Duduchava, D. Mitrea and M. Mitrea [Du2, Du3, Du4, DMM1]. As a result the 2-dimensional shell equation on a mid-surface $\mathscr{S}$ was written in terms of Günter's derivatives, unit normal vector field and the lamé constant, which coincides with the Lamé equation on the Hypersurface $\mathscr{S}$, investigated in [Du2, Du3, Du4, DMM1].

The present investigation is inspired by the paper of G. Friesecke, R. D. James & S. Mller [FJM1], where a hierarchy of Plate Models are derived from nonlinear elasticity by Γ-Convergence. The final goal of the present investigation is to derive 2D shell equations in terms of Günter's derivatives by Γ-Convergence.

As a first step to the final goal in the paper of T. Buchukuri, R. Duduchava & G. Tephnadze [BDT1] was studied a mixed boundary value problem for the stationary heat transfer equation in a thin layer around a surface $\mathscr{C}$ with the boundary. It was established what happens in Γ-limit when the thickness of the layer converges to zero. In particular, was shown that the Γ-limit of a mixed type Dirichlet-Neumann boundary value problem (BVP) for the Laplace equation in the initial thin layer is a Dirichlet BVP for the Laplace-Beltrami equation on the surface. For this was applied the variational formulation and the calculus of Günter's tangential differential operators on a hypersurface and layers. This approach allow global representation of basic differential operators and of corresponding BVPs in terms of the standard cartesian coordinates of the ambient Euclidean space $\mathbb{R}^n$.

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INTRODUCTION

Modern interest in shell theories has blossomed with the ubiquitous presence of thin films in science and technology. Thin structures encounter in engineering applications more and more often and there emerged numerous approaches proposed for modeling linearly elastic flexural shells. Started by the Cosserats pioneering work (1909), Goldenveiser (1961), Naghdi (1963), Vekua (1965), Novozhilov (1970), Koiter (1970) and many others contributed essentially the development of the shell theory. Ellipticity of the corresponding partial differential equations was proved later by Roug'e (1969) for cylindrical shells, by Coutris (1973) for the shell model proposed by Naghdi, Gordeziani (1974) for the shell model proposed by Vekua, Shoikhet (1974) for the shell model proposed by Novozhilov, Ciarlet & Miara (1992) for the model proposed by Koiter (cf. [Ci1], [Ci3]-[Ci6], [De1] for survey and further references).

Inspired by the books and papers of Sanchez-Palencia [Sa1, Sa2], Miara & Sanchez-Palencia [MS1], Ciarlet & Lods [CL1, CL2, CL3], Ciarlet, Lods & Miara [CLM1] and exposed in details in Ciarlet [Ci3, Ci5] we have developed in [Du2] asymptotic analysis of a linearly elastic shell based on the formal calculus of tangential Günter's derivatives, developed in the papers of the author with D. Mitrea and M. Mitrea [Du2, Du3, Du4, DMM1]. As a result the 2-dimensional shell equation on a middle surface $\mathscr{S}$ is derived written in terms of Gunter's derivatives, unit normal vector field and the lamé constant, which coincides with the Lamé equation on the Hypersurface $\mathscr{S}$, investigated in [Du2, Du3, Du4, DMM1].

The present investigation is inspired by the paper of G. Friesecke, R. D. James & S. Mller [FJM1], where a hierarchy of Plate Models are derived from nonlinear elasticity by Γ-Convergence. The final goal of the investigation is to derive 2D shell equations written in terms of Günter's derivatives by Γ-Convergence

Let us consider an example: a surface $\mathscr{S}$ be given by a local immersion

$$\Theta\,:\,\omega\to\mathscr{S}\,,\quad \omega\subset\mathbb{R}^{n-1}\,, \tag{0.1}$$

which means that the derivatives $\left\{\boldsymbol{g}_k := \partial_k\Theta\right\}_{k=1}^{n-1}$ are linearly independent, i.e., the Jakobi matrix $\nabla_x\Theta_x$ has the maximal rank $n-1$. Thus, $\left\{\boldsymbol{g}_k\right\}_{k=1}^{n-1}$ is a **basis** (or a **covariant frame** if the basis is enriched with 0) in the space $\omega(\mathscr{S})$ of all tangential vector fields on $\mathscr{S}$. The system $\left\{\boldsymbol{g}^k\right\}_{k=1}^{n-1}$ which is biorthogonal $\langle\boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk}$ forms the **contravariant basis** (the **contravariant frame**) in the same space $\omega(\mathscr{S})$ of all tangential vector fields on $\mathscr{S}$. Let $\boldsymbol{\nu}(\mathscr{X}) = (\nu_1(\mathscr{X}), \ldots, \nu_j(\mathscr{X}))^\top$ be the outer unit normal vector (the Gauß mapping) to $\mathscr{S}$ at $\mathscr{X} \in \mathscr{S}$ (see § 1.4 for details).

The Gram matrix $G_{\mathscr{S}}(\mathscr{X}) = [g_{jk}(\mathscr{X})]_{n-1\times n-1}$, $g_{jk} := \langle g_j, g_k\rangle$, is then positive definite, responsible for the Riemann metric on $\mathscr{S}$ and is called the **covariant metric tensor**. Moreover, it has the inverse matrix $G^{-1}_{\mathscr{S}}(\mathscr{X}) = [g^{jk}(\mathscr{X})]_{n-1\times n-1}$, , $g^{jk} := \langle\boldsymbol{g}^j, \boldsymbol{g}^k\rangle$ (cf. (1.19), (0.2)), which is called the **contravariant metric tensor**.

The Gram determinant

$$\mathscr{G}\big(\partial_1\Theta(x), \ldots, \partial_{n-1}\Theta)(x)\big) = \det G_{\mathscr{S}}(x)\,, \qquad x \in \omega \subset \mathbb{R}^{n-1} \tag{0.2}$$

is responsible for the volume element $d\sigma$ of the surface, which is the vector product of the tangential vectors

$$\begin{aligned} d\sigma := |\partial_1\Theta \wedge \ldots \wedge \partial_{n-1}\Theta| = \sqrt{\det G_{\mathscr{S}}}\, dx\,, \\ dx = dx_1 \cdots dx_{n-1}\,. \end{aligned} \tag{0.3}$$

The surface divergence and the surface gradient are defined in the intrinsic coordinates by the equalities

$$\begin{aligned} \operatorname{div}_{\mathscr{S}} \boldsymbol{U} &:= \big[\det G_{\mathscr{S}}\big]^{-1/2} \sum_{j=1}^{n} \partial_j\Big\{\big[\det G_{\mathscr{S}}\big]^{1/2} U^j\Big\}\,, \\ \nabla_{\mathscr{S}} f &= \sum_{j,k=1}^{n-1} (g^{jk}\partial_j f)\,\partial_k \end{aligned} \tag{0.4}$$

(see § 1.6 and [Ta2, Ch. 2, § 3]). Their composition is the **Laplace-Beltrami operator**

$$\boldsymbol{\Delta}_{\mathscr{S}} f := \operatorname{div}_{\mathscr{S}}\nabla_{\mathscr{S}} f = \big[\det G_{\mathscr{S}}\big]^{-1/2} \sum_{j,k=1}^{n-1} \partial_j\Big\{g^{jk}\big[\det G_{\mathscr{S}}\big]^{1/2}\partial_k f\Big\}, \quad f \in C^2(\mathscr{S})\,, \tag{0.5}$$

which is self-adjoint

$$\Delta^*_{\mathscr{S}} = \big(\nabla_{\mathscr{S}}\operatorname{div}_{\mathscr{S}}\big)^* = \big(\operatorname{div}_{\mathscr{S}}\big)^*\big(\nabla_{\mathscr{S}}\big)^* = \nabla_{\mathscr{S}}\operatorname{div}_{\mathscr{S}} = \boldsymbol{\Delta}_{\mathscr{S}}\,. \tag{0.6}$$

The intrinsic parameters enable generalization to arbitrary manifolds, not necessarily immersed in the Euclidean space $\mathbb{R}^n$.

On the other hand sometimes it is more convenient to record these operators in Cartesian coordinates. To set the conditions for precise formulations let us consider the **natural basis**

$$\boldsymbol{e}^1 := (1, 0, \ldots, 0)^\top, \ldots, \boldsymbol{e}^n := (0, \ldots, 0, 1)^\top \tag{0.7}$$

in the Euclidean space $\mathbb{R}^n$ ($\left\{e^j\right\}_{j=1}^n$ is also called the **Cartesian basis** since it is ordered). Each point $x = (x_1, \dots, x_n)^\top$ in the Euclidean space $\mathbb{R}^n$ is represented in the Cartesian basis $x = \sum_{j=1}^n x^j e_j$ in a unique way.

Let the operator (the matrix)

$$\pi_{\mathscr{S}} : \mathbb{R}^n \to \omega(\mathscr{S}), \quad \pi_{\mathscr{S}}(t) = I - \boldsymbol{\nu}(t)\boldsymbol{\nu}^\top(t) = \left[\delta_{jk} - \nu_j(t)\nu_k(t)\right]_{n\times n}, \quad t \in \mathscr{S} \tag{0.8}$$

denote the canonical orthogonal projection $\pi^2_{\mathscr{S}} = \pi_{\mathscr{S}}$ onto the space of tangential vector fields to $\mathscr{S}$ at the point $t \in \mathscr{S}$:

$$(\boldsymbol{\nu}, \pi_{\mathscr{S}} v) = \sum_j \nu_j v_j - \sum_{j,k} \nu_j^2 \nu_k v_k = 0 \qquad \text{for all} \quad v = (v_1, \dots, v_n)^\top \in \mathbb{R}^n .$$

It turns out that the surface gradient is nothing but the collection of the weakly tangential **Günter's derivatives** (cf. [Gu1], [KGBB1], [Du1])

$$\nabla_{\mathscr{S}} = \mathscr{D}_{\mathscr{S}} := (\mathscr{D}_1, \dots, \mathscr{D}_n)^\top, \quad \mathscr{D}_j := \partial_j - \nu_j(\mathscr{X})\partial_{\boldsymbol{\nu}} = \partial_{\boldsymbol{d}^j}, \tag{0.9}$$

where $\partial_{\boldsymbol{\nu}} := \sum_{j=1}^n \nu_j \partial_j$ denotes the normal derivative. The first-order differential operators

$$\mathscr{D}_j = \partial_{\boldsymbol{d}^j}, \qquad 1 \leqslant j \leqslant n, \tag{0.10}$$

are the directional derivative along the vector fields $\boldsymbol{d}^j := \pi_{\mathscr{S}} \boldsymbol{e}^j$, $j = 1, \dots, n$.

Moreover, the surface divergence coincides with the operator

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{U} = \sum_{j=1}^n \mathscr{D}_j U_j^0, \qquad \text{for} \quad \boldsymbol{U} = \sum_{j=1}^n U_j^0 \partial_j \in \omega(\mathscr{S}) \tag{0.11}$$

and the Laplace-Beltrami operator coincides with (see also [MM1, pp. 2ff and p. 8.])

$$\boldsymbol{\Delta}_{\mathscr{S}}\, \varphi := \operatorname{div}_{\mathscr{S}} \nabla_{\mathscr{S}} \varphi = \sum_{j=1}^n \mathscr{D}_j^2 \varphi, \quad \varphi \in C^2(\mathscr{S}). \tag{0.12}$$

Relatively simple form of recorded operators enables simplified treatment of corresponding boundary value problems, which require proofs of Korn's inequalities or similar.

The Laplace-Beltrami operator (0.12) is the natural operator associated with the Euler-Lagrange equations for a variational integral

$$\mathscr{E}[u] = -\frac{1}{2} \int_{\mathscr{S}} \|\mathscr{D}u\|^2\, dS. \tag{0.13}$$

A similar approach, based on the principle that, at equilibrium, the displacement minimizes the potential energy (Koiter's model), leads to the following form of the Lamé operator $\mathscr{L}_{\mathscr{S}}$ on $\mathscr{S}$ (cf. [DMM1])

$$\mathscr{L}_{\mathscr{S}} \boldsymbol{U} = \mu \pi_{\mathscr{S}} \operatorname{div}_{\mathscr{S}} \nabla_{\mathscr{S}} \boldsymbol{U} + (\lambda + \mu)\, \nabla_{\mathscr{S}} \operatorname{div}_{\mathscr{S}} \boldsymbol{U} + \mu\, \mathscr{H}^0_{\mathscr{S}} \mathscr{W}_{\mathscr{S}} \boldsymbol{U}, \tag{0.14}$$

(cf. (0.8) for the projection $\pi_{\mathscr{S}}$). Here $\boldsymbol{U}$ is an arbitrary (tangential) vector fields on $\mathscr{S}$, $\lambda, \mu \in \mathbb{R}$ are the Lamé moduli, whereas

$$\mathscr{H}^0_{\mathscr{S}} = -\mathrm{div}_{\mathscr{S}}\boldsymbol{\nu} := -\sum_{j=1}^{n} \mathscr{D}_j\nu_j = \mathrm{Tr}\,\mathscr{W}_{\mathscr{S}}\,, \qquad \mathscr{W}_{\mathscr{S}} = -\big[\mathscr{D}_j\nu_k\big]_{n\times n}. \tag{0.15}$$

Note, that $\mathscr{H}_{\mathscr{S}} := (n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}$ and $\mathscr{W}_{\mathscr{S}}$ represent, respectively, the **mean curvature** and the **Weingarten mapping** (cf. (1.34)) of $\mathscr{S}$. This identification ensures that the boundary-value problem

$$\begin{cases} \mathscr{L}_{\mathscr{S}}\boldsymbol{U} = 0 \quad \text{in} \quad \mathscr{S}\,, \\ \boldsymbol{U}\big|_{\Gamma} = f \in \mathbb{H}^s(\partial\mathscr{S})\,, \quad f\cdot\boldsymbol{\nu} = f\cdot\boldsymbol{\nu}_{\Gamma} = 0 \quad \text{on} \quad \Gamma := \partial\mathscr{S}, \end{cases} \tag{0.16}$$

where $\boldsymbol{U} = \sum_{j=1}^{n} U^0_j\boldsymbol{d}^j \in \omega(\mathscr{S})\cap\mathbb{H}^{s+1/2}(\partial\mathscr{S})$ is the (tangential) generalized displacement vector field of the elastic hypersurface $\mathscr{S}$, is well-posed, whenever $\mu > 0$, $2\mu+\lambda > 0$, and $0 \leqslant s \leqslant 1$. Here $\mathbb{H}^s$ stands for the usual L^2-based Sobolev scale, $\boldsymbol{\nu}$ is the normal vector to $\mathscr{S}$ and $\boldsymbol{\nu}_\Gamma(t)$ is the unit tangential vector to $\mathscr{S}$ at the boundary point $t \in \Gamma := \partial\mathscr{S}$ and outer normal vector to the boundary $\Gamma = \partial\mathscr{S}$.

1 AUXILIARY

In the present chapter we have collected, for the readers convenience, some auxiliary information, mostly from [Ci1, Ci2, Ci3, Ci4, DMM1, FJM1, Ta1].

1.1 Differentiation and implicit function theorem

In the present section we expose implicit and inverse function theorems, which are applied later.

Let us recall some standard notation: $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, \ldots\}$. For a natural number $n \in \mathbb{N}$ let $\mathbb{R}^n$ and C^n denote the n-dimensional spaces of vectors $x = (x_1, \ldots, x_n)^\top$ with real $x_j \in \mathbb{R}$ and complex $x_j \in \mathbb{C}$ entries and standard metrics, based on the scalar product

$$\langle x, y\rangle := x_1\overline{y_1} + \cdots + x_n\overline{y_n} \quad \text{for} \quad x, y \in C^n$$
$$\langle x, y\rangle := x_1 y_1 + \cdots + x_n y_n \quad \text{for} \quad x, y \in \mathbb{R}^n\,.$$

$\mathbb{N}_n$ and $\mathbb{N}_0^n$ denote the sets of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ with components from the corresponding sets

$$\partial^\alpha u(x) = \partial^\alpha_x u(x) := \frac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}\,, \qquad \partial_j := \frac{\partial}{\partial\, x_j}\,,\,, \quad j = 1, 2, \ldots, n \tag{1.1}$$
$$\alpha \in \mathbb{N}_0^n\,, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n\,.$$

Let $\Omega \subset \mathbb{R}^n$ be an open domain. A continuous function $\Phi : \Omega \to \mathbb{R}^m$ is called **differentiable** at a point $x \in \Omega$ with **derivative** $D\Phi(x) : \mathbb{R}^n \to \mathbb{R}^m$, if $D\Phi(x)$ is a linear mapping (i.e., a matrix) and

$$\Phi(x + y) = \Phi(x) + D\Phi(x)y + R(x, y)\,, \quad R(x, y) = \mathscr{o}(|y|) \tag{1.2}$$

for small $y \in \mathbb{R}^n$, $|y| \to 0$.

With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ the derivative $D\Phi(x)$ is the matrix of partial derivatives

$$D\Phi(x) = [\partial_j\Phi_k(x)]_{n\times m} \tag{1.3}$$

and transforms a column vector $u = (u_1, \ldots, u_n)^\top$ into a new column vector

$$D\Phi(x)u = \left(\sum_{j=1}^n \partial_j\Phi_k(x)u_j\right)^\top\,.$$

The matrix $D\Phi$ in (1.3) is called the **Jacobi matrix**. If $n = m$ the corresponding determinant is called **Jacoby determinant** or **Jacobian**.

Φ is differentiable whenever all the partial derivatives exist.

Let $\Omega \subset \mathbb{R}^n$ be an open domain (Ω can be non-compact, e.g. $\Omega = \mathbb{R}^n$). For $r, m \in \mathbb{N}_0$ by $C^r(\Omega, \mathbb{R}^m)$ (or by $C^r(\Omega)$) is denoted the r-times continuously diferentiable mappings $\Phi : \Omega \to \mathbb{R}^m$ and $C^\infty(\Omega, \mathbb{R}^m) := \bigcap_{r=1}^\infty C^r(\Omega, \mathbb{R}^m)$.

The set of complex valued mappings will be denoted by $C^r(\Omega, C^m)$ (or by $C^r(\Omega)$).

The subspace $\mathbb{C}_0^\infty(\Omega)$ consists of infinitely differentiable functions on Ω with compact supports.

A composition of functions

$$F = \Psi \circ \Phi \; : \; \Omega \to \mathbb{R}^k \,, \quad \Phi \; : \; \Omega \to \mathscr{M} \subset \mathbb{R}^m \,, \quad \Phi \; : \; \mathscr{M} \to \mathbb{R}^k \,,$$

where Φ is differentiable at a point $x \in \Omega$ and Ψ is differentiable at a point $z = \Phi(x) \in \mathscr{M}$, is differentiable at a point x and the **chain rule** holds:

$$D(\Psi \circ \Phi)(x) = (D\Psi)(\Phi(x))D\Phi(x) \,. \tag{1.4}$$

Let us recall that $\Omega \subset \mathbb{R}^n$ is called a **star-like domain** with respect to the point $x_0 \in \Omega$ if $y \in \Omega$ implies $x_0 + t(y - x_0) \in \Omega$ for all $0 \leqslant t \leqslant 1$.

The fundamental theorem of calculus, applied to $\varphi(t) = \Phi(x + ty)$ in a star-like domain with respect to $x \in \Omega$, gives the **Lagrange formula**

$$\Phi(x + y) = \Phi(x) + \int_0^1 D\Phi(x + ty)y \, dt = \Phi(x) + D\Phi(x + t_0 y)y \tag{1.5}$$

for $\Phi \in C^1(\Omega)$, all $y \in \Omega$ and some $0 \leq t_0 < 1$.

Let us consider a function

$$\Phi \; : \; \Omega \to \mathbb{R}^n \,, \qquad \Phi \in C^k \,, \tag{1.6}$$

which maps a domain $\Omega \subset \mathbb{R}^n$ to the same Euclidean space and $\Phi(x_0) = y_0$. It is important to know conditions ensuring the existence of the **inverse mapping**

$$\Phi^{-1} \; : \; \boldsymbol{V} \to \boldsymbol{U} \subset \Omega \,, \qquad \Phi(\Phi^{-1}(y)) \equiv y \,, \quad y \in \boldsymbol{V} \tag{1.7}$$

and its smoothness properties, at least locally, in a neighborhood of some y_0. The next inverse function theorem provides such conditions and, together with the Implicit function theorem (cf. Theorem 1.2), represent most fundamental results of multivariable analysis.

Theorem 1.1 (Inverse function theorem). *Let Ω be a domain in $\mathbb{R}^n$, $k \in \mathbb{N}$ and $\Phi \in C^k(\Omega, \mathbb{R}^n)$. Let the differential $D\Phi(x)$ be an invertible matrix at $x_0 \in \Omega$ and $\Phi(x_0) = y_0 \in \mathbb{R}^n$.*

*There exist neighborhoods $\boldsymbol{U} \subset \Omega$ of x_0 and $\boldsymbol{V} \subset \mathbb{R}^n$ of y_0 such that the mapping $\Phi \; : \; \boldsymbol{U} \to \boldsymbol{V}$ is one-to one and the inverse mapping $\Phi^{-1} \; : \; \boldsymbol{V} \to \boldsymbol{U}$ is C^k-smooth (i.e., Φ^{-1} is a C^k-**diffeomorphism**).*

Proof: Let

$$\Psi(x) := (D\Phi)(x_0)]^{-1}\big[\Phi(x_0 + x) - y_0\big] \,. \tag{1.8}$$

Then, obviously,

$$\Psi(0) = 0 \qquad \text{and} \quad (D\Psi)(0) = I \,.$$

Thus, the case reduces to $\Phi(0) = 0$, $(D\Phi)(0) = I$, $0 \in \Omega$, which we suppose fulfilled. Then we have to solve the equation $\Phi(u) = v$ for small v. Due to formula (1.2) this can be written as an equation

$$u + R(u) = v\,, \qquad R(0) = 0\,, \quad (DR)(0) = 0 \qquad (1.9)$$
$$\text{where} \qquad R(u) = o(|u|)\,.$$

with the mapping $R \in C^{k-1}(\Omega, \mathbb{R}^n)$. Solving (1.9) is equivalent to solving

$$T_v(u) = u\,, \qquad T_v(u) = v - R(u)\,. \qquad (1.10)$$

Thus, we look for a fixed point $u = K(v) = \Phi^{-1}(v)$ and will show that $(DK)(0) = I$ or, equivalently, $K(v) = v + o(|v|)$. The latter implies that for all x close to the origin (small enough)

$$(DK)(x) = \big(D\Psi(K(x))\big)^{-1} \qquad (1.11)$$

and taking further derivatives it follows by induction that $K \in C^k$. To implement this idea we consider a metric space

$$\mathfrak{M}_v := \{u \in \Omega \ : \ |u - v| \leqslant \mathscr{A}_v\}\ ,$$

where (cf. (1.2) and (1.9))

$$\mathscr{A}_v := \sup_{|w|\leqslant 2|v|} |R(w)| = o(|w|) = o(|v|)\,. \qquad (1.12)$$

Let us check that $\mathfrak{M}_v$ is invariant under the mapping

$$T_v \ : \ \mathfrak{M}_v \to \mathfrak{M}_v \qquad (1.13)$$

provided that v is small enough. Indeed, since $T_v(u) - v = -R(u)$ we only need to check that $|R(u)| \leqslant \mathscr{A}_v$ for all $u \in \mathfrak{M}_v$ provided that v is small enough. Indeed, if $u \in \mathfrak{M}_v$ than, due to (1.12), $|u| \leqslant |v| + \mathscr{A}_v \leqslant 2|v|$ for a v small enough and

$$|R(u)| \leqslant \sup_{|w|\leqslant 2|v|} |R(w)| = \mathscr{A}_v\,.$$

This completes the proof of the mapping property (1.13).

Due to the Lagrange formulae (1.5) and the property $(DR)(0) = 0$ (see (1.5), by taking v sufficiently small, the mapping (1.13) becomes a contraction

$$|T_v(u) - T_v(w)| = |R(u) - R(w)| = |(DR)(u + t_0(w - u))(u - w)|$$
$$\leqslant r|u - w|\,, \qquad 0 < r < 1\,.$$

Then, by virtue of the fixed point theorem there exists a unique fixed point $u = K(v) \in \mathfrak{M}_v$. Moreover, from $u \in \mathfrak{M}_v$ we conclude that

$$|K(v) - v| = |u - v| \leqslant \mathscr{A}_v = o\big(|v|\big)\,.$$

This completes the proof. ■

Theorem 1.2 (Implicit function theorem). *Let $\Omega \subset \mathbb{R}^m$, $\mathscr{E} \subset \mathbb{R}^n$ be domains and $k = 1, 2 \ldots$. Let $\Psi(x, y) : \Omega \times \mathscr{E} \to \mathbb{R}^n$ be a C^k-mapping, $\Psi(x_0, y_0) = 0$ and the partial $n \times n$ Jacoby matrix $D_y\Psi(x, y)$ be invertible at $(x_0, y_0) \in \Omega \times \mathscr{E}$.*

There exists a neighborhood $U_0 \subset \Omega$ of x_0 and a C^k-smooth mapping $y = \psi(x)$, $\psi : U_0 \to \mathscr{E}$ (called the **implicit function***) such that $\Psi(x, \psi(x)) \equiv 0$.*

The function $\psi(x)$ is unique: If there exists another continuous implicit function $\psi_1 : U^1 \to \mathscr{E}$, the functions coincide $\psi_1(x) = \psi(x)$ in the common neighborhood $x \in U^0 \cap U^1$ of x_0.

Proof: Consider the mapping $\Phi : \Omega \times \mathscr{E} \to \mathbb{R}^m \times \mathbb{R}^n$ defined by

$$\Phi(x, y) := \big(x, \Psi(x, y)\big) . \tag{1.14}$$

The corresponding differential (the Jacobi matrix)

$$\big(D_{(x,y))}\Phi\big) = \begin{pmatrix} I & D_x\Psi \\ 0 & D_y\Psi \end{pmatrix} \tag{1.15}$$

is, obviously, invertible. Therefore, by virtue of the foregoing Theorem 1.2, there exists the inverse function $\Phi^{-1} : V^0 \times U_0 \to \mathbb{R}^m \times \mathbb{R}^n$ and at the point (x, y_0) acquires the form

$$\Phi^{-1}(x, y_0) = \big(x, \psi(x, y_0)\big) .$$

The function $\psi(x) = \psi(x, y_0)$ is the desired implicit function.

The uniqueness of the implicit function follows since, according to Theorem 1.1, there exists only the unique inverse function to $\Phi(x, y) = (x, \Psi(x, y))$. ■

1.2 Calculus of tangential differential operators

The content of the present section follows [DMM1, § 4] with a slight modification.

Throughout the present section we keep the convention similar to that in § 1.6: $\mathscr{S}$ is a hypersurface in $\mathbb{R}^n$, given by an immersion

$$\Theta : \Omega \to \mathscr{S} , \quad \Omega \subset \mathbb{R}^{n-1} \tag{1.16}$$

with a boundary $\Gamma = \partial\mathscr{S}$, given by the corresponding immersion

$$\Theta_\Gamma : \omega \to \Gamma := \partial\mathscr{S} , \qquad \omega \subset \mathbb{R}^{n-2} , \tag{1.17}$$

such that the corresponding differentials

$$D\Theta_j(p) := \operatorname{matr}\left[\partial_1\Theta_j(p), \ldots, \partial_{n-1}\Theta_j(p)\right] , \tag{1.18}$$

have the full rank

$$\operatorname{rank} D\Theta_j(p) = n - 1 , \qquad \forall p \in Y_j , \quad k = 1, \ldots, n , \quad j = 1, \ldots, M ,$$

i.e. , all points of ω_j are regular for Θ_j for all $j = 1, \ldots, M$.

Let $\mathscr{S}$ be a hypersurface given by a collection of charts $\big\{(\mathscr{S}_j, \Theta_j)\big\}_{j=1}^{M}$, where

$$\Theta_j \ : \ \omega_j \to \mathscr{S}_j\,, \qquad \mathscr{S} = \bigcup_{j=1}^{M} \mathscr{S}_j\,, \qquad \omega_j \subset \mathbb{R}^{n-1}\,, \quad j = 1, \dots, M \tag{1.19}$$

(cf. (1.17)). The derivatives

$$\boldsymbol{g}_k = \partial_k \Theta_j, \qquad k = 1, \dots, n-1, \tag{1.20}$$

are then tangential vector fields on $\mathscr{S}$ and this system is a basis in the space of tangential vector fields $\omega(\mathscr{S})$. The symmetric **Gram matrix**

$$G_{\mathscr{S}}(x) := [\langle \boldsymbol{g}_k(\mathscr{x}), \boldsymbol{g}_m(\mathscr{x})\rangle]_{n-1\times n-1} = [\langle \partial_k\Theta_j(x), \partial_m\Theta_j(x)\rangle]_{n-1\times n-1}, \qquad x \in \omega_j \subset \mathbb{R}^{n-1} \tag{1.21}$$

defines the natural metric on the space of tangential vector fields $\omega(\mathscr{S})$, which is inherited from the ambient space $\mathbb{R}^n$. Namely, for arbitrary tangential vectors

$$u_k(x) = \alpha_k^1 \partial_1 \Theta_j(x) + \dots + \alpha_k^{n-1}\partial_{n-1}\Theta_j(x) \in \omega(\mathscr{S})\,, \qquad \alpha_k^m \in \mathbb{R}\,, \quad k = 1, 2\,,$$

the inner product is defined by the bilinear **first fundamental form**

$$\langle u_1, u_2\rangle = \langle G_{\mathscr{S}} a_1, a_2\rangle\,, \qquad a_k = (\alpha_k^1, \dots, \alpha_k^{n-1})^\top\,, \quad k = 1, 2\,. \tag{1.22}$$

$\boldsymbol{\nu}_\Gamma(t)$ is the outer normal vector field to the boundary Γ, which is tangential to $\mathscr{S}$ and $\boldsymbol{\nu}(\mathscr{x})$ is the outer unit normal vector field to $\mathscr{S}$, which has the most important role in the calculus of tangential differential operators we are going to apply. The **unit normal vector field** to the surface $\mathscr{S}$, also known as the **Gauß mapping**, is defined by the vector product of the covariant basis

$$\boldsymbol{\nu}(\mathscr{x}) := \pm\frac{\boldsymbol{g}_1(\mathscr{x}) \wedge \ldots \wedge \boldsymbol{g}_{n-1}(\mathscr{x})}{|\boldsymbol{g}_1(\mathscr{x}) \wedge \ldots \wedge \boldsymbol{g}_{n-1}(\mathscr{x})|}, \qquad \mathscr{x} \in \mathscr{S}. \tag{1.23}$$

The system of tangential vectors $\{\boldsymbol{g}_k\}_{k=1}^{n-1}$ to $\mathscr{S}$ (cf. (1.20)) is, known as the **covariant basis**. There exists the unique system $\big\{\boldsymbol{g}^k\big\}_{k=1}^{n-1}$ biorthogonal to it-the **contravariant basis**:

$$\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk} \qquad j, k = 1, \dots, n-1.$$

The contravariant basis is defined by the formula:

$$\boldsymbol{g}^k = \frac{1}{\det G_{\mathscr{S}}}\boldsymbol{g}_1 \wedge \dots \wedge \boldsymbol{g}_{k-1} \wedge \boldsymbol{\nu} \wedge \boldsymbol{g}_{k+1} \wedge \dots \wedge \boldsymbol{g}_{n-1}, \quad k = 1, \dots, n-1, \tag{1.24}$$

where $G_{\mathscr{S}}(\mathscr{x})$ is the Gram matrix (see (1.21)).

Next we expose yet another definition of a hypersurface-an **implicit** one.

Definition 1.3 *Let $k \geq 1$ an $\omega \subset \mathbb{R}^n$ be a compact domain. An implicit C^k-smooth (an implicit Lipschitz) hypersurface in $\mathbb{R}^n$ is defined as the set*

$$\mathscr{S} = \left\{ \mathscr{x} \in \omega \ : \ \Psi_{\mathscr{S}}(\mathscr{x}) = 0 \right\}, \tag{1.25}$$

where $\Psi_{\mathscr{S}} : \omega \to \mathbb{R}$ is a C^k-mapping (or is a Lipschitz mapping) which is regular $\nabla\,\Psi(\mathscr{x}) \neq 0$.

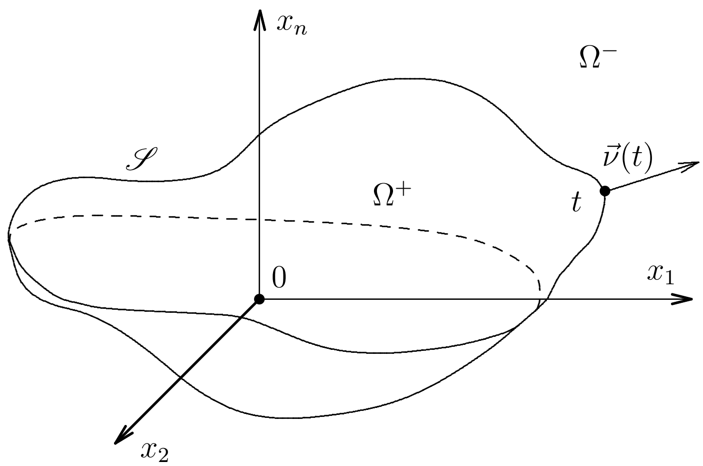

Fig. 1

Note, that Definition (1.16) and Definition 1.3 of a hypersurface $\mathscr{S}$ are equivalent and by taking a single function $\Psi_{\mathscr{S}}$ for the implicit definition of a hypersurface $\mathscr{S}$ we does not restrict the generality (see e.g., [Du4]).

It is well known that using implicit surface functions gradient (see (1.25)) we can write an alternative definition of the unit normal vector field on the surface (see (1.23)):

$$\boldsymbol{\nu}(t) := \lim_{x \to t} \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|}, \qquad t \in \mathscr{S}. \tag{1.26}$$

In applications it is necessary to extend the vector field $\boldsymbol{\nu}t$ in a neighborhood of $\mathscr{S}$, preserving some important features. Here is the precise definition of extension.

Definition 1.4 *Let $\mathscr{S}$ be a surface in $\mathbb{R}^n$ with unit normal $\boldsymbol{\nu}$. A vector filed $\mathscr{N} \in C^1(\Omega_{\mathscr{S}})$ in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, will be referred to as a* **proper extension** *if $\mathscr{N}\Big|_{\mathscr{S}} = \boldsymbol{\nu}$, it is unitary $|\mathscr{N}| = 1$ in $\Omega_{\mathscr{S}}$ and if $\mathscr{N}$ satisfies the condition in the neighborhood*

$$\partial_j \mathscr{N}_k(x) = \partial_k \mathscr{N}_j(x) \qquad \text{for all} \quad x \in \Omega_{\mathscr{S}}, \quad j,k = 1,\dots,n. \tag{1.27}$$

Such extension is needed, for example, to define correctly the normal derivative (the derivative along normal vector fields, outer or inner). It turned out that the "naive" extension (cf. (1.26))

$$\boldsymbol{\nu}(t) := \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|}, \qquad x \in \Omega_{\mathscr{S}} \tag{1.28}$$

is not proper. Indeed (see [DST1]), let $n = 2$ and $\mathscr{S}$ be the ellipse

$$\left\{x = (x_1, x_2) \in \mathbb{R}^2 | \Psi_{\mathscr{S}}(x_1, x_2) := x_1^2 + 2x_2^2 - 1 = 0\right\}.$$

Then

$$\begin{aligned}\mathscr{N}(x) := \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|} &= \frac{(x_1, 2x_2)}{\sqrt{x_1^2 + 4x_2^2}},\\ \partial_1\mathscr{N}_2(x) &= -\frac{2x_2\, x_1}{(x_1^2 + 4x_2^2)^{3/2}},\\ \partial_2\mathscr{N}_1(x) &= -\frac{4x_1x_2}{(x_1^2 + 4x_2^2)^{3/2}}.\end{aligned}$$

Hence $\partial_1\mathscr{N}_2(x) \neq \partial_2\mathscr{N}_1(x)$ unless $x_1 = 0$ or $x_2 = 0$.

For the proof of the next Proposition 1.5 and Corollary 1.6 on extension of the normal vector field we refer to [DST1].

Proposition 1.5 *Let $\mathscr{S} \subset \mathbb{R}^n$ be a hypersurface given by an implicit function*

$$\mathscr{S} = \left\{\mathscr{x} \in \mathbb{R}^n \ : \ \Psi_{\mathscr{S}}(\mathscr{x}) = 0\right\}.$$

Then the gradient $\nabla\Phi_{\mathscr{S}}(x)$ of the function

$$\Phi_{\mathscr{S}}(\mathscr{x} + t\boldsymbol{\nu}(\mathscr{x})) := t, \qquad \mathscr{x} + t\boldsymbol{\nu}(\mathscr{x}) \in \Omega_{\mathscr{S}}, \tag{1.29}$$

defined in the parameterized neighborhood

$$\Omega_{\mathscr{S}} := \{x = \mathscr{x} + t\boldsymbol{\nu}(\mathscr{x}) \ : \ \mathscr{x} \in \mathscr{S}, \quad -\varepsilon < t < \varepsilon\}$$

represents a unique proper extension of the unit normal vector field on the surface

$$\boldsymbol{\nu}(\mathscr{x}) = \lim_{x \to \mathscr{X}} \nabla\Phi_{\mathscr{S}}(x), \qquad \mathscr{x} \in \mathscr{S}.$$

Corollary 1.6 *For any proper extension $\mathscr{N}(x)$, $x \in \Omega_{\mathscr{S}} \subset \mathbb{R}^n$ of the unit normal vector field $\boldsymbol{\nu}$ to the surface $\mathscr{S} \subset \Omega_{\mathscr{S}}$ the equality*

$$\partial_{\mathscr{N}}\mathscr{N}(x) = 0 \qquad \textit{holds for all} \quad x \in \Omega_{\mathscr{S}}. \tag{1.30}$$

In particular, for the derivatives

$$\mathscr{D}_k = \partial_k - \mathscr{N}_k\partial_{\mathscr{N}}, \qquad k = 1, \dots, n, \tag{1.31}$$

which are extension into the domain $\Omega_{\mathscr{S}}$ of Günter's derivatives $\mathscr{D}_k = \partial_k - \nu_k\partial_{\boldsymbol{\nu}}$ on the surface $\mathscr{S}$, we have the equality:

$$\mathscr{D}_k\mathscr{N}_j = \partial_k\mathscr{N}_j - \mathscr{N}_k\partial_{\mathscr{N}} = \partial_k\mathscr{N}_j, \qquad \mathscr{D}_k\mathscr{N}_j = \mathscr{D}_j\mathscr{N}_k, \tag{1.32}$$
$$\textit{for all} \quad j, k = 1, \dots, n.$$

In the sequel we will dwell on a proper extension and apply the above properties of $\mathscr{N}$.

Lemma 1.7 (see [DMM1]) *For an arbitrary unitary extension $\mathscr{N}(x) \in C^1(\Omega_{\mathscr{S}})$, $|\mathscr{N}(x)| \equiv 1$, of $\boldsymbol{\nu}(\mathscr{x})$, in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, the following conditions are equivalent:*

i. $\partial_{\mathscr{N}}\mathscr{N}\big|_{\mathscr{S}} = 0$, *i.e.,* $\partial_{\mathscr{N}}\mathscr{N}_j(x) \to 0$ *for* $x \to \mathscr{x} \in \mathscr{S}$ *and* $j = 1, 2, ..., n$*;*

ii. $[\partial_k\mathscr{N}_j - \partial_j\mathscr{N}_k]\big|_{\mathscr{S}} = 0$ *for* $k,\ j = 1,\ 2, \ldots, n.$

The **second fundamental form** of $\mathscr{S}$ has the form

$$\begin{aligned} II(\boldsymbol{U}(\mathscr{x}), \boldsymbol{V}(\mathscr{x}))\boldsymbol{\nu}(\mathscr{x}) &:= \partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x}) - \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x}) = (I - \pi_{\mathscr{S}})\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x}) \\ &= \langle\boldsymbol{\nu}(\mathscr{x})\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x})\rangle\boldsymbol{\nu}(\mathscr{x}), \quad \forall\, \mathscr{x} \in \mathscr{S}\,, \quad \boldsymbol{U}, \boldsymbol{V} \in \omega(\mathscr{S}) \end{aligned} \tag{1.33}$$

and the **Weingarten matrix** (or the Weingarten mapping)

$$\mathscr{W}_{\mathscr{S}} : \omega(\mathscr{S}) \longrightarrow \omega(\mathscr{S}), \tag{1.34}$$

is defined uniquely by the requirement that

$$\begin{aligned} \langle\mathscr{W}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle = II(\boldsymbol{U}, \boldsymbol{V}) = \langle\boldsymbol{\nu}, \partial_{\boldsymbol{U}}\boldsymbol{V}\rangle = -\langle\partial_{\boldsymbol{U}}\boldsymbol{\nu}, \boldsymbol{V}\rangle = -\langle\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{\nu}, \boldsymbol{V}\rangle \\ \forall\, \boldsymbol{U}, \boldsymbol{V} \in \omega(\mathscr{S})\,. \end{aligned} \tag{1.35}$$

In the last equality in (1.35) we have applied the following: for a tangential vector field $\boldsymbol{V} \in \omega(\mathscr{S})$ holds $\langle\boldsymbol{\nu}(\mathscr{x}), \boldsymbol{V}(\mathscr{x})\rangle \equiv 0$, $\mathscr{x} \in \mathscr{S}$ and, by differentiating,

$$\langle\partial_{\boldsymbol{U}}\boldsymbol{\nu}(\mathscr{x}), \boldsymbol{V}(\mathscr{x})\rangle + \langle\boldsymbol{\nu}(\mathscr{x}), \partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x})\rangle \equiv 0\,, \qquad \mathscr{x} \in \mathscr{S}\,, \quad j = 1, \ldots, n\,, \tag{1.36}$$

$$\text{for all} \quad \boldsymbol{U} = \sum_{j=1}^n U_j d_j\,, \quad \boldsymbol{V} = \sum_{j=1}^n V_j d_j\,, \quad \boldsymbol{d}^j = \pi_{\mathscr{S}}\boldsymbol{e}^j\,, \quad \partial^{\mathscr{S}}_{\boldsymbol{U}} := \sum_{j=1}^n U_j\mathscr{D}_j\,.$$

We can extend the Weingarten matrix $\mathscr{W}_{\mathscr{S}}(x)$ from the surface $\mathscr{S}$ to a neighbourhood as follows:

$$\mathscr{W}_{\mathscr{S}}(x) := -\nabla\mathscr{N}(x) = -\big[\partial_j\mathscr{N}_k(x)\big]_{n\times n}\,, \qquad x \in \Omega_{\mathscr{S}}\,. \tag{1.37}$$

Lemma 1.8 *The extended Weingarten matrix $\mathscr{W}_{\mathscr{S}}(x)$ in* (1.37) *has the following properties:*

i. $\mathscr{W}_{\mathscr{S}}(x)\mathscr{N}(x) = 0$ *for all* $x \in \Omega_{\mathscr{S}}$*;*

ii. even if extension $\mathscr{N}(x)$ is not proper, the restriction to the hypersurface $\mathscr{W}_{\mathscr{S}}\big|_{\mathscr{S}}$ coincides with the Weingarten mapping of $\mathscr{S}$ and only depends on $\mathscr{S}$ (is independent of the choice of the extension $\mathscr{N}$);

iii. even if extension $\mathscr{N}(x)$ is not proper, $\mathrm{Tr}\,(\mathscr{W}_{\mathscr{S}})\big|_{\mathscr{S}} = \mathscr{H}^0_{\mathscr{S}}$, *where $\mathscr{H}^0_{\mathscr{S}}$ is the mean curvature of $\mathscr{S}$;*

iv. $\mathscr{W}_{\mathscr{S}}(x)\boldsymbol{V}(x)$, $x \in \mathscr{S}_C$, *is tangential to the level surface*

$$\mathscr{S}_C := \{y \in \mathbb{R}^n \ :\ \Psi_{\mathscr{S}}(y) = C := \Psi_{\mathscr{S}}(x)\} \tag{1.38}$$

for arbitrary vector field $\boldsymbol{V} : \mathscr{S} \to \mathbb{R}^n$.

Proof: First, $\mathscr{W}_{\mathscr{S}}\mathscr{N} = \nabla\|\mathscr{N}\|^2 = \nabla 1 = 0$ in $\Omega_{\mathscr{S}}$, justifying (i). Assertions (ii) and (iii) follow from Lemma 1.7.

Next, (iv) is proved as follows:

$$\langle \mathscr{N}(x), \mathscr{W}_{\mathscr{S}}\boldsymbol{V}(x)\rangle = -\sum_{j,k=1}^{n} \mathscr{N}_j(\partial_j\mathscr{N}_k)V_k = -\sum_{k=1}^{n}(\partial_{\mathscr{N}}\mathscr{N}_k)\omega = 0$$

due to (1.30), proved below. ■

We remind that

$$G_{\mathscr{S}}(\mathscr{X}) = G(\mathscr{X}) = [g_{jk}(\mathscr{X})]_{n-1\times n-1}\,, \qquad g_{jk} := \langle \boldsymbol{g}_j, \boldsymbol{g}_k\rangle$$

is the positive definite Gram matrix, which is known as the **covariant Riemannian metric tensor** and defines the metric on the surface $\mathscr{S}$ (cf. § 1.4).

Let $d\sigma = \sqrt{\det G_{\mathscr{S}}}dx$ and $d\mathfrak{s} = \sqrt{\det G_\Gamma}dx'$ stand for the volume elements on $\mathscr{S}$ and $\Gamma := \partial\mathscr{S}$, respectively ($x \in \mathbb{R}^{n-1}$, $x' \in \mathbb{R}^{n-2}$; cf. § 1.4).

Let

$$P(\nabla)u = \sum_{j=1}^{n} a_j\partial_j u + bu\,, \qquad a_j, b \in C^1(\mathbb{R}^{m\times m}) \tag{1.39}$$

be a first-order differential operator with real valued (variable) matrix coefficients, acting on vector-valued functions $u = (u_\beta)_\beta$ in $\mathbb{R}^n$ and its **principal symbol** is given by the matrix-valued function

$$\sigma(P;\xi) := \sum_{j=1}^{n} a_j\xi_j \qquad \xi = \big\{\xi_j\big\}_{j=1}^{n} \in \mathbb{R}^n\,. \tag{1.40}$$

Definition 1.9 *We say that P is a* **weakly tangential operator** *to the hypersurface $\mathscr{S}$, with unit normal $\boldsymbol{\nu}$, provided that*

$$\sigma(P;\boldsymbol{\nu}) = 0 \quad \textit{on the hypersurface } \mathscr{S}. \tag{1.41}$$

Next, call P a **strongly tangential operator** *to $\mathscr{S}$ provided that there exists an extended unit field $\mathscr{N}$ such that*

$$\sigma(P;\mathscr{N}) = 0 \quad \textit{in an open neighborhood of } \mathscr{S} \textit{ in } \mathbb{R}^n\,. \tag{1.42}$$

Note that in a strongly tangential operator the coordinate derivatives ∂_j can be replaced by the Günter's derivatives $\mathscr{D}_j$:

$$P(\nabla)u = \sum_{j=1}^{n} a_j\partial_j u + bu = \sum_{j=1}^{n} a_j\mathscr{D}_j u + bu = P(\mathscr{D})u\,, \qquad a_j, b \in C^1(\mathbb{R}^{m\times m}) \tag{1.43}$$

Most important tangential differential operators to the hypersurface are for us:

A. The weakly tangential Günter's derivatives (see (0.9))

$$\mathscr{D}_j := \partial_j - \nu_j\partial_{\boldsymbol{\nu}} = \partial_j - \nu_j\sum_{k=1}^{n}\nu_k\partial_k\,, \qquad j = 1,\dots,n.$$

B. The weakly tangential Stoke's derivatives $\mathscr{M}_{jk}=\nu_j\partial_k-\nu_k\partial_j$, introduced in § 1.5.

The Günter's and Stoke's derivatives are tangent since the corresponding vector fields are tangent

$$\begin{gathered}\mathscr{D}_j:=\partial_{\boldsymbol{d}^j}=\boldsymbol{d}^j\cdot\nabla\,,\qquad \mathscr{M}_{jk}:=\partial_{\mathfrak{m}_{jk}}=\mathfrak{m}_{jk}\cdot\nabla\,,\\ \boldsymbol{d}^j:=\pi_{\mathscr{S}}\boldsymbol{e}^j=\boldsymbol{e}^j-\nu_j\boldsymbol{\nu}=\boldsymbol{\nu}\wedge\left(\boldsymbol{\nu}\wedge\boldsymbol{e}^j\right)=\sum_{k=1}^n(\delta_{jk}-\nu_j\nu_k)\boldsymbol{e}^k\,,\\ \mathfrak{m}_{jk}:=\nu_je_k-\nu_ke_j\,,\quad \langle\boldsymbol{d}^j,\boldsymbol{\nu}\rangle=0\,,\qquad \langle\mathfrak{m}_{jk},\boldsymbol{\nu}\rangle=0\,,\quad j,k=1,\dots,n\,,\end{gathered}\tag{1.44}$$

where $\pi_{\mathscr{S}}$ is the projection on the tangential space to the surface (see (0.8)). Therefore $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ can be applied to functions which are defined on the surface $\mathscr{S}$ only.

The generating vector fields $\left\{\boldsymbol{d}^j\right\}_{j=1}^n$ $\left\{\mathfrak{m}_{jk}\right\}_{j,k=1}^n$ are not frame since they are linearly dependent

$$\sum_{j=1}^n\nu_j(\mathscr{x})\boldsymbol{d}^j(\mathscr{x})\equiv 0\,,\qquad \mathfrak{m}_{jj}=0\,,\tag{1.45}$$

but both systems $\left\{\boldsymbol{d}^j\right\}_{j=1}^n$ and $\left\{\mathfrak{m}_{jk}\right\}_{j,k=1}^n$ are full in the space of all tangential vector fields: any vector field $\boldsymbol{U}\in\omega(\mathscr{S})$ is represented as follows

$$\boldsymbol{U}(\mathscr{x})=\sum_{j=1}^n U^j(\mathscr{x})\boldsymbol{d}^j(\mathscr{x})=\sum_{0\leqslant j<k\leqslant 1}^n c_{jk}(\mathscr{x})\mathfrak{m}_{jk}(\mathscr{x})\,.\tag{1.46}$$

For example, the covariant vector fields $\boldsymbol{g}_1(\mathscr{x}):=\partial_1\Theta_k(\mathscr{x}),\dots,\boldsymbol{g}_{n-1}(\mathscr{x}):=\partial_{n-1}\Theta_k(\mathscr{x})$, $\mathscr{x}\in\mathscr{S}_k$, $k=1,\dots,N$ on $\mathscr{S}$, which generate the derivatives $\partial_j=\partial_{dx_j}$, are represented as follows

$$\boldsymbol{g}_j(\mathscr{x})=\sum_{m=1}^n g_j^m(\mathscr{x})\boldsymbol{e}^m=\sum_{m=1}^n g_j^m(\mathscr{x})\boldsymbol{d}^m(\mathscr{x})\tag{1.47}$$

and $\left\{\boldsymbol{e}^m\right\}_{m=1}^n$ is a Cartesian frame in $\mathbb{R}^n$. Indeed, by applying the derivative to Θ_k we get

$$\begin{aligned}\boldsymbol{g}_j=\sum_{m=1}^n g_j^m\boldsymbol{e}^m=\sum_{m=1}^n g_j^m\boldsymbol{d}^m\quad\text{since}\quad &\sum_{m=1}^n g_j^m[\boldsymbol{e}^m-\boldsymbol{d}^m]\\ &=\sum_{m=1}^n g_j^m\nu_m\boldsymbol{\nu}=\langle\boldsymbol{g}_j,\boldsymbol{\nu}\rangle\boldsymbol{\nu}=0\,,\quad j=1,\dots,n-1.\end{aligned}$$

Let us recall the following result about surface divergence $\mathrm{div}_{\mathscr{S}}$, the surface gradient $\nabla_{\mathscr{S}}$ and the surface Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$.

Theorem 1.10 ([DMM1]) *For any function $\varphi\in C^1(\mathscr{S})$ we have*

$$\nabla_{\mathscr{S}}\varphi=\left\{\mathscr{D}_1\varphi,\mathscr{D}_2\varphi,...,\mathscr{D}_n\varphi\right\}^\top.\tag{1.48}$$

Also, for a 1-smooth tangential vector field $\boldsymbol{V} = \sum_{j=1}^{n} V^j e_j \in \omega(\mathscr{S})$*,*

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{V} = -\nabla^*_{\mathscr{S}} \boldsymbol{V} := \sum_{j=1}^{n} \mathscr{D}_j V^j. \tag{1.49}$$

The Laplace-Beltrami operator $\Delta_{\mathscr{S}}$ *on* $\mathscr{S}$ *takes the form*

$$\boldsymbol{\Delta}_{\mathscr{S}}\, \psi = \operatorname{div}_{\mathscr{S}} \nabla_{\mathscr{S}}\, \psi = -\nabla^*_{\mathscr{S}} \Big(\nabla_{\mathscr{S}} \psi \Big) = \sum_{j=1}^{n} \mathscr{D}_j^2 \psi \tag{1.50}$$

$$= \sum_{j<k} \mathscr{M}_{jk}^2 \psi = \frac{1}{2} \sum_{j,k=1}^{n} \mathscr{M}_{jk}^2 \psi \qquad \forall\, \psi \in C^2(\mathscr{S})\,. \tag{1.51}$$

An important operator on forms is the **exterior derivative**. The **derivative of a 0-form**, i.e., of a scalar function

$$f \ :\ \mathscr{S} \to \mathbb{R}\,, \qquad f \in C^1(\mathscr{S})\,, \tag{1.52}$$

is a 1-form and maps

$$df(w)\ :\ \mathbb{T}_w\mathscr{S} \to \mathbb{R}\,. \tag{1.53}$$

Thus, $df(w)$ is a linear functional $df(w) \in \mathbb{T}^*_w\mathscr{S}$ over $\mathbb{T}_w\mathscr{S}$ for all $w \in \mathscr{S}$: being a vector $df(w) = Df(w) = (\partial_1 f(w), \dots, \partial_{n-1} f(w))^\top$ the differential assigns to a vector $\xi \in \mathbb{T}_w\mathscr{S}$ the number

$$df(x)\xi = \sum_{j=1}^{n-1} \partial_j f(x) \xi_j\,, \qquad \partial_j f(x) := \partial_{dx_j} f(x), \quad x \in \mathscr{S}_k, \tag{1.54}$$

where $\big\{dx_j = \partial_j \Theta_k\big\}_{j=1}^{n-1}$ is the covariant basis on $\mathscr{S}$ and $\Theta_k\ :\ \Omega_k \to \mathscr{S}_k$, $k = 1, \dots, N$ is the surface immersion.

From (1.47) and the definition of the derivative $\partial_j f(x) := \partial_{dx_j} f(x)$ in (1.54) follows that (see for the differential matrix $D\Theta_k$):

$$\begin{gathered}\partial_{\mathscr{S}} := \big(\partial_1, \dots, \partial_{n-1}\big)^\top := \big(\partial_{dx_1}, \dots, \partial_{dx_{n-1}}\big)^\top = \big(D\Theta_k\big)^\top \nabla_{\mathscr{S}}, \\ \nabla_{\mathscr{S}} := \big(\mathscr{D}_1, \dots, \mathscr{D}_n\big)^\top \quad \text{or} \quad \partial_{dx_j} = \sum_{m=1}^{n} (\partial_j \Theta_k^m) \mathscr{D}_m\,, \quad m = 1, \dots, n-1.\end{gathered} \tag{1.55}$$

Let $\mathscr{N}$ be a proper extension of the unit normal vector field $\boldsymbol{\nu}$ to $\mathscr{S}$ (cf. Definition 1.4). Then each operator $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ extends accordingly by setting

$$\mathscr{D}_j = \partial_j - \mathscr{N}_j \partial_{\mathscr{N}}\,, \qquad \mathscr{M}_{jk} := \mathscr{N}_j \partial_k - \mathscr{N}_k \partial_j\,, \qquad 1 \leqslant j, k \leqslant n \tag{1.56}$$

In the sequel, we shall make no distinction between the operator $\mathscr{D}_j$ or $\mathscr{M}_{jk}$ on $\mathscr{S}$ and the extended one in $\mathbb{R}^n$ given by (1.56). Note, that the extended operators $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ becomes even *strongly tangent*.

For further reference, below we collect some of the most basic properties of this system of differential operators.

Lemma 1.11 *Let $\mathscr{N}$ be a proper extension of the unit vector field of normal vectors $\boldsymbol{\nu}$ to $\mathscr{S}$. The following relations are valid for $j,k=1,\dots,n$:*

i. $\mathscr{M}_{jj}=0$, $\mathscr{M}_{jk}=-\mathscr{M}_{kj}$;

ii. $\partial_k=\sum_{j=1}^n \mathscr{N}_j\mathscr{M}_{jk}+\mathscr{N}_k\partial_{\mathscr{N}}=-\sum_{k=1}^n \mathscr{N}_k\mathscr{M}_{jk}+\mathscr{N}_j\partial_{\mathscr{N}}$;

iii. $\sum_{k=1}^n \mathscr{M}_{jk}\mathscr{N}_k=-\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}}$, *where* $\mathscr{H}^0_{\mathscr{S}}(\mathscr{x})=-\mathrm{div}_{\mathscr{S}}\,\boldsymbol{\nu}(\mathscr{x})$ *and* $\mathscr{H}_{\mathscr{S}}(\mathscr{x}):=(n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}(\mathscr{x})$ *is the mean curvature at* $\mathscr{x}\in\mathscr{S}$ *(see* (0.15)*);*

iv. $\mathscr{D}_j=\sum\limits_{k=1}^n \mathscr{N}_k\mathscr{M}_{kj}$;

v. $\mathscr{M}_{jk}=\mathscr{N}_j\mathscr{D}_k-\mathscr{N}_k\mathscr{D}_j$;

vi. $\sum\limits_{j=1}^n \mathscr{N}_j\mathscr{D}_j=0$;

vii. $\sum\limits_{r,j,k=m-1}^{m+1}\sigma(r,j,k)\mathscr{N}_i\mathscr{M}_{jk}=2\sum\limits_{\{r,j,k\}\subset\{(m-1),m,(m+1)\}}\sigma(r,j,k)\mathscr{N}_i\mathscr{M}_{jk}=0$ *for* $m=2,\dots,n-1$, *where* $\sigma(r,j,k)$ *is the permutation sign:*

$$\sigma(j_1,\dots,j_k)=\begin{cases}+1 & \text{if } (j_1,\dots,j_k)\text{is an even permutation of the strongly}\\ & \text{ordered row } (m_1,\dots,m_k)\,,\quad m_1<\dots<m_k\,,\\ 0 & \text{if } j_r=j_s \text{ for some } r,s=1\dots,k \text{ and } r\neq s\,,\\ -1 & \text{if } (j_1,\dots,j_k) \text{ is an odd permutation of the strongly}\\ & \text{ordered row } (m_1,\dots,m_k)\,,\quad m_1<\dots<m_k\,;\end{cases}\tag{1.57}$$

viii. $[\mathscr{D}_j,\mathscr{D}_k]=\sum\limits_{r=1}^n(\mathscr{M}_{jk}\mathscr{N}_r)\mathscr{D}_r+\big[\mathscr{N}_j\partial_{\mathscr{N}}\mathscr{N}_k-\mathscr{N}_k\partial_{\mathscr{N}}\mathscr{N}_j\big]\partial_{\mathscr{N}}$;

ix. $[\mathscr{D}_j,\mathscr{D}_k]=\sum\limits_{r=1}^n(\mathscr{M}_{jk}\mathscr{N}_r)\mathscr{D}_r=\mathscr{N}_k[\mathscr{D}_{\mathscr{N}},\partial_j]-\mathscr{N}_j[\mathscr{D}_{\mathscr{N}},\partial_k]$;

x. $\partial_j\mathscr{N}_k=\mathscr{D}_j\mathscr{N}_k=\mathscr{D}_k\mathscr{N}_j$.

Proof: The identities (i)-(ii) and (iv)-(vii) are simple consequences of the definitions. For the equality (iii) we have

$$\begin{aligned}\sum_{k=1}^n\mathscr{M}_{jk}\mathscr{N}_k=\sum_{k=1}^n\mathscr{M}_{jk}\mathscr{N}_k=\sum_{k=1}^n(\mathscr{N}_j\partial_k-\mathscr{N}_k\partial_j)\mathscr{N}_k\\ =\mathscr{N}_j\,\mathrm{div}\,\mathscr{N}-\frac{1}{2}\partial_j(\|\mathscr{N}\|^2)=-\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}},\end{aligned}$$

as claimed.

To prove (viii) we calculate

$$\begin{aligned}\mathscr{D}_j\mathscr{D}_k =& (\partial_j - \mathscr{N}_j\partial_{\mathscr{N}})(\partial_k - \mathscr{N}_k\partial_{\mathscr{N}}) = \partial_j\partial_k - (\partial_j\mathscr{N}_k)\partial_{\mathscr{N}}\\ & -\sum_{r=1}^{n}\Big[\mathscr{N}_k(\partial_j\mathscr{N}_r)\partial_r + \mathscr{N}_k\mathscr{N}_r\partial_r\partial_j + \mathscr{N}_j\mathscr{N}_r\partial_r\partial_k\Big] + \mathscr{N}_j\big(\partial_{\mathscr{N}}\mathscr{N}_k\big)\partial_{\mathscr{N}} + \mathscr{N}_j\mathscr{N}_k\partial^2_{\mathscr{N}}\\ =& -\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\partial_r + \mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}} + B_{jk}\\ =& -\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\mathscr{D}_r + \mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}} + B_{jk}\,, \end{aligned}\tag{1.58}$$

since

$$\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\mathscr{N}_r\partial_{\mathscr{N}} = \frac{1}{2}\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r^2)\partial_{\mathscr{N}} = \frac{1}{2}\mathscr{N}_k(\partial_j 1)\partial_{\mathscr{N}} = 0\,.$$

The operator

$$B_{jk} = \partial_j\partial_k - (\partial_j\mathscr{N}_k)\partial_{\mathscr{N}} - \sum_{r=1}^{n}\Big[\mathscr{N}_k\mathscr{N}_r\partial_r\partial_j + \mathscr{N}_j\mathscr{N}_r\partial_r\partial_k\Big] + \mathscr{N}_j\mathscr{N}_k\partial^2_{\mathscr{N}}$$

is symmetric $B_{jk} = B_{kj}$ and the desired commutator identity in (viii) follows from (1.58).

The first commutator identity in (ix) utilizes the facts that $\partial_{\mathscr{N}}\mathscr{N}_k = 0$ (cf. Lemma (1.27)) and follows from the identity in (viii). The second commutator identity in (ix) applies the same identity $\partial_{\mathscr{N}}\mathscr{N}_k = 0$, the identity $\partial_j\mathscr{N}_k = \partial_k\mathscr{N}_j$ (cf. (1.30)), and follows by a routine calculations.

The identities in (x) are already proved in (1.27) and (1.32). ■

The next Lemma 1.12 provides an useful and interesting example of restriction of the differential form to hypersurface and to it's boundary.

Lemma 1.12 *Let $\Theta\,:\,\Omega\to\mathscr{S}$ be a smooth hypersurface in $\mathbb{R}^n$ with a smooth boundary $\Gamma := \partial\mathscr{S}$, while $d\sigma$ and $d\mathfrak{s}$ designate the respective volume elements on $\mathscr{S}$ and on Γ. Let $\boldsymbol{\nu}(\mathscr{x}) = \big(\nu_1(\mathscr{x}),\ldots,\nu_n(\mathscr{x})\big)^\top$ be the outer unit normal vector to $\mathscr{S}$ at $\mathscr{x}\in\mathscr{S}$ and $\boldsymbol{\nu}_\Gamma(s) = \big(\nu^1_\Gamma(s),\ldots,\nu^n_\Gamma(s)\big)^\top$-the unit tangential vector to $\mathscr{S}$ at the boundary point $s\in\Gamma$, which is outward (unit) normal vector to the boundary $\mathscr{S}$. Then*

$$\nu_j dS = \beta_j\big|_{\mathscr{S}}\,, \tag{1.59}$$

$$\big[\nu_j\nu^k_\Gamma - \nu_k\nu^j_\Gamma\big]\,d\mathfrak{s} = \beta_{jk}\big|_\Gamma\,, \tag{1.60}$$

where

$$\begin{aligned}\beta_j :=& \big|dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge dx_n\big| = (-1)^{j-1}dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge dx_n\,,\\ \beta_{jk} :=& \big|dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge\widehat{dx_k}\wedge\ldots\wedge dx_n\big|\\ =& (-1)^{j+k-1}dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge\widehat{dx_k}\wedge\ldots\wedge dx_n\end{aligned}$$

and $\widehat{dx_m}$ denotes that the factor dx_m is dropped.

The next Theorem generalizes Stoke's formulae (see [Ta2, § 2.2, Theorem 2.1] for the version on compact Riemannian manifolds).

Theorem 1.13 *For any real-valued function $\varphi \in C^1(\mathscr{S})$ and any $1 \leqslant j < k \leqslant n$, there hold:*

$$\int_{\mathscr{S}} \mathscr{M}_{jk}\varphi\, d\sigma = \oint_{\Gamma} \big[\nu_j\nu^k_\Gamma - \nu_k\nu^j_\Gamma\big]\varphi\, d\mathfrak{s}\,, \tag{1.61}$$

where $\boldsymbol{\nu}_\Gamma(\xi) = \big(\nu^1_\Gamma(\xi), \ldots, \nu^n_\Gamma(\xi)\big)^\top$ is the unit tangential vector to $\mathscr{S}$ at the boundary point $\xi \in \Gamma := \partial\mathscr{S}$ and outward (unit) normal vector to the boundary $\Gamma = \partial\mathscr{S}$.

Proof: With formula (1.59) at hand the integrand in (1.61) can be represented as a total differential

$$(\mathscr{M}_{jk}\varphi)\, d\sigma = (\partial_k\varphi)\omega_j\big|_{\mathscr{S}} - (\partial_j\varphi)\omega_k\big|_{\mathscr{S}} = d\big[\varphi\omega_{jk}\big]\big|_{\mathscr{S}}\,.$$

Applying the well known Stoke's formula

$$\int_{\mathscr{S}} d\beta := \int_\Gamma \beta \tag{1.62}$$

(see, e.g., [Du2]) and formula (1.60) we get:

$$\int_{\mathscr{S}} \mathscr{M}_{jk}\varphi\, d\sigma = \int_{\mathscr{S}} d\big[\varphi\omega_{jk}\big]\big|_{\mathscr{S}} = \int_\Gamma \varphi\omega_{jk}\big|_\Gamma = \int_\Gamma \big[\nu_j\nu^k_\Gamma - \nu_k\nu^j_\Gamma\big]\varphi\, d\mathfrak{s}$$

and (1.61) is proved. ■

The formal adjoint (in $\mathbb{R}^n$) to P is defined by

$$P^*u = -\sum_j \partial_j a_j^\top u + b^\top u$$

If $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain, and if P is a first-order operator, weakly tangent to $\partial\Omega$, then, applying (1.69) (cf. § 1.5), P can be integrated by parts over Ω without boundary terms, i.e.

$$(Pu, v)_\Omega := \int_\Omega \langle Pu, v\rangle\, dx = \int_\Omega \langle u, P^*v\rangle\, dx =: (u, P^*v)_\Omega \tag{1.63}$$

for all vector-valued sections of vector fields $u, v \in C^1(\bar{\Omega})$.

For a weakly tangential differential operator Q on a closed hypersurface $\mathscr{S}$ let $Q^*_{\mathscr{S}}$ denote the "surface" adjoint:

$$(Q_{\mathscr{S}}\varphi, \psi)_{\mathscr{S}} := \oint_{\mathscr{S}} \langle Q_{\mathscr{S}}\varphi, \psi\rangle\, d\sigma = \oint_{\mathscr{S}} \langle \varphi, Q^*_{\mathscr{S}}\psi\rangle\, d\sigma = (\varphi, Q^*_{\mathscr{S}}\psi)_{\mathscr{S}} \tag{1.64}$$

for all vector-valued sections of vector fields $\varphi,\ \psi \in C^1(\bar{\Omega})$.

Corollary 1.14 *The surface-adjoint operator $P^*_{\mathscr{S}}$ to the weakly tangential differential operator P in* (1.39) *is equal to the formally adjoint one*

$$P^*_{\mathscr{S}}\varphi = P^*\varphi = -\sum_{j=1}^n \partial_j a_j^\top \varphi + b^\top \varphi \,. \tag{1.65}$$

In particular, the Stoke's derivatives are skew-symmetric

$$\big(\mathscr{M}^*_{jk}\big)_{\mathscr{S}} = \mathscr{M}^*_{jk} = -\mathscr{M}_{jk} = \mathscr{M}_{kj} \qquad \forall\, j,k = 1,\dots,n\,. \tag{1.66}$$

The adjoint operator to the operator $\mathscr{D}_j$ is

$$\big(\mathscr{D}_j\big)^*_{\mathscr{S}}\varphi = \mathscr{D}^*_j\varphi = -\mathscr{D}_j\varphi + \nu_j\mathscr{H}^0_{\mathscr{S}}\varphi\,, \qquad \varphi\in C^1(\mathscr{S})\,, \tag{1.67}$$

where $(n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}(\mathscr{x}) = \mathscr{H}_{\mathscr{S}}(\mathscr{x})$ is the mean curvature of the surface $\mathscr{S}$ (cf. (0.15)*).*

For any real-valued function $\varphi \in C^1(\mathscr{S})$, any $1 \leqslant j < k \leqslant n$ and for $\boldsymbol{\nu}_\Gamma = \big(\nu^1_\Gamma,\dots,\nu^n_\Gamma\big)^\top$ being the the same as in Theorem 1.13 the following integration by parts formula

$$\int_{\mathscr{S}} \Big[(\mathscr{D}_j\varphi)\psi - \varphi\big(\mathscr{D}^*_j\psi\big)\Big]\,d\sigma = \oint_\Gamma \nu^j_\Gamma\varphi\psi\,d\mathfrak{s}\,, \tag{1.68}$$

holds. It is an analogue of the classical Gaußian **integration by parts** *formula*

$$\int_\Omega \partial_j f(y)g(y)\,dy = \oint_{\mathscr{S}} \nu_j(\tau)f(\tau)g(\tau)\,d\sigma - \int_\Omega f(y)\partial_j g(y)\,dy, \tag{1.69}$$

which holds for arbitrary $f,\ g\in\mathbb{W}^1(\mathscr{S})$.

In particular, the following **Gauß formulae for open surfaces** *is valid:*

$$\int_{\mathscr{S}} \mathscr{D}_j\varphi\,d\sigma = \oint_\Gamma \nu^j_\Gamma\varphi\,d\mathfrak{s} + \int_{\mathscr{S}} \nu_j\mathscr{H}^0_{\mathscr{S}}\varphi\,d\sigma\,. \tag{1.70}$$

Proof: We start by proving (1.66): by applying the Stoke's formulae

$$\oint_{\mathscr{S}} (\mathscr{M}_{jk}f)(\tau)\,d\sigma = 0, \qquad j,k=1,\dots,n, \quad f\in\mathbb{W}^1_1(\mathscr{S}), \tag{1.71}$$

we get

$$\oint_{\mathscr{S}} (\mathscr{M}_{jk}\varphi)\psi\,d\sigma = \oint_{\mathscr{S}} (\mathscr{M}_{jk}\varphi\psi)\,d\sigma - \oint_{\mathscr{S}} \varphi(\mathscr{M}_{jk}\psi)\,d\sigma = -\oint_{\mathscr{S}} \varphi(\mathscr{M}_{jk}\psi)\,d\sigma$$

and the equality

$$\big(\mathscr{M}^*_{jk}\big)_{\mathscr{S}} = -\mathscr{M}_{jk} = \mathscr{M}_{kj} \tag{1.72}$$

follows. Moreover, note that the formal adjoint to $\mathscr{M}_{jk} = \mathscr{N}_j\mathscr{D}_k - \mathscr{N}_k\mathscr{D}_j$ is

$$\begin{aligned}\mathscr{M}^*_{jk}\varphi &= \big(\mathscr{N}_j\partial_k - \mathscr{N}_k\partial_j\big)^*\varphi = -\partial_j(\mathscr{N}_k\varphi) + \partial_k(\mathscr{N}_j\varphi)\\ &= \mathscr{N}_k\partial_j\varphi - \mathscr{N}_j\partial_k\varphi + (\partial_j\mathscr{N}_k)\varphi - (\partial_k\mathscr{N}_j)\varphi = -\mathscr{M}_{jk}\varphi\end{aligned}$$

(cf. (1.27)), where $\varphi \in C^1(\Omega_{\mathscr{S}})$ is defined in a neighborhood of $\mathscr{S}$. (1.66) is proved.

To prove (1.65) we note that, on $\mathscr{S}$,

$$\begin{aligned}P\varphi &= \sum_{j=1}^{n} a_j\partial_j\varphi + b\varphi = \sum_j a_j\big[\mathscr{D}_j + \nu_j\partial_{\boldsymbol{\nu}}\big]\varphi\\ &= \sum_{j=1}^{n} a_j\mathscr{D}_j\varphi + b\varphi + \sigma(P;\boldsymbol{\nu})\partial_{\boldsymbol{\nu}}\varphi = \sum_{j=1}^{n} a_j\mathscr{D}_j\varphi \qquad (1.73)\\ &= \sum_{j,k=1}^{n} a_j\nu_k\mathscr{M}_{kj}\varphi \qquad (1.74)\end{aligned}$$

due to Lemma 1.11.iv and the weak tangentiality of P. The property postulated in (1.65) follows from (1.74) and (1.66):

$$P^*_{\mathscr{S}}\varphi = \sum_{j,k=1}^{n}(\mathscr{M}_{kj})^*_{\mathscr{S}}a_j^\top\nu_k\varphi + b^\top\varphi = \sum_{j,k=1}^{n}(\mathscr{M}_{kj})^*a_j^\top\nu_k\varphi + b^\top\varphi = P^*\varphi\,.$$

With (1.65) and with (1.27) we get

$$\begin{aligned}(\mathscr{D}_j)^*_{\mathscr{S}}\varphi &= \mathscr{D}^*_j\varphi = -\partial_j\varphi + \sum_{k=1}^{n}\partial_k\big(\mathscr{N}_j\mathscr{N}_k\varphi\big)\\ &= -\partial_j\varphi + \sum_{k=1}^{n}\Big[\mathscr{N}_j\mathscr{N}_k\partial_k\varphi + \big(\mathscr{N}_k\partial_k\mathscr{N}_j\big)\varphi + \mathscr{N}_j\big(\partial_k\mathscr{N}_k\big)\varphi\Big]\\ &= -\mathscr{D}_j\varphi - \mathscr{N}_j\mathscr{H}^0_{\mathscr{S}}\varphi + (\partial_{\mathscr{N}}\mathscr{N}_j)\varphi\,, \qquad (1.75)\end{aligned}$$

where $\varphi \in C^1(\Omega_{\mathscr{S}})$ is defined in a neighborhood of $\mathscr{S}$ and

$$\mathscr{H}^0_{\mathscr{S}} := -\sum_{k=1}^{n}\mathscr{D}_k\mathscr{N}_k\,, \qquad \mathscr{H}^0_{\mathscr{S}}(\mathscr{x}) = -\sum_{k=1}^{n}\mathscr{D}_k\nu_k(\mathscr{x}) \qquad \text{for} \quad \mathscr{x}\in\mathscr{S}\,. \qquad (1.76)$$

(1.67) follows since (cf. (1.30)) $\partial_{\mathscr{N}}\mathscr{N}_j = 0$.

To prove (1.76) we apply

$$\begin{aligned}\partial_{\mathscr{N}}\mathscr{N}\big|_{\mathscr{S}} &= \left\{\sum_{j=1}^{n}\mathscr{N}_j\partial_j\mathscr{N}_k\right\}_{k=1}^{n}\Bigg|_{\mathscr{S}} = \left\{\sum_{j=1}^{n}\mathscr{N}_j\partial_k\mathscr{N}_j\right\}_{k=1}^{n}\Bigg|_{\mathscr{S}}\\ &= \frac{1}{2}\nabla_x|\mathscr{N}|^2\Big|_{\mathscr{S}} = \frac{1}{2}\nabla_x 1 = 0\,. \qquad (1.77)\end{aligned}$$

and proceed as follows

$$\sum_{k=1}^{n}\mathscr{D}_k\nu_k=\sum_{k=1}^{n}\left(\partial_k\nu_k-\nu_k\sum_{j=1}^{n}\nu_j\partial_j\nu_k\right)=-\mathscr{H}^0_{\mathscr{S}}-\sum_{j=1}^{n}\frac{\nu_j}{2}\partial_j 1=-\mathscr{H}^0_{\mathscr{S}}\,.$$

For the proof of the last formula (1.68) we apply Lemma 1.11.iv, (1.66), the equalities $\sum\limits_{k=1}^{n}\nu_k^2=1$, $\sum\limits_{k=1}^{n}\nu_k\nu_\Gamma^k=0$ and proceed as follows:

$$\begin{aligned}\oint_{\mathscr{S}}(\mathscr{D}_j\varphi)\psi\,d\sigma=&\sum_{k=1}^{n}\oint_{\mathscr{S}}\nu_k(\mathscr{M}_{jk}\varphi)\psi\,d\sigma-\sum_{k=1}^{n}\oint_{\mathscr{S}}\psi(\mathscr{M}_{jk}\nu_k\psi)\,d\sigma\\&+\sum_{k=1}^{n}\oint_{\Gamma}(\nu_k^2\nu_\Gamma^j-\nu_k\nu_j\nu_\Gamma^k)\varphi\psi\,d\mathfrak{s}=\oint_{\mathscr{S}}\psi(\mathscr{D}_j^*\psi)\,d\sigma+\oint_{\Gamma}\nu_\Gamma^j\varphi\psi\,d\mathfrak{s}\,.\end{aligned}$$

Concerning the formula (1.70): it follows from formulae (1.68) and (1.67), if we insert $\psi(t)\equiv 1$ in (1.68) and note, that $\mathscr{D}_j 1=0$. ■

Lemma 1.15 *Let P be, as in ((1.39)), a first-order differential operator with C^1-smooth coefficients. P is weakly/strongly tangent if and only if the formally adjoint P^* is so.*

If P is weakly tangent to $\mathscr{S}$ and P is defined in a neighborhood of $\mathscr{S}$, then

$$(P\varphi)\Big|_{\mathscr{S}}=P\Big(\varphi|_{\mathscr{S}}\Big) \tag{1.78}$$

for every C^1 function φ defined in a neighborhood of $\mathscr{S}$. In particular,

$$\mathscr{D}_j\varphi\big|_{\mathscr{S}}=\mathscr{D}_j\big(\varphi\big|_{\mathscr{S}}\big)\,,\qquad \mathscr{M}_{jk}\varphi\big|_{\mathscr{S}}=\mathscr{M}_{jk}\big(\varphi\big|_{\mathscr{S}}\big)\,,\qquad j,k=1,\dots,n\,. \tag{1.79}$$

Furthermore, (1.78) *is true for the adjoint P^*, and*

$$\int_{\mathscr{S}}\langle P\varphi,\psi\rangle\,d\sigma=\int_{\mathscr{S}}\langle\varphi,P^*\psi\rangle\,d\sigma+\oint_{\Gamma}\langle\sigma(P;\boldsymbol{\nu}_\Gamma)\varphi,\psi\rangle\,d\mathfrak{s} \tag{1.80}$$

for any vector-valued functions $\varphi,\psi\in\mathscr{S}$.

Proof: The first assertion follows since $\sigma(P^*;\xi)=-\sigma(P;\xi)^\top$, for each $\xi\in\mathbb{R}^n$.

Due to the representation (1.73) it suffices to prove (1.78) for only the operator $\mathscr{D}_j=\boldsymbol{d}^j\cdot\nabla$, where $\boldsymbol{d}^j=\pi_{\mathscr{S}}\boldsymbol{e}^j=\mathscr{N}\wedge\big(\mathscr{N}\wedge\boldsymbol{e}^j\big)$ is at least C^1-smooth vector field in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, tangent to the surface $\mathscr{S}$ at surface points (cf. (1.44)). Thus, we have to justify the following equality:

$$\mathscr{D}_j\varphi\big|_{\mathscr{S}}=(\boldsymbol{d}^j\cdot\nabla)\varphi\Big|_{\mathscr{S}}=\boldsymbol{d}^j\cdot\nabla\left(\varphi\Big|_{\mathscr{S}}\right)=\mathscr{D}_j\left(\varphi\Big|_{\mathscr{S}}\right)\,. \tag{1.81}$$

The vector field $\boldsymbol{d}^j(x) = \boldsymbol{d}^j(\theta, \mathscr{x})$ depends on the signed distance $\theta = \operatorname{dist}(x, \mathscr{S}) = \pm|x - \mathscr{x}|$ continuously ($\theta > 0$ for the outer domain and $\theta > 0$ for the inner one). Let $\mathscr{F}^t_{\boldsymbol{d}^j(\cdot)}$ be the integral curve of the vector field $\boldsymbol{d}^j$ and

$$\mathscr{F}^t_{\boldsymbol{d}^j(\cdot)} \ : \ \Omega_{\mathscr{S}} \to \Omega_{\mathscr{S}} \,, \qquad \mathscr{F}^t_{\boldsymbol{d}^j(0,\cdot)} = \mathscr{F}^t_{\boldsymbol{d}^j}(\cdot) \ : \ \mathscr{S} \to \mathscr{S} \tag{1.82}$$

be the flow generated by this vector field ℓ_θ in the neighborhood $\Omega_{\mathscr{S}}$ (cf. § 1.2). Since the flow depends continuously on the parameter θ, we get

$$\begin{aligned}\big(\boldsymbol{d}^j(\theta, \mathscr{x}) \cdot \nabla\big)\varphi\Big|_{\mathscr{S}} &= \lim_{\theta\to 0} \frac{d}{dt}\, \varphi\left(\mathscr{F}^t_{\boldsymbol{d}^j(\theta,\mathscr{x})}\right)\Big|_{t=0} = \frac{d}{dt}\, \varphi\left(\mathscr{F}^t_{\boldsymbol{d}^j}\right)\big|_{t=0} \\ &= \boldsymbol{d}^j \cdot \nabla\big(\varphi\big|_{\mathscr{S}}\big) = \mathscr{D}_j\left(\varphi\Big|_{\mathscr{S}}\right)\end{aligned}$$

and (1.81) is proved.

Next, using (1.73), (1.68) and integrating by parts we get

$$\begin{aligned}\int_{\mathscr{S}} \langle P\varphi, \psi\rangle\, d\sigma &= \sum_{j=1}^n \int_{\mathscr{S}} \langle a_j \mathscr{D}_j \varphi, \psi\rangle\, d\sigma + \int_{\mathscr{S}} \langle b\varphi, \psi\rangle\, d\sigma \\ &= \sum_{j=1}^n \int_{\mathscr{S}} \langle \varphi, \mathscr{D}^*_j a^\top_j \psi\rangle\, d\sigma + \int_{\mathscr{S}} \langle \varphi, b^\top \psi\rangle\, d\sigma + \sum_{j=1}^n \oint_\Gamma \langle \varphi, \nu^j_\Gamma a^\top_j \psi\rangle\, d\sigma \\ &= \int_{\mathscr{S}} \langle \varphi, P^*\psi\rangle\, d\sigma + \oint_\Gamma \langle \sigma(P; \boldsymbol{\nu}_\Gamma)\varphi, \psi\rangle\, d\mathfrak{s}\end{aligned}$$

and this completes the proof. ■

Remark 1.16 *By iteration, an identity similar in spirit to* (1.80) *holds for higher order weakly tangential differential operators which are higher order polynomials of Günter's or Stoke's derivatives (cf. Lemma 1.17).*

In this connection, let us also point out that the strongly tangential operator-the Stoke's gradient

$$\mathscr{M}_{\mathscr{S}} := \mathscr{N} \wedge \nabla_{\mathscr{S}} = \mathscr{N} \wedge \nabla = \big\{\mathscr{M}_{23}, -\mathscr{M}_{13}, \mathscr{M}_{12}\big\}\,, \qquad \mathscr{M}_{\mathscr{S}}\big|_{\mathscr{S}} = \boldsymbol{\nu} \wedge \nabla_{\mathscr{S}} \tag{1.83}$$

in $\mathbb{R}^3$ acting on scalar functions on $\mathscr{S}$, is naturally identified with the skew-symmetric matrix whose entries are the Stoke's derivatives, in the sense that

$$\boldsymbol{\nu} \wedge d = \frac{1}{2} \sum_{j,k=1}^3 \mathscr{M}_{jk}\, dx_j \wedge dx_k = \sum_{1\leqslant j<k\leqslant 3} \mathscr{M}_{jk}\, dx_j \wedge dx_k\,. \tag{1.84}$$

Further important examples of strongly tangential, first-order differential operators are offered by

$$\begin{aligned} P_1\boldsymbol{U} &:= \operatorname{div}\boldsymbol{U} - \partial_{\boldsymbol{\nu}}\boldsymbol{U}\,\langle \boldsymbol{U}, \boldsymbol{\nu}\rangle, &&\text{with} && P^*_1\varphi = -\nabla\varphi + (\partial_{\boldsymbol{\nu}}\varphi + \mathscr{H}^0_{\mathscr{S}}\varphi)\boldsymbol{\nu}\,, \\ P_2\boldsymbol{U} &:= \operatorname{div}_{\mathscr{S}} \pi_{\mathscr{S}}\boldsymbol{U}, &&\text{with} && P^*_2\varphi = -\pi_{\mathscr{S}}\nabla_{\mathscr{S}}\varphi\,, \\ P_3\boldsymbol{U} &:= \partial_{\boldsymbol{\nu}}\pi_{\mathscr{S}}\boldsymbol{U} - \boldsymbol{\nu} \vee dU, &&\text{with} && P^*_3\varphi = -\pi_{\mathscr{S}}\partial_{\boldsymbol{\nu}}\varphi - \mathscr{H}^0_{\mathscr{S}}\pi_{\mathscr{S}}\varphi - \delta(\boldsymbol{\nu} \wedge \varphi)\,. \end{aligned} \tag{1.85}$$

Indeed,

$$\sigma(P_1;\xi) = \langle\xi,\cdot\rangle - \langle\boldsymbol{\nu},\xi\rangle\langle\boldsymbol{\nu},\cdot\rangle\,, \quad \sigma(P_2;\xi) = \langle\xi,\pi_{\mathscr{S}}(\cdot)\rangle\,, \quad \sigma(P_3;\xi) = \langle\xi,\boldsymbol{\nu}\rangle\,\pi_{\mathscr{S}} - \boldsymbol{\nu}\vee(\xi\wedge\cdot)\,,$$

so that (1.42) is easily verified in each case.

In the sequel we use the following standard notation

$$\begin{aligned} \nabla^\alpha_{\mathscr{S}} &:= \mathscr{D}_1^{\alpha_1}\dots\mathscr{D}_n^{\alpha_n}\,, \qquad \alpha\in\mathbb{N}_0^n\,,\\ \mathscr{M}^\beta_{\mathscr{S}} &:= \mathscr{M}_1^{\beta_1}\dots\mathscr{M}_m^{\beta_m}\,, \quad \beta\in\mathbb{N}_0^m\,, \quad m=\frac{n(n-1)}{2}\,, \end{aligned} \tag{1.86}$$

where

$$\nabla_{\mathscr{S}} := (\mathscr{D}_1,\dots,\mathscr{D}_n)^\top\,, \qquad \mathscr{M}_{\mathscr{S}} := (\mathscr{M}_{12},\dots,\mathscr{M}_{n-1,n})^\top \tag{1.87}$$

and the selected Stoke's derivatives $\mathscr{M}_1 := \mathscr{M}_{1,2},\dots,\mathscr{M}_m := \mathscr{M}_{n-1,n}$ are non–vanishing, while the remaining non-vanishing Stoke's derivatives differ from the selected ones only by the sign. In contrast to the case of the usual derivatives ∂^α it does really matters in which sequence we apply the derivatives $\mathscr{D}_j^{\alpha_j}$ and $\mathscr{M}_k^{\beta_k}$ in (1.86), because they have variable coefficients. In this connection let us write precisely what is meant under the dual operators:

$$\begin{aligned} (\mathscr{D}_x^*)^\alpha &:= (\mathscr{D}_n^*)^{\alpha_n}\dots(\mathscr{D}_1^*)^{\alpha_1}\,, \quad \alpha\in\mathbb{N}_0^n\,,\\ (\mathscr{M}_x^*)^\beta &:= (-1)^{|\beta|}(\mathscr{M}_m)^{\beta_m}\dots(\mathscr{M}_1)^{\beta_1}\,, \quad \beta\in\mathbb{N}_0^m\,, \end{aligned} \tag{1.88}$$

Note, that we use the same operators $\mathscr{M}_1^* = -\mathscr{M}_1 = -\mathscr{M}_{1,2},\dots,\mathscr{M}_m^* = -\mathscr{M}_m := -\mathscr{M}_{n-1,n}$ for the adjoint operators to the Stokes derivatives, because these operators are skew-symmetric $(\mathscr{M}_{j,k})^* = -\mathscr{M}_{j,k}$ (cf. (1.66)).

Lemma 1.17 *Let $\mathbf{G}(\mathscr{D})$ be a tangential differential operator of the form*

$$\mathbf{G}(\mathscr{D}) = \sum_{|\alpha|\leqslant k}\boldsymbol{g}_\alpha(t)\mathscr{D}_t^\alpha = \sum_{|\beta|\leqslant k} f_\beta(t)\mathscr{M}_t^\beta, \qquad t\in\mathscr{S}\,. \tag{1.89}$$

Then

$$\oint_{\mathscr{S}}\langle\mathbf{G}(\mathscr{D})\varphi,\psi\rangle\,d\sigma = \oint_{\mathscr{S}}\langle\varphi,\mathbf{G}^*(\mathscr{D})\psi\rangle\,d\sigma\,, \tag{1.90}$$

where

$$\mathbf{G}^*(\mathscr{D}) = \sum_{|\alpha|\leqslant k}(\mathscr{D}^*)^\alpha g_\alpha^\top I = \sum_{|\beta|\leqslant k}(-1)^{|\beta|}\mathscr{M}^\beta f_\beta^\top I \tag{1.91}$$

and $\mathscr{D}^*$ and $\mathscr{M}^*$ are the adjoint operators (cf. (1.67) *and* (1.66)*).*

Remark 1.18 *Note that the operators $i\mathscr{M}_j,\ j=1,\dots,m$ with variable coefficients*

$$\mathbf{A}(x,\mathscr{M}_x)u=\sum_{j=1}^{M} b_j(x)(i\mathscr{M}_j)^{m_j}\overline{b_j^\top(x)}u\,,\qquad b_j\in[C^\infty(\mathscr{S})]^{N\times N} \tag{1.92}$$

and polynomials with constant self adjoint $N\times N$ matrix coefficients

$$\mathbf{B}(\mathscr{M}_x)u=\sum_{j=1}^{M} a_j\mathscr{M}_j^{m_j}u\,,\qquad \overline{a_j^\top}=a_j=\mathrm{const}\quad \forall j=1,\dots,M\,,\quad \forall m_j\in\mathbb{N}_0\,, \tag{1.93}$$

are all self adjoint on the hypersurface $\mathbf{A}^*_{\mathscr{S}}(\mathscr{M}_x)=\mathbf{A}(\mathscr{M}_x)$.

1.3 Equation of elastic hypersurface

One way of understanding the genesis of the Laplace-Beltrami operator $\mathbf{\Delta}_{\mathscr{S}}$ on the surface $\mathscr{S}$ (see (1.50)) is to consider the energy functional

$$\mathscr{E}[u]:=\int_{\mathscr{S}}\|\nabla u\|^2\,d\sigma,\qquad u\in C^\infty(\mathscr{S}). \tag{1.94}$$

Then any minimizer u of the functional (1.94) should satisfy

$$\begin{aligned}0&=\frac{d}{dt}\mathscr{E}[u+tv]\Big|_{t=0}=\int_{\mathscr{S}}\left[\langle\nabla u,\nabla v\rangle+\langle\nabla v,\nabla u\rangle\right]\,d\sigma\\ &=2\mathrm{Re}\int_{\mathscr{S}}\langle\nabla u,\nabla v\rangle\,d\sigma\qquad u\in C^\infty(\mathscr{S}),\qquad \forall\, v\in C_0^\infty(\mathscr{S}),\end{aligned} \tag{1.95}$$

which implies

$$\Delta u=0\quad \text{on}\quad \mathscr{S}. \tag{1.96}$$

In other words, (1.96) is the Euler-Lagrange equation associated with the integral functional (1.94).

Similarly, minimizers of the energy functional

$$\mathscr{E}[\boldsymbol{U}]:=\int_{\mathscr{S}}\Big[\|dU\|^2+\|\delta\boldsymbol{U}\|^2\Big]\,d\sigma,\qquad \boldsymbol{U}\in\Lambda^\ell\omega(\mathscr{S}), \tag{1.97}$$

are null-solutions to the Hodge-Laplacian (cf. later (1.147)), while minimizers of the energy functional

$$\mathscr{E}[\boldsymbol{U}]:=\int_{\mathscr{S}}\|\nabla\boldsymbol{U}\|^2\,d\sigma,\qquad \boldsymbol{U}\in\omega(\mathscr{S}), \tag{1.98}$$

are null-solutions to the Bochner-Laplacian (cf. later (1.148)).

Our aim is to adopt a similar point of view in the case of anisotropic and isotropic (Lamé) system of elasticity on $\mathscr{S}$.

The Günter's derivatives $\left\{\mathscr{D}_j\right\}_{j=1}^n$ are tangent and represent a full system (cf. (1.44)-(1.46)). But the derivative $\mathscr{D}_j\boldsymbol{V}$ is not covariant and maps the tangential vectors to non-tangential ones $\mathscr{D}_j\,:\,\omega(\mathscr{S})\not\to\omega(\mathscr{S})$. To improve this we just eliminate the normal component of the vector by applying the canonical orthogonal projection $\pi_{\mathscr{S}}$ onto $\omega(\mathscr{S})$ (cf. (0.8))

$$\mathscr{D}_j^{\mathscr{S}}\boldsymbol{V}:=\pi_{\mathscr{S}}\mathscr{D}_j\boldsymbol{V}=\mathscr{D}_j\boldsymbol{V}-\langle\boldsymbol{\nu},\mathscr{D}_j\boldsymbol{V}\rangle\boldsymbol{\nu}=\mathscr{D}_j\boldsymbol{V}+(\partial_{\boldsymbol{V}}\nu_j)\boldsymbol{\nu}\,, \tag{1.99}$$
$$\text{where}\quad \partial_{\boldsymbol{V}}\varphi:=\sum_{k=1}^n V_k^0\partial_k\varphi=\sum_{k=1}^n V_k^0\mathscr{D}_k\varphi$$

and obtain an automorphisms of the space of tangential vector fields

$$\mathscr{D}_j^{\mathscr{S}}\,:\,\omega(\mathscr{S})\to\omega(\mathscr{S})\,. \tag{1.100}$$

The starting point is to consider the total free (elastic) energy

$$\mathscr{E}[\boldsymbol{U}]:=\int_{\mathscr{S}}E(y,\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y))\,d\sigma,\quad \mathscr{D}^{\mathscr{S}}\boldsymbol{U}:=\left[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0\right]_{n\times n},\quad \boldsymbol{U}\in\omega(\mathscr{S}) \tag{1.101}$$

(cf. (1.99), (1.100)), ignoring at the moment the displacement boundary conditions (Koiter's model). As before, equilibria states correspond to minimizers of the above variational integral (see [NH1, § 5.2]). First we should identify the correct form of the stored energy density $E(x,\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x))$. We shall restrict attention to the case of linear elasticity. In this scenario, $E=(\mathfrak{S}_{\mathscr{S}},\mathrm{Def}_{\mathscr{S}})$ depends bi-linearly on the stress tensor $\mathfrak{S}_{\mathscr{S}}=\left[\mathfrak{S}^{jk}\right]_{n\times n}$ and the deformation (strain) tensor

$$\mathrm{Def}_{\mathscr{S}}=\left[\mathfrak{D}_{jk}\right]_{n\times n},\qquad \mathfrak{D}_{jk}\boldsymbol{U}:=\frac{1}{2}\Big[\big(\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U}\big)_j+\big(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U}\big)_k\Big],\quad j,k=1,\dots,n \tag{1.102}$$

which, according to Hooke's law, satisfy $\mathfrak{S}_{\mathscr{S}}=\mathbb{T}\mathrm{Def}_{\mathscr{S}}$, for some linear, fourth-order tensor $\mathbb{T}$. If the medium is also homogeneous (i.e. the density and elastic parameters are position-independent), it follows that *E depends quadratically on the covariant derivative $\mathscr{D}^{\mathscr{S}}\boldsymbol{U}$*, i.e.

$$E(x,\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x))=\langle\mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x),\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\rangle \tag{1.103}$$

for a linear operator

$$\mathbb{T}\,:\,\mathbb{M}^{n\times n}(\mathbb{R})\longrightarrow\mathbb{M}^{n\times n}(\mathbb{R}), \tag{1.104}$$

where $\mathbb{M}^{n\times n}(\mathbb{R})$ stands for the vector space of all $n\times n$ matrices with real entries. Hereafter, we organize $\mathbb{M}^{n\times n}(\mathbb{R})$ as a real Hilbert space with respect to the inner product

$$\langle A,B\rangle:=\mathrm{Tr}(AB^\top)=\sum_{i,j}a_{ij}b_{ij},\qquad \forall\,A=[a_{ij}]_{i,j}\,,\ B=[b_{ij}]_{i,j}\in\mathbb{M}^{n\times n}(\mathbb{R}), \tag{1.105}$$

where $B^\top$ denotes transposed matrix, and Tr is the usual trace operator for square matrices.

A linear operator (1.104) is a tensor of order 4, i.e., $\mathbb{T}=\left[c_{ijk\ell}\right]_{ijk\ell}$, and

$$\mathbb{T}A=\left[\sum_{k,\ell}c_{ijk\ell}a_{k\ell}\right]_{ij},\qquad\text{for}\ \ A=[a_{k\ell}]_{k\ell}\in\mathbb{M}^{n\times n}(\mathbb{R})\,. \tag{1.106}$$

$\mathbb{T}$ will be referred to in the sequel as the **elasticity tensor**. It is customary to assume that the elasticity tensor (1.104) is self-adjoint

$$\langle \mathbb{T}A, B\rangle = \langle A, \mathbb{T}B\rangle\,, \qquad A, B \in \mathbb{M}^{n\times n}(\mathbb{R})\,. \tag{1.107}$$

The condition rescaling (1.107), written in coordinate notation, is equivalent to the following equality

$$c_{ijk\ell} = c_{k\ell ij}, \qquad \forall\, i, j, k, \ell\,. \tag{1.108}$$

Indeed, the equality

$$\mathrm{Tr}((\mathbb{T}A)B^\top) = \sum_{i,j,k,\ell} c_{ijk\ell}a_{k\ell}b_{ij} = \sum_{i,j,k,\ell} c_{k\ell ij}a_{k\ell}b_{ij} = \mathrm{Tr}(A(\mathbb{T}B)^\top)$$

holds, for arbitrary $A = [a_{k\ell}]_{k\ell}$ and $B = [b_{k\ell}]_{k\ell}$, if and only if (1.108) holds: by inserting the delta functions $a_{k\ell} = \delta_{k\ell}$, $b_{ij} = \delta_{ij}$ we get the equality (1.108).

It is also customary to impose a symmetry condition, presented with two natural options:

$$\mathbb{T}(A^\top) = \mathbb{T}A \quad \text{and} \quad (\mathbb{T}A)^\top = \mathbb{T}A \qquad \forall\, A \in \mathbb{M}^{n\times n}(\mathbb{R})\,. \tag{1.109}$$

Then (1.109) amounts to the following symmetry in the indices of the elastic tensor:

$$c_{ijk\ell} = c_{ij\ell k} \quad \text{and} \quad c_{ijk\ell} = c_{jik\ell} \qquad \forall\, i, j, k, \ell\,. \tag{1.110}$$

Remark 1.19 *The conditions* (1.107) *and the first equality in* (1.109) *imply the second equality in* (1.109) *as well as the conditions* (1.107) *and the second equality in* (1.109) *imply the first equality in* (1.109). *This is evident if we apply an equivalent formulation for corresponding tensors and matrices:* (1.108) *and* (1.110).

A linear operator $\mathbb{T}$ in the energy functional of anisotropic elasticity (1.103) *satisfies the symmetry conditions* (1.107)*, and* (1.109)*. Equivalently, the corresponding elasticity tensor $\mathbb{T} = \left[c_{ijk\ell}\right]_{ijk\ell}$ has the symmetries* (1.108)*,* (1.110) *and, therefore, might have $n + n^2(n-1)^2/2$ different entries only.* ■

Remark 1.20 *It is rather natural to introduce the* **deformation tensor** *as the symmetrized covariant derivative (cf., e.g., [Ta2, V. I, Ch. 5, § 12]).*

$$\begin{aligned}(\mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U})(\boldsymbol{V},\boldsymbol{W}) &= \frac{1}{2}\Big\{\langle\partial_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle + \langle\partial_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\}\\ &= \frac{1}{2}\Big\{\langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle + \langle\partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\}, \qquad \forall\,\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}).\end{aligned} \tag{1.111}$$

It is also worth of mentioning that the antisymmetric part of the covariant derivative $\partial^{\mathscr{S}}_{\boldsymbol{U}}$

$$d\boldsymbol{U}(\boldsymbol{V},\boldsymbol{W}) = \langle d\boldsymbol{U},\boldsymbol{V}\wedge\boldsymbol{W}\rangle = \frac{1}{2}\Big\{\langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle - \langle\partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\}, \qquad \forall\,\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}), \tag{1.112}$$

is the exterior differential.

By inserting the value (1.102) of deformation tensor $\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}$ and applying the symmetry properties (1.110), we obtain

$$4\langle\mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(x),\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(x)\rangle=\langle\mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x),\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\rangle=E(x,\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)) \tag{1.113}$$

(cf. (1.103)) which means that *the density of the elastic energy functional depends quadratically also on the deformation tensor*.

The density of the potential energy of an elastic medium should be strictly positive for the non-vanishing deformation tensor $\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}\neq 0$ (the energy conservation law!). This leads to the following.

Lemma 1.21 *There exists a constant $C_0>0$ such that*

$$\langle\mathbb{T}\zeta,\zeta\rangle:=\sum_{i,j,k,\ell}c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell}\geqslant C_0\sum_{i,j}|\zeta_{i,j}|^2:=C_0|\zeta|^2 \tag{1.114}$$

for all symmetric and complex valued $\zeta_{ij}=\zeta_{ji}\in\mathbb{C}$ tensors $\zeta:=\big[\zeta_{ij}\big]_{n\times n}$.

Proof: The sum in the left hand side of (1.114) is real $\langle\mathbb{T}\zeta,\zeta\rangle=\overline{\langle\mathbb{T}\zeta,\zeta\rangle}$ (easy to check applying the symmetry properties (1.110) of the real valued coefficients). Dividing equality in (1.114) by $|\zeta|^2=\sum_{lm}|\zeta_{lm}|^2$ we find that it suffices to prove

$$\inf_{|\zeta|=1}\sum_{i,j,k,\ell}c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell}\geqslant C_0>0\,. \tag{1.115}$$

If otherwise $C_0=0$, we select a sequence $\zeta^{(q)}_{jk}=\zeta^{(q)}_{kj}\in\mathbb{C}$, $q=1,2,\dots$ such that

$$\lim_{m\to\infty}\sum_{i,j,k,\ell}c_{ijk\ell}\zeta^{(q)}_{ij}\overline{\zeta^{(q)}_{k\ell}}=0\,,\qquad|\zeta^{(q)}|=1\,.$$

Since the space of tensors $[\zeta^{(q)}_{jk}]_{n\times n}$ is finite dimensional, there exists a convergent subsequence $\zeta^{(q_r)}_{k\ell}\to\zeta^{(0)}_{k\ell}$ as $r\to\infty$. Then we get an obvious contradiction

$$\sum_{i,j,k,\ell}c_{ijk\ell}\zeta^{(0)}_{ij}\overline{\zeta^{(0)}_{k\ell}}=0\,,\qquad|\zeta^{(0)}|=1\,.$$

which proves that $C_0>0$. ■

Theorem 1.22 *The total free (elastic) energy functional (cf.* (1.101)*) acquires the form*

$$\mathscr{E}[\boldsymbol{U}]:=\int_{\mathscr{S}}\langle\mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y),\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y)\rangle\,d\sigma=4\int_{\mathscr{S}}\langle\mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y),\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y)\rangle\,d\sigma\,, \tag{1.116}$$
$$\boldsymbol{U}\in\omega(\mathscr{S})$$

and the Euler-Lagrange equation associated with the energy functional (1.116) *for a linear anisotropic elastic medium, reads*

$$\mathscr{L}_{\mathscr{S}}\boldsymbol{U}=\mathrm{Def}^*_{\mathscr{S}}\mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}\,,\qquad\boldsymbol{U}\in\omega(\mathscr{S})\,. \tag{1.117}$$

Here again $\mathbb{T}=\big[c_{ijk\ell}\big]_{ijk\ell}$ is the elasticity tensor which is positive definite (cf. (1.114)*) and has the symmetry properties* (1.108), (1.110).

Proof: The representation (1.116) follows from (1.101) and (1.113).

The Euler-Lagrange equation (1.117) is derived from (1.116) as a similar equation e3.3 is derived from (1.94):

$$\begin{aligned}\mathscr{E}[\boldsymbol{U}] &:= 4\int_{\mathscr{S}} \langle \mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y), \mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y)\rangle\, d\sigma \\ &= 4\int_{\mathscr{S}} \langle \mathrm{Def}^*_{\mathscr{S}}\mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y), \boldsymbol{U}(y)\rangle\, d\sigma = 0\end{aligned}$$

if and only if $\boldsymbol{U} \in \omega(\mathscr{S})$ is a solution of equation (1.117) due to the positive definiteness of the elasticity tensor $\mathbb{T}$ (cf. (1.114)). ■

Next we will find the Euler-Lagrange equation associated with the energy functional (1.101) for a linear isotropic elastic medium (Lamé equation) which is simpler. Such energy functional should be invariant with respect to any rotation. For the elasticity tensor $\mathbb{T}$ this results into the requirement that

$$\mathbb{T}(BAB^{-1}) = B(\mathbb{T}A)B^{-1}, \qquad \forall\, A, B \in \mathbb{M}^{n\times n}(\mathbb{R}) \text{ and unitary } B^\top = B^{-1}. \tag{1.118}$$

Examples of linear operators (1.104) satisfying (1.109) and (1.118) include

$$\mathbb{T} = \mathbb{T}A := (\mathrm{Tr}\, A)I \quad \text{and} \quad \mathbb{T}A := A + A^\top, \tag{1.119}$$

where I denotes the identity. The incisive step in the direction of identifying all such operators is the observation that any other operator of the type is a linear combination of these two. Namely, we have the following.

Lemma 1.23 *Let a linear operator $\mathbb{T}$ in* (1.104) *be frame indifferent (cf.* (1.118)*)*

$$\mathbb{T}(BAB^\top) = B(\mathbb{T}A)B^\top, \quad \textit{for all} \quad A \in \mathbb{M}^{3\times 3} \quad \textit{and for all orthogonal} \quad B \in \mathbb{SO}(3)$$

and have the symmetry property: one of conditions in (1.109) *holds.*

Then $\mathbb{T}$ has the form

$$\mathbb{T}A = \lambda\,(\mathrm{Tr}\, A)I + \mu\,(A + A^\top), \qquad A \in \mathbb{M}_{n,n}(\mathbb{R}), \tag{1.120}$$

where $\lambda, \mu \in \mathbb{R}$ are some constants and it has both symmetry properties from (1.109).

Proof: Let us first show that any linear operator (1.104) satisfying (1.109), (1.118) is represented in the form (1.120). By the previous discussion (cf. (1.119)), it suffices to prove that the space of linear operators (1.104) satisfying (1.109), (1.118) has dimension two.

It suffices to show that

$$\mathbb{T}D = aD + b(I - D) \quad \text{where} \quad D := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{1.121}$$

for the identity matrix I and two numbers $a, b \in \mathbb{R}$. Indeed, consider the following types of unitary matrices:

$$U_{j,k} := \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ & & \dots & \dots & \dots & & \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ & & \dots & \dots & \dots & & \\ 0 & 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 \end{bmatrix}, \quad W_{j,k} := \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ & & \dots & \dots & \dots & & \\ 0 & \dots & -1 & \dots & 0 & \dots & 0 \\ & & \dots & \dots & \dots & & \\ 0 & 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & \dots & \dots & 1 \end{bmatrix}$$

where the only non-zero, off the diagonal entries are at (j, k) and (k, j). By multiplication $U_{j,k}A$ exchanges j-th with k-th rows in A, while $W_{j,k}A$, $j < k$, makes the same but changes the sign of j-th row before shifting it to k-th row.

By applying the unitary operator $U_{1,k}$, we get

$$\begin{aligned} \mathbb{T}E &= \sum_{j=1}^{n} \boldsymbol{e}^k \mathbb{T} U_{1,k} D U_{1,k}^{-1} = \sum_{j=1}^{n} \boldsymbol{e}^k U_{1,k} (\mathbb{T}D) U_{1,k}^{-1} \\ &= \sum_{j=1}^{n} \boldsymbol{e}^k U_{1,k} [aD - b(I - D)] U_{1,k}^{-1} = aE + b(I - E) \end{aligned} \tag{1.122}$$

for arbitrary diagonal matrix $E = [\delta_{jk}\boldsymbol{e}^k] = \sum_{j=1}^{n} \boldsymbol{e}^k U_{1,k} D U_{1,k}^{-1}$. Since for any $A \in \mathbb{M}^{n\times n}(\mathbb{R})$ we have $\mathbb{T}A = \frac{1}{2}\mathbb{T}(A + A^\top)$, thanks to (1.109), and since a self adjoint matrix can be diagonalized $\frac{1}{2}(A + A^\top) = UEU^{-1}$ with a suitable unitary matrix $\boldsymbol{U}$, the equality (1.122) holds for arbitrary A:

$$\mathbb{T}A = \mathbb{T}UEU^{-1} = \boldsymbol{U}(\mathbb{T}E)U^{-1} = \boldsymbol{U}[aE + b(I - E)]U^{-1} = aA + b(I - A)\,.$$

To check (1.121) we again apply the unitary matrices U_{i_o,j_o} and W_{i_o,j_o}. Set

$$A := \mathbb{T}D, \qquad A = \left[a_{ij}\right]_{1\leqslant i,j\leqslant n}$$

and observe that D is invariant under conjugation by W_{i_o,j_o}, i.e. $W_{i_o,j_o} D W_{i_o,j_o}^\top = D$, as long as $i_o \neq 1$ and $j_o \neq 1$. Thus, by (1.118), the same is true for $A = \mathbb{T}D$ which, in turn, eventually implies that

$$a_{i_o i_o} = a_{j_o j_o}, \qquad \forall\, i_o, j_o \neq 1. \tag{1.123}$$

The next observation is that D is invariant under conjugation by the product $U_{i_o j_o} W_{i_o,j_o}$, i.e. $U_{i_o j_o} W_{i_o,j_o} D W_{i_o,j_o}^\top U_{i_o j_o}^\top = D$, this time for every $1 \leqslant i_o \neq j_o \leqslant n$. Hence, by (1.118), the same holds for $A = \mathbb{T}D$, which ultimately implies that $a_{i_o j_o} = -a_{j_o i_o}$ for every pair of indices $1 \leqslant i_o \neq j_o \leqslant n$. Consequently,

$$a_{i_o j_o} = 0, \quad \text{for every } \ 1 \leqslant i_o \neq j_o \leqslant n. \tag{1.124}$$

Under the current assumptions, i.e. (1.118), the first condition in (1.109), the desired conclusion, i.e. that $\mathbb{T}D$ has the two-parameter diagonal form indicated above, now follows readily from (1.123) and (1.124).

Let us analyze the case when the linear operator $\mathbb{T}$ satisfies (1.118) along with the second condition in (1.109). In this situation, let us consider the adjoint $\mathbb{T}^*$ to the tensor $\mathbb{T}$ with respect to the inner product (1.105) $\langle \mathbb{T}A, B\rangle = \langle A, \mathbb{T}^*B\rangle$. It can be readily checked that the adjoint $\mathbb{T}^*$ satisfies (1.118) and the first condition in (1.109), so the previous reasoning applies. Consequently, $\mathbb{T}^*$ can be represented in the form (1.120), which is invariant under the adjunction. Hence $\mathbb{T}$ can be written in the form (1.120) also. In particular, (1.120) holds in this case as well.

Concerning the equivalence of the first and the second condition in (1.109). Each of two conditions in (1.109) along with the condition (1.118) imply that the linear operator (1.104) has the form (1.120). Then, in particular, $\mathbb{T}$ is self adjoint. Since conditions in (1.109) are obtained by taking the adjoint, they are equivalent and the proof is completed. ■

Remark 1.24 *A posteriori, the conditions* (1.109) *and* (1.118) *imply that the linear operator* (1.104) *has the form* (1.120) *and, in particular, is self adjoint, i.e., imply the condition* (1.107).

Remark 1.25 *The above proof can be modified to hold in the case when* (1.118) *is (seemingly) weakened to allow only orientation preserving unitary matrices $\boldsymbol{U}$. All one has to do in this later case is to employ the invariance of D under conjugation by $U_{k_o\ell_o}U_{i_oj_o}W_{i_o,j_o}$ (with $k_o, \ell_o \neq 1$), in place of conjugation by (the inversion) $U_{i_oj_o}W_{i_o,j_o}$ as in the original proof.*

We are now ready to derive the Lamé equations of elasticity on a hypersurface.

Theorem 1.26 *On a smooth hypersurface $\mathscr{S}$ in $\mathbb{R}^n$, modeling a homogeneous, linear, isotropic, elastic medium, the Lamé operator $\mathscr{L}_{\mathscr{S}}$ is given by*

$$\mathscr{L}_{\mathscr{S}} = -\lambda\,\nabla_{\mathscr{S}}\operatorname{div}_{\mathscr{S}} + 2\mu\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}} = \lambda\operatorname{div}^*_{\mathscr{S}}\operatorname{div}_{\mathscr{S}} + 2\mu\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}. \tag{1.125}$$

In particular, $\mathscr{L}_{\mathscr{S}}$ is a formally self-adjoint differential operator of second order.

Proof: According to the discussion in the first part of this section, the elasticity tensor in the case of linear, isotropic, elastic medium is given by (1.120), where λ, μ are the Lamé moduli. Applying the following properties of the trace

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)\,, \qquad \mathrm{Tr}(A^\top) = \mathrm{Tr}(A)\,,$$
$$\langle A+A^\top, A\rangle = Tr\big[(A+A^\top)A^\top\big] = \frac{1}{2}Tr\big[A^2 + 2AA^\top + (A^\top)^2\big]^2 = \frac{1}{2}Tr(A+A^\top)^2,$$

which are easy to verify directly, due to (1.103) the stored energy density is of the form

$$\begin{aligned} E(A) = \langle \mathbb{T}A, A\rangle &= \langle \lambda\mathrm{Tr}(A)I + \mu(A+A^\top), A\rangle = \lambda\mathrm{Tr}(A)\langle I, A\rangle + \mu\langle A+A^\top, A\rangle \\ &= \lambda\,(\mathrm{Tr}\,A)^2 + \frac{\mu}{2}\,\mathrm{Tr}\,((A+A^\top)^2)\,. \end{aligned} \tag{1.126}$$

Further, by inserting $A := \mathscr{D}^{\mathscr{S}}\boldsymbol{U}$ in (1.126) and recalling (1.49), we get

$$E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)) = \lambda\,(\operatorname{div}_{\mathscr{S}}\boldsymbol{U})^2(x) + 2\mu\,\langle(\mathrm{Def}_{\mathscr{S}}\boldsymbol{U})(x), (\mathrm{Def}_{\mathscr{S}}\boldsymbol{U})(x)\rangle \tag{1.127}$$

by (1.111) and since the trace

$$\mathrm{Tr}(\nabla_{\mathscr{S}}\boldsymbol{U}) = \sum_{j=1}^{n}\langle\partial_{h_j}\boldsymbol{U}, h_j\rangle = \mathrm{Div}_{\mathscr{S}}\boldsymbol{U} \tag{1.128}$$

is the divergence and is independent of a basis $\{h_j\}_{j=1}^{n}$. Thus, we are led to considering the variational integral

$$\mathscr{E}[\boldsymbol{U}] = \int_{\mathscr{S}}\Big[\lambda\,(\mathrm{div}_{\mathscr{S}}\,\boldsymbol{U})^2 + 2\mu\,\langle\mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U}, \mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U}\rangle\Big]\,d\sigma, \qquad \boldsymbol{U}\in\omega(\mathscr{S}). \tag{1.129}$$

To determine the associated Euler-Lagrange equation, for a smooth and compactly supported vector field $\boldsymbol{V}\in\omega(\mathscr{S})\bigcap C_0^1(\mathscr{S})$ we compute

$$\frac{d}{dt}\mathscr{E}[\boldsymbol{U}+tV]\Big|_{t=0} = 2\int_{\mathscr{S}}\Big[\lambda\,\mathrm{div}_{\mathscr{S}}\boldsymbol{U}\,\mathrm{div}_{\mathscr{S}}\boldsymbol{V} + 2\mu\,\langle\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}, \mathrm{Def}_{\mathscr{S}}\boldsymbol{V}\rangle\Big]\,d\sigma$$

By applying the Gaußtheorem on the divergence $\mathrm{div}_{\mathscr{S}}$

$$\int_{\Omega}\mathrm{div}\,F(y)\,dy = \oint_{\mathscr{S}}\langle\boldsymbol{\nu}(\tau), F(\tau)\rangle\,d\sigma \tag{1.130}$$

and taking into account that $\boldsymbol{V}$ vanishes on the boundary $\Gamma = \partial\mathscr{S}$ we get

$$\begin{aligned}\frac{d}{dt}\mathscr{E}[\boldsymbol{U}+tV]\Big|_{t=0} &= 2\int_{\mathscr{S}}\langle(-\lambda\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}} + 2\mu\,\mathrm{Def}^{*}_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}})\boldsymbol{U}, \boldsymbol{V}\rangle\,d\sigma\\ &= 2\int_{\mathscr{S}}\langle\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle\,d\sigma = 0\,.\end{aligned} \tag{1.131}$$

Since the vector field $\boldsymbol{V}\in\omega(\mathscr{S})\bigcap C_0^1(\mathscr{S})$ is arbitrary, from (1.131) follows that the displacement vector field $\boldsymbol{U}$ satisfies the equality $\mathscr{L}_{\mathscr{S}}\boldsymbol{U} = 0$.

That the operator $\mathscr{L}_{\mathscr{S}} = \lambda\,\mathrm{div}^{*}_{\mathscr{S}}\,\mathrm{div}_{\mathscr{S}} + 2\mu\,\mathrm{Def}^{*}_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}$ is formally self adjoint, is obvious from its structure:

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V})_{\mathscr{S}} = \lambda(\mathrm{div}^{*}_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V})_{\mathscr{S}} + \mu(\mathrm{Def}^{*}_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V})_{\mathscr{S}} = (\boldsymbol{U}, \mathscr{L}_{\mathscr{S}}\boldsymbol{V})_{\mathscr{S}}. \qquad \blacksquare$$

1.4 The surface Lamé operator and related PDO's

The present section deals mostly with the identification of the deformation tensor

$$\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})(\boldsymbol{V}, \boldsymbol{W}) := \frac{1}{2}\{\langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U}, \boldsymbol{W}\rangle + \langle\partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U}, \boldsymbol{V}\rangle\}, \qquad \forall\,\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\in\omega(\mathscr{S}), \tag{1.132}$$

and the Lamé operator (1.125).

Theorem 1.27 *For the deformation tensor and the Lamé operator on $\mathscr{S}$ the following identities are valid:*

$$\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := \big[\mathfrak{D}_{jk}(\boldsymbol{U})\big]_{n\times n}\,, \tag{1.133}$$

$$\mathfrak{D}_{jk}(\boldsymbol{U}) = \frac{1}{2}\Big[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k + (\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})_j\Big] = \frac{1}{2}\Big[\mathscr{D}_jU_k + \mathscr{D}_kU_j + \partial_{\boldsymbol{U}}\big(\nu_j\nu_k\big)\Big]\,, \tag{1.134}$$

$$[\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})]^\top = \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \quad \textit{and} \quad \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{\nu} = 0\,, \tag{1.135}$$

$$\begin{aligned}\mathscr{L}_{\mathscr{S}} &= \mu\,\pi_{\mathscr{S}}\nabla^*_{\mathscr{S}}\,\nabla_{\mathscr{S}} + (\lambda+\mu)\,\nabla_{\mathscr{S}}\,\nabla^*_{\mathscr{S}} - \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\\ &= -\mu\,\boldsymbol{\Delta}_{\mathscr{S}} - (\lambda+\mu)\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}} - \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\,.\end{aligned} \tag{1.136}$$

Proof: Given the local nature of the identities we seek to prove, it suffices to work locally, in a small open subset $\mathscr{O}$ of $\mathscr{S}$, where an orthonormal basis $T_1,\dots,T_{n-1}$ to $\omega(\mathscr{S})$ has been fixed. We extend the basis by the outer unit normal vector field $T_n := \boldsymbol{\nu}$ so that $\{T_j\}_{1\leqslant j\leqslant n}$ becomes an orthonormal basis for $\mathbb{R}^n$, at points in $\mathscr{O}$.

Since $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})$ is a linear operator (see (1.132)) it is represented by an $n\times n$ matrix in the fixed basis $\{T_j\}_{1\leqslant j\leqslant n}$ and the first equality in (1.133) follows. The symmetry property of the matrix, recorded as the first equality in (1.135), follows from (1.132) since by interchanging vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ does not affect the definition (1.132).

For a tangential field $\boldsymbol{U}$ to $\mathscr{S}$ with $\mathrm{supp}\,\boldsymbol{U}\subset\mathscr{O}$ and arbitrary $\boldsymbol{V},\boldsymbol{W}\in\mathbb{R}^n$ we have

$$\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U} = \partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}}\boldsymbol{U}\,,\qquad \langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle = \langle\partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}}\boldsymbol{U},\pi_{\mathscr{S}}\boldsymbol{W}\rangle$$

and, by the definition of the deformation tensor (cf. (1.132)) obtain

$$\langle\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{V},\boldsymbol{W}\rangle := \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})(\pi_{\mathscr{S}}\boldsymbol{V},\pi_{\mathscr{S}}\boldsymbol{W}),\qquad \forall\,\boldsymbol{V},\boldsymbol{W}\in\mathbb{R}^n. \tag{1.137}$$

Equality (1.137) implies the second equality in (1.135). Applying (1.111) and (1.99) we eventually obtain the second equality in (1.133):

$$\begin{aligned}\mathfrak{D}_{jk}(\boldsymbol{U}) &= \frac{1}{2}\Big[\big(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U}\big)_j + \big(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U}\big)_k\Big] = \frac{1}{2}\Big[\mathscr{D}_kU_j + \mathscr{D}_jU_k + \partial_{\boldsymbol{U}}\big(\nu_j\nu_k\big)\Big]\\ &= \frac{1}{2}\Big[\mathscr{D}_kU_j + \mathscr{D}_jU_k + \sum_{r=1}^n U_r(\mathscr{D}_r\nu_k)\nu_j + \sum_{r=1}^n U_r(\mathscr{D}_r\nu_j)\nu_k\Big]\,.\end{aligned}$$

We proceed with the proof of the last remaining equality (1.136). If $\boldsymbol{V}$ is also a smooth vector field, tangential to $\mathscr{S}$, applying (1.133) we get

$$\begin{aligned}&\int_{\mathscr{S}}\langle\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\boldsymbol{V}\rangle\,d\sigma = \int_{\mathscr{S}}\langle\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\mathrm{Def}_{\mathscr{S}}(\boldsymbol{V})\rangle\,d\sigma\\ &\qquad = \sum_{j,k=1}^n\frac{1}{4}\int_{\mathscr{S}}\Big[\mathscr{D}_kU_j + \mathscr{D}_jU_k + \partial_{\boldsymbol{U}}(\nu_j\nu_k)\Big]\Big[\mathscr{D}_kV_j + \mathscr{D}_jV_k + \partial_{\boldsymbol{V}}(\nu_j\nu_k)\Big]\,d\sigma\,.\end{aligned} \tag{1.138}$$

Next consider

$$\sum_{j,k=1}^{n}\int_{\mathscr S}(\mathscr D_jU_k+\mathscr D_kU_j)(\mathscr D_jV_k+\mathscr D_kV_j)\,d\sigma=2\sum_{j,k=1}^{n}\int_{\mathscr S}\mathscr D_j^*(\mathscr D_jU_k+\mathscr D_kU_j)V_k\,d\sigma$$

$$=2\sum_{j,k=1}^{n}\int_{\mathscr S}\Big[-V_k\mathscr D_j^2U_k-V_k\mathscr D_j\mathscr D_kU_j-\mathscr H^0_{\mathscr S}\nu_j(\mathscr D_jU_k)V_k-\mathscr H^0_{\mathscr S}\nu_j(\mathscr D_kU_j)V_k\Big]\,d\sigma$$

$$=-2\int_{\mathscr S}\langle\boldsymbol\Delta_{\mathscr S}\boldsymbol U,\boldsymbol V\rangle\,d\sigma-2\sum_{j,k=1}^{n}\int_{\mathscr S}\Big[V_k\mathscr D_j\mathscr D_kU_j+\mathscr H^0_{\mathscr S}\nu_j(\mathscr D_kU_j)V_k\Big]\,d\sigma\,,\quad(1.139)$$

since $\sum\limits_{j=1}^{n}\nu_j\mathscr D_j=0$ on $\mathscr S$.

To proceed in the second integrand in (1.139): we employ the commutator identity from Lemma 1.11.ix and recall that the fields $\boldsymbol U$ and $\boldsymbol V$ are tangential to write

$$\sum_{j,k=1}^{n}\int_{\mathscr S}V_k\mathscr D_j\mathscr D_kU_j\,d\sigma=\sum_{j,k=1}^{n}\int_{\mathscr S}\Big[V_k\mathscr D_k\mathscr D_jU_j+V_k[\mathscr D_j,\mathscr D_k]U_j\Big]d\sigma$$

$$=\int_{\mathscr S}\langle\nabla_{\mathscr S}\mathrm{div}_{\mathscr S}\boldsymbol U,\boldsymbol V\rangle d\sigma+\sum_{j,k,l=1}^{n}\int_{\mathscr S}\big[V_k\nu_j\mathscr D_k\nu_l-\nu_kV_k\mathscr D_j\nu_l\big]\mathscr D_lU_jdS$$

$$=\int_{\mathscr S}\langle\nabla_{\mathscr S}\mathrm{div}_{\mathscr S}\boldsymbol U,\boldsymbol V\rangle d\sigma+\sum_{j,k,l=1}^{n}\int_{\mathscr S}V_k(\mathscr D_k\nu_l)\big[\mathscr D_l(\nu_jU_j)-(\mathscr D_l\nu_j)U_j\big]d\sigma$$

$$=\int_{\mathscr S}\langle\nabla_{\mathscr S}\mathrm{div}_{\mathscr S}\boldsymbol U,\boldsymbol V\rangle d\sigma-\sum_{j,k,l=1}^{n}\int_{\mathscr S}(\partial_k\nu_l)(\partial_l\nu_j)U_jV_k\,d\sigma$$

$$=\int_{\mathscr S}\langle\nabla_{\mathscr S}\mathrm{div}_{\mathscr S}\boldsymbol U,\boldsymbol V\rangle\,d\sigma-\int_{\mathscr S}\langle\mathscr W^2_{\mathscr S}\boldsymbol U,\boldsymbol V\rangle\,d\sigma\qquad(1.140)$$

on $\mathscr S$, because $\sum\limits_{j=1}^{n}\nu_jU_j=\sum\limits_{k=1}^{n}\nu_kV_k=0$ and, due to (1.36)

$$\sum_{j,l,k=1}^{n}(\partial_k\nu_l)(\partial_l\nu_j)U_jV_k=\sum_{j,l,k=1}^{n}(\partial_l\nu_j)U_j(\partial_j\nu_k)V_k=\langle\mathscr W_{\mathscr S}\boldsymbol U,\mathscr W_{\mathscr S}\boldsymbol V\rangle=\langle\mathscr W^2_{\mathscr S}\boldsymbol U,\boldsymbol V\rangle.$$

For the third integrand in (1.139) we use Lemma 1.7.i and that the field $\boldsymbol U$ is tangential:

$$\sum_{j,k=1}^{n}\int_{\mathscr S}\mathscr H^0_{\mathscr S}\nu_j(\mathscr D_kU_j)V_k\,d\sigma=\mathscr H^0_{\mathscr S}\sum_{j,k=1}^{n}\int_{\mathscr S}V_k\left[\mathscr D_k\big(\nu_jU_j\big)-\big(\mathscr D_k\nu_j\big)U_j\right]\,d\sigma$$

$$=\int_{\mathscr S}\mathscr H^0_{\mathscr S}\langle\mathscr W_{\mathscr S}\boldsymbol U,\boldsymbol V\rangle\,d\sigma.\quad(1.141)$$

At this point, we may therefore conclude that

$$\sum_{j,k=1}^{n}\int_{\mathscr{S}}(\mathscr{D}_jU_k+\mathscr{D}_kU_j)(\mathscr{D}_jV_k+\mathscr{D}_kV_j)\,d\sigma$$
$$=2\int_{\mathscr{S}}\langle-\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}+\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U}-\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma. \tag{1.142}$$

We now proceed to analyze the remaining terms in (1.138). More precisely, we still have to take into account the terms containing either $\partial_{\boldsymbol{U}}(\nu_j\nu_k)$ or $\partial_{\boldsymbol{V}}(\nu_j\nu_k)$. We start with the identity

$$\sum_{j,k=1}^{n}(\mathscr{D}_kU_j)\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)=\sum_{j,k=1}^{n}\nu_k(\mathscr{D}_kU_j)\mathscr{D}_{\boldsymbol{V}}\nu_j+\sum_{j,k=1}^{n}(\mathscr{D}_{\boldsymbol{V}}\nu_k)\big[\mathscr{D}_k\big(\nu_jU_j\big)-U_j\mathscr{D}_k\nu_j\big]$$
$$=-\sum_{k,j=1}^{n}(\mathscr{D}_{\boldsymbol{V}}\nu_k)(\mathscr{D}_{\boldsymbol{U}}\nu_k)=-\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle, \tag{1.143}$$

valid at points on $\mathscr{S}$, because $\sum_k\nu_k\mathscr{D}_k=0$, $\sum_j\nu_jU_j=0$ and $\mathscr{D}_k\nu_j=\mathscr{D}_j\nu_k$. There are four such terms in (1.138), i.e. containing either $\mathscr{D}_{\boldsymbol{U}}(\nu_j\nu_k)$ or $\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)$. An inspection of the above calculation shows that, on $\mathscr{S}$, they are all equal to $-\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle$. We still have to compute the last integrand in (1.138):

$$\sum_{j,k=1}^{n}\mathscr{D}_{\boldsymbol{U}}(\nu_j\nu_k)\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)$$
$$=\sum_{j,k,r,l=1}^{n}\Big[U_r(\mathscr{D}_r\nu_j)\nu_k+U_r(\mathscr{D}_r\nu_k)\nu_j\Big]\Big[V_l(\mathscr{D}_l\nu_j)\nu_k+V_l(\mathscr{D}_l\nu_k)\nu_j\Big]$$
$$=2\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle+2\sum_{k,r,l=1}^{n}U_r(\mathscr{D}_r\nu_k)V_l\nu_k\frac{1}{2}\mathscr{D}_l\Big(\sum_{j=1}^{n}(\nu_j)^2\Big)=2\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle,$$

on $\mathscr{S}$. At this point we combine all the above to get

$$4\sum_{j,k=1}^{n}\int_{\mathscr{S}}\mathfrak{D}_{jk}(\boldsymbol{U})\mathfrak{D}_{jk}(\boldsymbol{V})\,d\sigma=2\int_{\mathscr{S}}\langle-\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}-\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma\,. \tag{1.144}$$

Having deduced (1.144), we may now compute

$$4\int_{\mathscr{S}}\langle\mathrm{Def}_{\mathscr{S}}^*\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\boldsymbol{V}\rangle\,d\sigma=\int_{\mathscr{S}}\langle\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\mathrm{Def}_{\mathscr{S}}(\boldsymbol{V})\rangle\,d\sigma$$
$$=4\sum_{j,k=1}^{n}\int_{\mathscr{S}}\mathfrak{D}_{jk}(\boldsymbol{U})\mathfrak{D}_{jk}(\boldsymbol{V})\,d\sigma=2\int_{\mathscr{S}}\langle-\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}-\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}\boldsymbol{U}\,,\,\boldsymbol{V}\rangle\,d\sigma$$
$$=2\int_{\mathscr{S}}\langle-\pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}-\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}\boldsymbol{U}\,,\,\boldsymbol{V}\rangle\,d\sigma, \tag{1.145}$$

since $\langle \boldsymbol{W}, \boldsymbol{V}\rangle = \langle \pi_{\mathscr{S}} \boldsymbol{W}, \boldsymbol{V}\rangle$ for a tangential vector field $\boldsymbol{V}$ and an arbitrary vector field $\boldsymbol{W}$ (also note that the original operator $\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}} : \omega(\mathscr{S}) \to \omega(\mathscr{S})$ is tangential). We have also applied that the vector $\mathscr{W}_{\mathscr{S}}\boldsymbol{W} \in \omega(\mathscr{S})$ is tangential or an arbitrary vector field $\boldsymbol{W}$. Thus,

$$4\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}} = -2\pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}} - 2\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}} - 2\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}, \tag{1.146}$$

since the tangential vectors fields $\boldsymbol{U}$, $\boldsymbol{V}$ are arbitrary.

The first identity in (1.136) now follows easily from (1.146) and (1.125). The remaining identity in (1.136) then follows from what we have just proved and from Theorem 1.10. ■

Next recall the definition of the Hodge-Laplacian acting on 1-forms, i.e.

$$\Delta_{HL} := -\boldsymbol{d}^{\mathscr{S}}\boldsymbol{d}^*_{\mathscr{S}} - \boldsymbol{d}^*_{\mathscr{S}}\boldsymbol{d}^{\mathscr{S}} : \Lambda^1\omega(\mathscr{S}) \longrightarrow \Lambda^1\omega(\mathscr{S}) \tag{1.147}$$

where $\boldsymbol{d}^{\mathscr{S}}$ is the exterior derivative operator on $\mathscr{S}$, and $\boldsymbol{d}^*_{\mathscr{S}}$ its formal adjoint. As explained in §2, 1-forms on $\mathscr{S}$ are naturally identified with tangential fields to $\mathscr{S}$ so, from now on, we shall think of Δ_{HL} as mapping $\omega(\mathscr{S})$ into itself.

As pointed out in §2, the Hodge-Laplacian (1.147) is related to the Bochner-Laplacian on $\mathscr{S}$

$$\Delta_{BL} := -(\nabla^{\mathscr{S}})^*\nabla^{\mathscr{S}}, \tag{1.148}$$

via the Weitzenbock identity

$$\Delta_{BL} = \Delta_{HL} + \mathrm{Ric}_{\mathscr{S}}. \tag{1.149}$$

Our aim is to find alternative expressions for all these objects, starting with the Ricci tensor.

The Ricci curvature $\mathrm{Ric}_{\mathscr{S}}$ on $\mathscr{S}$ is a $(0,2)$-tensor defined as a contraction of $\boldsymbol{R}_{\mathscr{S}}$:

$$\mathrm{Ric}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) := \sum_{j=1}^{n}\langle \boldsymbol{R}_{\mathscr{S}}(h_j,\boldsymbol{V})\boldsymbol{U}, h_j\rangle = \sum_{j=1}^{n}\langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{V},h_j)h_j,\boldsymbol{U}\rangle, \qquad \forall\ \boldsymbol{U},\boldsymbol{V}\in\omega(\mathscr{S}), \tag{1.150}$$

where $h_1,\dots,h_n$ is an orthonormal basis (of unit vectors) in $\omega(\mathscr{S})$. Thus, $\mathrm{Ric}_{\mathscr{S}}$ is a symmetric bilinear form.

Theorem 1.28 *For the Ricci tensor* $\mathrm{Ric}_{\mathscr{S}}$ *(cf.* (1.150)*) on* $\mathscr{S}$ *there holds*

$$\mathrm{Ric}_{\mathscr{S}} = -\mathscr{W}^2_{\mathscr{S}} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}. \tag{1.151}$$

In particular, when $n=3$ *–i.e. for a two dimensional hypersurface* $\mathscr{S}$ *in* $\mathbb{R}^3$*– the above identity reduces to*

$$\mathrm{Ric}_{\mathscr{S}} = -\det\mathscr{W}_{\mathscr{S}} = -\mathscr{K}_{\mathscr{S}}, \tag{1.152}$$

where $\mathscr{K}_{\mathscr{S}}$ *is the Gaußian curvature of the hypersurface* $\mathscr{S}$*.*

Proof: The Riemannian curvature tensor $\boldsymbol{R}_{\mathscr{S}}$ of $\mathscr{S}$ is given by

$$\boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W} = [\partial^{\mathscr{S}}_{\boldsymbol{U}},\partial^{\mathscr{S}}_{\boldsymbol{V}}]\boldsymbol{W} - \partial^{\mathscr{S}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{W}, \qquad \boldsymbol{U},\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}), \tag{1.153}$$

where $[\boldsymbol{U},\boldsymbol{V}] := \partial_{\boldsymbol{U}}\boldsymbol{Y} - \partial_{\boldsymbol{Y}}\boldsymbol{U}$ is the usual commutator bracket. It is convenient to change this into a $(0,4)$-tensor by setting

$$\boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z}) := \langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle, \qquad \boldsymbol{U},\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z}\in\omega(\mathscr{S}). \tag{1.154}$$

Since $\mathbb{R}^n$ has zero curvature, it follows from Gauß's Theorema Egregium that, if $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$, $\boldsymbol{W}$ are tangential vector fields to $\mathscr{S}$, then

$$\langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle = \langle II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{Z}), II_{\mathscr{S}}(\boldsymbol{V},\boldsymbol{W})\rangle - \langle II_{\mathscr{S}}(\boldsymbol{V},\boldsymbol{Z}), II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{W})\rangle \tag{1.155}$$

(see, e.g., [Ta2], Vol. II, p. 481). By inserting the second fundamental form $II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) = \langle \partial_{\boldsymbol{U}}\boldsymbol{V} - \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V}, \boldsymbol{\nu}\rangle = \langle \partial_{\boldsymbol{U}}\boldsymbol{V}, \boldsymbol{\nu}\rangle$ (cf. (1.33)), we obtain:

$$\begin{aligned}\langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle &= \langle \partial_{\boldsymbol{U}}\boldsymbol{Z},\boldsymbol{\nu}\rangle\langle \partial_{\boldsymbol{V}}\boldsymbol{W},\boldsymbol{\nu}\rangle - \langle \partial_{\boldsymbol{V}}\boldsymbol{Z},\boldsymbol{\nu}\rangle\langle \partial_{\boldsymbol{U}}\boldsymbol{W},\boldsymbol{\nu}\rangle\\ &= \langle \boldsymbol{Z},\partial_{\boldsymbol{U}}\boldsymbol{\nu}\rangle\langle \boldsymbol{W},\partial_{\boldsymbol{V}}\boldsymbol{\nu}\rangle - \langle \boldsymbol{Z},\partial_{\boldsymbol{V}}\boldsymbol{\nu}\rangle\langle \boldsymbol{W},\partial_{\boldsymbol{U}}\boldsymbol{\nu}\rangle\\ &= \langle \boldsymbol{R}_{\mathscr{S}}\boldsymbol{Z},\boldsymbol{U}\rangle\langle \boldsymbol{R}_{\mathscr{S}}\boldsymbol{W},\boldsymbol{V}\rangle - \langle \boldsymbol{R}_{\mathscr{S}}\boldsymbol{Z},\boldsymbol{V}\rangle\langle \boldsymbol{R}_{\mathscr{S}}\boldsymbol{W},\boldsymbol{U}\rangle.\end{aligned} \tag{1.156}$$

For the second equality in (1.156) we have used the fact that $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$, and $\boldsymbol{Z}$ are tangential, so in particular, $\partial_{\boldsymbol{U}}\langle \boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, $\partial_{\boldsymbol{V}}\langle \boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, $\partial_{\boldsymbol{V}}\langle \boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, and $\partial_{\boldsymbol{U}}\langle \boldsymbol{W},\boldsymbol{\nu}\rangle = 0$ on $\mathscr{S}$.

Next, recall from (1.150) the definition of the Ricci tensor, i.e.

$$\mathrm{Ric}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) = \sum_{j=1}^{n-1}\langle \boldsymbol{R}_{\mathscr{S}}(h_j,\boldsymbol{V})\boldsymbol{U},h_j\rangle,$$

where $h_1,\dots,h_{n-1}$ is, locally, an orthonormal basis in $\omega(\mathscr{S})$, and $\boldsymbol{U}$, $\boldsymbol{V}$ are arbitrary tangential vector fields to $\mathscr{S}$. If we set $h_n := \boldsymbol{\nu}$, and employ (1.156) together with $\mathscr{W}_{\mathscr{S}}\boldsymbol{\nu} = 0$, we obtain

$$\sum_{j=1}^{n-1}\langle \boldsymbol{R}_{\mathscr{S}}(T_j,\boldsymbol{V})\boldsymbol{U},T_j\rangle = \sum_{j=1}^{n}[\langle \boldsymbol{R}_{\mathscr{S}}T_j,T_j\rangle\langle \boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle - \langle \boldsymbol{R}_{\mathscr{S}}T_j,\boldsymbol{V}\rangle\langle \boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},T_j\rangle]$$

$$= -\mathscr{H}^0_{\mathscr{S}}\langle \boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle - \langle \boldsymbol{R}_{\mathscr{S}}\boldsymbol{V},\sum_{j=1}^{n}\langle T_j,\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U}\rangle T_j\rangle - \langle(\mathscr{W}^2_{\mathscr{S}} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}})\boldsymbol{U},\boldsymbol{V}\rangle, \tag{1.157}$$

which takes care of (1.151).

Finally, (1.152) is a consequence of what we have proved so far, in Lemma 1.8.ii, and the elementary identity $A^2 - (\mathrm{Tr}\,A)A = -(\det A)I$, valid for any 2×2 matrix A. ∎

Lemma 1.29 *Let $H := \{h_j\}_{j=1}^n$, $|h_j| = 1$, be a basis in n-dimensional Banach space $\mathfrak{B}$. Consider the hyperspace $\mathfrak{B}_{\boldsymbol{\nu}} := \{u\in\mathfrak{B} : \langle u,\boldsymbol{\nu}\rangle = 0\}$ orthogonal to some vector $\boldsymbol{\nu}\in\mathfrak{B}$, $|\boldsymbol{\nu}|\neq 0$. Consider the system*

$$\hat{h}_j := h_j - \nu_j\boldsymbol{\nu}\,,\quad \nu_j := \langle\boldsymbol{\nu},h_j\rangle \qquad j = 1,\dots,n\,, \tag{1.158}$$

which is full in $\mathfrak{B}_{\boldsymbol{\nu}}$ but **linearly dependent** *and thus can not be a basis. Nevertheless, for a linear operator $A = [a_{jk}]_{n\times n} : \mathfrak{B}\to\mathfrak{B}$ with $A\boldsymbol{\nu} = 0$ and $A\mathfrak{B}_{\boldsymbol{\nu}}\subset\mathfrak{B}_{\boldsymbol{\nu}}$ (i.e., it maps $A : \mathfrak{B}_{\boldsymbol{\nu}}\to\mathfrak{B}$) we have*

$$\widehat{A} := [\widehat{a}_{jk}]_{n\times n} = [a_{jk}]_{n\times n} := A\,, \tag{1.159}$$

where $\widehat{A} := [\widehat{a}_{jk}]_{n\times n}$ is the matrix representations of A in the systems $\widehat{H} := \{\hat{h}_j\}_{j=1}^n\subset\mathfrak{B}_{\boldsymbol{\nu}}$.

Proof: Let us note that

$$\sum_{k=1}^{n} a_{jk}\nu_k = \sum_{k=1}^{n} a_{kj}\nu_k = 0 \qquad \text{for all} \quad j = 1, \dots, n\,,$$

where the first equality is equivalent to $A\boldsymbol{\nu} = 0$ and the second one-to $\langle \boldsymbol{\nu}, A\xi \rangle = 0$ for all $\xi \in \mathfrak{B}$. Applying the obtained equalities we find that

$$A\hat{h}_j = Ah_j - \nu_j A\boldsymbol{\nu} = \sum_{k=1}^{n} a_{kj}h_k = \sum_{k=1}^{n} a_{kj}\hat{h}_k + \sum_{k=1}^{n} a_{kj}\nu_k\boldsymbol{\nu} = \sum_{k=1}^{n} a_{kj}\hat{h}_k$$

which entails $\widetilde{a}_{kj} = a_{kj}$. ■

Theorem 1.30 *The following identities are valid:*

$$\Delta_{BL} = \pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}} + \mathscr{W}_{\mathscr{S}}^2, \tag{1.160}$$

$$\Delta_{HL} = \pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}} + 2\mathscr{W}_{\mathscr{S}}^2 - \mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}. \tag{1.161}$$

Proof: In order to identify the Bochner-Laplacian operator Δ_{BL} on $\mathscr{S}$ we observe that, with tangential field $\boldsymbol{U}$ fixed, if the matrix $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})$ satisfies $\langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{V}, \boldsymbol{W} \rangle = \langle \partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}}\boldsymbol{U}, \pi_{\mathscr{S}}\boldsymbol{W} \rangle$, for each $\boldsymbol{V}, \boldsymbol{W} \in \mathbb{R}^n$ then, much as in the proof of Theorem 1.10

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{e}^k, \boldsymbol{e}^j \rangle = \langle \partial_{\bar{e}_k}\boldsymbol{U}, \bar{e}_j \rangle = \mathscr{D}_k U_j - \sum_{r=1}^{n} \nu_j\nu_r\mathscr{D}_k(U_r). \tag{1.162}$$

On account of this we can now write

$$\begin{aligned}
&\int_{\mathscr{S}} \langle (\nabla^{\mathscr{S}})^*\nabla^{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V} \rangle\, d\sigma = \int_{\mathscr{S}} \langle \nabla^{\mathscr{S}}\boldsymbol{U}, \nabla^{\mathscr{S}}\boldsymbol{V} \rangle\, d\sigma = \sum_{j,k=1}^{n-1} \int_{\mathscr{S}} \langle \nabla^{\mathscr{S}}_{T_j}\boldsymbol{U}, T_k \rangle \langle \nabla^{\mathscr{S}}_{T_j}\boldsymbol{V}, T_k \rangle\, d\sigma \\
&\quad = \sum_{j,k=1}^{n} \int_{\mathscr{S}} \langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})T_j, T_k \rangle \langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{V})T_j, T_k \rangle\, d\sigma = \sum_{j,k=1}^{n} \int_{\mathscr{S}} \mathfrak{D}_{jk}(\boldsymbol{U})\mathfrak{D}_{jk}(\boldsymbol{V})\, d\sigma \\
&\quad = \sum_{j,k=1}^{n} \int_{\mathscr{S}} \Big[\mathscr{D}_k U_j \mathscr{D}_k V_j - \sum_{r=1}^{n} \nu_j\nu_r \mathscr{D}_j U_r \mathscr{D}_k V_j - \sum_{l=1}^{n} \nu_j\nu_l \mathscr{D}_k U_j \mathscr{D}_k V_l \\
&\quad + \sum_{r,l=1}^{n} \nu_r\nu_l \mathscr{D}_k U_r \mathscr{D}_k V_l \Big]\, d\sigma = \sum_{j,k=1}^{n} \int_{\mathscr{S}} \Big[(\mathscr{D}_k^*\mathscr{D}_k U_j)V_j - \sum_{r=1}^{n} U_r V_j (\partial_k\nu_r)(\partial_k\nu_j) \Big]\, d\sigma \\
&\quad = \int_{\mathscr{S}} \langle -\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U} - \mathscr{W}_{\mathscr{S}}^2\boldsymbol{U}, \boldsymbol{V} \rangle\, d\sigma.
\end{aligned} \tag{1.163}$$

In the next-to-the-last equality, we have applied the following identity to the terms under the integral sign in the fourth line above:

$$\sum_{r=1}^{n} \nu_r\mathscr{D}_s W_r = \mathscr{D}_s\Big(\sum_{r=1}^{n} \nu_r W_r\Big) - \sum_{r=1}^{n} W_r\mathscr{D}_s\nu_r = -\sum_{r=1}^{n} W_r\partial_s\nu_r, \quad \text{on} \quad \mathscr{S}, \tag{1.164}$$

valid for any tangential vector field $\boldsymbol{W}$, and any index $s \in \{1, \dots, n\}$. In turn, the identity (1.164) can be seen from a direct computation (recall that $\partial_{\boldsymbol{\nu}}\nu_r = 0$ on $\mathscr{S}$). Finally, to justify the last equality in (1.163), it suffices to recall (1.50), (1.67) and the fact that $\sum_{k=1}^{n} \nu_k \mathscr{D}_k = 0$.

The conclusion is that (1.160) holds. Finally, the identity (1.160) in concert with (1.149) and (1.151) implies (1.161). ■

Recall now from [EM1, *Note Added in Proof*, pp.161-162], [Ta1] (cf. also the remark at the end of this paper), and [Ta2, Vol. III], that the Navier-Stokes system for a velocity field $\boldsymbol{U}$, tangential to $\mathscr{S}$, and a (scalar-valued) pressure function p on $\mathscr{S}$ reads

$$\begin{aligned} &\frac{\partial \boldsymbol{U}}{\partial t} - 2\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) + \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{U} - \nabla_{\mathscr{S}} p = f \quad \text{in} \quad \mathscr{S}\times(0,\infty),\\ &\mathrm{div}_{\mathscr{S}}\boldsymbol{U} = 0, \quad \text{in} \quad \mathscr{S}. \end{aligned} \tag{1.165}$$

If $\mathscr{S}$ is embedded in $\mathbb{R}^n$ and the Riemannian metric is inherited from $\mathbb{R}^n$, a directional derivative $\partial_{\boldsymbol{U}}$ along a tangential vector field $\boldsymbol{U} \in \omega(\mathscr{S})$ maps the space of tangential vector fields to the space of possibly non-tangential vector fields

$$\partial_{\boldsymbol{U}} \,:\, \omega(\mathscr{S}) \longrightarrow \omega(\mathscr{S})\,.$$

If composed with the projection

$$\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V} := \pi_{\mathscr{S}}\partial_{\boldsymbol{U}}\boldsymbol{V} = \partial_{\boldsymbol{U}}\boldsymbol{V} - \langle \boldsymbol{\nu}, \partial_{\boldsymbol{U}}\boldsymbol{V}\rangle\boldsymbol{\nu} \tag{1.166}$$

(cf. (0.8)), it becomes an automorphism of the space of tangential vector fields. Such derivatives are compatible with the Riemannian metric on $\mathscr{S}$ and are torsion free as well. Therefore, they represent the natural Levi-Civita connection on $\mathscr{S}$.

Theorem 1.31 *The Navier-Stokes system* (1.165) *is equivalent to*

$$\begin{aligned} &\frac{\partial \boldsymbol{U}}{\partial t} + \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{U} + \pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U} - \nabla_{\mathscr{S}} p = f \quad \textit{in} \quad \mathscr{S}\times(0,\infty)\,,\\ &\mathrm{div}_{\mathscr{S}}\boldsymbol{U} = 0 \quad \textit{in} \quad \mathscr{S}\,. \end{aligned} \tag{1.167}$$

Proof: This is a direct consequence of (1.146) and (1.166). ■

1.5 LIONS' LEMMA AND KORN'S INEQUALITIES

For $1 \leqslant p < \infty$, an integer $m = 1, 2, \dots$ and a closed C^{m+1}-smooth hypersurface $\mathscr{S}$ by $\mathbb{W}^m_p(\mathscr{S})$, $\mathbb{W}^m(\mathscr{S}) := \mathbb{W}^m_2(\mathscr{S})$ we denote the Sobolev spaces. The space $\mathbb{W}^{-m}_p(\mathscr{S})$ is defined as the dual to $\mathbb{W}^m_{p'}(\mathscr{S})$, $p' := \dfrac{p}{p-1}$, with respect to the sesquilinear form $(\varphi, \psi)_{\mathscr{S}}$ (cf. (1.64)) on functions $\varphi, \psi \in C^m(\mathscr{S})$ and extended by continuity to pairs $\varphi \in \mathbb{W}^m_{p'}(\mathscr{S})$ and $\psi \in \mathbb{W}^{-m}_p(\mathscr{S})$.

The embeddings $\mathbb{W}^m_p(\mathscr{S}) \subset \mathbb{L}_p(\mathscr{S}) \subset \mathbb{W}^{-m}_p(\mathscr{S})$ are continuous, even compact, and

$$\mathbb{W}^{-m}_p(\mathscr{S}) := \{\mathscr{D}^\alpha\varphi \,:\, \varphi \in \mathbb{L}_p(\mathscr{S}) \text{ for all } \mathscr{D}^\alpha = \mathscr{D}_1^{\alpha_1}\cdots\mathscr{D}_n^{\alpha_n}\,, \quad |\alpha| = m\}\,.$$

If $\mathscr{S}$ is an open surface with the Lipschitz boundary $\Gamma = \partial\mathscr{S} \neq \emptyset$, $\widetilde{\mathbb{W}}^m_p(\mathscr{S})$ denotes the space of functions obtained by closing the space $C_0^\infty(\mathscr{S})$ of smooth functions with compact support in the norm of $\mathbb{W}^m_p(\widetilde{\mathscr{S}})$, where $\widetilde{\mathscr{S}} \supset \mathscr{S}$ is a closed surface which extends the surface $\mathscr{S}$. The notation $\mathbb{W}^m_p(\mathscr{S})$ is used for the factor space $\mathbb{W}^m_p(\widetilde{\mathscr{S}})/\widetilde{\mathbb{W}}^m_p(\widetilde{\mathscr{S}} \setminus \mathscr{S})$; the space $\mathbb{W}^m_p(\mathscr{S})$ can also be viewed as the restriction of all functions $\varphi\big|_{\mathscr{S}}$ of the space $\mathbb{W}^m_p(\widetilde{\mathscr{S}})$ to the subsurface $\mathscr{S}$ (cf. [Tr1] and [DS1] for details about these spaces spaces).

The following generalizes essentially J. L. Lions' Lemma (cf. [DaL1, p.111], [Ta1], [AG1, Proposition 2.10], [Ci3, § 1.7], [Mc1]).

Lemma 1.32 *Let $\mathscr{S}$ be a 2-smooth closed hypersurface in $\mathbb{R}^n$. Then the inclusions $\varphi \in \mathbb{W}^{-1}_p(\mathscr{S})$, $\mathscr{D}_j\varphi \in \mathbb{W}^{-1}_p(\mathscr{S})$, for all $j = 1, \ldots, n$ imply $\varphi \in \mathbb{L}_p(\mathscr{S})$.*

Moreover, the assertion holds for a hypersurface $\mathscr{S}$ with the Lipschitz boundary $\Gamma := \partial\mathscr{S}$ and the spaces $\mathbb{W}^{-1}_p(\mathscr{S})$ and $\widetilde{\mathbb{W}}^{-1}_p(\mathscr{S})$.

Proof: First we assume that $\mathscr{S}$ is a closed surface. The proof is based on the following facts from [DS1, Hr1, Ta2], which we recall without proofs.

A. There exists a "lifting operator" (a Bessel potential operator) $\Lambda(\mathscr{x}, D)$, which has the inverse $\Lambda^{-1}(\mathscr{x}, D)$ and they mapping isometrically the spaces

$$\Lambda^{-1}(\mathscr{x}, D)\ :\ \mathbb{W}^{m-1}_p(\mathscr{S}) \to \mathbb{W}^m_p(\mathscr{S})\,, \qquad \Lambda(\mathscr{x}, D)\ :\ \mathbb{W}^m_p(\mathscr{S}) \to \mathbb{W}^{m-1}_p(\mathscr{S}) \tag{1.168}$$

for arbitrary $m = 0, \pm 1, \ldots$.

B. $\Lambda^{-1}(\mathscr{x}, D)$ is a pseudodifferential operator of order -1 and the commutant

$$[\mathscr{D}_j, \Lambda^{-1}(\mathscr{x}, D)] := \mathscr{D}_j\Lambda^{-1}(\mathscr{x}, D) - \Lambda^{-1}(\mathscr{x}, D)\mathscr{D}_j \tag{1.169}$$

with the pseudodifferential operator $\mathscr{D}_j$ has order -1, i.e., maps continuously the spaces

$$[\mathscr{D}_j, \Lambda^{-1}(\mathscr{x}, D)]\ :\ \mathbb{W}^{-1}_p(\mathscr{S}) \to \mathbb{L}_p(\mathscr{S}).$$

Let $\varphi \in \mathbb{W}^{-1}_p(\mathscr{S})$, $\mathscr{D}_j\varphi \in \mathbb{W}^{-1}_p(\mathscr{S})$, for all $j = 1, \ldots, n$. Then, due to (1.168), $\psi := \Lambda^{-1}(\mathscr{x}, D)\varphi \in \mathbb{L}_p(\mathscr{S})$ and, due to (1.169), $\mathscr{D}_j\psi = [\mathscr{D}_j, \Lambda^{-1}(\mathscr{x}, D)]\varphi + \Lambda^{-1}(\mathscr{x}, D)\mathscr{D}_j\varphi \in \mathbb{L}_p(\mathscr{S})$ for all $j = 1, \ldots, n$. From the definition of the space $\mathbb{W}^1_p(\mathscr{S})$ follows that $\psi \in \mathbb{W}^1_p(\mathscr{S})$. Due to (1.169) we get finally $\varphi = \Lambda(\mathscr{x}, D)\psi \in \mathbb{L}_p(\mathscr{S})$.

If $\mathscr{S}$ has non-empty Lipschitz boundary $\Gamma \neq \emptyset$, there exist pseudodifferential operators

$$\begin{aligned} \Lambda^{-1}_-(\mathscr{x}, D)\ &:\ \mathbb{W}^m_p(\mathscr{S}) \to \mathbb{W}^{m+1}_p(\mathscr{S}), \\ \Lambda^{-1}_+(\mathscr{x}, D)\ &:\ \widetilde{\mathbb{W}}^m_p(\mathscr{S}) \to \widetilde{\mathbb{W}}^{m+1}_p(\mathscr{S}), \end{aligned} \tag{1.170}$$

arranging isomorphisms between the indicated spaces, and having the inverses $\Lambda^{-r}_-(\mathscr{x}, D)$, $\Lambda^{-r}_+(\mathscr{x}, D)$ (cf. [DS1]).

Moreover, pseudodifferential operators $\Lambda^{-1}_\pm(\mathscr{x}, D)$ have order -1 and the commutants $[\mathscr{D}_j, \Lambda^{-1}_\pm(\mathscr{x}, D)] := \mathscr{D}_j\Lambda^{-1}_\pm(\mathscr{x}, D) - \Lambda^{-1}_\pm(\mathscr{x}, D)\mathscr{D}_j$ have order -1, i.e., map continuously the spaces $\mathbb{W}^{-1}_p(\mathscr{S}) \to \mathbb{L}_p(\mathscr{S})$.

By using the formulated assertions the proof is completed as in the case of a closed surface $\mathscr{S}$. ■

The foregoing Lemma 1.32 has the following generalization for the Bessel potential spaces $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$ and $\mathbb{H}^s_p(\mathscr{S})$ (see [Tr1] and [DS1] for details about these spaces spaces).

Lemma 1.33 *If $\mathscr{S}$ is closed, sufficiently smooth, $1<p<\infty$, $s\in\mathbb{R}$, $m=1,2,\dots$ and*

$$\varphi\in\mathbb{H}^{s-m}_p(\mathscr{S})\,,\quad \mathscr{D}^\alpha\varphi=\mathscr{D}_1^{\alpha_1}\cdots\mathscr{D}_n^{\alpha_n}\varphi\in\mathbb{H}^{s-m}_p(\mathscr{S})\quad \textit{for all}\quad |\alpha|\leqslant m,$$

then $\varphi\in\mathbb{H}^s_p(\mathscr{S})$.

Moreover, the assertion holds for a hypersurface $\mathscr{S}$ with the Lipschitz boundary $\Gamma:=\partial\mathscr{S}$ and the spaces $\mathbb{H}^s_p(\mathscr{S})$ and $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$.

Proof: Assume first $\mathscr{S}$ has no boundary. The proof is based, as in the foregoing case, on the following facts from [Hr1, Ta2, Tr1], which we recall without proofs.

A. There exists a “lifting operator” (the Bessel potential operator),

$$\Lambda^r(\mathscr{x},D)\;:\;\mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s-r}_p(\mathscr{S})\,,\quad r\in\mathbb{R} \tag{1.171}$$

arranging isomorphism between the indicated spaces, and having the inverse $\Lambda^{-r}(\mathscr{x},D)$.

B. $\Lambda^r(\mathscr{x},D)$ is a pseudodifferential operator of order $-r$ and the commutant

$$[\mathscr{D}^\alpha,\Lambda^r(\mathscr{x},D)]:=\mathscr{D}^\alpha\Lambda^r(\mathscr{x},D)-\Lambda^r(\mathscr{x},D)\mathscr{D}^\alpha \tag{1.172}$$

with the pseudodifferential operator $\mathscr{D}^\alpha=\mathscr{D}_1^{\alpha_1}\cdots\mathscr{D}_n^{\alpha_n}$ has order $|\alpha|+r-1$, i.e., maps continuously the spaces $\mathbb{H}^\gamma_p(\mathscr{S})\to\mathbb{H}^{\gamma-|\alpha|-r+1}_p(\mathscr{S})$, $\forall\,\gamma\in\mathbb{R}$.

Assume that $m=1$. Then $\varphi\in\mathbb{H}^{s-1}_p(\mathscr{S})$ and, due to (1.171), (1.172), it follows that $\psi:=\Lambda^{s-1}_{\mathscr{S}}(\mathscr{x},D)\varphi\in\mathbb{L}_p(\mathscr{S})$, $\mathscr{D}_j\psi=[\mathscr{D}_j,\Lambda^{s-1}_{\mathscr{S}}(\mathscr{x},D)]\varphi+\Lambda^{s-1}_{\mathscr{S}}(\mathscr{x},D)\mathscr{D}_j\varphi\in\mathbb{L}_p(\mathscr{S})$ for all $j=1,\dots,n$. By the definition of the space $\mathbb{W}^1_p(\mathscr{S})$ we conclude that $\psi\in\mathbb{W}^1_p(\mathscr{S})$. Due to (1.169) we get finally $\varphi=\Lambda^{1-s}(\mathscr{x},D)\psi\in\mathbb{H}^s_p(\mathscr{S})$.

Now assume: $m=2,3,\dots$ and the assertion is valid for $m-1$. Then, due to the hypothesis, $\psi_j:=\mathscr{D}_j\varphi\in\mathbb{H}^{s-m}_p(\mathscr{S})$ for $j=1,\dots,n$. Moreover, due to the same hypothesis,

$$\mathscr{D}^\alpha\psi_j:=\mathscr{D}^\alpha\mathscr{D}_j\varphi\in\mathbb{H}^{s-m}_p(\mathscr{S})\quad\text{for all}\quad|\alpha|\leqslant m-1\quad\text{and all}\quad j=1,\dots,n.$$

Hence the induction hypothesis implies that $\psi_j:=\mathscr{D}_j\varphi\in\mathbb{H}^{s-1}_p(\mathscr{S})$ for $j=1,\dots,n$. Now it follows from the already considered case $m=1$ that $\varphi\in\mathbb{H}^s_p(\mathscr{S})$.

If $\mathscr{S}$ has non-empty Lipschitz boundary $\Gamma\neq\emptyset$, there exist pseudodifferential operators

$$\Lambda^r_-(\mathscr{x},D)\;:\;\mathbb{H}^s_p(\mathscr{S})\to\mathbb{H}^{s-r}_p(\mathscr{S})\,,\qquad \Lambda^r_+(\mathscr{x},D)\;:\;\widetilde{\mathbb{H}}^s_p(\mathscr{S})\to\widetilde{\mathbb{H}}^{s-r}_p(\mathscr{S})\,, \tag{1.173}$$

arranging isomorphisms between the indicated spaces, and having the inverses $\Lambda^{-r}_-(\mathscr{x},D)$, $\Lambda^{-r}_+(\mathscr{x},D)$ (cf. [DS1]).

Moreover, the pseudodifferential operators $\Lambda^{-r}_\pm(\mathscr{x},D)$ have order $-r$ and the commutants $[\mathscr{D}^\alpha,\Lambda^{-r}_\pm(\mathscr{x},D)]:=\mathscr{D}^\alpha\Lambda^{-1}_\pm(\mathscr{x},D)-\Lambda^{-r}_\pm(\mathscr{x},D)\mathscr{D}^\alpha$ have order $|\alpha|-r-1$, i.e., map continuously the spaces $\mathbb{H}^\gamma_p(\mathscr{S})\to\mathbb{H}^{\gamma+r+1-|\alpha|}_p(\mathscr{S})$.

By using the formulated assertions the proof is completed as in the foregoing cases. ■

Theorem 1.34 (Korn's I inequality "without boundary condition"). *Let $\mathscr{S} \subset \mathbb{R}^n$ be a Lipshitz hypersurface without boundary,* $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := [\mathfrak{D}_{jk}(\boldsymbol{U})]_{n\times n}$ *be the deformation tensor*

$$\mathfrak{D}_{jk}(\boldsymbol{U}) = \frac{1}{2}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \partial_{\boldsymbol{U}}\big(\nu_j\nu_k\big)\Big] = \frac{1}{2}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \sum_{m=1}^{n} U_m \mathscr{D}_m\big(\nu_j\nu_k\big)\Big]$$

(cf. (1.133)*) and*

$$\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\| := \left[\sum_{j,k=1}^{n}\big\|\mathfrak{D}_{jk}\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p\right]^{1/p}, \qquad \boldsymbol{U}\in\mathbb{W}^1_p(\mathscr{S}) \tag{1.174}$$

for $1 < p < \infty$*. Then*

$$\big\|\boldsymbol{U}\big|\mathbb{W}^1_p(\mathscr{S})\big\| \leqslant M\,\Big[\big\|\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p} \tag{1.175}$$

for some constant $M > 0$ *or, equivalently, the mapping*

$$\boldsymbol{U} \mapsto \Big[\big\|\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p} \tag{1.176}$$

is an equivalent norm on the space $\mathbb{W}^1_p(\mathscr{S})$*.*

Proof: Consider the space

$$\widehat{\mathbb{W}}^1_p(\mathscr{S}) := \Big\{\boldsymbol{U} = \big(U^0_1\ldots,U^0_n\big)^\top\,:\,U^0_j\,,\,\mathfrak{D}_{jk}(\boldsymbol{U})\in\mathbb{L}_p(\mathscr{S})\ \textit{for all}\ j,k=1,\ldots n\Big\} \tag{1.177}$$

endowed with the norm (cf. (1.175) and (1.176)):

$$\big\|\boldsymbol{U}\big|\widehat{\mathbb{W}}^1_p(\mathscr{S})\big\| := \Big[\big\|\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p}. \tag{1.178}$$

The derivatives here are understood in the sense of distributions: $\mathfrak{D}_{jk}(\boldsymbol{U})\in\mathbb{L}_p(\mathscr{S})$ means that there exists a function in $\mathbb{L}_p(\mathscr{S})$ denoted by $\mathfrak{D}_{jk}(\boldsymbol{U})$ such that

$$\begin{aligned}(\mathfrak{D}_{jk}(\boldsymbol{U}),\boldsymbol{V})_{\mathscr{S}} := \int_{\mathscr{S}}\Big[&U_j(\mathscr{x})\overline{\mathscr{D}^*_k V(\mathscr{x})} + U_k(\mathscr{x})\overline{\mathscr{D}^*_j V(\mathscr{x})}\\ &+\sum_{m=1}^{n} V(\mathscr{x})U_m(\mathscr{x})\mathscr{D}^*_m\big(\nu_j(\mathscr{x})\nu_k(\mathscr{x})\big)\Big]\,d\sigma \qquad \forall\,\boldsymbol{V}\in\mathbb{L}_p(\mathscr{S}),\end{aligned}$$

(cf. (1.67) for the formal dual $\mathscr{D}^*_m$).

It is obviously sufficient to prove, that the spaces $\mathbb{W}^1_p(\mathscr{S})$ and $\widehat{\mathbb{W}}^1_p(\mathscr{S})$ are identical. The inclusion $\mathbb{W}^1_p(\mathscr{S})\subset\widehat{\mathbb{W}}^1_p(\mathscr{S})$ is trivial and we concentrate on the proof of the inverse inclusion $\widehat{\mathbb{W}}^1_p(\mathscr{S})\subset\mathbb{W}^1_p(\mathscr{S})$.

To this end take $\boldsymbol{U} \in \widehat{\mathbb{W}}^1_p(\mathscr{S})$ and note that the inclusions $\boldsymbol{U} \in \mathbb{L}_p(\mathscr{S})$, $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \in \mathbb{L}_p(\mathscr{S})$ (i.e., $\mathfrak{D}_{jk}\boldsymbol{U} \in \mathbb{L}_p(\mathscr{S})$ for all $j,k=1,\dots,n$) imply

$$\mathfrak{D}^0_{jk}(\boldsymbol{U}) = \frac{1}{2}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k\Big] = \mathfrak{D}_{jk}(\boldsymbol{U}) - \frac{1}{2}\sum_{r=1}^n \partial_r\big(\nu_j\nu_k\big)U_r \in \mathbb{L}_p(\mathscr{S}) \tag{1.179}$$
$$\text{for all} \quad j,k=1,\dots,n\,.$$

Then (cf. [DMM1, Proposition 4.4.iv] for the commutator $[\mathscr{D}_j,\mathscr{D}_k]$):

$$\mathscr{D}_j U_k \in \mathbb{H}^{-1}_p(\mathscr{S}) \qquad \big[\mathscr{D}_j,\mathscr{D}_k\big]U_m = \sum_{r=1}^n \big[\nu_j\mathscr{D}_k\nu_r - \nu_k\mathscr{D}_j\nu_r\big]\mathscr{D}_r U_m \in \mathbb{H}^{-1}_p(\mathscr{S}),$$
$$\mathscr{D}_k\mathscr{D}_j U_m = \mathscr{D}_j\widetilde{\mathfrak{D}}_{km}(\boldsymbol{U}) + \mathscr{D}_k\widetilde{\mathfrak{D}}_{jm}(\boldsymbol{U}) - \mathscr{D}_m\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}) - \frac{1}{2}\big[\mathscr{D}_j,\mathscr{D}_k\big]U_m$$
$$-\frac{1}{2}\big[\mathscr{D}_j,\mathscr{D}_m\big]U_k - \frac{1}{2}\big[\mathscr{D}_k,\mathscr{D}_m\big]U_j \in \mathbb{H}^{-1}_p(\mathscr{S}) \quad \text{for} \quad j,k,m=1,\dots,n\,,$$

Due to Lemma 1.32 of J. L. Lions this implies $\mathscr{D}_j U_m \in \mathbb{L}_p(\mathscr{S})$ for all $j,m=1,\dots,n$ and the claimed result $\boldsymbol{U} \in \mathbb{W}^1_p(\mathscr{S})$ follows. ■

Remark 1.35 *The foregoing Theorem 1.34 is proved by P. Ciarlet in [Ci3] for the case $p=2$, for curvilinear coordinates and covariant derivatives.*

A remarkable consequence of Korn's inequality (1.175) *is that the space*

$$\mathbb{W}^1_p(\mathscr{S}) := \Big\{\boldsymbol{U} = \big(U_1\dots,U_n\big)^\top \,:\, U_j\,,\,\mathscr{D}_k U_j \in \mathbb{L}_p(\mathscr{S}) \ \text{ for all } j,k=1,\dots n\Big\}$$

and the space $\widehat{\mathbb{W}}^1_p(\mathscr{S})$ (cf. (1.177)*) are isomorphic (i.e. can be identified), although only $\dfrac{n(n+1)}{2} < n^2$ linear combinations of the n^2 derivatives $\mathscr{D}_j U_k$, $j,k=1,\dots n$ participate in the definition of the space $\widehat{\mathbb{H}}^1_p(\mathscr{S})$.*

1.6 Killing's vector fields and further Korn's inequalities

Definition 1.36 *Let $\mathscr{S}$ be a hypersurface in the Euclidean space $\mathbb{R}^n$. The space $\mathscr{R}(\mathscr{S})$ of solutions to the deformation equations*

$$\begin{aligned}\mathfrak{D}_{jk}(\boldsymbol{U}) &:= \frac{1}{2}\big[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})^0_k + (\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})^0_j\big]\\ &= \frac{1}{2}\Big[\mathscr{D}_k U^0_j + \mathscr{D}_j U^0_k + \sum_{m=1}^n U^0_m\mathscr{D}_m\big(\nu_j\nu_k\big)\Big] = 0\,,\end{aligned} \tag{1.180}$$
$$\boldsymbol{U} = \sum_{j=1}^n U^0_j\boldsymbol{d}^j \in \omega(\mathscr{S}), \qquad j,k=1,\dots,n$$

is called the space of **Killing's vector fields***.*

Killing's vector fields on a domain in the Euclidean space $\Omega\subset\mathbb{R}^n$ are known as the **rigid motions** and we start with this simplest class.

The space of rigid motions $\mathscr{R}(\Omega)$ extends naturally to the entire $\mathbb{R}^n$ and consists of linear vector-functions

$$\boldsymbol{V}(x)=a+\mathscr{B}x\,,\qquad \mathscr{B}=\big[b_{jk}\big]_{n\times n}\,,\qquad a\in\mathbb{R}^n\,,\quad x\in\mathbb{R}^n\,,\tag{1.181}$$

where the matrix $\mathscr{B}$ is skew symmetric

$$\mathscr{B}:=\begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1(n-2)} & b_{1(n-1)}\\ -b_{12} & 0 & b_{21} & \cdots & b_{1(n-3)} & b_{2(n-2)}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ -b_{1(n-2)} & -b_{2(n-3)} & -b_{3(n-4)} & \cdots & 0 & b_{(n-1)1}\\ -b_{1(n-1)} & -b_{2(n-2)} & -b_{3(n-3)} & \cdots & -b_{(n-1)1} & 0\end{bmatrix}=-\mathscr{B}^\top\tag{1.182}$$

with real valued entries $b_{jk}\in\mathbb{R}$. For $n=3,4,\dots$ the space $\mathscr{R}(\mathbb{R}^n)$ is finite dimensional and $\dim\mathscr{R}(\mathbb{R}^n)=n+\dfrac{n(n-1)}{2}=\dfrac{n(n+1)}{2}$.

Note that for $n=3$ the vector field $\boldsymbol{V}\in\mathscr{R}(\Omega)$, $\Omega\subset\mathbb{R}^3$, is the classical rigid displacement

$$\begin{aligned}&\boldsymbol{V}(x)=a+\mathscr{B}x=a+b\wedge x\,,\\ &b:=(b_1,b_2,b_3)^\top\in\mathbb{R}^3\,,\quad x\in\Omega\,,\end{aligned}\qquad \mathscr{B}:=\begin{bmatrix}0 & -b_3 & b_2\\ b_3 & 0 & -b_1\\ -b_2 & b_1 & 0\end{bmatrix}.\tag{1.183}$$

Definition 1.37 *We call a subset $\mathscr{M}\subset\mathbb{R}^n$* **essentially m-dimensional** *and write $\operatorname{ess\,dim}\mathscr{M}=m$, if there exist $m+1$ points $\mathscr{x}^0,\mathscr{x}^1,\cdots,\mathscr{x}^m\in\mathscr{M}$ such that the vectors $\big\{\mathscr{x}^j-\mathscr{x}^0\big\}_{j=1}^m$ are linearly independent.*

Note, that any m-dimensional subset $\mathscr{M}\subset\mathbb{R}^m$ is essentially m-dimensional, because contains m linearly independent vectors. Moreover, any collection of $m+1$ points in $\mathbb{R}^m$ (a 0-dimensional subset!) is essentially m-dimensional, provided these points does not belong to any $m-1$ dimensional hyperplane.

Lemma 1.38 *Let*

$$\mathrm{Def}(\boldsymbol{U}):=\big[\mathfrak{D}^0_{jk}(\boldsymbol{U})\big]_{n\times n}\,,\tag{1.184}$$

$$\mathfrak{D}^0_{jk}(\boldsymbol{U})=\frac12\Big[\partial_k U^0_j+\partial_j U^0_k\Big],\qquad \boldsymbol{U}=\sum_{j=1}^n U^0_j\boldsymbol{e}^j.$$

be the deformation tensor in Cartesian coordinates.

The linear space $\mathscr{R}(\mathbb{R}^n)$ of rigid motions (of Killing's vector fields) in $\mathbb{R}^n$ consists of vector fields $\mathbf{K}=(K^0_1,\dots,K^0_n)^\top$ which are solutions to the system

$$2\mathfrak{D}^0_{jk}(\mathbf{K})(x)=\partial_k K^0_j(x)+\partial_j K^0_k(x)=0\qquad x\in\mathscr{S}\quad\text{for all}\quad j,k=1,\dots,n.\tag{1.185}$$

If a rigid motion vanishes on an essentially $(n-1)$-dimensional subset $\mathbf{K}(\mathscr{x})=0$ for all $\mathscr{x}\in\mathscr{M}$, $\operatorname{ess\,dim}\mathscr{M}=n-1$, or at infinity $\mathbf{K}(x)=\mathscr{o}(1)$ as $|x|\to\infty$, then $\mathbf{K}$ vanishes identically $\mathbf{K}(x)\equiv0$ on $\mathbb{R}^n$.

Proof: By differentiating (1.185) and recalling that $\partial_k\partial_l K^0_j = \partial_l\partial_k K^0_j$, we get

$$\partial_j\partial_k K^0_m = \partial_j\mathfrak{D}^0_{km}(\mathbf{K}) + \partial_k\mathfrak{D}^0_{jm}(\mathbf{K}) - \partial_m\mathfrak{D}^0_{jk}(\mathbf{K}) = 0 \quad \text{for all } j,k,m = 1,2,\dots,n-1\,.$$

Therefore,

$$K^0_j(x) = a_j + b_{j1}x_1 + \dots + b_{jn}x_n \qquad j = 1,2,\dots,n$$

or

$$\mathbf{K}(x) = a + \mathscr{B}\,x \qquad \text{with} \quad \mathscr{B} = [b_{jk}]_{n\times n}\,. \tag{1.186}$$

From (1.185) we derive that $\mathscr{B}$ is a skew symmetric matrix (cf. (1.182)):

$$\partial_j K^0_k(x) = -\partial_k K^0_j(x) \equiv 0 \implies b_{jk} = -b_{kj} \quad j,k = 1,2,\dots,n \implies \mathscr{B} = -\mathscr{B}^\top.$$

The inclusion $\mathbf{K} \in \mathscr{R}(\mathbb{R}^n)$ is proved.

The inverse statement, that any vector field $\mathbf{K} \in \mathscr{R}(\mathbb{R}^n)$ (of the form (1.181)) is a solution of the system (1.185), is easy to verify.

Let us prove the second assertion: for any linearly independent vectors $x^0,\dots,x^{n-1}$ the condition

$$\mathbf{K}(x^k) = 0 \quad \implies \quad a + \mathscr{B}x^k = \mathbf{K}(x^k) = 0 \tag{1.187}$$

implies $a = 0$ and $\mathscr{B} = 0$, i.e., $\mathbf{K}(x) = 0$ for all $x \in \mathbb{R}^n$. Indeed, if $\mathscr{B} = 0$ then, obviously, $a = 0$. Accepting $\mathscr{B} \neq 0$, for rank of $\mathscr{B}$ we have the estimate $2 \leqslant \operatorname{rank}\mathscr{B} < n$ (if $\mathscr{B} \neq 0$ then, due to the symmetry $\mathscr{B} = -\mathscr{B}^\top$, there exists a non-degenerate minor of order at least 2). On the other hand, from (1.187) follows

$$\mathscr{B}(x^k - x^0) = 0 \qquad \forall\, k = 1,\dots,n-1\,,$$

which contradicts the estimate $2 \leqslant \operatorname{rank}\mathscr{B} < n$ since $\left\{x^1 - x^0,\dots,x^{n-1} - x^0\right\}$ are linearly independent.

If a rigid motion $\mathbf{K}(x)$ in (1.186) vanishes at infinity $\mathbf{K}(x) = o(1)$ as $|x| \to \infty$, then obviously $a = 0$, $\mathscr{B} = 0$ and, therefore, $\mathbf{K}(x) = 0$ for all $x \in \mathbb{R}^n$. ■

Remark 1.39 *For the deformation tensor in Cartesian coordinates* $\operatorname{Def}(\boldsymbol{U})$ *(cf.* (1.184)*) in a domain* $\Omega \subset \mathbb{R}^n$ *Korn's inequality*

$$\left\|\boldsymbol{U}\middle|\mathbb{H}^1_p(\Omega)\right\| \leqslant M\left[\left\|\boldsymbol{U}\middle|\mathbb{L}_p(\Omega)\right\|^P + \left\|\operatorname{Def}(\boldsymbol{U})\middle|\mathbb{L}_p(\Omega)\right\|^p\right]^{1/p}, \quad 1 < p < \infty \tag{1.188}$$

with some constant $M > 0$ *is well known and is proved e.g. in [Ci2] (cf.* (1.174) *for a similar norm).*

In contrast to the rigid motions in $\mathbb{R}^n$ nobody can identify Killing's vector fields on hypersurfaces explicitly so far. The next Theorem 1.40 underlines importance of Killing's vector fields for the Lamé equation on hypersurfaces. Later we investigate properties of Killing's vector fields to prepare tools for investigations of boundary value problems for the Lamé equation.

Theorem 1.40 *Let $\mathscr{S}$ be an ℓ-smooth closed hypersurface in $\mathbb{R}^n$ and $\ell \geqslant 2$. The Lamé operator $\mathscr{L}_{\mathscr{S}}$ for an isotropic media*

$$\begin{gathered}\mathscr{L}_{\mathscr{S}} \;:\; \mathbb{H}^{s+1}_p(\mathscr{S}) \to \mathbb{H}^{s-1}_p(\mathscr{S}), \\ \mathscr{L}_{\mathscr{S}}\boldsymbol{U} = \mu\pi_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\,\nabla_{\mathscr{S}}\boldsymbol{U} + (\lambda+\mu)\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U} + \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U}\,,\end{gathered} \tag{1.189}$$

is self adjoint $\mathscr{L}^*_{\mathscr{S}} = \mathscr{L}_{\mathscr{S}}$, elliptic, Fredholm and index $\mathrm{Ind}\,\mathscr{L}_{\mathscr{S}} = 0$ for all $1 < p < \infty$ and all $s \in \mathbb{R}$, provided that $|s| \leqslant \ell$.

The kernel of the operator $\mathrm{Ker}\mathscr{L}_{\mathscr{S}} \subset \mathbb{H}^s_p(\mathscr{S})$ is independent of the parameters p and s, coincides with the space of Killing's vector fields

$$\mathrm{Ker}\,\mathscr{L}_{\mathscr{S}} = \{\boldsymbol{U} \in \omega(\mathscr{S}) \;:\; \mathscr{L}_{\mathscr{S}}\boldsymbol{U} = 0\} = \mathscr{R}(\mathscr{S}), \tag{1.190}$$

is finite dimensional and $\dim\mathscr{R}(\mathscr{S}) = \dim\mathrm{Ker}\,\mathscr{L}_{\mathscr{S}} < \infty$.

If $\mathscr{S}$ is C^∞ smooth, then the Killing's vector fields are smooth as well $\mathscr{R}(\mathscr{S}) \subset C^\infty(\mathscr{S})$.

$\mathscr{L}_{\mathscr{S}}$ is non-negative on the space $\mathbb{H}^1(\mathscr{S})$ and positive definite on the orthogonal complement $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ to the kernel

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} \geqslant 0 \qquad \text{for all} \quad \boldsymbol{U} \in \mathbb{H}^1(\mathscr{S})\,, \tag{1.191}$$

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} \geqslant C\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 \qquad \text{for all} \quad \boldsymbol{U} \in \mathbb{H}^1_{\mathscr{R}}(\mathscr{S})\,, \qquad C > 0\,, \tag{1.192}$$

where $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ is the orthogonally complemented subspace to $\mathscr{R}(\mathscr{S})$ in $\mathbb{H}^1(\mathscr{S})$.

Moreover, the following Gåarding's inequality

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{U})_{\mathscr{S}} \geqslant C_1\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{S})\|^2 - C_0\|\boldsymbol{U}|\mathbb{H}^{-r}(\mathscr{S})\|^2 \tag{1.193}$$

holds for all $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{S})$, with any $-1 < r \leqslant \ell$ and some positive constants $C_0 > 0$, $C_1 > 0$.

Proof: The proof is exposed in [Du2]. Here we draw the following consequence.

Corollary 1.41 *Let $\mathscr{S} \subset \mathbb{R}^n$ be a Lipshitz hypersurface without boundary, $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := \big[\mathfrak{D}_{jk}(\boldsymbol{U})\big]_{n\times n}$ be the deformation tensor*

$$\begin{aligned}\mathfrak{D}^0_{jk}(\boldsymbol{U}) = (\mathrm{Def}_{\mathscr{S}}\,(\boldsymbol{U}))_{jk} &= \frac{1}{2}\big[(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})_j + (\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k\big] \\ &= \frac{1}{2}\big[\mathscr{D}_jU^0_k + \mathscr{D}_kU^0_j + \partial_{\boldsymbol{U}}(\nu_j\nu_k)\big]\,, \quad \forall\, j,k = 1,\ldots,n.\end{aligned} \tag{1.194}$$

where $(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k$ denotes the $k-th$ component of the covariant derivative $\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U}$. The norm $\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\|$ is defined by (1.174).

Then the following Korn's inequality

$$\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\| \geqslant c\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\| \qquad \forall\,\boldsymbol{U} \in \mathbb{H}^1_{\mathscr{R}}(\mathscr{S}) \tag{1.195}$$

holds for some constant $c > 0$ or, equivalently, the mapping $\boldsymbol{U} \mapsto \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\|$ is an equivalent norm on the orthogonal complement $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ to the space of Killing's vector fields.

Proof: Due to Korn's inequality (1.175) for $p=2$

$$\left\|\boldsymbol{U}\middle|\mathbb{L}_2(\mathscr{S})\right\|^2 \geqslant M_1\Big[\left\|\boldsymbol{U}\middle|\mathbb{H}^1(\mathscr{S})\right\|^2 - \left\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\middle|\mathbb{L}_2(\mathscr{S})\right\|^2\Big]$$

the mapping $\mathrm{Def}_{\mathscr{S}} : \mathbb{H}^1_{\mathscr{R}}(\mathscr{S}) \to \mathbb{L}_2(\mathscr{S})$ is Fredholm and has index 0. The inequality (1.195) follows since the mapping is injective (has an empty kernel). ■

Let us recall some results related to the uniqueness of solutions to arbitrary elliptic equation.

Definition 1.42 *Let Ω be an open subset with the Lipschitz boundary $\partial\Omega \neq \emptyset$ either on a Lipschitz hypersurface $\mathscr{S} \subset \mathbb{R}^n$ or in the Euclidean space $\mathbb{R}^{n-1}$.*

A class of functions $\mathscr{U}(\Omega)$ defined in a domain Ω in $\mathbb{R}^n$, is said to have the **strong unique continuation property***, if every $u \in \mathscr{U}(\Omega)$ in this class which vanishes to infinite order at one point must vanish identically.*

If a surface $\mathscr{S}$ is C^∞-smooth, any elliptic operator on $\mathscr{S}$ has the strong unique continuation property due to Holmgren's theorem. But we can have more.

Lemma 1.43 *Let $\mathscr{S}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$. The class of solutions to a second order elliptic equation $\mathbb{A}(\mathscr{X},\mathscr{D})u=0$, with Lipschitz continuous top order coefficients on a surface $\mathscr{S}$ has the strong unique continuation property.*

In particular, if the solution $u(\mathscr{X})=0$ vanishes in any open subset of $\mathscr{S}$ it vanishes identically on entire $\mathscr{S}$.

Proof: The result was proved in [AKS1] for a domain $\Omega \subset \mathbb{R}^n$ by the method of "Carleman estimates" (also see [Hr1, Volume 3, Theorem 17.2.6]). Another proof, involving monotonicity of the frequency function was discovered by N. Garofalo and F. Lin (see [GL1, GL2]). A differential equation $\mathbb{A}(\mathscr{X},\mathscr{D})u(\mathscr{X})=0$ with Lipschitz continuous top order coefficients on a $\mathbb{W}^2_\infty$-smooth surface $\mathscr{S}$ is locally equivalent to a differential equation with Lipschitz continuous top order coefficients on a domain $\Omega \subset \mathbb{R}^{n-1}$. Therefore a solution $u(\mathscr{X})$ has the strong unique continuation property locally (on each coordinate chart) on $\mathscr{S}$.

Since $\mathscr{S}$ is covered by a finite number of local coordinate charts which intersect on open neighborhoods, a solution $u(\mathscr{X})$ has the strong unique continuation property globally on $\mathscr{S}$. ■

Remark 1.44 *If the top order coefficients of a second order elliptic equation $\mathbb{A}(\mathscr{X},\mathscr{D})u=0$ in open subsets $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, are merely Hölder continuous, with exponent less than 1, examples due to A. Plis [Pl1] and K. Miller [Mi1] show that a solution $u(x)$ does not have the strong unique continuation property.*

Lemma 1.45 *Let $\mathscr{C}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$ with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$ and $\gamma \subset \Gamma$ be an open part of the boundary Γ. Let $\mathbb{A}(\mathscr{X},\mathscr{D})$ be a second order elliptic system with Lipschitz continuous top order matrix coefficients on a surface $\mathscr{S}$.*

The Cauchy problem

$$\begin{cases} \mathbb{A}(\mathscr{X},\mathscr{D})u=0 & \text{on} \quad \mathscr{C}, \quad u \in \mathbb{H}^1(\Omega), \\ u(\mathfrak{s})=0 & \textit{for all} \quad \mathfrak{s}\in\gamma, \\ (\partial_{\boldsymbol{V}}u)(\mathfrak{s})=0 & \textit{for all} \quad \mathfrak{s}\in\gamma, \end{cases} \tag{1.196}$$

where the vector $\boldsymbol{V}$ is a non-tangential to Γ, but tangential to $\mathscr{S}$, has only a trivial solution $u(\mathscr{x}) = 0$ on entire $\mathscr{S}$.

Proof: With a local diffeomorphism the Cauchy problem (1.196) is transformed into a similar problem on a domain $\Omega \subset \mathbb{R}^{n-1}$ with the Cauchy data vanishing on some open subset of the boundary $\gamma \subset \Gamma := \partial\Omega$.

Let us, for simplicity, use the same notation $\gamma \subset \Gamma = \partial\Omega$, the non-tangential vector $\boldsymbol{V}$ to γ, the function u and the differential operator $\mathbb{A}(x, \mathscr{D})$ for the transformed Cauchy problem in the transformed domain Ω. Moreover, we will suppose that γ is a part of the hypersurface $x_1 = 0$ (otherwise we can transform the domain Ω again). We also use new variables $t = x_1$ and $x := (x_2, \dots, x_{n-1})$. Then $(0, x) \in \gamma$ while $(t, x) \in \Omega$ for all small $0 < t < \varepsilon$ and some $x \in \Omega'$.

Thus, the natural basis element $\boldsymbol{e}^1$ (cf. (0.7)) is orthogonal to γ and, therefore, $\boldsymbol{e}^1 = c_1(x)\boldsymbol{V}(x) + c_2(x)\boldsymbol{g}(x)$ for some unit tangential vector $\boldsymbol{g}(x)$ to γ for some scalar functions $c_1(x)$, $c_2(x)$ and all $x \in \Omega'$. Then, due to the third line in (1.196),

$$(\partial_t u)(0, x) = \partial_{\boldsymbol{e}^j} u(0, x) = c_1(x)\partial_{\boldsymbol{V}} u(0, x) + c_2(x)\partial_{\boldsymbol{g}} u(0, x) = 0$$

because any derivative along tangential vector to γ vanishes $\partial_{\boldsymbol{g}} u(0, x) = 0$ due to the second line in (1.196).

The second order equation $\mathbb{A}(t, x; \mathscr{D})$ can be written in the form

$$\mathbb{A}(t, x, D)u = \mathbb{A}(t, x; \boldsymbol{e}^1)\partial_t^2 u + \mathbb{A}_1(t, x; D)\partial_t u + \mathbb{A}_2(t, x; D)u, \qquad D := -i\partial_x,$$

where $\mathbb{A}(t, x; \boldsymbol{e}^1)$ is the (invertible) matrix function, $\mathbb{A}_1(t, x; D)$ and $\mathbb{A}_2(t, x; D)$ are differential operators of orders 1 and 2 respectively, compiled of derivatives ∂_x, $x \in \Omega'$. Therefore, if $\mathbb{A}_j^0(t, x; D) := \mathbb{A}^{-1}(t, x; \boldsymbol{e}^1)\mathbb{A}_j(t, x; D)$, $j = 1, 2$, the Cauchy problem (1.196) transforms into

$$\begin{cases} \partial_t^2 u(t, x) + \mathbb{A}_1^0(t, x; D)\partial_t u(t, x) + \mathbb{A}_2^0(t, x; D)u(t, x) = 0 & \text{on} \quad (t, x) \in \Omega_\varepsilon, \\ u(0, x) = 0 & \text{for all} \quad x \in \Omega', \\ (\partial_t u)(0, x) = 0 & \text{for all} \quad x \in \Omega', \end{cases} \tag{1.197}$$

where $\Omega_\varepsilon := (0, \varepsilon) \times \Omega' \subset \Omega$, $u \in \mathbb{H}^1(\Omega_\varepsilon)$ and $\gamma := \big\{(0, x) \ : \ x \in \Omega'\big\}$.

Now let us recall the inequality (see [Miz1, § 4.3, Theorem 4.3, § 6.14], [Sc1, § 4-7, Lemma 4-21]): There is a constant C which depends on ε and $\mathbb{A}(t, x; D)$ only and such that the inequality

$$\int_{\Omega_\varepsilon} e^{-\lambda t}|v(t, x)|^2 dt\, dx \leqslant C \int_{\Omega_\varepsilon} e^{-\lambda t}|(\mathbb{A}(t, x; D)v)(t, x)|^2 dt\, dx, \tag{1.198}$$

holds for $\mathbb{A}(t, x; D)v \in \mathbb{L}_2(\Omega_\varepsilon)$, $v \in C^\infty(\Omega_\varepsilon)$; moreover, $v(t, x)$ should vanishe near $t = \varepsilon$ and should have vanishing Cauchy data $v(0, x) = (\partial_t v)(0, x) = 0$ for all $x \in \Omega'$.

Let $\rho \in C^2(0, \varepsilon)$ be a cut-off function: $\rho(t) = 1$ for $0 \leqslant t < \varepsilon/2$ and $\rho(t) = 0$ for $3\varepsilon/4 \leqslant t < \varepsilon$. Then $v := \rho u \in \mathbb{H}^1(\Omega_\varepsilon)$ and since $\mathbb{A}(t, x; D)u = 0$ on Ω_ε, we get

$$\begin{aligned} \mathbb{A}(t, x; D)(\rho u) &= \rho\mathbb{A}(t, x; D)u + (\partial_t^2\rho)u + (\partial_t\rho)\partial_t u + (\partial_t\rho)\mathbb{A}_1^0(t, x; D)u \\ &= (\partial_t^2\rho)u + (\partial_t\rho)\partial_t u + (\partial_t\rho)\mathbb{A}_1^0(t, x; D)u. \end{aligned}$$

We have asserted $u \in \mathbb{H}^1(\Omega_\varepsilon)$, $\rho \in C^2$ and this implies $(\partial_t^2\rho)u \in \mathbb{L}_2(\Omega_\varepsilon)$, $(\partial_t\rho)\partial_t u \in \mathbb{L}_2(\Omega_\varepsilon)$. Note, that $\partial_t\rho(t)$ vanishes for $0 < t < \varepsilon/2$. Therefore $(\partial_t\rho)\mathbb{A}_1^0(t,x;D)u$ vanishes in a neighborhood of the boundary $\gamma \subset \Gamma$. Due to a priori regularity result (cf. [LM1, Ch. 2, § 3.2, § 3.3]), a solution to an elliptic equation in (1.197) has additional regularity $u \in \mathbb{H}^2(\Omega_\varepsilon^0)$ for arbitrary Ω_ε^0 properly imbedded into Ω_ε. This implies $(\partial_t\rho)\mathbb{A}_1^0(t,x;D)u \in \mathbb{L}_2(\Omega_\varepsilon)$ and we conclude

$$\mathbb{A}(t,x;D)(\rho u) \in \mathbb{L}_2(\Omega_\varepsilon). \tag{1.199}$$

Introducing $v = \rho u$ into the inequality (1.198) we get

$$\begin{aligned}\int_{\Omega'}\int_0^{\varepsilon/4} e^{-\lambda t}|\rho(t)u(t,x)|^2 dt\,dx &\leqslant \int_{\Omega_\varepsilon} e^{-\lambda t}|\rho(t)u(t,x)|^2 dt\,dx\\ &\leqslant C\int_{\Omega'}\int_{\varepsilon/2}^{3\varepsilon/4} e^{-\lambda t}|(\mathbb{A}(t,x;D))\rho(t)u(t,x)|^2 dt\,dx.\end{aligned}$$

This implies for $\lambda > 0$

$$\int_{\Omega'}\int_0^{\varepsilon/4}|\rho(t)u(t,x)|^2 dt\,dx \leqslant e^{-\lambda\varepsilon/4}\int_{\Omega_\varepsilon}\left|(\mathbb{A}(t,x;D))\rho(t)u(t,x)\right|^2 dt\,dx \leqslant C_1 e^{-\lambda\varepsilon/4}.$$

where, due to (1.196), $C_1 > 0$ is a finite constant. By sending $\lambda \to \infty$ we get the desired result $u(t,x) = 0$ for all $0 \leqslant t \leqslant \varepsilon/4$ and all $x \in \Omega'$. Since $u(x)$ vanishes in a subset of the domain Ω, bordering γ, due to Lemma 1.43 the solution vanishes on entire Ω (on entire $\mathscr{C}$). ■

Due to our specific interest (see the next Lemma 1.47) and many applications, for example to control theory, the following boundary unique continuation property is of a special interest.

Definition 1.46 *Let $\mathscr{S}$ be a Lipschitz hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be an open subsurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.*

We say that a class of functions $\mathscr{U}(\Omega)$ has the **strong unique continuation property from the boundary** *if a vector-function $\boldsymbol{U} \in \mathscr{U}(\Omega)$ which vanishes $\boldsymbol{U}(\mathfrak{s}) = 0$, $\forall\,\mathfrak{s} \in \gamma$ on an open subset of the boundary $\gamma \subset \Gamma$, vanishes on the entire $\mathscr{C}$.*

Lemma 1.47 *Let $\mathscr{S}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be an open $\mathbb{W}^2_\infty$-smooth subsurface.*

The set of Killing's vector fields $\mathscr{R}(\mathscr{S})$ on the open surface $\mathscr{C}$ has the strong unique continuation property from the boundary.

Proof: Let $\gamma \subset \Gamma := \partial\mathscr{C}$, $\operatorname{mes}\gamma > 0$ and $\boldsymbol{U}(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \gamma \subset \Gamma := \partial\mathscr{C}$. Then (cf. (1.39))

$$\begin{cases} (\mathscr{D}_j U_k^0)(\mathfrak{s}) + (\mathscr{D}_k U_j^0)(\mathfrak{s}) = -\sum\limits_{m=1}^n U_m^0(\mathfrak{s})\mathscr{D}_m\big(\nu_j(\mathfrak{s})\nu_k(\mathfrak{s})\big) = 0,\\ U_k^0(\mathfrak{s}) = 0 \qquad \forall\,\mathfrak{s} \in \gamma, \quad j,k = 1,\dots,n.\end{cases} \tag{1.200}$$

Among tangential vector fields generating the Gunter's derivatives $\left\{\boldsymbol{d}^j(\mathfrak{s})\right\}_{j=1}^{n-1}$ only $n-1$ are linearly independent. One of vectors might collapse at a point $\boldsymbol{d}^j(\mathfrak{s})=0$ if the corresponding basis vector $\boldsymbol{e}^j$ is orthogonal to the surface at $\mathfrak{s}\in\mathscr{S}$, while others might be tangential to the subsurface Γ, except at least one, say $\boldsymbol{d}^n(\mathfrak{s})$, which is non-tangential to γ. Then from (1.200) follows

$$2(\mathscr{D}_n U_n^0)(\mathfrak{s})=0 \quad \text{and implies} \quad (\mathscr{D}_j U_n^0)(\mathfrak{s})=0 \tag{1.201}$$
$$\text{for all} \quad \mathfrak{s}\in\gamma \quad \text{and all} \quad j=1,\dots,n.$$

Indeed, the vector $\boldsymbol{d}^j$, $1\leqslant j\leqslant n-1$, is a linear combination $\boldsymbol{d}^j(\mathfrak{s})=c_1(\mathfrak{s})\boldsymbol{d}^n(\mathfrak{s})+c_2(\mathfrak{s})\boldsymbol{\tau}^j(\mathfrak{s})$ of the non-tangential vector $\boldsymbol{d}^n(\mathfrak{s})$ and of the projection $\boldsymbol{\tau}^j(\mathfrak{s}):=\pi_\gamma\boldsymbol{d}^j(\mathfrak{s})$ of $\boldsymbol{d}^j(\mathfrak{s})$ to the subsurface γ at the point $\mathfrak{s}\in\gamma$. Since U^n vanishes identically on γ, the derivative $(\partial_{\boldsymbol{\tau}^j}U_n^0)(\mathfrak{s})=0$ vanishes as well and (1.201) follows:

$$(\mathscr{D}_j U_n^0)(\mathfrak{s})=c_1(\mathfrak{s})(\partial_{\boldsymbol{d}^n}U_n^0)(\mathfrak{s})+c_2(\mathfrak{s})(\partial_{\boldsymbol{\tau}^j}U_n^0)(\mathfrak{s})=c_1(\mathfrak{s})(\mathscr{D}_n U_n^0)(\mathfrak{s})=0 \qquad \forall\,\mathfrak{s}\in\gamma.$$

Equalities (1.200) and (1.201) imply

$$(\mathscr{D}_n U_j^0(\mathfrak{s})=-(\mathscr{D}_j U_n^0)(\mathfrak{s})=0 \qquad \forall\,\mathfrak{s}\in\gamma, \quad \forall\,j=1,\dots,n. \tag{1.202}$$

Thus, we have the following Cauchy problem

$$\begin{cases} \mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U}(\mathscr{X})=0 & \text{on} \quad \mathscr{C}, \\ \boldsymbol{U}(\mathfrak{s})=0 & \text{for all} \quad \mathfrak{s}\in\gamma, \\ (\mathscr{D}_n\boldsymbol{U})(\mathfrak{s})=(\partial_{\boldsymbol{d}^n}\boldsymbol{U})(\mathfrak{s})=0 & \text{for all} \quad \mathfrak{s}\in\gamma, \end{cases} \tag{1.203}$$

where $\boldsymbol{d}^n$ is a vector filed non-tangential to Γ. Due to Lemma 41.45, $\boldsymbol{U}(\mathscr{X})=0$ for all $\mathscr{X}\in\mathscr{C}$. ■

Before we draw some consequences from the proved unique continuation property, we should make some comments. The finite dimensionality of the linear space $\mathscr{R}(\mathscr{C})$ when the surface $\mathscr{C}$ is 2-smooth, was proved in the papers [CLM1, GS1, Ge1].

The foregoing Lemma 1.47 generalizes essentially the "infinitesimal rigid displacement lemma" (see [Ci3, Theorem 2.7-2] the following conditions are imposed:

i. $\mathscr{C}\subset\mathscr{S}$ is C^3-smooth, **elliptic** in $\mathbb{R}^3$, i.e., if

$$\sum_{k=1}^{2}|\xi^k|^2\leqslant C\sum_{k,j=1}^{2}\left|b_{jk}(\mathscr{X})\xi^j\xi^k\right| \qquad \forall\,\mathscr{X}\in\mathscr{S}, \quad \forall(\xi^1,\xi^2)^\top\in\mathbb{R}^2, \tag{1.204}$$

where $b_{jk}(\mathscr{X})\ :\ \mathscr{S}\to\mathbb{R}$ are the covariant components of the curvature tensor of $\mathscr{S}$; the equivalent condition is that the Gaußian curvature is positive on the entire surface $\mathscr{S}$ or that tho principal curvatures of the surface $\mathscr{S}$ have the same sign everywhere on $\mathscr{S}$.

ii. The Killing's vector field $\boldsymbol{U}$ vanishes on the entire boundary $\partial\mathscr{S}$, i.e.,

$$\mathscr{R}_0(\mathscr{C})=\left\{\boldsymbol{U}\in\mathscr{R}\ :\ \boldsymbol{U}\big|_{\partial\mathscr{C}}=0\right\}=\{0\}. \tag{1.205}$$

A similar assertion is proved by Lods & Mardare in [LoM1], but for $C^{2,1}$-smooth hypersurface with the Lipschitz boundary $\partial\mathscr{S}$ and when a Killing's vector field expires on the entire boundary $\partial\mathscr{S}$. An earlier version of the "infinitesimal rigid displacement lemma" is due to I. Vekua [Ve1], who proved it using the theory of "generalized analytic functions".

Corollary 1.48 (Korn's I inequality "with boundary condition"). *Let $\mathscr{C}\subset\mathbb{R}^n$ be a C^ℓ-smooth hypersurface with the Lipschitz boundary $\Gamma:=\partial\mathscr{C}\neq\emptyset$ and $\ell\geqslant 2$, $|s|\leqslant\ell$. Then*

$$\big\|\boldsymbol{U}\big|\mathbb{H}^s_p(\mathscr{C})\big\|\leqslant M\big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U})\big|\mathbb{H}^{s-1}_p(\mathscr{C})\big\|\qquad\forall\,\boldsymbol{U}\in\widetilde{\mathbb{H}}^s_p(\mathscr{C})$$

for some constant $M>0$. In other words: the mapping

$$\boldsymbol{U}\mapsto\big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U})\big|\mathbb{H}^{s-1}_p(\mathscr{C})\big\| \tag{1.206}$$

is an equivalent norm on the space $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$.

Proof: If the claimed inequality (1.206) is false, there exists a sequence $\boldsymbol{U}^j\in\widetilde{\mathbb{H}}^s_p(\mathscr{C})$, $j=1,2,\dots$ such that

$$\big\|\boldsymbol{U}^j\big|\mathbb{H}^s_p(\mathscr{C})\big\|=1\quad\forall j=1,2,\dots\quad\lim_{j\to\infty}\big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^j)\big|\mathbb{H}^{s-1}_p(\mathscr{C})\big\|=0.$$

Due to the compact embedding $\widetilde{\mathbb{H}}^s_p(\mathscr{C})\subset\mathbb{H}^s_p(\mathscr{C})\subset\mathbb{H}^{s-1}_p(\mathscr{C})$, a convergent subsequence $\boldsymbol{U}^{j_1},\boldsymbol{U}^{j_2},\dots$ in $\mathbb{H}^{s-1}_p(\mathscr{C})$ can be selected. Let $\boldsymbol{U}^0=\lim_{k\to\infty}\boldsymbol{U}^{j_k}$. Then

$$\big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^0)\big|\mathbb{H}^{s-1}_p(\mathscr{C})\big\|=\lim_{k\to\infty}\big\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^{j_k})\big|\mathbb{H}^{s-1}_p(\mathscr{C})\big\|=0$$

and $\boldsymbol{U}^0$ is a Killing's vector field. Since $\boldsymbol{U}(x)=0$ on Γ, due to Lemma 1.47 $\boldsymbol{U}^0(x)=0$ for all $x\in\mathscr{C}$ which contradicts to $\big\|\boldsymbol{U}^0\big|\mathbb{H}^s_p(\mathscr{C})\big\|=\lim_{k\to\infty}\big\|\boldsymbol{U}^{j_k}\big|\mathbb{H}^s_p(\mathscr{C})\big\|=1$. ■

Let us check the following equalities for a later use:

$$\nabla_{\Omega^\varepsilon}\boldsymbol{U}=\big[\mathscr{D}_jU^0_k\big]_{n+1\times n+1}+\langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{W}_{\Omega^\varepsilon}, \tag{1.207}$$

where

$$\boldsymbol{U}:=\sum_{m=1}^{n+1}U^0_m\boldsymbol{d}^m=\sum_{m=1}^{n}U_m\boldsymbol{e}^m,\quad U^0_{n+1}=\sum_{m=1}^{n}\mathscr{N}_mU_m,\quad\mathscr{D}_{n+1}:=\partial_{\mathscr{N}},\quad\boldsymbol{d}^{n+1}:=\mathscr{N}.$$

$\mathscr{W}_{\Omega^\varepsilon}$ is the extended Weingarten matrix

$$\mathscr{W}_{\Omega^\varepsilon}:=\big[\mathscr{D}_j\mathscr{N}_k\big]_{n+1\times n+1} \tag{1.208}$$

and its last column and last row are 0, because $\mathscr{D}_j\mathscr{N}_{n+1}=\mathscr{D}_{n+1}\mathscr{N}_j=\mathscr{D}_{n+1}\mathscr{N}_{n+1}=0$ for $j=1,\dots,n$.

In fact (see (3.13) fore some further details of calculation):

$$
\begin{aligned}
\nabla_{\Omega^\varepsilon}\boldsymbol{U} := \big[\partial_j U_k\big]_{n\times n} = \sum_{j,ky=1}^{n} \partial_j U_k \boldsymbol{e}^j \otimes \boldsymbol{e}^k \\
:= \sum_{j,k=1}^{n} [\mathscr{D}_j + \mathscr{N}_j \partial_{\mathscr{N}}][U_k^0 + \mathscr{N}_k \langle \mathscr{N}, \boldsymbol{U}\rangle][\boldsymbol{d}^j + \mathscr{N}_j \mathscr{N}] \otimes [\boldsymbol{d}^k + \mathscr{N}_k \mathscr{N}] \\
= \sum_{j,k=1}^{n} (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes [\boldsymbol{d}^k + \mathscr{N}_k \mathscr{N}] + \sum_{j,k=1}^{n} \mathscr{D}_j [\mathscr{N}_k \langle \mathscr{N}, \boldsymbol{U}\rangle]\boldsymbol{d}^j \otimes [\boldsymbol{d}^k + \mathscr{N}_k \mathscr{N}] \\
+ \sum_{j,k=1}^{n} \mathscr{N}_j^2 (\partial_{\mathscr{N}} U_k^0)\mathscr{N} \otimes [\boldsymbol{d}^k + \mathscr{N}_k \mathscr{N}] + \sum_{j,k=1}^{n} \mathscr{N}_j^2 \mathscr{N}_k^2 \partial_{\mathscr{N}} \langle \mathscr{N}, \boldsymbol{U}\rangle \mathscr{N} \otimes \mathscr{N} \\
= \sum_{j,k=1}^{n} (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes \boldsymbol{d}^k + \sum_{j,k=1}^{n} \mathscr{N}_k (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
+ \sum_{j,k=1}^{n} \langle \mathscr{N}, \boldsymbol{U}\rangle (\mathscr{D}_j \mathscr{N}_k)\boldsymbol{d}^j \otimes [\boldsymbol{d}^k + \mathscr{N}_k \mathscr{N}] + \sum_{j,k=1}^{n} \mathscr{N}_k^2 \mathscr{D}_j \langle \mathscr{N}, \boldsymbol{U}\rangle \boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
+ \sum_{k=1}^{n} (\mathscr{D}_{n+1} U_k^0)\boldsymbol{d}^{n+1} \otimes \boldsymbol{d}^k + \sum_{k=1}^{n} \left[\mathscr{N}_k \mathscr{D}_{n+1} U_k^0 + \mathscr{D}_{n+1} U_{n+1}^0\right] \boldsymbol{d}^{n+1} \otimes \boldsymbol{d}^{n+1} \\
= \sum_{j,k=1}^{n} (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes \boldsymbol{d}^k + \sum_{j,k=1}^{n} \left[\mathscr{D}_j (\mathscr{N}_k U_k^0) - U_k^0 \mathscr{D}_j \mathscr{N}_k\right] \boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
+ \langle \mathscr{N}, \boldsymbol{U}\rangle \sum_{j,k=1}^{n} (\mathscr{D}_j \mathscr{N}_k)\boldsymbol{d}^j \otimes \boldsymbol{d}^k + \sum_{j=1}^{n} \mathscr{D}_j \langle \mathscr{N}, \boldsymbol{U}\rangle \boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
+ \sum_{k=1}^{n} (\mathscr{D}_{n+1} U_k^0)\boldsymbol{d}^{n+1} \otimes \boldsymbol{d}^k + (\mathscr{D}_{n+1} U_{n+1}^0)\boldsymbol{d}^{n+1} \otimes \boldsymbol{d}^{n+1} \\
= \sum_{j,k=1}^{n+1} (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes \boldsymbol{d}^k - \sum_{j,k=1}^{n} U_k^0 (\mathscr{D}_j \mathscr{N}_k)\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} + \langle \mathscr{N}, \boldsymbol{U}\rangle \sum_{j,k=1}^{n} (\mathscr{D}_j \mathscr{N}_k)\boldsymbol{d}^j \otimes \boldsymbol{d}^k \\
= \big[\mathscr{D}_j U_k\big]_{(n+1)\times(n+1)} + \langle \mathscr{N}, \boldsymbol{U}\rangle \mathscr{W}_{\Omega^\varepsilon} - \sum_{j,k=1}^{n} U_k^0 (\mathscr{D}_j \mathscr{N}_k)\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
= \big[\mathscr{D}_j U_k\big]_{(n+1)\times(n+1)} + \langle \mathscr{N}, \boldsymbol{U}\rangle \mathscr{W}_{\Omega^\varepsilon} - \big[(\mathscr{W}_{\Omega^\varepsilon} \boldsymbol{U}^0)_j \delta_{j,n+1}\big]_{(n+1)\times(n+1)},
\end{aligned}
$$

since

$$
\partial_{\mathscr{N}} \mathscr{N}_j = 0, \quad \sum_{j,k=1}^{n} \mathscr{N}_j^2 = 1, \quad \sum_{j=1}^{n} \mathscr{N}_j \mathscr{D}_j = 0, \quad \sum_{j=1}^{n} \mathscr{N}_j \boldsymbol{d}^j = 0,
$$

$$
\sum_{k=1}^{n} \mathscr{N}_k U_k^0 = 0, \quad \sum_{k=1}^{n} \mathscr{N}_k \mathscr{D}_j \mathscr{N}_k = \frac{1}{2}\mathscr{D}_j \sum_{k=1}^{n} \mathscr{N}_k^2 = \frac{1}{2}\mathscr{D}_j 1 = 0, \quad j = 1, 2 \ldots, n+1.
$$

For a domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\mathscr{M} := \partial\Omega$ and $\mathscr{M}_0 \subset \mathscr{M}$-a subsurface of non-zero measure let $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ denote a subspace of functions $\varphi \in \mathbb{W}^1(\Omega, \mathscr{M}_0)$ which is the closure of the set $C^\infty(\Omega, \mathscr{M}_0)$ of smooth functions $\varphi(x)$ which have vanishing trace on $\mathscr{M}_0$, i.e. $\varphi^+(\mathscr{x}) = 0$ for all $\mathscr{x} \in \mathscr{M}_0$. The space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ inherits the standard norm from $\mathbb{W}^1(\Omega)$:

$$\|\varphi \,|\, \mathbb{W}^1(\Omega)\,\| := \left[\|\varphi \,|\, \mathbb{L}_p(\Omega)\| + \sum_{j=1}^{n} \|\partial_j\varphi \,|\, \mathbb{L}_p(\Omega)\|\right]^{1/p}.$$

Since the space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ does not contain constants, it is easy to prove the following.

Lemma 1.49 *The formula*

$$\|\varphi \,|\, \widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)\,\| := \left[\sum_{j=1}^{n} \|\partial_j\varphi \,|\, \mathbb{L}_p(\Omega)\|\right]^{1/p}. \tag{1.209}$$

defines an equivalent norm in the space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$.

If ε is sufficiently small, the boundary $\mathscr{M}_\varepsilon := \partial\Omega_\varepsilon$ is represented as the union of three C^1-smooth surfaces $\mathscr{M}_\varepsilon = \mathscr{M}_{\varepsilon,D} \cup \mathscr{M}^-_{\varepsilon,N} \cup \mathscr{M}^+_{\varepsilon,N}$, where $\mathscr{M}_{\varepsilon,D} = \partial\mathscr{C} \times [-\varepsilon, \varepsilon]$ is the lateral surface, $\mathscr{M}^+_{\varepsilon,N} = \mathscr{C} \times \{+\varepsilon\}$ is the upper surface and $\mathscr{M}^-_{\varepsilon,N} = \mathscr{C} \times \{-\varepsilon\}$ is the lower surface of the of the boundary $\mathscr{M}_\varepsilon$ of the layer domain Ω_ε.

The next Lemma 1.50 is proved for a later use in § 3.

Lemma 1.50 $T \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M})$, *Let* $\mathscr{M}_0 := \gamma \times [-\varepsilon, \varepsilon]$, *where* $\gamma \subset \Gamma := \partial\mathscr{C}$ *is a subset of the boundary of the surface* $\mathscr{C}$ *of non-trivial measure. If* $g \in \mathbb{L}_2(\Omega_\varepsilon)$, *for the linear functional*

$$E_\varepsilon(u) = \int\limits_{\Omega_\varepsilon} g(x)u(x)\,dx, \qquad u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0) \tag{1.210}$$

we have the following estimate:

$$E_\varepsilon(u) \leqslant C\|g|\mathbb{L}_2(\Omega_\varepsilon)\|\;\|\mathscr{D}_\mathscr{C} u|\mathbb{L}_2(\Omega_\varepsilon)\| \tag{1.211}$$

for some constant $C > 0$ *independent of* $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$.

Proof: To prove (1.211) we recall that $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$ vanishes on the lateral subsurface $\mathscr{x} \in \mathscr{M}_0 \subset \mathscr{M}_D := \partial\mathscr{C} \times (-\varepsilon, \varepsilon)$.

Let $\mathscr{C}_t$ be the "parallel" surface to the mid-surface $\mathscr{C}$ on a distance $|t|$ and for negative $t < 0$ the surface $\mathscr{C}_t$ is "below" $\mathscr{C}$, while for positive $t > 0$ is "above" $\mathscr{C}$, i.e., in the direction of the normal vector filed $\boldsymbol{\nu}(\mathscr{x})$, $\mathscr{x} \in \mathscr{C}$. Note, that $\mathscr{C}_{\pm 1\varepsilon} = \mathscr{M}^\pm_D$. Taking $u(\mathscr{x}, t)$, $\mathscr{x} \in \mathscr{C}$, $-\varepsilon < t < \varepsilon$ from a dense subset of the space $\widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$ we can assume that

$u(\cdot,t) \in \widetilde{\mathbb{W}}^1(\mathscr{C}_t)$ for all fixed $-\varepsilon \leqslant t \leqslant \varepsilon$. Since $u(\mathscr{x},t)$ vanishes on the part of the boundary $\mathscr{M}_0 \cap \partial\mathscr{C}_t$, the Sobolev semi-norm

$$\|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\|_0 := \|\mathscr{D}_{\mathscr{C}} u(\cdot,t)|\mathbb{L}_2(\mathscr{C}_t)\| = \left[\sum_{j=1}^{3}\int_{\mathscr{C}_t} |\mathscr{D}_j u(\mathscr{x},t)|^2 d\sigma\right]^{1/2}$$

turns into the norm and is equivalent to the standard Sobolev norm

$$\|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\| := \left[\int_{\mathscr{C}_t} |u(\mathscr{x},t)|^2 d\sigma + \sum_{j=1}^{3}\int_{\mathscr{C}_t} |\mathscr{D}_j u(\mathscr{x},t)|^2 d\sigma\right]^{1/2}$$

for all $t \in [-\varepsilon, \varepsilon]$, which means

$$M\|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\| \leqslant \|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\|_0 \leqslant \|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\|$$

for some constant $0 < M < 1$, independent of t and u. From this equivalence we get the estimate

$$\|u(\cdot,t)|\mathbb{L}_2(\mathscr{C}_t)\|^2 \leqslant \frac{1-M^2}{M^2}\|\mathscr{D}_{\mathscr{C}} u(\cdot,t)|\mathbb{L}_2(\mathscr{C}_t)\|^2. \tag{1.212}$$

By integrating the obtained inequality with respect to the variable t we get the following final estimate

$$\|u|\mathbb{L}_2(\Omega_\varepsilon)\| \leqslant \frac{\sqrt{1-M^2}}{M}\|\mathscr{D}_{\mathscr{C}} u|\mathbb{L}_2(\Omega_\varepsilon)\| \qquad \forall u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0). \tag{1.213}$$

The estimate in (1.211) follows with the help of the Cauchy inequality and inequality (1.213):

$$\int_{\Omega_\varepsilon} g(x)u(x)dx \leqslant \|g|\mathbb{L}_2(\Omega_\varepsilon)\|\|u|\mathbb{L}_2(\Omega_\varepsilon)\| \leqslant \frac{\sqrt{1-M^2}}{M}\|g|\mathbb{L}_2(\Omega_\varepsilon)\|\|\mathscr{D}_{\mathscr{C}} u|\mathbb{L}_2(\Omega_\varepsilon)\|. \qquad \blacksquare$$

Remark 1.51 *Let us stress that in estimate* (1.211) *we only need the surface derivatives* $\mathscr{D}_1, \mathscr{D}_2$ *and* $\mathscr{D}_3$. *If we would have* $g \in \mathbb{W}^{-1}(\Omega_\varepsilon)$, *then we should assume* $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon)$. *These spaces are dual and, therefore if the integral in the functional* E_ε *in* (1.210), *is understood as the duality, the functional* E_ε *is bounded, but then estimate writes*

$$E_\varepsilon(u) \leqslant C\|g|\mathbb{L}_2(\Omega_\varepsilon)\|\;\|\mathscr{D}_{\Omega_\varepsilon} u|\mathbb{L}_2(\Omega_\varepsilon)\|, \qquad \mathscr{D}_{\Omega_\varepsilon} = (\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3, \mathscr{D}_4)^\top. \tag{1.214}$$

In this estimate all derivatives, the surface and the transversal $\partial_t = \partial_{\boldsymbol{\nu}} = \mathscr{D}_4$ *(the normal to the surface* $\mathscr{C}$*) are participating.*

1.7 AUXILIARY FROM THE OPERATOR THEORY

The results exposed in the present section will be applied to complex valued matrices, which are identified with operators in the finite dimensional space $\mathbb{C}$. Nevertheless, we will formulate results in general setting of operators in a Hilbert space.

Throughout this section we assume that $\mathfrak{H}$ is a Hilbert space with respect to some continuous scalar product, a bilinear form $(\cdot,\cdot)\ :\ \mathfrak{H}\times\mathfrak{H}\to\mathbb{C}$, i.e.,

$$(\lambda\, u+\mu\, w, v)=\overline{\lambda}(u,v)+\overline{\mu}(w,v)\,,\qquad (u,\lambda\, v+\mu\, z)=\lambda(u,v)+\mu(u,z)\,,$$

$$\big|(u,v)\big|\leqslant C\|u\|_{\mathfrak{H}}\|v\|_{\mathfrak{H}}\,,\quad \forall\, u,w\in\mathfrak{H}\,,\quad \forall\, v,z\in\mathfrak{H}\,,$$

$$(\varphi,\psi)=\overline{(\psi,\varphi)}\qquad \forall\,\varphi,\psi\in\mathfrak{H}.$$

Recall, that the dual operator $(\boldsymbol{A}^*\varphi,\psi)=(\varphi,\boldsymbol{A}\psi)$ maps continuously the same space $\boldsymbol{A}^*\ :\ \mathfrak{H}\longrightarrow\mathfrak{H}$ and $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is **self-adjoint** operator if

$$(\boldsymbol{A}\varphi,\psi)=(\varphi,\boldsymbol{A}\psi)\qquad \forall\,\varphi,\ \psi\in\mathfrak{H}. \tag{1.215}$$

$\boldsymbol{A}\in\mathscr{L}(\mathfrak{H},\mathfrak{H})$ is **positive definite** (or **coercive**) if the inequality

$$(\boldsymbol{A}\varphi,\varphi)\geqslant C\|\varphi\big|\mathfrak{H}\|^2 \tag{1.216}$$

holds for some constant $C>0$ and all $\varphi\in\mathfrak{H}$.

Lemma 1.52 *Let $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$. The inequality*

$$\|\boldsymbol{A}\varphi\big|\mathfrak{H}\|\geqslant C\|\varphi\big|\mathfrak{H}\| \tag{1.217}$$

with some constant $C>0$ holds if and only if the operator $\boldsymbol{A}$ is normally solvable $\Im\, A=\overline{\Im\, A}$ and injective $\operatorname{Ker}\boldsymbol{A}=\{0\}$.

Proof. If the inequality (1.217) holds, then $\boldsymbol{A}\varphi=0$, $\varphi\in\mathfrak{H}$ implies $\varphi=0$ and $\operatorname{Ker}\boldsymbol{A}=\{0\}$. Now let $\psi_j=\boldsymbol{A}\varphi_j\to\psi_0$ (convergence in the norm). From (1.217) follows the convergence $\varphi_j\to\varphi_0$. Due to continuity of $\boldsymbol{A}$ this implies $\boldsymbol{A}\varphi_0=\psi_0\in\Im\,\boldsymbol{A}$ and $\Im\,\boldsymbol{A}$ is closed.

Vice versa, let $\boldsymbol{A}$ be normally solvable and $\operatorname{Ker}\boldsymbol{A}=\{0\}$. Then $\Im\,\boldsymbol{A}$ is a Hilbert space, subspace of $\mathfrak{H}$ and the operator $\boldsymbol{A}\ :\ \mathfrak{H}\to\Im\,\boldsymbol{A}$ is bijective. Due to the Banach's Inverse mapping theorem $\boldsymbol{A}$ is invertible: there exists $\mathbf{B}\in\mathscr{L}(\Im\,\boldsymbol{A})$ such that $\mathbf{AB}x=x$ $\mathbf{BA}y=y$ for all $x\in\Im\,\boldsymbol{A}$ and all $y\in\mathfrak{H}$. Inserting in $\|\mathbf{B}\psi\big|\mathfrak{H}\|\leqslant C\|\psi\big|\Im\,\boldsymbol{A}\|:=\|\psi\big|\mathfrak{H}\|$ the equality $\psi=\boldsymbol{A}\varphi$, $\varphi\in\mathfrak{H}$, we get (1.217). ∎

Definition 1.53 *For an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ the closed set*

$$\Sigma(\boldsymbol{A}):=\overline{\{(\boldsymbol{A}\varphi,\varphi)\ :\ \varphi\in\mathfrak{H}\}}, \tag{1.218}$$

where the overbar denotes closing of the set, is called the **spectral set** *of $\boldsymbol{A}$.*

Lemma 1.54 *If the spectral set $\Sigma(\boldsymbol{A})$ of an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is real valued $\Sigma(\boldsymbol{A})\subset\mathbb{R}$, then $\boldsymbol{A}$ is self-adjoint.*

Proof. We proceed as follows:

$$\begin{aligned}
(\boldsymbol{A}\varphi,\psi) &= \frac{1}{4}\Big\{(\boldsymbol{A}[\varphi+\psi],\varphi+\psi)-(\boldsymbol{A}[\varphi-\psi],\varphi-\psi)\\
&\quad +i(\boldsymbol{A}[\varphi+i\psi],\varphi+i\psi)-i(\boldsymbol{A}[\varphi-i\psi],\varphi-i\psi)\Big\}\\
&= \frac{1}{4}\Big\{\overline{(\boldsymbol{A}[\varphi+\psi],\varphi+\psi)}-\overline{(\boldsymbol{A}[\varphi-\psi],\varphi-\psi)}\\
&\quad +i\overline{(\boldsymbol{A}[\varphi+i\psi],\varphi+i\psi)}-i\overline{(\boldsymbol{A}[\varphi-i\psi],\varphi-i\psi)}\Big\}\\
&= \frac{1}{4}\Big\{(\varphi+\psi,\boldsymbol{A}[\varphi+\psi])-(\varphi-\psi,\boldsymbol{A}[\varphi-\psi])\\
&\quad +i(\varphi+i\psi,\boldsymbol{A}[\varphi+i\psi])-i(\varphi-i\psi,\boldsymbol{A}[\varphi-i\psi])\Big\}\\
&= (\varphi,\boldsymbol{A}\psi)\,, \qquad \varphi,\psi\in\mathfrak{H}
\end{aligned}$$

since $(\boldsymbol{A}u,u)=\overline{(\boldsymbol{A}u,u)}$ by the condition $\Sigma(\boldsymbol{A})\subset\mathbb{R}$ and $\overline{(\boldsymbol{A}u,u)}=(u,\boldsymbol{A}u)$ by the definition. ■

Corollary 1.55 *If an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is positive definite, it is self-adjoint and invertible.*

Proof. If $\boldsymbol{A}$ is positive definite, its spectral set is real valued and $\boldsymbol{A}$ is self-adjoint.

From (1.216) we get

$$\|\boldsymbol{A}\varphi|\mathfrak{H}\|\|\varphi|\mathfrak{H}\|\geqslant(\boldsymbol{A}\varphi,\varphi)\geqslant C\|\varphi|\mathfrak{H}\|^2$$

and further

$$\|\boldsymbol{A}\varphi|\mathfrak{H}\|\|\geqslant C\|\varphi|\mathfrak{H}\|\,, \qquad \varphi\in\mathfrak{H}\,. \tag{1.219}$$

Due to Lemma 1.52 the inequality (1.216) implies that $\boldsymbol{A}$ is normally solvable and has a trivial kernel $\operatorname{Ker}\boldsymbol{A}=\{0\}$. Being self-adjoint $\boldsymbol{A}^*=\boldsymbol{A}$ the operator has the trivial cokernel $\dim\operatorname{Coker}\boldsymbol{A}=\dim Ker\,\boldsymbol{A}=0$ (due to (1.216) $\boldsymbol{A}\varphi=0$ implies that $\varphi=0$). Therefore, $\boldsymbol{A}$ is invertible. ■

Let $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ and $\boldsymbol{A}=\mathbf{R}\mathbf{H}_{\boldsymbol{A}}$ be its left polar decomposition, where $\mathbf{R}\in\mathbb{SO}(\mathfrak{H})$ is the orthogonal (unitary) operator $\mathbf{R}^*=\mathbf{R}^{-1}$ and $\mathbf{H}_{\boldsymbol{A}}$ is positive, self adjoint (Hermitian) operator

$$\langle\mathbf{H}_{\boldsymbol{A}}\varphi,\varphi\rangle\geqslant C_0\|\varphi\|^2, \qquad C_0>0, \qquad \mathbf{H}^*_{\boldsymbol{A}}=\mathbf{H}_{\boldsymbol{A}}, \quad \forall\,\varphi\in\mathfrak{H}.$$

Let us check, that $\mathbf{H}_{\boldsymbol{A}}=\sqrt{\boldsymbol{A}^*\boldsymbol{A}}$. Indeed, if $\boldsymbol{A}=\mathbf{R}\mathbf{U}_{\boldsymbol{A}}$, then $\boldsymbol{A}^*=\mathbf{H}^*_{\boldsymbol{A}}\mathbf{R}^*=\mathbf{H}_{\boldsymbol{A}}\mathbf{R}^{-1}$ and $\sqrt{\boldsymbol{A}^*\boldsymbol{A}}=\sqrt{\mathbf{H}_{\boldsymbol{A}}\mathbf{R}^{-1}\mathbf{R}\mathbf{H}_{\boldsymbol{A}}}=\sqrt{\mathbf{H}^2_{\boldsymbol{A}}}=\mathbf{H}_{\boldsymbol{A}}$. ■

Similarly, for the right polar decomposition $\boldsymbol{A}=\mathbf{H}'_{\boldsymbol{A}}\mathbf{R}'$ we get $\mathbf{H}'_{\boldsymbol{A}}=\sqrt{\boldsymbol{A}\,\boldsymbol{A}^*}$.

Note, that if $\boldsymbol{A}$ is positive definite (or, at least, has a real valued spectral set), then $\boldsymbol{A}$ is self adjoint $\boldsymbol{A}^*=\boldsymbol{A}$ and the polar decomposition is trivial $\mathbf{H}_A=\mathbf{H}'_A=\sqrt{\boldsymbol{A}\boldsymbol{A}}=\boldsymbol{A}$, $\mathbf{R}=\mathbf{R}'=\mathbf{I}$.

Note, that the norm has the following property:

$$\|\mathbf{RAR}\| = \|\mathbf{RA}\| = \|\mathbf{AR}\| = \|\boldsymbol{A}\| \qquad \forall\, \boldsymbol{A} \in \mathscr{L}(\mathfrak{H}), \quad \text{and} \quad \forall\, \mathbf{R} \in \mathbb{SO}(\mathfrak{H}). \tag{1.220}$$

Indeed,

$$\|\mathbf{RA}\| = \inf_{\varphi\in\mathfrak{H}} \sqrt{((\mathbf{RA})^*\mathbf{RA}\varphi,\varphi)} = \inf_{\varphi\in\mathfrak{H}} \sqrt{(\boldsymbol{A}^*\mathbf{R}^*\mathbf{RA}\varphi,\varphi)} = \inf_{\varphi\in\mathfrak{H}} \sqrt{(\boldsymbol{A}^*\boldsymbol{A}\varphi,\varphi)} = \|\boldsymbol{A}\|\,.$$

By using the obtained equality and recalling that $\|\boldsymbol{A}\| = \|\boldsymbol{A}^*\|$ and $\mathbf{R} \in \mathbb{SO}(\mathfrak{H})$ implies $\mathbf{R}^* \in \mathbb{SO}(\mathfrak{H})$, we prove the following

$$\|\mathbf{AR}\| = \|(\mathbf{AR})^*\| = \|\mathbf{R}^*\boldsymbol{A}^*\| = \|\boldsymbol{A}\|\,.$$

As a consequence, $\|\mathbf{RAR}\| = \|\mathbf{RA}\| = \|\mathbf{AR}\| = \|\boldsymbol{A}\|$. ■

Next we will prove, that if $\boldsymbol{A} = \mathbf{R}\mathbf{H}_{\boldsymbol{A}}$ is the left polar decomposition, then

$$\begin{aligned} &\operatorname{dist}(\boldsymbol{A}, \mathbb{SO}(\mathfrak{H})) = \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\| && \text{if } \boldsymbol{A} \text{ is invertible,}\\ &\operatorname{dist}(\boldsymbol{A}, \mathbb{SO}(\mathfrak{H})) \leqslant \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\| && \text{otherwise.}\end{aligned} \tag{1.221}$$

Indeed, due to (1.220),

$$\begin{aligned}\operatorname{dist}(\boldsymbol{A}, \mathbb{SO}(\mathfrak{H}) &= \inf_{\boldsymbol{V}\in\mathbb{SO}(\mathfrak{H})} \|\mathbf{R}\mathbf{H}_{\boldsymbol{A}} - \boldsymbol{V}\| = \inf_{\boldsymbol{V}\in\mathbb{SO}(fH)} \|\mathbf{R}^*(\mathbf{R}\mathbf{H}_{\boldsymbol{A}} - \boldsymbol{V})\|\\ &= \inf_{\boldsymbol{V}\in\mathbb{SO}(\mathfrak{H})} \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{R}^*\boldsymbol{V}\| \leqslant \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\|\,,\end{aligned}$$

since $\mathbf{I}, \mathbf{R}^*\boldsymbol{V} \in \mathbb{SO}(\mathfrak{H})$.

The second inequality in (1.221) is proved. To prove the first equality in (1.221) we assume $\boldsymbol{A}$ is invertible and...

The subsequent proof has to be modified later!

Let us show that a small perturbation $\mathbf{R}(t) := \mathbf{R} + t\mathbf{U}$, $|t| < \varepsilon$, of $\mathbf{R}$ by arbitrary matrix $\mathbf{U}$ with the constraint $\mathbf{R}(t) \in \mathbb{SO}(\mathfrak{H})$ (i.e., $\mathbf{R}(t)\mathbf{R}^*(t) = \mathbf{I}$ for all $|t| < \varepsilon$) gives

$$\begin{aligned}\inf_{t\in\mathbb{R}} \|\boldsymbol{A} - \mathbf{R}(t)\| &= \inf_{t\in\mathbb{R}} \inf_{\varphi\in\mathfrak{H}} \sqrt{((\boldsymbol{A} - \mathbf{R}(t))\varphi, (\boldsymbol{A} - \mathbf{R}(t))\varphi)}\\ &= \inf_{t\in\mathbb{R}} \inf_{\varphi\in\mathfrak{H}} \sqrt{((\boldsymbol{A} - \mathbf{R}(t))^*(\boldsymbol{A} - \mathbf{R}(t))\varphi, \varphi)}\\ &= \inf_{t\in\mathbb{R}} \inf_{\varphi\in\mathfrak{H}} \sqrt{(\,[(\mathbf{H}_{\boldsymbol{A}} - \mathbf{I})^2 - t(\mathbf{U}^*\boldsymbol{A} + \boldsymbol{A}^*\mathbf{U})]\,\varphi, \varphi)}\\ &= \inf_{t\in\mathbb{R}} \inf_{\varphi\in\mathfrak{H}} \sqrt{((\mathbf{H}_{\boldsymbol{A}} - \mathbf{I})^2\varphi, \varphi)} = \inf_{\varphi\in\mathfrak{H}} \sqrt{((\mathbf{H}_{\boldsymbol{A}} - \mathbf{I})\varphi, (\mathbf{H}_{\boldsymbol{A}} - \mathbf{I})\varphi)}\\ &= \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\|\end{aligned}$$

because the minimization by t shows, that the norm minimizes at $(\mathbf{U}^*\boldsymbol{A} + \boldsymbol{A}^*\mathbf{U}) = (\mathbf{U}^*\boldsymbol{A} + \boldsymbol{A}^*\mathbf{U})^* = 0$. ■

1.8 GEOMETRIC RIGIDITY

The basic rigidity result relevant to passage to the thin plate limit is the following.

Proposition 1.56 (see [FJM1]) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$ and $1 < p < \infty$. There exists a constant $C(\Omega)$ with the following property: For each $U \in \mathbb{W}^1(\Omega)$ there is an associated rotation $R_U \in \mathbb{SO}(n)$ such that,*

$$\left\|\nabla U - R_U \middle| \mathbb{L}_p(\Omega)\right\| \leqslant C(\Omega) \left\|\operatorname{dist}(\nabla U, \mathbb{SO}(n)) \middle| \mathbb{L}_p(\Omega)\right\|. \tag{1.222}$$

The result is sharp in the sense that neither the norm on the right hand side nor the power with which it appears can be improved.

By considering the special case when the right hand side in (1.222) is zero, Proposition 1.56 reduces to the following.

Corollary 1.57 (Liouville theorem) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$. If U is a $\mathbb{W}^1(\Omega)$ map which satisfies the partial differential equation*

$$\nabla U(x) \in \mathbb{SO}(n) \qquad \text{a.e. in} \quad \Omega, \tag{1.223}$$

then it is affine $U(x) = Rx + c$, $R \in \mathbb{SO}(n)$, $c = const$ or, equivalently, $\nabla U = R \in \mathbb{SO}(n)$.

Proof: In the setting of Sobolev maps this was first proved by Reshetnyak [Re1]. A short modern proof belongs to G. Friesecke, R.D. James & S. Müller [FJM1] and consists of three observations.

First, for $n \times n$ matrix $A = [a_{jk}]_{n\times n}$ let $\operatorname{cof} A$ denote the matrix of cofactors of A, i.e.,

$$\operatorname{cof} A = \left[(-1)^{j+k} \det A_{jk}\right]_{n\times n}, \tag{1.224}$$

where A_{jk} is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the j-th row and the k-th column. It is well-known that

$$\operatorname{div} \operatorname{cof} \nabla U = 0 \qquad \text{for all} \quad U \in \mathbb{W}^1(\Omega). \tag{1.225}$$

Note first that if the equality (1.225) is proved for $U \in C^2(\Omega)$, it can be extended to arbitrary $U \in \mathbb{W}^1(\Omega)$.

We have to prove

$$C_i := \sum_{k=1}^n \partial_k (\operatorname{cof} \nabla U)_{ki} = 0, \quad i = 1, \ldots, n. \tag{1.226}$$

Note, that C_i can be formally written as

$$C_i = \det \begin{pmatrix} \partial_1 & \partial_2 & \cdots & \partial_n \\ \partial_1 v_1^{(i)} & \partial_2 v_1^{(i)} & \cdots & \partial_n v_1^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 v_{n-1}^{(i)} & \partial_2 v_{n-1}^{(i)} & \cdots & \partial_n v_{n-1}^{(i)} \end{pmatrix}, \tag{1.227}$$

where $v^{(i)} = (U_1, \cdots, U_{i-1}, U_{i+1}, \cdots, U_n)$. The equality (1.226) follows from the following assertion: For any $u = (u_1, \cdots u_{n-1}) \in C^2(\mathbb{R}^{n-1})$

$$\det \begin{pmatrix} \partial_1 & \partial_2 & \cdots & \partial_n \\ \partial_1 u_1 & \partial_2 u_1 & \cdots & \partial_n u_1 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 u_{n-1} & \partial_2 u_{n-1} & \cdots & \partial_n u_{n-1} \end{pmatrix} = 0, \tag{1.228}$$

which can be easily proved by induction, expanding the determinant with respect to the last row.

Second, (1.223) implies that U is harmonic, and in particular smooth. To prove this recall, that if $A \in \mathbb{GL}(n)$ is an invertible matrix, $A^{-1} = \det A(\operatorname{cof} A)^\top$. In particular, for $B \in \mathbb{SO}(n)$, which means $B^{-1} = B^\top$, $\det B = 1$, we get $\operatorname{cof} B = B$. Then from the asserted inclusion $\nabla U \in \mathbb{SO}(n)$ we get $\nabla(U)(x) = \operatorname{cof} \nabla U(x)$ and by taking the divergence we get the following:

$$\Delta U = \operatorname{div} \nabla \boldsymbol{U} = \operatorname{div} \operatorname{cof} \nabla U(x) = 0.$$

Third, the second gradient squared of any harmonic map can be expressed pointwise via derivatives of the inner products,

$$\frac{1}{2}\left(|\nabla U|^2 - n\right) = \langle \nabla U, \Delta \nabla U \rangle + |\nabla^2 U|^2 = |\nabla^2 U|^2; \tag{1.229}$$

but $|\nabla U|^2 - n = 0$ when U satisfies (1.223). ■

An estimate in terms of $\varepsilon + \sqrt{\varepsilon}$, where $\varepsilon := \big\|\operatorname{dist}(\nabla U, \mathbb{SO}(n))\big|\mathbb{L}_p(\Omega)\big\|$, is much easier to prove, but is insufficient for the application to plate theory, where one needs to sum the estimate over many small cubes of size h.

Corollary 1.58 (see [Re1]) *If $U_j \to U$ in $\mathbb{W}^1(\Omega)$ and $\operatorname{dist}(\nabla U_j, \mathbb{SO}(n)) \to 0$ in measure, then $\nabla U_j \to R$ in $\mathbb{L}_2(\Omega)$ for some constant rotation matrix $R \in \mathbb{SO}(n)$).*

Let us remind, that the space of $n \times n$ matrices $\mathbb{M}^{n\times n}(\mathbb{R})$ is a real Hilbert space with respect to the inner product (1.105) and the norm

$$\|A\| := \sqrt{\operatorname{Tr}(AA^\top)} = \sqrt{\operatorname{Tr}(A^\top A)} = \sqrt{\sum_{i,j} a_{ij}^2}, \qquad \forall\, A = [a_{ij}]_{n\times n}. \tag{1.230}$$

Note, that the norm has the following property:

$$\|RAR\| = \|RA\| = \|AR\| = \|A\| \qquad \forall\, A \in \mathbb{M}^{n\times n}, \quad \text{and} \quad \forall\, R \in \mathbb{SO}(n). \tag{1.231}$$

Indeed,

$$\|RA\| = \sqrt{\operatorname{Tr}[(RA)^\top (RA)]} = \sqrt{\operatorname{Tr}(A^\top R^\top RA)} = \sqrt{\operatorname{Tr}(A^\top A)} = \|A\|,$$
$$\|AR\| = \sqrt{\operatorname{Tr}[(AR)(AR)^\top]} = \sqrt{\operatorname{Tr}(ARR^\top A^\top)} = \sqrt{\operatorname{Tr}(AA^\top)} = \|A\|$$

and, as a consequence, $\|RAR\| = \|RA\| = \|AR\| = \|A\|$. ∎

Let $A = RH_A$ be the left polar decomposition of a matrix A, where $R \in \mathbb{SO}(n)$ is the orthogonal (unitary) matrix $R^\top = R^{-1}$ and H_A is positive, self adjoint (Hermitian) matrix

$$\langle H_A \xi, \xi \rangle \geqslant C_0 \|\xi\|^2, \qquad C_0 > 0, \qquad H_A^* = H_A, \qquad \forall\, \xi \in \mathbb{C}.$$

Let us check, that $H_A = \sqrt{A^\top A}$. Indeed, if $A = RU_A$, then $A^\top = H_A^\top R^\top = H_A R^{-1}$ and $\sqrt{A^\top A} = \sqrt{H_A R^{-1} R H_A} = \sqrt{H_A^2} = H_A$. ∎

Similarly, for the right polar decomposition $A = H'_A R'$ we get $H'_A = \sqrt{A\,A^\top}$.

By analogue with (1.221) is proved, that if $A = RH_A$ is the left polar decomposition of A, then

$$\begin{aligned} &\operatorname{dist}(A, \mathbb{SO}(n)) = \|H_A - I\| &&\text{if } \det A \neq 0 \quad \text{(i.e., } A \text{ is invertible)},\\ &\operatorname{dist}(A, \mathbb{SO}(n)) \leqslant \|H_A - I\| &&\text{otherwise.} \end{aligned} \tag{1.232}$$

2 Γ-CONVERGENCE BY NUMBERS

2.1 Some preliminaries

The main purpose of the present chapter is to introduce Γ-convergence, discuss it's properties and demonstrate its application on a simplest version of dimension-reduced problems. Γ-convergence was introduced by De Giorgi in [DF1] and represents powerful toolkit for the investigation of various dimension-reduction problems.

Our exposition follows mostly the book [Br1]. Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ and $\mathbb{R}$ the set of real numbers.

Definition 2.1 *Let $f : X \to \mathbb{R}$. We define the lower limit (*$\liminf$ *for short) of f at x as*

$$\begin{aligned}\liminf_{y\to x} f(y) &= \inf\left\{\liminf_{j} f(x_j) : x_j \in X,\ x_j \to x\right\}\\ &= \inf\left\{\lim_{j} f(x_j) : x_j \in X,\ x_j \to x,\ \exists \lim_{j} f(x_j)\right\}.\end{aligned}$$

and upper limit ($\limsup$ *for short) of f at x as*

$$\begin{aligned}\limsup_{y\to x} f(y) &= \sup\left\{\limsup_{j} f(x_j) : x_j \in X,\ x_j \to x\right\}\\ &= \sup\left\{\lim_{j} f(x_j) : x_j \in X,\ x_j \to x,\ \exists \lim_{j} f(x_j)\right\}.\end{aligned}$$

The lower limit is linked to our minimum problems much more then the upper limit. The first notion will be preferred in our statements, but many results will obviously hold for the limsup, with the due changes. Definition (2.1) can also be given if f is not defined in the whole X (in this case the x_j must be taken in the domain of f); in particular, we can have $X = \mathbb{N}$ and $x = \infty$ and recover the usual definition of $\liminf$ and $\limsup$ for sequences.

By taking $x_j = x$ we always get $\liminf_{y\to x} f(y) \leqslant f(x)$. Moreover, it can easily checked that

$$\begin{aligned}\liminf_{y\to x}(-f(y)) &= -\liminf_{y\to x} f(y),\\ \liminf_{y\to x}(f(y)+g(y)) &\geq \liminf_{y\to x} f(y) + \liminf_{y\to x} g(y),\\ \liminf_{y\to x}(f(y)+g(y)) &\leqslant \limsup_{y\to x} f(y) + \liminf_{y\to x} g(y).\end{aligned} \tag{2.1}$$

A way to interpret these limit operators is that they give the sharpest upper and lower bounds for the behaviour of f close to x : that is, for all $\varepsilon > 0$ we will have

$$\liminf_{y\to x} f(y) - \varepsilon < f(x') < \liminf_{y\to x} f(y) + \varepsilon,$$

provided that $d(x, x') < \delta = \delta(\varepsilon)$. With this observation in mind it can be easily checked that we have the equivalent topological definitions:

$$\liminf_{y\to x} f(y) = \sup_{U\in N(x)} \inf_{y\in U} f(y), \quad \limsup_{y\to x} f(y) \inf_{U\in N(x)} \sup_{y\in U} f(y), \tag{2.2}$$

where we have used the notation $N(x)$ for the family of all open sets containing a point $x \in X$.

2.2 Lower semicontinuity

Definition 2.2 *A function $f : X \to \overline{\mathbb{R}}$ will be said to be (sequentially) lower semicontinuous functions (l.s.c for short) at $x \in X$, if for every sequence (x_j) converging to x we have*

$$f(x) \leqslant \liminf_j f(x_j),$$

or, in other words,

$$f(x) = \min\left\{\liminf_j f(x_j) : \ x_j \to x\right\}.$$

We will say that f is lower semicontinuous functions (l.s.c for short) (on X) if it is l.s.c at all $x \in X$.

Remark 2.3 *The following conditions are equivalent:*

(i) *f is lower semicontinuous.*

(ii) *we have $f(x) = \liminf_{y \to x} f(y)$, for all $x \in X$.*

(iii) *for all $t \in \mathbb{R}$ the sublevel set $\{f \leqslant t\}$ is closed.*

Indeed the equivalence of (i) and (ii) is given by (2.2). Note that (i) implies that if $f(x_j) \leqslant t$ and $x_j \to x$ then $f(x) \leqslant t$, while if there exists x and $x_j \to x$ such that $f(x) > t > \liminf_j f(x_j)$ then (iii) is violated for such a t.

Remark 2.4 (i) *If f and g are l.s.c at x, then so is $f + g$ by (2.1).*

(ii) *Let $\{f_j : i \in I\}$ be a family of l.s.c functions (I an arbitrary set of indices, not necessarily countable). Then the function defined by $f(x) = \sup_i f_i(x)$* is l.s.c. *In fact, for fixed $x \in X$ and $x_j \to x$, we have*

$$f_i(x) \leqslant \liminf_j f_i(x_j) \leqslant \liminf_j f(x_j).$$

By taking the supremum for $i \in I$ we obtain $f(x) \leqslant \liminf_j f(x_j)$. In particular, the supremum of a family of continues functions is l.s.c.

(iii) *If $f = \chi_E$ is the characteristic function of the set E, then f is l.s.c, if and only if E is open, by Remark (2.3) (iii).*

(iv) *A function $f : X \to \overline{\mathbb{R}}$ is called upper semicontinuous if $-f$ is l.s.c. All the results of this section have an obvious counterpart for upper semicontinuous functions. In particular $f = \chi_E$ is upper semicontinuous if and only if E closed.*

2.3 Convexity

Definition 2.5 *We recall that a function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is convex if we have*

$$f(tz_1 + (1-t) z_2) \leqslant tf(z_1) + (1-t) f(z_2)$$

for all $z_1, z_2 \in \mathbb{R}^n$ and $t \in (0,1)$.

Remark 2.6 *(a) The convexity of f is equivalent to requiring that Jensen's inequality holds:*

$$f\left(\int_X g d\mu\right) \leqslant \int_X f(g)\, d\mu \tag{2.3}$$

for all probability spaces (X, μ) and measurable $f : X \to \mathbb{R}^n$.

(b) If $f \in C^1(\mathbb{R}^n)$, then it is convex if and only if

$$f(z) \leqslant f(w) + \left\langle f^{'}(z), z - w\right\rangle \tag{2.4}$$

for all $z, w \in \mathbb{R}^n$.

(c) The supremum of the family of convex functions is convex.

(d) If f is *a convex function and f is finite at every point of en open set Ω, then f is continuous on Ω and locally Lipschitz continuous on Ω.*

(e) If f is convex and there exists $1 \leqslant p < \infty$ and $c > 0$ such that

$$0 \leqslant f(z) \leqslant c\left(1 + |z|^p\right)$$

for all $z \in \mathbb{R}^n$, then f satisfies the local Lipschitz condition

$$|f(z) - f(w)| \leqslant c^{'}\left(1 + |z|^{p-1} + |w|^{p-1}\right)|z - w| \tag{2.5}$$

for all $z, w \in \mathbb{R}^n$ for some $c^{'}$ defending only c and p.

(f) If $f_j : \mathbb{R}^n \to \mathbb{R}$ is a sequence of locally equi-bounded convex functions then there exists a subsequence of (f_j) converging uniformly on all compact subsets of $\mathbb{R}^n$.

We can now recall the definition of Γ-convergence and make some first remark.

2.4 Γ-CONVERGENCE

Definition 2.7 *(Γ-convergence) We say that a sequence $f_j : X \to \overline{\mathbb{R}}$ Γ-converges in X to $f_\infty : X \to \overline{\mathbb{R}}$ if for all $x \in X$ we have*

(i) ($\liminf$ inequality) For every sequence (x_j) converging to x,

$$f_\infty(x) \leqslant \liminf_j f_j(x_j); \tag{2.6}$$

(ii) ($\limsup$ inequality) There exists a sequence $\left(x_j^0\right)$ converging to x, such that

$$f_\infty(x) \geq \limsup_j f_j\left(x_j^0\right); \tag{2.1}$$

The function f_∞ is called the Γ-limit of (f_j) and we write $f_\infty = \Gamma\text{-}\lim_j f_j$.

Pointwise definition. The definition above can be also given a fixed point $x \in X :$ we say that (f_j) Γ-convergence at x to the value $f_\infty(x)$ if $(i), (ii)$ above hold. In this case we write

$$f_\infty(x) = \Gamma\text{-}\lim_j f_j(x).$$

In this notation, f_j Γ-convergence to f_∞ if and only if

$$f_\infty(x) = \Gamma\text{-}\lim_j f_j(x)$$

at all $x \in X$.

If we want to highlight the role of the metric, we can add the dependence on the distance d, and write $\Gamma(d)$-$\lim_j$ $\Gamma(d)$-convergence and so on.

Different ways of writing the $\limsup$ **inequality.** Note that if (x_j) satisfies the $\limsup$ inequality, then by (2.6) we have

$$f_\infty(x) \leqslant \liminf_j f_j(x) \leqslant \limsup_j f_j(x) \leqslant f_\infty(x),$$

so that indeed

$$f_\infty(x) = \lim_j f_j(x_j),$$

hence (ii) can be substituted by

$(ii)'$ (exitance of a recovery sequence) there exists a sequence (x_j) converging to x, such that

$$f_\infty(x) = \lim_j f_j(x_j); \tag{2.8}$$

On the other hand, sometimes it is more convenient to prove (ii) with a small error and then deduce its validity by an approximation argument: that is, (ii) can be replaced by

$(ii)''$ (approximate $\limsup$ inequality) for all $\varepsilon > 0$ there exists a sequence (x_j) converging to x, such that

$$f_\infty(x) \geq \limsup_j f_j(x_j) - \varepsilon. \tag{2.9}$$

In the following (and in the literature) all conditions (ii), $(ii)'$, $(ii)''$ are equally referred to as the $\limsup$ inequality or as the exitance of a recovery sequence.

Note that the $\liminf$ inequality (i) can be rewritten as

$$f_\infty(x) \leqslant \inf\left\{\liminf_j f_j(x_j) :\ x_j \to x\right\}.$$

Trivially, we always have

$$\inf\left\{\liminf_j f_j(x_j) :\ x_j \to x\right\} \leqslant \inf\left\{\limsup_j f_j(x_j) :\ x_j \to x\right\}$$

and if $(\overline{x_j})$ is a recovery sequence for (ii) we have

$$\inf\left\{\limsup_j f_j(x_j) :\ x_j \to x\right\} \leqslant \limsup_j f_j(\overline{x_j}) \leqslant f_\infty(x),$$

so that (i) and (ii) imply that we have

$$f_\infty(x) = \min\left\{\liminf_j f_j(x_j) :\ x_j \to x\right\} = \min\left\{\limsup_j f_j(x_j) :\ x_j \to x\right\}. \tag{2.10}$$

(and actually both minima are obtained as limits along a recovery sequence). It is important to keep in mind this characterization as many properties of the Γ-limit will be easily explained from it.

Remark 2.8 ***(Γ-convergence as an equality of upper and lower bounds)*** *It is sometimes convenient to state the equality in (2.10) as an equality of imfima*

$$f_{\infty}\left(x\right)=\inf\left\{\liminf_{j} f_{j}\left(x_{j}\right):\ x_{j}\rightarrow x\right\}=\inf\left\{\limsup_{j} f_{j}\left(x_{j}\right):\ x_{j}\rightarrow x\right\}. \qquad (2.11)$$

This equality is indeed equivalent to the definition of Γ-limit, that is, the Γ-limit exists if and only if the two infima in (2.11) are equal. This characterization will be important in that in this way the exitance of the Γ-limit (which not always exists) is expressed as the equality of two quantities which are always defined and which can (and will) be studied separately. The first quantity can be thought as the lower bound for the Γ-limit, the second as an upper bound.

By (2.11) we obtain in particular that the Γ-limit, if it exists, is unique.

Remark 2.9 ***(stability under continuous perturbations).*** *An important property of Γ-converges is its stability under continuous perturbations: if (f_j) Γ-converges to f_∞ and $g : X \rightarrow [-\infty, +\infty]$ is a d-continuous function then $(f_j + g)$ Γ-converges to $f_\infty + g$.*

This is an immediate consequence of the definition, since if (i) holds then for all $x \in X$ and $x_j \rightarrow x$ we get

$$f_{\infty}\left(x\right)+g\left(x\right)\leqslant\liminf_{j} f_{j}\left(x_{j}\right)+\lim_{j} g\left(x_{j}\right)=\liminf_{j}\left(f_{j}\left(x_{j}\right)+g\left(x_{j}\right)\right),$$

while if $(ii)^{'}$ above holds then we get

$$f_{\infty}\left(x\right)+g\left(x\right)=\lim_{j} f_{j}\left(x_{j}\right)+\lim_{j} g\left(x_{j}\right)=\lim_{j}\left(f_{j}\left(x_{j}\right)+g\left(x_{j}\right)\right).$$

Remark 2.10 ***(Γ-limit of a constant sequence).*** *Consider the simplest case $f_j = f$ for all $j \in \mathbb{N}$. In this case it will be easily seen that f_j Γ-convergence. By the $\liminf$ inequality, the limit f_∞ must satisfy*

$$f_{\infty}\left(x\right)\leqslant\liminf_{j} f\left(x_{j}\right)$$

for all x and $x_j \rightarrow x$. If f is not lower semicontinuous then there exists $\overline{x}$ and a sequence $\overline{x_j} \rightarrow \overline{x}$ such that

$$\liminf_{j} f_{j}\left(\overline{x_{j}}\right)<f\left(\overline{x}\right),$$

hence, in particular $f_\infty\left(\overline{x}\right) \neq f\left(\overline{x}\right)$. This shows that Γ-convergence does not satisfy the requirement that a constant sequence $f_j = f$ converges to f (if f is not lower semicontinuous). We will see however that this holds true in the family of lower semicontinuous functions (see Remark (2.12))

Remark 2.11 ***(dependence on the metric).*** *The choice of the metric on X is clearly a fundamental step in problems involving Γ-limit. In general, even when two distance d and d' are comfortable; That is*

$$\lim_{j} d'\left(x_{j},x\right)=0\Rightarrow\lim_{j} d\left(x_{j},x\right)=0. \qquad (2.12)$$

The exitance of Γ*-limit in one metric does not imply the existence of the* Γ*-limit in the second (see examples in Section 1.3). However, in this situation, if both* Γ*-limits exist then we have*

$$\Gamma(d) - \lim_j f_j \leqslant \Gamma\left(d^{'}\right) - \lim_j f_j.$$

This is clear, for example, from the characterization (2.10) *since the set of converging sequences for* d *is larger than that is for* $d^{'}$.

Remark 2.12 ***(comparison with pointwise and uniform limits).*** *As a very particular case, we can consider the metric* $d^{'}$ *of the discrete topology (where the only converging sequences are constant sequences). In this case the* Γ*-limit coincides with the pointwise limit (if it exists). If* d *is any other metric then (2.12) holds trivially, so that we obtain*

$$\Gamma(d) - \lim_j f_j \leqslant \lim_j f_j$$

as a particular case of the previous remark.

If f_j converges uniformly to a f on an open set U (in particular if $f_j = f$) and f is l.s.c. then we have also that f_j Γ-convergence to f. Indeed, the $\limsup$ inequality is obtained by the constant sequence, while the liminf inequality is immediately verified once we remark that if $x_j \to x \in U$, then $x_j \in U$, for j large enough, so that

$$\liminf_j f_j(x_j) = \lim_j (f_j(x_j) - f(x_j)) + \liminf_j f_j(x) \geq f(x).$$

2.5 Some examples on the real line

In this section we will compute some simple Γ-limit of functions defined on the real line (equipped with the usual Euclidean distance) and will also compare it with the point-wise convergence, which can be thought of as a Γ-limit with respect to the discrete metric, as explained in Remark 2.12.

We have seen that a constant sequence $f_j = f$ Γ-converges to f if and only if f is lower semicontinuous. Hence, if f is not l.s.c the point-wise limit and Γ-limit are different. Now we construct an example where these two limits differ even if the pointwise limit is lower semicontinuous.

Example 1. Let

$$f_j(t) = f_1(jt),$$

where

$$f_1(t) = \sqrt{2}t \exp\left(-\left(2t^2 - 1\right)/2\right)$$

or

$$f_1(t) = \begin{cases} \pm 1, & \text{if} \quad t = \pm 1, \\ 0, & otherwise. \end{cases}$$

Then $f_j \to 0$ pointwise, but $\Gamma - \lim_j f_j = f$, where

$$f(t) = \begin{cases} 0, & \text{if} \quad t \neq 0, \\ -1, & \text{if} \quad t = 0. \end{cases}$$

Indeed, the sequence f_j converges locally uniformly (and hence also Γ-converges) to 0 in $\mathbb{R}\setminus\{0\}$, while clearly the optimal sequence for $x = 0$ is $-1/j$, for which $f_j(x_j) = -1$. In this case the pointwise and Γ-limits both exist and are different at one point.

Example 2. Take

$$f_j(t) = -f_1(jt),$$

where f_1 is as in the previous example. Clearly, the Γ-limit remains unchanged. This shows that in general

$$\Gamma\text{-}\lim_j(-f_j) \neq -\Gamma\text{-}\lim_j f_j,$$

$$\Gamma\text{-}\lim_j(f_j + g_j) \neq \Gamma\text{-}\lim_j f_j + \Gamma\text{-}\lim_j g_j.$$

(taking in the example $g_j = -f_j$) even if all functions are continuous.

The pointwise and Γ-limits may exist and be different at every point. Take $g_j = f_j$, where

$$g_j(t) = \begin{cases} 0, & \text{if} \quad t \notin \mathbb{Q} \text{ or } t = \frac{k}{n}, \text{ with } k \in \mathbb{Z} \text{ and } n \in \{1, ..., j\} \\ & -1, \quad otherwise. \end{cases}$$

We then have $f_j \to -1$. The liminf inequality is trivial and limsup inequality is easily obtained by remarking that $\{g_j = -1\}$ is dance for all $j \in \mathbb{N}$.

Example 3. There may be no pointwise converging subsequence of (f_j) but the Γ-$\lim_j f_j$ may exist all the same. Take, for example, $f_j(t) = -\cos(jt)$. In this case Γ-$\lim_j f_j = -1$. Again, the liminf inequality is trivial, while the $\limsup$ inequality is easily obtained by taking, for example, $x_j = [jx/2\pi]\, 2\pi/j$ ($[t]$ the integer part of t).

The sequence f_j may be converging pointwise, but may not Γ-converge. Take for example

$$f_j = (-1)^j g_j,$$

where g_j, is defined in Example 2. In this case $f_j \to 0$ pointwise, but the Γ-$\lim_j f_j$ does not exist at any point.

2.6 Function spaces and their properties

In all the follows (a, b) is bounded open interval of $\mathbb{R}$.

The norm (or quasi norm) of the space $L_p(a, b)$ $(0 < p \leqslant \infty)$ is defined by

$$\|f\|_{L_p(a,b)} := \left(\int_a^b |f(x)|^p \, d\mu(x)\right)^{1/p}, \qquad (0 < p < \infty) \tag{2.13}$$

and

$$\|f\|_{L_\infty(a,b)} = \inf_M (\mu\{x : |f(x)| \geq M\} = 0) \tag{2.14}$$

It is well known that

$$\|f\|_{L_\infty(a,b)} = \lim_{p\to\infty} \left(\int_{(a,b)} |f(x)|^p \, d\mu(x)\right)^{1/p}. \tag{2.15}$$

Definition 2.13 ***(Weak derivative).*** *We say that* $u \in L_1(a,b)$ *is weakly differentiable if a function* $g \in L_1(a,b)$ *exists such that the following integration by parts formula holds*

$$\int_a^b u\varphi^{'} dt = -\int_a^b g\varphi dt$$

for all $\varphi \in C_0^1(a,b)$. *If such* g *exists then it is called the weak derivative of* u *and is denoted* $u^{'}$.

Remark 2.14 *The notion of weak derivative is an extension of notion of classical derivative: If* $u \in C^1(a,b)$ *and its classical derivative belongs to* $L_1(a,b)$ *then the classical derivative coincides with its weak derivative. The function* $x \to |x|$ *is weakly differentiable in any* (a,b), *but* $u \notin C^1(-1,1)$ *and its weak derivative is the function* $x \to x/|x|$, *which in turn is not weakly differentiable in* $(-1,1)$.

Definition 2.15 ***(Sobolev spaces)*** *Let* $p \in [0.+\infty]$. *The Sobolev spaces* $\mathbb{W}^{1,p}(a,b)$ *is defined as the space of all weakly differentiable* $u \in L_p(a,b)$ *such that* $u^{'} \in L_p(a,b)$. *The norm of* u *in* $\mathbb{W}^{1,p}(a,b)$ *is defined as*

$$\|u\|^p_{\mathbb{W}^{1,p}(a,b)} = \|u\|^p_{L_p(a,b)} + \left\|u^{'}\right\|^p_{L_p(a,b)}.$$

The space $W^{1,p}_{loc}(\mathbb{R})$ *consists of such functions* u *for which* $u \in W^{1,p}_{loc}(I)$ *for all bounded open intervals* $I \subset \mathbb{R}$.

Remark 2.16 *The Sobolev spaces* $\mathbb{W}^{1,p}(a,b)$ *equipped with the natural norm, indicated in Definition 2.15, is a Banach space. This is easily checked upon identifying* $\mathbb{W}^{1,p}(a,b)$ *with the subspace of* $L_p(a,b) \times L_p(a,b)$ *of all pairs* $\left(u,u^{'}\right)$ *with* $u \in \mathbb{W}^{1,p}(a,b)$. *The same identification shows that* $\mathbb{W}^{1,p}(a,b)$ *is separable if* $1 \leqslant p < \infty$.

Theorem 2.17 ***(pointwise value of Sobolev spaces).*** *Let* $u \in \mathbb{W}^{1,p}(a,b)$. *Then there exists* $\dot{u} \in C_0^1[(a,b)]$ *such, that* $\dot{u} = u$ *a.e. on* (a,b) *and*

$$\dot{u}(y) - \dot{u}(x) = \int_x^y u^{'}(t)\,dt \qquad (2.16)$$

for all $x,y \in [a,b]$. *We commontly identify* u *with its continuous reprensentative* $u^{'}$ *whenever pointwise values are taken into account.*

Remark 2.18 ***(boundary values).*** *If* $\mathbb{W}^{1,p}(a,b)$ *then the boundary values* $u(a)$ *and* $u(b)$ *are uniquely defined by the values* $\dot{u}(a)$ *and* $\dot{u}(b)$, *respectively. We may then extand a function* $\mathbb{W}^{1,p}(a,b)$ *to the function* $W^{1,p}_{loc}(\mathbb{R})$ *by simply setting* $u(t) = u(a)$ *for* $t \leqslant a$ *and* $u(t) = b$, *for* $t \geq b$.

Theorem 2.19 ***(equivalent definitions of Sobolev spaces).*** *Let* $1 \leqslant p \leqslant \infty$. *Then the following statements are equivalent:*

(i) $u \in \mathbb{W}^{1,p}(a,b)$.

(ii) There exists $C \geq 0$, *such that*

$$\left|\int_a^b u\varphi' dt\right| \leqslant C \|\varphi\|_{L_{p'}(a,b)},$$

for arbitrary $\varphi \in C_0^1(a,b)$.

(iii) There exists $C \geq 0$, *such that for all for all* $I \subset (a,b)$ *and for all* $h \in \mathbb{R}$ *such that* $|h| \leqslant dist(I, \{a,b\})$ *we have*

$$\|\tau_h u - u\|_{L_p(I)} \leqslant C|h|,$$

where $\tau_h u = u(t-h)$.

(iv) There exists a sequence (u_j) *in* $C^\infty([a,b])$ *such that*

$$\lim_j \|u_j - u\|_{\mathbb{W}^{1,p}(a,b)} = 0. \tag{2.17}$$

(v) There exists a sequence (u_j) *in* $C^\infty(\mathbb{R})$ *such that (2.17) holds.*

(vi) There exists a sequence (u_j) *in* $C^\infty(\mathbb{R})$ *such that* $\sup_j \|u_j\|_{\mathbb{W}^{1,p}(a,b)} < \infty$ *and*

$$\lim_j \|u_j - u\|_{L_p(a,b)} = 0.$$

Remark 2.20 *(a) The best constant* C *in (ii) and (iii) above is* $\|u'\|_{L_p(a,b)}$.

(b) If $p = 1$ *then (i) ⇒(ii)⇔(iii). Note that the function* $x \to x/|x|$ *satisfies (ii)-(vi) with* $p = 1$ *but does not belongs to* $\mathbb{W}^{1,1}(-1,1)$.

(c) By (iii) we easily see that $\mathbb{W}^{1,\infty}(a,b)$ *coincides with the space* $lip(a,b)$ *of all Lipschitz functions on* (a,b) *and* $\|u'\|_{L_\infty(a,b)}$ *is the best Lipschitz constant for* u.

Theorem 2.21 ***(embedding results).*** *There exists a constant* $C = C(a,b)$ *such that*

$$\|u\|_{L_\infty(a,b)} \leqslant C\|u\|_{\mathbb{W}^{1,p}(a,b)} \tag{2.18}$$

Moreover, we have compact embeddings

$$\mathbb{W}^{1,p}(a,b) \subset C^0([a,b]) \tag{2.19}$$

for $1 < p \leqslant \infty$ *and*

$$\mathbb{W}^{1,1}(a,b) \subset L_q(a,b)$$

for all $q \geq 1$.

Definition 2.22 *The space* $W_0^{1,p}(a,b)$ *is defined as the closure of* $C_0^\infty(a,b)$ *in the* $W_0^{1,p}$*-norm, or equivalently, as the set of those* $u \in \mathbb{W}^{1,p}(a,b)$ *with boundary values* $u(a) = u(b) = 0$.

Theorem 2.23 ***(Poincare's inequality).*** *There exists a constant $C = C(a,b)$ such that*

$$\|u\|_{\mathbb{W}^{1,p}(a,b)} \leqslant C \left\| u^{'} \right\|_{L_p(a,b)}, \tag{2.20}$$

for all $u \in \mathbb{W}^{1,p}(a,b)$ such that $\dot{u}(x) = 0$, for some $x \in [a,b]$. In particular this holds for $u \in W_0^{1,p}(a,b)$.

Definition 2.24 *Let $u : (a,b) \to \mathbb{R}$ be measurable function. The total variation of u on (a,b) is defined as*

$$\begin{aligned} &Var\,(u,(a,b)) := \\ &= \inf_{v=u,\ a.e.\ on\ (a,b)} \sup \left\{ \sum_{i=1}^{N} |v(t_{i+1}) - v(t_i)| : \ a < t_0 < ... < t_N < b,\ N \in \mathbb{N} \right\} \end{aligned}$$

If $Var\,(u,(a,b)) < \infty$ then we say that u is a function of bounded variation. We simply write $Var\,(u)$ if (a,b) is fixed.

Remark 2.25 *If $u \in \mathbb{W}^{1,1}(a,b)$, then*

$$Var\,(u,(a,b)) = \int_a^b \left| u^{'} \right| dt.$$

In particular, u is a function of bounded variation. Note that also

$$v(x) = x/|x|$$

is a function of bounded variation with $Var\,(v,(-1,1)) = 2$.

2.7 MORE PROPERTIES OF Γ-LIMITS

From the definition of Γ-convergence we immediately obtain the following properties.

Remark 2.26 *If $\{f_{j_k}\}$ is a subsequence of $\{f_j\}$, then*

$$\Gamma - \lim\inf_j f_j \leqslant \Gamma - \lim\inf_k f_{j_k}, \qquad \Gamma - \lim\sup_k f_{j_k} \leqslant \Gamma - \lim\sup_j f_j.$$

In particular, if $f_\infty = \Gamma - \lim_j f_j$ exists then for every increasing sequence of integers j_k $f_\infty = \Gamma - \lim_k f_{j_k}$.

Remark 2.27 *If g is a continuous function then $f_\infty + g = \Gamma - \lim_j (f_j + g)$; more in general, if $g_j \to g$ uniformly then $f_\infty + g = \Gamma - \lim_j (f_j + g_j)$. In particular, if $f_j \to f$ uniformly on an open set U, then $\Gamma - \lim_j f_j = sc\, f$ On U.*

Remark 2.28 *If $f_j \to f$ pointwise, then $\Gamma - \lim\sup_j f_j \leqslant f$ and, hence, $\Gamma - \lim\sup_j f_j \leqslant sc\, f$.*

We can state some simple but important cases when Γ-limit does exist and is computed easily.

Proposition 2.29 (Γ-limit of monotone sequaences) *(i) (decreasing sequences) If $f_{j+1} \leqslant f_j$ for all $j \in \mathbb{N}$, then*

$$\Gamma - \lim_j f_j = sc(\inf_j f_j) = sc(\lim_j f_j). \tag{2.2}$$

(ii) (increasing sequences) If $f_j \leqslant f_{j+1}$ for all $j \in \mathbb{N}$, then

$$\Gamma - \lim_j f_j = sc(\sup_j sc\, f_j) = \lim_j sc f_j. \tag{2.3}$$

In particular, if f_j is l.s.c. for every $j \in \mathbb{N}$, then

$$\Gamma - \lim_j f_j = \lim_j f_j. \tag{2.4}$$

In particular, if f_j is l.s.c. for every $j \in \mathbb{N}$, then

Proof: As $f_j \to \inf_k f_k$ pointwise, by Remark 2.28 we have $\Gamma - \lim\sup_j f_j \leqslant sc(\inf_k f_k)$, while the other inequality is trivially derived from the inequality $sc(\inf_k f_k) \leqslant \inf_k f_k \leqslant f_j$ and (i) is proved.

To prove (ii) note that since $sc\, f_j \to \sup_k sc\, f_k$ pointwise, by Remark 2.28

$$\Gamma - \lim\sup_j f_j = \Gamma - \lim\sup_j sc\, f_j \leqslant \sup_k sc\, f_k.$$

On the other hand $sc\, f_k \leqslant f_j$ for all $j \geqslant k$ so that the converse inequality follows easily. ■

Remark 2.30 *By Proposition 2.29.(ii), if f_j is a equi-mildly coercive non-decreasing sequence of l.s.c. functions, then $\sup_j \min_X f_j = \min_X \sup_j f_j$.*

Proposition 2.31 (Compactness of Γ-convergence) *Let (X, d) be a separable metric space. and for all $j \in \mathbb{N}$ let $f_j : X \to \overline{\mathbb{R}}$ be a function. Then there exists a subsequence f_{j_k} such that the Γ-limit $\Gamma - \lim_k f_{j_k}$ exists for all $x \in X$.*

Proof: Let $\{U_k\}$ be a countable base of open sets in the topology of X. Since $\overline{\mathbb{R}}$ is compact, there exists an increasing sequence of integers $\{\sigma_j^0\}_j$ along which the limit

$$\lim_j \inf_{y \in U_0} f_{\sigma^0 j}(y)$$

exists, and for all $k \geqslant 1$ we define $\{\sigma_j^k\}_j$ recurrently as a subsequence of $\{\sigma_j^{k-1}\}_j$ along which the limit

$$\lim_j \inf_{y \in U_0} f_{\sigma^k j}(y)$$

exists. The "duiagonal" sequence $j+k := \sigma_k^k$ being a subsequence of $\{\sigma_j^j\}_j$, has the property that the limit

$$\lim_j \inf_{y \in U_\ell} f_{j_k}(y)$$

exists for all $\ell \in \mathbb{N}$. In particular, we have

$$\liminf_k \inf_{y \in U_\ell} f_{j_k}(y) = \limsup_k \inf_{y \in U_\ell} f_{j_k}(y)$$

for all $\ell \in \mathbb{N}$, and the claimed convergence follows. ■

Remark 2.32 *If (X, d) is not a separable metric space, then Proposition 2.31 fails. As an example we can take $X = \{-1, 1\}^{\mathbb{N}}$ equipped with the discrete topology. X is metrizable and Γ-convergence on X is equivalent to pointwise convergence. We take the sequence $f_j : X \to \{-1, 1\}$ defined by $f_j(x) = x_j$ if $x = (x_0, x_1, \ldots)$. If $\{f_{j_k}\}$ is a subsequence of $\{f_j\}$ and we define x by $x_{j_k} = (-1)^k$ and $x_j = 1$ if $j \notin \{j_k : k \in \mathbb{N}\}$, then the limit $\lim_k f_{j_k}(x)$ does not exist. Hence no subsequence of $\{f_j\}$ Γ-converges.*

Γ-convergence enjoys the following useful property.

Proposition 2.33 (Urisohn property of Γ-convergence) *We have $f_\infty = \Gamma - \lim_j f_j$ if and only if for every subsequence $\{f_{j_k}\}$ there exists a further subsequence which Γ-converges to f_∞.*

Proof: Clearly, if f_j Γ-converges to f_∞, then every subsequence of f_j Γ-converges to the same limit (see Remark 2.26).

For an increasing sequence of integers $\{j_k\}$ we have

$$\Gamma - \liminf_j f_j \leqslant \Gamma - \liminf_k f_{j_k} \leqslant \Gamma - \limsup_k f_{j_k} \leqslant \Gamma - \limsup_j f_j.$$

Hence if $\Gamma - \liminf_k f_{j_k}(x) = f_\infty(x)$ but $\Gamma - \lim\ f_j(x)$ does not exist we have

$$\text{either} \quad f_\infty(x) \leqslant \Gamma - \limsup_j f_j(x) \quad \text{or} \quad f_\infty(x) \geqslant \Gamma - \liminf_j f_j(x).$$

In the first case we have

$$f_\infty(x) < \sup_{U \in \mathscr{N}(x)} \limsup_j \inf_{y \in U} f_j(y),$$

so that there exists $U \in \mathscr{N}(x)$ with the property

$$f_\infty(x) < \limsup_j \inf_{y \in U} f_j(y).$$

This means that there exists a subsequence $\{f_{j_k}\}$ of $\{f_j\}$ along which

$$f_\infty(x) < \liminf_k \inf_{y \in U} f_{j_k}(y),$$

so that $f_\infty(x) < \Gamma - \liminf_k \inf_{y \in U} f_{j_k}(y)$ leads to a contradiction. In the second case a sequence x_j converging to x exists such that $\liminf_j f_j(x_j) < f_\infty(x)$. This means that $\Gamma - \limsup_k f_{j_k}, f_\infty(x)$, thus giving a contradiction. ■

Proposition 2.34 *Let X be a topological vector space. If $\{f_j\}$ is the sequence of convex functions, the Γ-limit $f := \Gamma - \limsup_j f_j$ is also a convex function.*

The statement fails in general case.

Proof: We leave the proof to the reader as an exercise (see [Br1], Exercise 1.6). ■

3 Γ-CONVERGENCE OF HEAT TRANSFER EQUATION

A mixed boundary value problem for the stationary heat transfer equation in a thin layer around a surface $\mathscr{C}$ with the boundary is investigated. The main object is to trace what happens in Γ-limit when the thickness of the layer converges to zero. The limit Dirichlet BVP for the Laplace-Beltrami equation on the surface is described explicitly and we show how the Neumann boundary conditions in the initial BVP transform in the Γ-limit. For this we apply the variational formulation and the calculus of Günter's tangential differential operators on a hypersurface and layers, which allow global representation of basic differential operators and of corresponding boundary value problems in terms of the standard Euclidean coordinates of the ambient space $\mathbb{R}^n$.

The exposition follows the paper of T. Buchukuri, R. Duduchava & G. Tephnadze [BDT1].

3.1 Introduction

The main object of the paper is to demonstrate what happens with a boundary value problem for the Laplace equation in a thin layer Ω^ε around a surface $\mathscr{C}$ in $\mathbb{R}^3$ when the thickness of the layer ε diminishes to zero: $\varepsilon \to 0$. We impose the Neumann boundary conditions on the upper and lower faces of the layer $\mathscr{C} \times \{\pm\varepsilon\}$ and the Dirichlet boundary conditions on the lateral surface $\partial\mathscr{C} \times (-\varepsilon, \varepsilon)$.

The limit of the associated functionals is understood in the sense of Γ-convergence and the main tool is the representation of differential operators with the help of Gunter's derivatives-the system of tangential derivatives on the surface $\mathscr{D}_j := \partial_j - \nu_j\partial_{\boldsymbol{\nu}}$, $j = 1, 2, 3$ and the normal derivative $\partial_{\boldsymbol{\nu}} := \sum_{j=1}^3 \nu_j\partial_j$, where $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^\top$ is the unit normal vector field on the mid surface $\mathscr{C}$. The first-order differential operator $\mathscr{D}_j$ is the directional derivative along $\pi\, e^j$, where $\pi : \mathbb{R}^3 \to T\mathscr{C}$ is the orthogonal projection onto the tangent plane to $\mathscr{C}$ and $e^1, \dots, e^n$ is the canonical basis in the Euclidean space $e^j = (\delta_{jk})_{1\leq k\leq 3} \in \mathbb{R}^3$, with δ_{jk} denoting the Kronecker symbol (cf. [Gu1], [KGBB1], [Du1]).

Calculus of Gunter's derivatives on a hypersurface allows representation of the most basic partial differential operators (PDO's), as well as their associated boundary value problems, on a hypersurface $\mathscr{C}$ in global form, in terms of the standard spatial coordinates in $\mathbb{R}^n$. Such BVPs arise in a variety of situations and have many practical applications. See, for example, [Ha1, §72] for the heat conduction by surfaces, [Ar1, §10] for the equations of surface flow, [Ci1], [AC1] for the vacuum Einstein equations describing gravitational fields, [TZ1] for the Navier-Stokes equations on spherical domains, as well as the references therein.

A hypersurface $\mathscr{C}$ in $\mathbb{R}^3$ has the natural structure of a 2-dimensional Riemannian manifold and the aforementioned PDE's are not the immediate analogues of the ones corresponding to the flat, Euclidean case, since they have to take into consideration geometric characteristics of $\mathscr{C}$ such as curvature. Inherently, these PDE's are originally written in local coordinates, intrinsic to the manifold structure of $\mathscr{C}$.

The surface gradient

$$\mathscr{D} := (\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3)^\top \tag{3.1}$$

is defined on $\mathscr{C}$, and has a relatively simple structure. In terms of (3.1), the Laplace-Beltrami operator on $\mathscr{C}$ simply becomes (see [MM1, pp. 2ff and p. 8.])

$$\Delta_{\mathscr{C}} = \mathscr{D}^*\mathscr{D} \quad \text{on} \quad \mathscr{C}.$$

Alternatively, this is the natural operator associated with the Euler-Lagrange equations for the variational integral

$$\mathscr{E}[u] = -\frac{1}{2}\int_{\mathscr{C}} \langle \mathscr{D}u, \mathscr{D}u\rangle \, dS, \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

A similar approach, based on the principle that, at equilibrium, the displacement minimizes the potential energy, leads to the derivation of the equation for the elastic hypersurface (cf. [DMM1, Du3] for the isotropic case).

These results are useful in numerical and engineering applications (cf. [AN1], [Be1], [Ce1], [Co1], [DaL1], [BGS1], [Sm1]) and we plan to treat a number of special surfaces in greater detail in a subsequent publication.

We consider heat conduction by an "isotropic" medium, governed by the Laplace equations, with the classical mixed Dirichlet-Neumann boundary conditions on the boundary in the layer domain $\Omega^\varepsilon := \mathscr{C} \times (-\varepsilon, \varepsilon)$ of thickness 2ε, where $\mathscr{C} \subset \mathscr{S}$ is a smooth subsurface of a closed hypersurface $\mathscr{S}$ with smooth nonempty boundary $\partial\mathscr{C}$. In particular, we confine ourselves with zero Dirichlet and non-zero Neumann data (see Remark 3.5 for the case of non-zero Dirichlet data):

$$\begin{aligned}
&\Delta_{\Omega^\varepsilon}\widetilde{T}(\mathscr{x}, t) = f(\mathscr{x}, t), && (\mathscr{x}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon),\\
&\widetilde{T}^+(\mathscr{x}, t) = 0, && (\mathscr{x}, t) \in \partial\mathscr{C} \times (-\varepsilon, \varepsilon),\\
&(\partial_t\widetilde{T})^+(\mathscr{x}, \pm\varepsilon) = q_\varepsilon^\pm(\mathscr{x}), && \mathscr{x} \in \mathscr{C}.
\end{aligned} \tag{3.3}$$

In the investigation we apply that the Laplace operator $\Delta_{\Omega^\varepsilon} = \partial_1^2 + \partial_2^2 + \partial_3^2$ is represented as the sum of the Laplace-Beltrami operator on the mid-surface, the square of the transversal derivative and the lower order term

$$\Delta_{\Omega^\varepsilon}\widetilde{T} = \Delta_{\mathscr{C}}\widetilde{T} + \partial_t^2\widetilde{T} + 2\mathscr{H}_{\mathscr{C}}\partial_t\widetilde{T}, \tag{3.4}$$

where $\mathscr{D}_4 = \partial_t$. The Laplace-Beltrami operator $\Delta_{\mathscr{C}}$ defined in (0.12) and the mean curvature $\mathscr{H}_{\mathscr{C}}(\mathscr{x}) = \sum_{k=1}^{3} \mathscr{D}_k\mathscr{N}_k(\mathscr{x})$ of the surface are extended properly from $\mathscr{C}$ (see the forthcoming Lemma 3.4).

Introducing the function $G(\mathscr{x}, t)$ which has the same Dirichlet and Neumann traces as T on the $\partial\mathscr{C} \times (-\varepsilon, \varepsilon)$ and on $\mathscr{C} \times \{\pm\varepsilon\}$ respectively

$$G(\mathscr{x}, t) = \frac{1}{4\varepsilon}(t+\varepsilon)^2 q_\varepsilon^+(\mathscr{x}) - \frac{1}{4\varepsilon}(t-\varepsilon)^2 q_\varepsilon^-(\mathscr{x}), \tag{3.5}$$

we can reduce the problem (3.3) to the following boundary value problem with respect to unknown function $T = \widetilde{T} - G$

$$\Delta_{\Omega^\varepsilon}T(\mathscr{x}, t) = F(\mathscr{x}, t), \qquad (\mathscr{x}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon), \tag{3.6}$$

$$T^+(\mathscr{x}, t) = 0, \qquad (\mathscr{x}, t) \in \partial\mathscr{C} \times (-\varepsilon, \varepsilon), \tag{3.7}$$

$$(\partial_t T)^+(\mathscr{x}, \pm\varepsilon) = 0, \qquad \mathscr{x} \in \mathscr{C}. \tag{3.8}$$

where

$$\begin{aligned} F(\mathscr{x},t) &:= f(\mathscr{x},t)-\frac{1}{4\varepsilon}\big((t+\varepsilon)^2\Delta_{\mathscr{C}}q^+_\varepsilon(\mathscr{x})-(t-\varepsilon)^2\Delta_{\mathscr{C}}q^-_\varepsilon(\mathscr{x})\big)\\ &- \frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{x})}{2\varepsilon}\big((t+\varepsilon)q^+_\varepsilon(\mathscr{x})-(t-\varepsilon)q^-_\varepsilon(\mathscr{x})\big)-\frac{1}{2\varepsilon}\big(q^+_\varepsilon(\mathscr{x})-q^-_\varepsilon(\mathscr{x})\big), \quad (3.9)\\ &\qquad (\mathscr{x},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon).\end{aligned}$$

The BVP (3.6)-(3.8) is reformulated as the minimization problem for the functional which, after scaling (stretching the variable $t=\varepsilon\tau$ and dividing the entire functional by ε) has the following form

$$\begin{aligned} E_\varepsilon(T_\varepsilon):&= \int_{-1}^{1}\int_{\mathscr{C}}\left[\frac{1}{2}(\mathscr{D}_{\mathscr{C}}T_\varepsilon)^2(\mathscr{x},\tau)+\frac{1}{2\varepsilon^2}(\partial_\tau T_\varepsilon)^2(\mathscr{x},\tau)+F_\varepsilon(\mathscr{x},\tau)T_\varepsilon(\mathscr{x},\tau)\right]d\sigma d\tau \quad (3.10)\\ F_\varepsilon(\mathscr{x},t) &:= F(\mathscr{x},\varepsilon t)=f(\mathscr{x},\varepsilon t)-\frac{\varepsilon}{4}\big((t+1)^2\Delta_{\mathscr{C}}q^+_\varepsilon(\mathscr{x})-\frac{\varepsilon}{4}(t-1)^2\Delta_{\mathscr{C}}q^-_\varepsilon(\mathscr{x})\big)\\ &- \frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{x})}{2}\big((t+1)q^+_\varepsilon(\mathscr{x})-(t-1)q^-_\varepsilon(\mathscr{x})\big)-\frac{1}{2\varepsilon}\big(q^+_\varepsilon(\mathscr{x})-q^-_\varepsilon(\mathscr{x})\big), \quad (3.11)\\ T_\varepsilon(\mathscr{x},\tau) &:= T(\mathscr{x},\varepsilon\tau), \quad T_\varepsilon\in\widetilde{\mathbb{H}}^1(\Omega^1,\partial\mathscr{C}\times(-1,1)), \quad F_\varepsilon\in\widetilde{\mathbb{H}}^{-1}(\Omega^1), \quad q^\pm_\varepsilon\in\widetilde{\mathbb{H}}^2(\mathscr{C}),\\ &\qquad (\mathscr{x},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon).\end{aligned}$$

(For the definition of $\widetilde{\mathbb{H}}^1(\Omega^1,\partial\mathscr{C}\times(-1,1))$ see (3.9).)

Let

$$\mathscr{P}(\mathscr{C}):=\left\{T\in\mathbb{H}^1(\Omega^1):T(\mathscr{x},\tau)=T_{\mathscr{C}}(\mathscr{x}), \qquad T_{\mathscr{C}}\in\widetilde{\mathbb{H}}^1(\mathscr{C}), \quad \tau\in[-1,1]\right\}. \quad (3.12)$$

The main result of the present investigation is the following Theorem 3.1.

Theorem 3.1 *Let*

$$f_\varepsilon(\mathscr{x},t):=f(\mathscr{x},\varepsilon t)\underset{\varepsilon\to0}{\to}f^0(\mathscr{x}) \quad in \quad \mathbb{L}_2(\Omega^1),$$

$q^\pm_\varepsilon\in\widetilde{\mathbb{H}}^2(\mathscr{C})$ *be uniformly bounded (with respect to* ε*) in* $\mathbb{H}^2(\mathscr{C})$*, and*

$$\begin{aligned}&\lim_{\varepsilon\to0}q^+_\varepsilon=\lim_{\varepsilon\to0}q^-_\varepsilon=q_0, \quad q_0\in\mathbb{L}_2(\mathscr{C}),\\ &\frac{1}{2\varepsilon}(q^+_\varepsilon-q^-_\varepsilon)\underset{\varepsilon\to0}{\to}q_1 \quad in \quad \mathbb{L}_2(\mathscr{C}).\end{aligned}$$

Then the functional in (3.10) Γ*-converges to the functional*

$$E^{(0)}(T)=\begin{cases}\displaystyle\int_{\mathscr{C}}\left[\langle\mathscr{D}_{\mathscr{C}}T_{\mathscr{C}}(\mathscr{x}),\mathscr{D}_{\mathscr{C}}T_{\mathscr{C}}(\mathscr{x})\rangle+2\big(f^0(\mathscr{x})-\mathscr{H}^0_{\mathscr{C}}q_0(\mathscr{x})-q_1(\mathscr{x})\big)T_{\mathscr{C}}(\mathscr{x})\right]d\sigma, & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad if \quad T\in\mathscr{P}(\mathscr{C}); & \\ +\infty, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad if \quad T\notin\mathscr{P}(\mathscr{C}). & \end{cases} \quad (3.13)$$

The following Dirichlet boundary value problem for Laplace-Beltrami equation on the mid surface $\mathscr{C}$

$$\begin{aligned}
&\Delta_{\mathscr{C}} T(\mathscr{X}) = f^0(\mathscr{X}) - \mathscr{H}^0_{\mathscr{C}} q_0(\mathscr{X}) - q_1(\mathscr{X}), \quad \mathscr{X} \in \mathscr{C},\\
&T^+(\mathscr{X}) = 0, \qquad \mathscr{X} \in \partial\mathscr{C},\\
&T \in \mathbb{H}^1(\mathscr{C}), \quad f^0, q_0, q_1 \in \mathbb{L}_2(\mathscr{C}),
\end{aligned} \tag{3.14}$$

is an equivalent reformulation of the minimization problem with the energy functional (3.13).

Remark 3.2 *The BVP* (3.14) *is the "*Γ *limit" of the initial BVP* (3.3) *in the following sense: The corresponding functional* (3.13) *is the* Γ*-limit of the functional* (3.10)*, corresponding to the BVP* (3.6)-(3.8)*.*

It is remarkable to note that the weak derivative q^0 *of the Neumann condition from the initial BVP* (3.3) *migrated into the right hand side of the limit equation.*

Note as well that the Γ*-limit* $T_{\mathscr{C}}(\mathscr{X})$ *of a solution* $T(\mathscr{X}, \varepsilon\tau)$*.* $T \in \mathbb{H}^1(\Omega_\varepsilon)$ *to the BVP* (3.6)-(3.8) *has better smoothness* $T_{\mathscr{C}} \in \mathbb{H}^1(\mathscr{C})$ *than expected.*

Γ-limits of boundary value problems in thin structures, reformulated as a minimization problem for the associated energy functional, were studied by many authors (see, e.g., [FJM1, FJMM1, Ve1, Br1] and the literature cited therein). But mostly the Lamé equations for elastic plates $\mathscr{C} \subset \mathbb{R}^2$ and zero boundary conditions were treated (the Laplace equation for a plate is studied in [Br1]). In the papers [FJMM1, Ve1] the case of shells is treated, but with a different technique. Our approach is based on the calculus of Günter's derivatives, which we find more appropriate for such problems.

The layout of the paper is as follows. In § 1-§ 2 we review some basic differential-geometric concepts which are relevant for the work at hand (e.g., hypersurfaces and different methods of their identification). In § 3 we identify the most important partial differential operators on hypersurfaces, such as gradient, divergence, Laplace-Beltrami operator. In § 4 we consider the energy functional (3.2) and the associated Euler-Lagrange equation. In sections § 5, § 6 the aforementioned approach is applied and proved main theorems of the present paper, including Theorem 3.1.

3.2 Laplace operator in a layer domain

We will keep the notation of § 1: Θ, ω, $\mathscr{S}$ and $\mathscr{C}$. We consider a **layer domain**

$$\Omega^\varepsilon := \Big\{ \mathscr{X}_t \in \mathbb{R}^n \,:\, \mathscr{X}_t = \mathscr{X} + t\boldsymbol{\nu}(\mathscr{X}) = \Theta(x) + t\boldsymbol{\nu}\big(\Theta(x)\big), \quad x \in \omega\,,\; -\varepsilon < t < \varepsilon \Big\}$$
$$= \mathscr{C} \times (-\varepsilon, \varepsilon), \tag{3.1}$$

where $\boldsymbol{\nu}(\mathscr{X}) = \boldsymbol{\nu}(\Theta(y))$ for $\mathscr{X} = \Theta(y) \in \mathscr{S}$, is the outer unit normal vector field (see (1.23) and (1.26)). The surface $\mathscr{C}$ is a mid-surface for the layer domain.

We will also use the notation $\boldsymbol{\nu}(y) := \boldsymbol{\nu}(\Theta(y))$ for brevity unless this leads to a confusion. The coordinate t will be referred to as the **transverse variable**.

Without going into detail let us remark only that if the hypersurface $\mathscr{S}$ is C^2-smooth and $1/\varepsilon$ is more than the maximum of modules of all principal curvatures of the surface $\mathscr{S}$ (i.e., of all eigenvalues $|\lambda_1(\mathscr{X})|,\ldots,|\lambda_{n-1}(\mathscr{X})|,\lambda_n(\mathscr{X})\equiv 0$ of the Weingarten matrix $\mathscr{W}_{\mathscr{S}}(\mathscr{X})$, $\mathscr{X}\in\mathscr{S}$), then the mapping

$$\begin{aligned}&\Theta^\varepsilon\,:\,\omega^\varepsilon:=\omega\times(-\varepsilon,\varepsilon)\to\Omega^\varepsilon\,,\quad \omega^\varepsilon\subset\mathbb{R}^n\,,\\ &\Theta^\varepsilon(y,t):=\Theta(y)+t\boldsymbol{\nu}(y)\,,\qquad (y,t)\in\omega^\varepsilon\end{aligned}\tag{3.2}$$

is a diffeomorphism.

We will also suppose that $\mathscr{N}$ is a proper extension of the outer unit normal vector field $\boldsymbol{\nu}$ into the layer neighborhood Ω^ε (cf. Definition 1.4).

The n-tuple $\boldsymbol{g}_1:=\partial_1\Theta,\ldots,\boldsymbol{g}_{n-1}:=\partial_{n-1}\Theta,\boldsymbol{g}_n:=\mathscr{N}$, where $\mathscr{N}$ is the proper extension of $\boldsymbol{\nu}$ in the neighborhood Ω^ε, is a basis in Ω^ε and arbitrary vector field $\boldsymbol{U}=\sum_{j=1}^n U_j^0\boldsymbol{e}^j$ on Ω^ε is represented with this basis in "curvilinear coordinates".

Let us consider the system of $(n+1)$-vectors

$$\boldsymbol{d}^{\,j}:=\boldsymbol{e}^j-\mathscr{N}_j\mathscr{N}\,,\qquad j=1,\ldots,n\quad\text{and}\quad \boldsymbol{d}^{\,n+1}:=\mathscr{N},\tag{3.3}$$

where $\boldsymbol{e}^1,\ldots,\boldsymbol{e}^n$ is the Cartesian basis in $\mathbb{R}^n$ (cf. (0.7)); the first n vectors $\boldsymbol{d}^{\,1},\ldots,\boldsymbol{d}^{\,n}$ are tangential to the surface $\mathscr{C}$, while the last one $\boldsymbol{d}^{\,n+1}=\mathscr{N}$ is orthogonal to all $\boldsymbol{d}^{\,1},\ldots,\boldsymbol{d}^{\,n}$. This system is, obviously, linearly dependent, but full and any vector field $\boldsymbol{U}\in\mathscr{W}(\Omega^\varepsilon)$ is written in the following form:

$$\boldsymbol{U}=\sum_{j=1}^n U_j\boldsymbol{e}^j=\sum_{j=1}^{n+1}U_j^0\boldsymbol{d}^{\,j}.\tag{3.4}$$

Since the system $\left\{\boldsymbol{d}^{\,j}\right\}_{j=1}^{n+1}$ is linearly dependent

$$\sum_{j=1}^n\mathscr{N}_j\boldsymbol{d}^{\,j}=0,\qquad \langle\mathscr{N},\boldsymbol{d}^j\rangle=0,\quad j=1,\ldots,n,\tag{3.5}$$

the representation (3.4) is not unique. To fix the unique representation in (3.4) we will keep the following convention:

$$U_j^0:=U_j-\langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{N}_j,\quad j=1,\ldots,n,\quad U_{n+1}^0=\langle\mathscr{N},\boldsymbol{U}\rangle=\sum_{j=1}^n U_j\mathscr{N}_j.\tag{3.6}$$

The convention (3.6) is natural because if the vector $\boldsymbol{U}(\mathscr{X})$ is tangent to $\mathscr{C}$ for $\mathscr{X}\in\mathscr{C}$, then $U_j^0(\mathscr{X}):=U_j(\mathscr{X})$ for $j=1,\ldots,n$ and $U_{n+1}^0(\mathscr{X})=0$.

Moreover, if the scalar product of vectors

$$\boldsymbol{U}:=\sum_{j=1}^n U_j\boldsymbol{e}^{\,j}=\sum_{j=1}^{n+1}U_j^0\boldsymbol{d}^{\,j},\quad \boldsymbol{V}:=\sum_{j=1}^n V_j\boldsymbol{e}^{\,j}=\sum_{j=1}^{n+1}V_j^0\boldsymbol{d}^{\,j}\tag{3.7}$$

is defined by the equality

$$\langle\boldsymbol{U},\boldsymbol{V}\rangle^0:=\sum_{j=1}^{n+1}U_j^0V_j^0,$$

then the "new" and the "old" scalar products coincide:

$$\langle \boldsymbol{U}, \boldsymbol{V}\rangle^0 = \sum_{j=1}^{n+1} U_j^0 V_j^0 = \sum_{j=1}^{n}(U_j - \mathscr{N}_j\langle \mathscr{N}, \boldsymbol{U}\rangle)(V_j - \mathscr{N}_j\langle \mathscr{N}, \boldsymbol{V}\rangle) + \langle \mathscr{N}, \boldsymbol{U}\rangle\langle \mathscr{N}, \boldsymbol{V}\rangle$$
$$= \sum_{j=1}^{n} U_j V_j = \langle \boldsymbol{U}, \boldsymbol{V}\rangle. \tag{3.8}$$

In particular,

$$\|\boldsymbol{U}\|^0 := \sum_{j=1}^{n+1} |U_j^0|^2 = \sum_{j=1}^{n} |U_j|^2 = \|\boldsymbol{U}\|. \tag{3.9}$$

Note for a later use, that due to the equalities (3.5) and the convention (3.6) we get

$$\partial_{\boldsymbol{U}} = \sum_{j=1}^{n} U_j \partial_j = \sum_{j=1}^{n}[U_j^0 \partial_j + \langle \mathscr{N}, \boldsymbol{U}\rangle \mathscr{N}_j \partial_j] = \sum_{j=1}^{n} U_j^0(\partial_j - \mathscr{N}_j \partial_{\mathscr{N}}) + \langle \mathscr{N}, \boldsymbol{U}\rangle \partial_{\mathscr{N}}$$
$$= \sum_{j=1}^{n} U_j^0 \mathscr{D}_j + U_{n+1}^0 \mathscr{D}_{n+1} = \sum_{j=1}^{n+1} U_j^0 \mathscr{D}_j =: \mathscr{D}_{\boldsymbol{U}}.$$

Definition 3.3 *For a function $\varphi \in \mathbb{H}^1(\Omega^\varepsilon)$ the extended gradient is*

$$\mathscr{D}_{\Omega^\varepsilon}\varphi = \Big\{\mathscr{D}_1\varphi, ..., \mathscr{D}_n\varphi, \mathscr{D}_{n+1}\varphi\Big\}^\top = \sum_{j=1}^{n+1}(\mathscr{D}_j\varphi)\boldsymbol{d}^j, \qquad \mathscr{D}_{n+1}\varphi := \partial_{\mathscr{N}}\varphi \tag{3.10}$$

and for a smooth vector field $\boldsymbol{U} = \sum_{j=1}^{n+1} U_j^0 \boldsymbol{d}^j \in \mathscr{W}(\Omega^\varepsilon)$ (see (3.4)*,* (3.6)*) the extended divergence is*

$$\operatorname{div}_{\Omega^\varepsilon} \boldsymbol{U} := \sum_{j=1}^{n+1} \mathscr{D}_j U_j^0 + \mathscr{H}_{\mathscr{C}}^0 \langle \mathscr{N}, \boldsymbol{U}\rangle = -\nabla_{\Omega^\varepsilon}^* \boldsymbol{U}, \tag{3.11}$$

since

$$\mathscr{H}_{\Omega^\varepsilon}^0(x) := \sum_{j=1}^{n} \partial_j \mathscr{N}_j(x) = \sum_{j=1}^{n+1} \mathscr{D}_j \mathscr{N}_j(x) = \sum_{j=1}^{n} \mathscr{D}_j \nu_j(\mathscr{x}) = \mathscr{H}_{\mathscr{C}}^0(\mathscr{x}),$$
$$x \in \Omega^\varepsilon, \qquad \mathscr{x} = \pi_{\mathscr{S}} x$$

and $\mathscr{H}_{\mathscr{C}}^0(\mathscr{x})$ differs from the mean curvature $\mathscr{H}_{\mathscr{C}}(\mathscr{x})$ (see (1.183)*) by the constant multiplier $\mathscr{H}_{\mathscr{C}}^0(\mathscr{x}) = (n-1)\mathscr{H}_{\mathscr{C}}(\mathscr{x})$.*

Lemma 3.4 *The classical gradient* $\nabla\varphi := \left\{\partial_1\varphi, ..., \partial_n\varphi\right\}^\top$*, written in the full system of vectors* $\left\{\boldsymbol{d}^j\right\}_{j=1}^{n+1}$ *in* (3.3) *coincides with the extended gradient* $\mathscr{D}_{\Omega^\varepsilon}\varphi$ *in* (3.10).

Similarly: the classical divergence $\operatorname{div}\boldsymbol{U} := \sum\limits_{j=1}^{n}\partial_j U_j$ *of a vector field* $\boldsymbol{U} := \sum\limits_{j=1}^{n} U_j\boldsymbol{e}^j$*, written in the full system* (3.3)*, coincides with the extended divergence* $\operatorname{div}\boldsymbol{U} = \operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}$ *in* (3.11).

The extended gradient and the negative extended divergence are dual $\mathscr{D}^*_{\Omega^\varepsilon} = -\operatorname{div}_{\Omega^\varepsilon}$ *and* $\operatorname{div}^*_{\Omega^\varepsilon} = -\mathscr{D}_{\Omega^\varepsilon}$.

The Laplace-Beltrami operator $\Delta_{\Omega^\varepsilon} := \operatorname{div}_{\Omega^\varepsilon}\mathscr{D}_{\Omega^\varepsilon}\varphi = -\mathscr{D}^*_{\Omega^\varepsilon}\left(\mathscr{D}_{\Omega^\varepsilon}\varphi\right)$ *on* Ω^ε*, written in the full system* (3.3)*, acquires the following form*

$$\Delta_{\Omega^\varepsilon}\varphi = \sum_{j=1}^{n}\mathscr{D}_j^2\varphi + \partial^2_{\mathscr{N}}\varphi + \mathscr{H}^0_{\mathscr{C}}\partial_{\mathscr{N}}\varphi = \sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi + \mathscr{H}^0_{\mathscr{C}}\mathscr{D}_{n+1}\varphi\,, \quad \varphi\in\mathbb{H}^2(\Omega^\varepsilon)\,. \tag{3.12}$$

Proof: A similar lemma is proved in [Du5, Lemma 4.3], but definition of the divergence $\operatorname{div}_{\Omega^\varepsilon}$ is different there. Therefore we expose the full proof below.

That the gradients coincide follows from the choice of the full system (3.3):

$$\begin{aligned}\nabla\varphi := \left\{\partial_1\varphi, ..., \partial_n\varphi\right\}^\top &= \sum_{j=1}^{n}(\partial_j\varphi)\boldsymbol{e}^j = \sum_{j=1}^{n}(\mathscr{D}_j\varphi + \mathscr{N}_j\mathscr{D}_{n+1}\varphi)\boldsymbol{e}^j\\ &= \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^j + (\mathscr{D}_{n+1}\varphi)\mathscr{N} = \sum_{j=1}^{n+1}(\mathscr{D}_j\varphi)\boldsymbol{d}^j = \mathscr{D}_{\Omega^\varepsilon}\varphi\end{aligned} \tag{3.13}$$

since

$$\begin{gathered}\boldsymbol{e}^j = \boldsymbol{d}^j + \mathscr{N}_j\mathscr{N}, \quad \partial_j = \mathscr{D}_j + \mathscr{N}_j\partial_{\mathscr{N}},\\ \sum_{j=1}^{n}\mathscr{N}_j\mathscr{D}_j = 0, \quad \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{e}^j = \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^j.\end{gathered} \tag{3.14}$$

By applying (3.6) and (3.14) we proceed as follows:

$$\begin{aligned}\operatorname{div}\boldsymbol{U} &= \sum_{j=1}^{n}\partial_j U_j = \sum_{j=1}^{n}\mathscr{D}_j U_j + \sum_{j=1}^{n}\mathscr{N}_j\partial_{\mathscr{N}}U_j = \sum_{j=1}^{n}\mathscr{D}_j\left[U_j^0 + \mathscr{N}_j\langle\mathscr{N},\boldsymbol{U}\rangle\right]\\ &\quad + \sum_{j=1}^{n}\partial_{\mathscr{N}}\left(\mathscr{N}_j U_j\right) = \sum_{j=1}^{n}\mathscr{D}_j U_j^0 + \sum_{j=1}^{n}(\mathscr{D}_j\mathscr{N}_j)\langle\mathscr{N},\boldsymbol{U}\rangle + \mathscr{D}_{n+1}U^0_{n+1}\\ &= \sum_{j=1}^{n+1}\mathscr{D}_j U_j^0 + \mathscr{H}^0_{\mathscr{C}}\langle\mathscr{N},\boldsymbol{U}\rangle = \operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}.\end{aligned} \tag{3.15}$$

The proved equality and the classical equality $\nabla^* = -\mathrm{div}$, ensure the both claimed equalities $\mathscr{D}^*_{\Omega^\varepsilon} = -\mathrm{div}_{\Omega^\varepsilon}$ and $\mathrm{div}^*_{\Omega^\varepsilon} = -\mathscr{D}_{\Omega^\varepsilon}$:

$$(\mathscr{D}_{\Omega^\varepsilon}\varphi, \boldsymbol{U}) = (\nabla\,\varphi, \boldsymbol{U}) = -(\varphi, \mathrm{div}\,\boldsymbol{U}) = -(\varphi, \mathrm{div}_{\Omega^\varepsilon}\boldsymbol{U}).$$

Formula (3.12) for the Laplace-Beltrami operator is a direct consequence of equalities (3.13), (3.15) and definitions. Indeed, the first n components of the gradient

$$\nabla\varphi = \mathscr{D}_{\Omega^\varepsilon}\varphi = \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^{\,j} + (\mathscr{D}_{n+1}\varphi)\mathscr{N}$$

have the property $(\mathscr{D}_j\varphi)^0 = \mathscr{D}_j\varphi - \langle\mathscr{N}, \mathscr{D}_{\Omega^\varepsilon}\varphi\rangle\mathscr{N}_j = \mathscr{D}_j\varphi$ because (see the third formula in (3.14)) $\langle\mathscr{N}, \mathscr{D}_{\Omega^\varepsilon}\varphi\rangle = \sum_{j=1}^{n}\mathscr{N}_j\mathscr{D}_j\varphi = 0$ and we can write

$$\begin{aligned}\boldsymbol{\Delta}\varphi = \mathrm{div}\,\nabla\varphi = \mathrm{div}_{\Omega^\varepsilon}\mathscr{D}_{\Omega^\varepsilon}\varphi &= \sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi + \mathscr{H}^0_{\mathscr{C}}\langle\mathscr{N}, \nabla\varphi\rangle\\ &= \sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi + \mathscr{H}^0_{\mathscr{C}}\mathscr{D}_{n+1}\varphi = \Delta_{\Omega^\varepsilon}\varphi.\end{aligned}$$

□

3.3 CONVEX ENERGIES

Let again Ω^ε be a layer domain of width 2ε in the direction transversal to the mid-surface $\mathscr{C}$ (cf. § 3).

Any minimizer u of the energy functional

$$\mathscr{E}^\varepsilon(u) := \int_{\Omega^\varepsilon}\langle\nabla\,u, \nabla\,u\rangle\,dy, \qquad u \in \mathbb{H}^1(\Omega^\varepsilon) \tag{3.1}$$

should satisfy

$$\begin{aligned}0 &= \frac{d}{dt}\mathscr{E}^\varepsilon(u+tv)\Big|_{t=0} = \int_{\Omega^\varepsilon}\left[\langle\nabla\,u, \nabla\,v\rangle + \langle\nabla\,v, \nabla\,u\rangle\right]\,dy\\ &= 2\mathrm{Re}\int_{\Omega^\varepsilon}\langle\nabla\,u, \nabla\,v\rangle\,dy = -2\mathrm{Re}\int_{\Omega^\varepsilon}\langle\mathrm{div}\nabla\,u, v\rangle\,dy = -2\mathrm{Re}\int_{\Omega^\varepsilon}\langle\Delta\,u, v\rangle\,dy\end{aligned}$$

for arbitrary $v \in \widetilde{\mathbb{H}}^1(\Omega^\varepsilon)$, which implies

$$\Delta\,u = 0 \qquad \text{on} \quad \Omega^\varepsilon. \tag{3.2}$$

In other words, (3.2) is the Euler-Lagrange equation associated with the energy functional (3.1).

Similarly, minimizers of the energy functional

$$\mathscr{E}_0(u) := \int_{\mathscr{C}} \langle \nabla_{\mathscr{C}} u, \nabla_{\mathscr{C}} u \rangle \, d\sigma, \qquad u \in H^1(\mathscr{C})$$

on the hypersurface $\mathscr{C}$ should satisfy the following Laplace-Beltrami equation

$$\Delta_{\mathscr{C}} u := \mathrm{div}_{\mathscr{C}} \nabla_{\mathscr{C}} u = 0 \qquad \text{on} \quad \mathscr{C}. \tag{3.3}$$

To treat the dimension reduction problem for the Laplace equation (see [Br1] for a similar consideration in case of a flat 3D body), we assume, without restricting generality, that Ω^1 (i.e., for $\varepsilon = 1$) is still a layer domain. Otherwise we can first change the variable $\mathscr{x}_n = \varepsilon_0 \bar{\mathscr{x}}_n$, $0 < \bar{\mathscr{x}}_n < 1$, where $0 < \varepsilon_0 < 1$ is such that Ω^{ε_0} is still a layer domain.

Next we introduce a new coordinate system (cf. (3.6))

$$\begin{gathered} x := \sum_{m=1}^{n} x_m \boldsymbol{e}^m = \sum_{m=1}^{n} \mathscr{x}_m \boldsymbol{d}^m + t \boldsymbol{d}^{n+1}, \\ \mathscr{x}_k := x_k - \mathscr{N}_k \langle \mathscr{N}, x \rangle, \quad k = 1, \dots, n, \quad t = \mathscr{x}_{n+1} := \langle x, \mathscr{N} \rangle = \sum_{m=1}^{n} x_m \mathscr{N}_m \end{gathered} \tag{3.4}$$

and define the scalar product of elements as follows (cf. similar in (3.7)):

$$\langle \mathscr{x}, \mathscr{y} \rangle := \sum_{j=1}^{n+1} \mathscr{x}_j \mathscr{y}_j \quad \text{for} \quad \mathscr{x} := \sum_{m=1}^{n+1} \mathscr{x}_m \boldsymbol{d}^m, \quad \mathscr{y} := \sum_{m=1}^{n+1} \mathscr{y}_m \boldsymbol{d}^m.$$

Then (cf. (3.8)-(3.9))

$$\begin{aligned} \langle \mathscr{x}, \mathscr{y} \rangle = \sum_{j=1}^{n+1} \mathscr{x}_j \mathscr{y}_j = \sum_{j=1}^{n} (x_j - \mathscr{N}_j \langle \mathscr{N}, x \rangle)((y_j - \mathscr{N}_j \langle \mathscr{N}, y \rangle)) + \langle \mathscr{N}, x \rangle \langle \mathscr{N}, y \rangle \\ = \sum_{j=1}^{n} x_j \overline{y_j} = \langle x, y \rangle. \end{aligned}$$

In particular,

$$\| \mathscr{x} \| := \sum_{j=1}^{n+1} |\mathscr{x}_j|^2 = \sum_{j=1}^{n} |x_j|^2 = \|x\|. \tag{3.5}$$

Due to Lemma 3.4 the classical gradient in the energy functional (3.1) can be replaced by the extended gradient

$$\mathscr{E}^\varepsilon(u) := \int_{\Omega^\varepsilon} \langle \mathscr{D}_{\Omega^\varepsilon} u(y), \mathscr{D}_{\Omega^\varepsilon} u(y) \rangle \, dy = \int_{-\varepsilon}^{\varepsilon} \int_{\mathscr{C}} \left[|\mathscr{D}_{\mathscr{C}} u(\mathscr{x}, t)|^2 + |\partial_t u(\mathscr{x}, t)|^2 \right] d\sigma \, dt, \tag{3.6}$$

where $\mathscr{D}_{\mathscr{C}} := (\mathscr{D}_1, \dots, \mathscr{D}_n)^\top$ is the surface gradient and $u \in \mathbb{H}^1(\Omega^\varepsilon)$ is arbitrary, because $\mathscr{D}_{n+1} = \partial_{\mathscr{N}} = \partial_t$. Here $\mathscr{C}$ is the mid surface of the layer domain $\Omega^\varepsilon = \mathscr{C} \times (-\varepsilon, \varepsilon)$ and $d\sigma$ is the surface measure on $\mathscr{C}$.

Due to the representation (3.6) and the new coordinate system (3.4) we can apply the scaling with respect to the variable t and study the scaled energy. The approach is based on Γ-convergence (see [Br1, FJM1]) and can be applied to a general energy functional which is convex and has square growth. The problem we have in mind is the following: *Do these energies defined on thin n-dimensional domains Ω^ε converge (and in which sense) to an energy defend on the $n-1$ dimensional Hypersurface $\mathscr{C}$ (the mid-surface of Ω^ε) when the domain Ω^ε is "squeezed" infinitely in the transversal direction to $\mathscr{C}$?*

In the next two sections we apply the results developed in the present paper to boundary value problems for the heat conduction by a hypersurface. In particular we shall show, that if the thickness of the layer domain Ω^ε, with the mid-surface $\mathscr{C}$, tends to zero, the functionals in variational formulation of the linear heat conduction equation, Gamma-converge to the functional corresponding to some explicit boundary value problem for the Laplace-Beltrami equation on the mid-surface $\mathscr{C}$.

3.4 Variational reformulation of heat transfer problems

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ with the piecewise smooth boundary $\partial\Omega = \overline{\mathscr{C}}_D \cup \overline{\mathscr{C}}_N$, where $\mathscr{C}_D$ and $\mathscr{C}_N$ are open non-intersecting surfaces $\mathscr{C}_D \cap \mathscr{C}_N = \varnothing$ and their common boundary is a smooth arc. Denote by $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^\top$ the unit normal on $\mathscr{C}$, external with respect to Ω.

We consider the general steady-state, linear heat transfer problem for a medium occupying domain Ω. We assume that on the $\mathscr{C}_D$ part of the boundary $\partial\Omega$ the temperature g is prescribed, while on the $\mathscr{C}_N$ part of $\partial\Omega$ is prescribed the heat flux q.

We look for a temperature distribution $T(x)$ in Ω, which satisfies the linear heat conduction equation

$$\operatorname{div}(\mathscr{A}(x)\nabla T)(x) = f(x), \qquad x \in \Omega \tag{3.1}$$

and boundary conditions

$$T^+(y) = g(y) \qquad \text{on } \mathscr{C}_D, \tag{3.2}$$

$$-\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\rangle = q(y) \qquad \text{on } \mathscr{C}_N, \tag{3.3}$$

where $\mathscr{A}$ is the thermal conductivity, f is the heat source, g is the distribution of temperature and q is the heat flux. All these quantities are supposed known.

We assume, that $\mathscr{A}(x)$ is a bounded measurable and positive definite 3×3 matrix-function (cf. a similar condition (1.131))

$$\langle \mathscr{A}(x)\xi, \xi\rangle \geqslant C\|\xi\|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^3.$$

The following inequality is an obvious consequence of the positive definiteness of $\mathscr{A}$:

$$(\mathscr{A}\boldsymbol{U}, \boldsymbol{U}) \geqslant C\|\boldsymbol{U}|\mathbb{L}_2(\Omega)\|^2$$

for all 3-vectors $\boldsymbol{U} = (U_1, U_2, U_3)^\top \in \mathbb{L}_2(\Omega)$. Further we assume that the traces $\mathscr{A}^+(y)$ at the boundary $\mathscr{C}$ exist. Then $\mathscr{A}^+$ has the same properties as $\mathscr{A}$ on Ω, namely, is a bounded, measurable positive definite matrix function.

We impose the following natural constraints on the solution T and on the prescribed data f, g, q:

$$T \in \mathbb{H}^1(\Omega), \quad f \in \widetilde{\mathbb{H}}^{-1}(\Omega), \quad g \in \mathbb{H}^{1/2}(\mathscr{C}_D), \quad q \in \mathbb{H}^{-1/2}(\mathscr{C}_N). \tag{3.4}$$

The existence of the traces $\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+\rangle \in \mathbb{H}^{-1/2}(\mathscr{C}_3)$, which is not ensured by the trace theorem, follows from the Green formula

$$\int_\Omega (\operatorname{div} \mathscr{A}(x)\nabla T)(x)\psi(x)dx = \int_{\mathscr{C}} \langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\rangle \psi^+(y)\, d\sigma - \int_\Omega \langle \mathscr{A}(x)\nabla T(x), \nabla\psi(x)\rangle\, dx \tag{3.5}$$

by the duality between the spaces $\mathbb{H}^{1/2}(\mathscr{C})$ and $\widetilde{\mathbb{H}}^{-1/2}(\mathscr{C})$ due to the fact that T is a solution to the equation (3.1). For this we rewrite (3.5) in the form

$$\int_{\mathscr{C}} \langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\rangle \psi^+(y)\, d\sigma = \int_\Omega f(x)\psi(x)dx + \int_\Omega \langle \mathscr{A}(x)\nabla T(x), \nabla\psi(x)\rangle\, dx,$$

and note that $\psi \in \mathbb{H}^1(\Omega)$ is arbitrary and, therefore, $\psi^+ \in \mathbb{H}^{1/2}(\mathscr{C})$ is arbitrary.

First we will reduce the BVP (3.1)–(3.3) to the equivalent BVP with vanishing Dirichlet data.

Remark 3.5 *Let us assume the subsurface $\mathscr{C}_D$ is smooth and $g \in \mathbb{H}^s(\mathscr{C}_D)$, $s \geqslant \frac{1}{2}$. There exists a domain Ω' with a smooth boundary $\mathscr{C}' := \partial\Omega'$, with the properties: $\Omega \subset \Omega'$ and $\mathscr{C}_D \subset \mathscr{C}'$. Let $g^0 \in \mathbb{H}^s(\mathscr{C}')$ be such extension of g which maintains the space.*

The Dirichlet BVP

$$\begin{array}{ll} \operatorname{div}(\mathscr{A}(x)\nabla G)(x) = 0, & x \in \Omega', \\ G^+(y) = g^0(y) \quad \text{on } \mathscr{C}' & \end{array} \tag{3.6}$$

has a unique solution

$$G(x) = W\left(\frac{1}{2}I + W_0\right)^{-1} g^0(x), \qquad x \in \Omega', \qquad G \in \mathbb{H}^{s+1/2}(\Omega'),$$

where W is the double layer potential for the operator $\operatorname{div}\mathscr{A}(x)\nabla$ and W_0 is its direct value (a singular integral operator) on the surface $\mathscr{C}'$ $I : \mathbb{H}^s(\mathscr{C}') \to \mathbb{H}^s(\mathscr{C}')$ is a unit operator). Then the BVP

$$\begin{array}{l} \operatorname{div}(\mathscr{A}(x)\nabla T_0)(x) = f(x), \qquad x \in \Omega, \\ T_0^+(y) = 0 \qquad \text{on } \mathscr{C}_D, \\ -\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T_0)^+(y)\rangle = q_0(y) \qquad \text{on } \mathscr{C}_N \end{array} \tag{3.7}$$

is an equivalent reformulation of the BVP (3.1)–(3.3)*, now with vanishing Dirichlet traces. The solutions and Neumann datae are related as follows:*

$$\begin{array}{c} T_0(x) := T(x) - G(x), \quad x \in \Omega, \\ q_0(y) := q(y) - \left(\partial_{\boldsymbol{\nu}} W\left(\tfrac{1}{2}I + W_0\right)^{-1} g^0\right)^+(y), \quad x \in \mathscr{C}. \end{array} \tag{3.8}$$

Note, that if we require higher smoothness for the Neumann data $q \in \mathbb{H}^r(\mathscr{C}_N)$, $r > -1/2$ and take $g \in \mathbb{H}^{r+1}(\mathscr{C}_D)$ (i.e., $s = r + 1$ in (3.6)*), the Neumann data in the BVP* (3.7) *inherits the same smoothness $q_0 \in \mathbb{H}^r(\mathscr{C}_N)$.*

Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipshitz boundary $\mathscr{M} := \partial\Omega$ and $\mathscr{M}_0 \subset \partial\Omega$-be a subsurface of the boundary surface which has the non-zero measure. By $\widetilde{\mathbb{H}}^1(\Omega, \mathscr{M}_0)$ we denote a subspace of $\mathbb{H}^1(\Omega)$ of those functions which have vanishing traces on the part of the boundary

$$\widetilde{\mathbb{H}}^1(\Omega, \mathscr{M}_0) := \left\{\varphi \in \mathbb{H}^1(\Omega) \ : \ \varphi^+(y) = 0 \quad \forall\, y \in \mathscr{M}_0\right\}. \tag{3.9}$$

This space inherits the standard norm from $\mathbb{H}^1(\Omega)$:

$$\|\varphi \,|\, \mathbb{H}^1(\Omega)\,\| : \ = \left[\|\varphi \,|\, \mathbb{L}_2(\Omega)\,\|^2 + \sum_{j=1}^{n} \|\partial_j \varphi \,|\, \mathbb{L}_2(\Omega)\|^2\right]^{1/2}.$$

Consider the functional

$$\Phi(T) = \int\limits_{\Omega} \left[\frac{1}{2}\langle \mathscr{A}(x)\nabla T(x), \nabla T(x)\rangle + f(x)T(x)\right] dx + \int\limits_{\mathscr{C}_N} q(y)T^+(y)d\sigma \tag{3.10}$$

where f and q satisfy conditions (3.4) and $T \in \mathbb{H}^1(\Omega)$ has vanishing traces on $\mathscr{C}_D$, i.e., $T \in \widetilde{\mathbb{H}}^1(\Omega, \mathscr{C}_D)$ (see (3.9)).

The second summand in the integral on Ω is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{-1}(\Omega)$ and $\mathbb{H}^1(\Omega)$. Concerning the integral on $\mathscr{C}_N$: it is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}_N)$ and $\mathbb{H}^{-1/2}(\mathscr{C}_N)$ because $q \in \mathbb{H}^{-1/2}(\mathscr{C}_N)$ and the conditions $T \in \widetilde{\mathbb{H}}^1(\Omega, \mathscr{C}_D)$, $\operatorname{supp} T^+ \subset \mathscr{C}_N$ imply the inclusion $T^+ \in \widetilde{\mathbb{H}}^{1/2}(\mathscr{C}_N)$.

Theorem 3.6 *The problem* (3.1)-(3.3) *with vanishing Dirichlet condition* $T^+(y) = g(y) = 0$ *for all* $y \in \mathscr{C}_D$ *is reformulated into the following equivalent variational problem: Let* f *and* q *satisfy conditions* (3.4) *and look for a temperature distribution* $T \in \widetilde{\mathbb{H}}^1(\Omega, \mathscr{C}_D)$ *(see* (3.9)*) which is a stationary point of the functional* (3.10)*.*

Proof: Let $T(x)$ be a stationary point of the functional (3.10). Consider the variation

$$\delta\Phi = \frac{d}{d\varepsilon}\Phi(T + \varepsilon \boldsymbol{V})|_{\varepsilon=0} = \int\limits_{\Omega} \left[\langle \mathscr{A}(x)\nabla T(x), \nabla \boldsymbol{V}(x)\rangle + f(x)\boldsymbol{V}(x)\right]dx + \int\limits_{\mathscr{C}_N} q(y)\boldsymbol{V}^+(y)d\sigma. \tag{3.11}$$

The trial function $\boldsymbol{V} \in \mathbb{H}^1(\Omega)$ is such that $T + \varepsilon\boldsymbol{V}$ satisfies the boundary conditions. Then from the equalities $T^+(y) + \boldsymbol{V}^+(y) = 0 = T^+(y)$ on $\mathscr{C}_D$ follows that $T^+(y) = \boldsymbol{V}^+(y) = 0$ on $\mathscr{C}_D$, i.e., T and $\boldsymbol{V}$ have the traces vanishing on the part $\mathscr{C}_D$ of the boundary:

It is clear, that for those $\boldsymbol{V}$ for which the functional $\Phi(T + \varepsilon\boldsymbol{V})$ has a stationary point, we have $\delta\Phi = 0$. By applying the Gauß theorem to the first summand under the integral on

Ω in (3.11), we obtain the associated Euler-Lagrange equation

$$\int\limits_{\Omega}\big[-\operatorname{div}\mathscr{A}(x)\nabla T(x)+f(x)\big]\boldsymbol{V}(x)\,dx+\int\limits_{\mathscr{C}_D}\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T)^+(y)\rangle\boldsymbol{V}^+(y)d\sigma$$
$$+\int\limits_{\mathscr{C}_N}\Big[q(y)+\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T)^+(y)\rangle\Big]\boldsymbol{V}^+(y)d\sigma=0. \quad (3.12)$$

Since the trial function $\boldsymbol{V}$ vanishes on $\mathscr{C}_D$ (see (3.9)), the integral on $\mathscr{C}_D$ in (3.12) vanishes. Now taking arbitrary function $\boldsymbol{V}\in C_0^\infty(\Omega)$ (vanishing in the vicinity of the boundary $\mathscr{C}$), all summands in (3.12) except the first one vanish and we obtain

$$\int\limits_{\Omega}\big[-\operatorname{div}\mathscr{A}(x)\nabla T(x)+f(x)\big]\boldsymbol{V}(x)\,dx=0,$$

which is equivalent to the basic differential equation in (3.1).

Therefore from (3.12) follows that

$$\int\limits_{\mathscr{C}_N}\Big[q(y)+\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T)^+(y)\rangle\Big]\boldsymbol{V}^+(y)\,d\sigma=0. \quad (3.13)$$

The trace $\boldsymbol{V}^+$ of a trial function in (3.13) is arbitrary, we derive, the boundary condition (3.3).

Vice versa: Let T be a solution to the mixed problem (3.1)-(3.3) with vanishing Dirichlet traces $T^+(y)=g(y)=0$ on $\mathscr{C}$, by taking the scalar product of the basic equation in (3.1) with the solution T, by applying the Green formulae and the boundary conditions (3.2) with $g=0$, we get the following equality:

$$0=\int\limits_{\Omega}\big[-\operatorname{div}\mathscr{A}(x)\nabla T(x)+f(x)\big]T(x)\,dx=\int\limits_{\Omega}\big[\mathscr{A}(x)\nabla T(x)+f(x)\big]\nabla T(x)\,dx$$
$$+\int\limits_{\mathscr{C}_D\cup\mathscr{C}_N}\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T)^+(y)\rangle T^+(y)d\sigma$$
$$=\int\limits_{\Omega}\big[\mathscr{A}(x)\nabla T(x)+f(x)\big]\nabla T(x)\,dx\int\limits_{\mathscr{C}_N}q(y)T^+(y)d\sigma.$$

Therefore, T is a stationary point of the functional Φ in (3.10). □

If $\mathscr{C}_D=\mathscr{C},\ \mathscr{C}_N=\emptyset$, the problem (3.1)-(3.3) reduces to the problem with a Dirichlet boundary condition

$$T^+(y)=0 \qquad \text{on} \quad \mathscr{C}$$

and the corresponding functional Φ in variational formulation (see (3.10)) takes the form

$$\Phi_D(T)=\frac{1}{2}\int\limits_{\Omega}\Big[\langle\mathscr{A}(x)\nabla T(x),\nabla T(x)\rangle+f(x)T(x)\Big]\,dx.$$

If $\mathscr{C}_D = \emptyset,\ \mathscr{C}_N = \mathscr{C}$, from (3.1)-(3.3) we get the problem with Neumann boundary condition

$$-\langle \mathscr{A}^+(y)\boldsymbol{\nu}(y), (\nabla T)^+(y) = q(y) \qquad \text{on} \quad \mathscr{C}$$

and the corresponding functional in variational formulation (see (3.10)) takes the form

$$\Phi_N(T) = \frac{1}{2}\int\limits_{\Omega} \Big[\langle \mathscr{A}(x)\nabla T(x), \nabla T(x)\rangle + f(x)T(x)\Big]\, dx + \int\limits_{\mathscr{C}} q(y)T^+(y)d\sigma.$$

We conclude the section with some auxiliary results on Lebesgue points of integrable functions which is important in the next section.

Let $B(x)$ be a ball in the Euclidean space $B \subset \mathbb{R}^n$ centered at x. *The derivative of the integral at* x is defined to be

$$\lim_{B(x)\to x} \frac{1}{|B(x)|}\int_{B(x)} f(y)\,\mathrm{d}y, \tag{3.14}$$

where $|B(x)|$ denotes the volume (i.e., the Lebesgue measure) of $B(x)$, and $B(x) \to x$ means that the diameter of $B(x)$ tends to 0. Note that

$$\begin{aligned}\left|\frac{1}{|B(x)|}\int_{B(x)} f(y)\,\mathrm{d}y - f(x)\right| &= \left|\frac{1}{|B(x)|}\int_{B(x)} [f(y) - f(x)]\,\mathrm{d}y\right| \\ &\leqslant \frac{1}{|B(x)|}\int_{B(x)} |f(y) - f(x)|\,\mathrm{d}y. \end{aligned}\tag{3.15}$$

The points x for which the right hand side tends to zero are called the *Lebesgue points of* f.

Theorem 3.7 (Lebesgue Differentiation Theorem, Lebesgue 1910.) *For an integrable function* $f \in \mathbb{L}_1(\Omega)$ *the derivative of the integral* (3.14) *exists and is equal to* $f(x)$ *at almost every point* $x \in \Omega$.

Moreover, almost every point $x \in \Omega$ *is a Lebesgue point of* f *(see* (3.15)*).*

Corollary 3.8 *If* $g \in \mathbb{L}_2(\Omega)$, $f \in \mathbb{L}_2(\Omega \times (-1, 1))$, *then*

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} (g(\cdot), f(\cdot, \tau)))_\Omega d\tau = (g(\cdot), f(\cdot, t)))_\Omega \tag{3.16}$$

for almost all $t \in (-1, 1)$.

Proof: It is clear, that $g \cdot f \in \mathbb{L}_1(\Omega \times (-1, 1))$ and for the function $h(t) := (g(\cdot), f(\cdot, t))_\Omega$ the inclusion $h \in \mathbb{L}_1((-1, 1))$ is true. Thence we can apply Theorem 3.7 to the function $h(t)$ and get (3.16). □

3.5 Heat transfer in thin Layers

Let $\mathscr{C}$ be a C^2 smooth orientable surface in $\mathbb{R}^3$ given by a single chart (immersion)

$$\theta \,:\, \omega \to \mathscr{C}, \qquad \omega \subset \mathbb{R}^2$$

and let $\boldsymbol{\nu}(\mathscr{x}),\ \mathscr{x} \in \mathscr{C}$ be the unit normal vector field on $\mathscr{C}$ with the fixed orientation. Chart is supposed to be single just for convenience and multi-chart case can be considered similarly. Denote by Ω^ε the layer domain i.e. the set of all points in $\mathbb{R}^3$ in the distance less then ε from $\mathscr{C}$. Then for sufficiently small ε the map $\Theta \,:\, \mathscr{C} \times (-\varepsilon, \varepsilon) \to \Omega^\varepsilon$

$$\Theta(\mathscr{x}, t) = \mathscr{x} + t\boldsymbol{\nu}(\mathscr{x}) = \theta(x) + t\boldsymbol{\nu}(\theta(x)), \qquad x \in \omega$$

is C^1 homeomorphism and $\Theta(\mathscr{C} \times \{0\}) = \mathscr{C}$.

As noted above we can extend unit normal vector field to the entire Ω^ε properly by assuming

$$\boldsymbol{\nu}(\mathscr{x} + t\boldsymbol{\nu}(\mathscr{x})) = \boldsymbol{\nu}(\mathscr{x}), \qquad \mathscr{x} \in \mathscr{C}, \qquad -\varepsilon < t < \varepsilon.$$

If ε is sufficiently small, the boundary $\mathscr{M}^\varepsilon := \partial\Omega^\varepsilon$ is represented as the union of three C^1-smooth surfaces $\mathscr{M}^\varepsilon = \mathscr{M}_{\varepsilon,D} \cup \mathscr{M}^-_{\varepsilon,N} \cup \mathscr{M}^+_{\varepsilon,N}$, where $\mathscr{M}_{\varepsilon,D} = \partial\mathscr{C} \times [-\varepsilon, \varepsilon]$ is the lateral surface, $\mathscr{M}^+_{\varepsilon,N} = \mathscr{C} \times \{+\varepsilon\}$ is the upper surface and $\mathscr{M}^-_{\varepsilon,N} = \mathscr{C} \times \{-\varepsilon\}$ is the lower surface of the of the boundary $\mathscr{M}^\varepsilon$ of layer domain Ω^ε.

In the present section will be considered the heat conduction problem by an "isotropic" medium, governed by the BVP (cf. (3.4) for $\Delta_{\Omega^\varepsilon}$)

$$\begin{aligned}
&\Delta_{\Omega^\varepsilon} T(\mathscr{x}, t) = f(\mathscr{x}, t), && (\mathscr{x}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon),\\
&T^+(\mathscr{x}, t) = 0, && (\mathscr{x}, t) \in \partial\mathscr{C} \times (-\varepsilon, \varepsilon),\\
&(\partial_t T)^+(\mathscr{x}, \pm\varepsilon) = q_\varepsilon^\pm(\mathscr{x}), && \mathscr{x} \in \mathscr{C}.
\end{aligned} \tag{3.1}$$

The case of an "anisotropic" medium will be treated in a forthcoming publication.

We impose the following constraints

$$\begin{gathered}
T \in \mathbb{H}^1(\Omega^\varepsilon), \qquad q_\varepsilon^\pm \in \widetilde{\mathbb{H}}^2(\mathscr{C}), \qquad f \in \mathbb{L}_2(\Omega^1),\\
0 \ \text{ is the Lebesgue point for the function } \ \widetilde{f}(t) := \int_{\mathscr{C}} |f(\mathscr{x}, t)|^2 d\sigma
\end{gathered} \tag{3.2}$$

(see (3.15) and note that $\|\widetilde{f}\big|\mathbb{L}_1(-1,1)\| \leqslant \|f\big|\mathbb{L}_2(\Omega^1)\|^2$). The latter constraint implies that $\widetilde{f}(0)$ exists and, due to Theorem 3.7,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \widetilde{f}(t)dt = \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_{\mathscr{C}} |f(\mathscr{x}, t)|^2 d\sigma dt = \widetilde{f}(0).$$

The formulated BVP (3.1) governs a heat transfer in the body Ω^ε when there are thermal sources or sinks in Ω^ε. The temperature on the lateral surface $\partial\mathscr{C} \times (-\varepsilon, \varepsilon)$ is zero, the heat fluxes are fixed on the upper and lower surfaces $\mathscr{C}^\pm := \mathscr{C} \times \{\pm\varepsilon\}$. It is well known, that the boundary value problem (3.1) as well as it's equivalent problem (3.6)-(3.8) have the unique solution $T \in \mathbb{H}^1(\Omega^\varepsilon)$ (respectively, $T_0 \in \mathbb{H}^1(\Omega^\varepsilon)$; see, e.g., [DTT1]).

The energy functional associated with the problem (3.1) reads (cf. Theorem 3.6)

$$E(T_\varepsilon) : = \int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{\mathscr{C}} \left[\frac{1}{2}(\mathscr{D}_{\mathscr{C}}T^2(\mathscr{x},\tau) + \frac{1}{2\varepsilon^2}(\partial_\tau T^2(\mathscr{x},\tau) + F(\mathscr{x},\tau)T_\varepsilon(\mathscr{x},\tau)\right] d\sigma d\tau, \tag{3.3}$$

$$\begin{aligned} F(\mathscr{x},t) &:= f(\mathscr{x},t) - \frac{1}{4\varepsilon}\big((t+\varepsilon)^2\Delta_{\mathscr{C}}q_\varepsilon^+(\mathscr{x}) - (t-\varepsilon)^2\Delta_{\mathscr{C}}q_\varepsilon^-(\mathscr{x})\big) \\ &- \frac{\mathscr{H}^0_{\mathscr{C}}}{2\varepsilon}\big((t+\varepsilon)q_\varepsilon^+(\mathscr{x}) - (t-\varepsilon)q_\varepsilon^-(\mathscr{x})\big) - \frac{1}{2\varepsilon}\big(q_\varepsilon^+(\mathscr{x}) - q_\varepsilon^-(\mathscr{x})\big), \\ &\qquad (\mathscr{x},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon). \end{aligned} \tag{3.4}$$

More generally, we consider the non-linear functional

$$E_\varepsilon(T) = \int\limits_{\Omega^\varepsilon} \left[\mathscr{K}_0(\mathscr{D}_{\Omega^\varepsilon}T(x),\, T(x)) + F_\varepsilon(x)T(x)\right] dx, \tag{3.5}$$

where $\mathscr{K}_0(\mathscr{D}_{\Omega^\varepsilon}T,\, T)$ is strictly convex and has quadratic estimate. In the case of the functional (3.3),

$$\mathscr{K}_0(\mathscr{D}_{\Omega^\varepsilon}T,\, T) = \frac{1}{2}\langle\mathscr{D}_{\Omega^\varepsilon}T, \mathscr{D}_{\Omega^\varepsilon}T\rangle = \frac{1}{2}(\mathscr{D}_{\Omega^\varepsilon}T)^2 = \frac{1}{2}(\mathscr{D}_{\mathscr{C}}T_\varepsilon)^2(\mathscr{x},\tau) + \frac{1}{2\varepsilon^2}(\partial_\tau T_\varepsilon)^2(\mathscr{x},\tau) \tag{3.6}$$

and it is clear that the kernel is strictly convex because the quadratic function $F(x) = x^2$ is strictly convex $[\theta x_1 + (1-\theta)x_2]^2 < \theta x_1^2 + (1-\theta)x_2^2$ for all $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, $0 < \theta < 1$. The kernel has a trivial quadratic estimate, because is a quadratic function.

A nice proof of the next Lemma 1.49 is exposed in [AMR1, Example 3.6]

Lemma 3.9 *Let Ω be a domain in $\mathbb{R}^n$ with the Lipshitz boundary $\mathscr{M} := \partial\Omega$ and $\mathscr{M}_0 \subset \mathscr{M}$ be a subsurface of non-zero measure. Then the inequality*

$$\|\varphi \,|\, \mathbb{L}_2(\Omega)\| \leqslant C\|\nabla\varphi \,|\, \mathbb{L}_2(\Omega)\| = C\left[\sum_{j=1}^n \|\partial_j\varphi \,|\, \mathbb{L}_2(\Omega)\|^2\right]^{1/2} \tag{3.7}$$

holds for all functions $\varphi \in \widetilde{\mathbb{H}}^1(\Omega, \mathscr{M}_0)$ and the constant C is independent of φ.

Now we perform the scaling of the variable $t = \varepsilon\tau$, $-1 < \tau < 1$, and study the following functionals in the scaled domain $\Omega^1 = \mathscr{C} \times (-1, 1)$

$$E_\varepsilon(T_\varepsilon) = \int_{-1}^1 \int_{\mathscr{C}} \left[\mathscr{K}_0\left(\mathscr{D}_{\mathscr{C}}T_\varepsilon,\, \frac{1}{\varepsilon}\partial_t T_\varepsilon, T_\varepsilon\right) + F_\varepsilon T_\varepsilon\right] d\sigma d\tau \tag{3.8}$$

where $\mathscr{D}_{\mathscr{C}} = (\mathscr{D}_1,\, \mathscr{D}_2,\, \mathscr{D}_3)$, $\mathscr{D}_4 = \partial_t$. The functionals $E_\varepsilon(T_\varepsilon)$ are related to the original functional $E\,(T)$ by the equality

$$E_\varepsilon\,(T_\varepsilon) = \frac{1}{\varepsilon}E\,(T)\,, \quad \text{where} \quad T_\varepsilon(\mathscr{x},t) = T\,(\mathscr{x}_1,\mathscr{x}_2,\mathscr{x}_3,\varepsilon t)\,, \quad \text{and} \tag{3.9}$$

$$\begin{aligned} F_\varepsilon(\mathscr{x},t) = F(\mathscr{x},\varepsilon t) &= f(\mathscr{x},\varepsilon t) - \frac{\varepsilon}{4}\big((t+1)^2\Delta_{\mathscr{C}} q^+_\varepsilon(\mathscr{x}) - \frac{\varepsilon}{4}(t-1)^2\Delta_{\mathscr{C}} q^-_\varepsilon(\mathscr{x})\big) \\ &- \frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{x})}{2}\big((t+1)q^+_\varepsilon(\mathscr{x}) - (t-1)q^-_\varepsilon(\mathscr{x})\big) - \frac{1}{2\varepsilon}\big(q^+_\varepsilon(\mathscr{x}) - q^-_\varepsilon(\mathscr{x})\big), \\ &\qquad (\mathscr{x},t) \in \mathscr{C} \times (-\varepsilon,\varepsilon). \end{aligned} \tag{3.10}$$

Lemma 3.10 *Let F_ε be uniformly bounded in $\mathbb{L}_2(\Omega^1)$*

$$\sup_{\varepsilon<\varepsilon_0} \|F_\varepsilon|\mathbb{L}_2(\Omega^1)\| < \infty. \tag{3.11}$$

Then the energy functional $E_\varepsilon(T)$ in (3.8) is correctly defined on the space $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1,1))$, is strictly convex and has the following quadratic estimate

$$\begin{aligned} &E_\varepsilon(\theta T_1 + (1-\theta)T_2) < \theta E_\varepsilon(T_1) + (1-\theta)E_\varepsilon(T_2), \qquad 0<\theta<1 \\ &C_1 \int\limits_{\Omega^1} \mathscr{K}_0\left(\mathscr{D}_{\mathscr{C}}T,\, \frac{1}{\varepsilon}\partial_t T, T\right) d\sigma dt - C_2 \leqslant E_\varepsilon(T) \\ &\leqslant C_3\left[1 + \int\limits_{\Omega^1} \mathscr{K}_0\left(\mathscr{D}_{\mathscr{C}}T,\, \frac{1}{\varepsilon}\partial_t T, T\right) d\sigma dt\right], \\ &\qquad \forall\, T_1,\ T_2, T \in \widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1,1)) \end{aligned} \tag{3.12}$$

for some positive constants C_1, C_2 and C_3 not depending on ε.

Proof: Let us decompose the functional $E_\varepsilon(T)$ in (3.8) into the sum of non-linear and linear parts

$$\begin{aligned} &E_\varepsilon(T) = E^{(1)}_\varepsilon(T) + E^{(2)}_\varepsilon(T) \\ &E^{(1)}_\varepsilon(T) := \int_{\Omega^1} \mathscr{K}_0\left(\mathscr{D}_{\mathscr{C}}T,\, \frac{1}{\varepsilon}\partial_t T, T\right) dx, \\ &E^{(2)}_\varepsilon(T) := \int_{\Omega^1} F_\varepsilon(x)T(x)dx. \end{aligned} \tag{3.13}$$

By the conditions imposed on $\mathscr{K}_0$ in (3.5), the first (non-linear) functional $E^{(1)}_\varepsilon(T)$ is strictly convex and has a quadratic estimate:

$$\begin{aligned} &C^0_1 \int\limits_{\Omega^1} \left(\langle \mathscr{D}_{\mathscr{C}}T_j, \mathscr{D}_{\mathscr{C}}T_j\rangle + \frac{1}{\varepsilon_j^2}|\partial_t T_j|^2\right) dx - C^0_2 \leqslant E^{(1)}_\varepsilon(T) \\ &\qquad \leqslant C^0_3\left[1 + \int\limits_{\Omega^1} \left(\langle \mathscr{D}_{\mathscr{C}}T_j, \mathscr{D}_{\mathscr{C}}T_j\rangle + \frac{1}{\varepsilon_j^2}|\partial_t T_j|^2\right) dx\right]. \end{aligned} \tag{3.14}$$

On the other hand $E^{(2)}_\varepsilon(T)$ is linear and, therefore, strictly convex (see the first inequality in (3.12)). Thus, we only have to prove the two-sided quadratic estimate in (3.12) for the linear functional $E^{(2)}_\varepsilon(T)$. Due to Lemma 3.9 and the equality (3.13) we can write:

$$
\begin{aligned}
|E^{(2)}_\varepsilon(T)| \leqslant \left|\int_{\Omega^1} F_\varepsilon(x)T(x)dx\right| &\leqslant \|F_\varepsilon|\mathbb{L}_2(\Omega^1)\|\,\|T|\mathbb{L}_2(\Omega^1)\| \leqslant M\|\nabla T|\mathbb{L}_2(\Omega^1)\| \\
&\leqslant M\Big(\frac{1}{\eta}+\eta\|\nabla T|\mathbb{L}_2(\Omega^1)\|^2\Big) \leqslant M\Big(\frac{1}{\eta}+\eta\|\mathscr{D}_{\Omega^1}T|\mathbb{L}_2(\Omega^1)\|^2\Big). \quad (3.15)
\end{aligned}
$$

Choosing $\eta = 1$ in (3.15) and taking into account (3.14) we get the right inequality in the second line of (3.12), whereas taking η sufficiently small we obtain

$$
E_\varepsilon(T) \geqslant |E^{(1)}_\varepsilon(T)| - |E^{(2)}_\varepsilon(T) \geqslant C_1\|\mathscr{D}_{\Omega^\varepsilon}T|\mathbb{L}_2(\Omega^\varepsilon)\|^2 - C_2. \qquad \square
$$

Let $F_j = F_{\varepsilon_j}$, $0 < \varepsilon_j \leq 1$, $\lim\limits_{j\to\infty}\varepsilon_j = 0$ and F_{ε_j} be uniformly bounded (see (3.11)). Further let $T_j = T_{\varepsilon_j} \in \widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C}\times(-1,1)),\quad j = 1,2,\dots$ be the sequence of functions with "finite energy":

$$
\sup_j E_{\varepsilon_j}(T_j) < +\infty. \tag{3.16}
$$

Then from (3.14)–(3.15) we get

$$
\begin{aligned}
C^0_1\|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^1)\|^2 &= \int_{\Omega^1}\left(\frac{1}{2}\langle\mathscr{D}_\mathscr{C}T_j,\mathscr{D}_\mathscr{C}T_j\rangle + \frac{1}{2\varepsilon_j^2}|\partial_t T_j|^2\right)dx \\
&= C^0_1 E_{\varepsilon_j}(T_j) - C^0_1\int_{\Omega^1} F_j(\mathscr{x},t)T_j(\mathscr{x},t)d\sigma dt \\
&\leqslant C^0_2\big(1+\|F_j|\mathbb{L}_2(\Omega^1)\|\|T_j|\mathbb{L}_2(\Omega^1)\|\big) \\
&\leqslant C^0_3\Big(1+\|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^\varepsilon)\|^2\Big)^{\frac{1}{2}},
\end{aligned} \tag{3.17}
$$

since, due to Lemma 3.9,

$$
\|T_j|\mathbb{L}_2(\Omega^1)\| \leqslant C_0\|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^1)\|. \tag{3.18}
$$

Consequently,

$$
\sup_j\|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^1)\| = \sup_j\left(\int_{\Omega^1}\left(\frac{1}{2}\langle\mathscr{D}_\mathscr{C}T_j,\mathscr{D}_\mathscr{C}T_j\rangle + \frac{1}{2\varepsilon_j^2}|\partial_t T_j|^2\right)dx\right)^{1/2} < +\infty. \tag{3.19}
$$

From (3.17)-(3.19) follows

$$
\sup_j\int_{\Omega^1}|T_j|^2\,dx < \infty, \quad \sup_j\int_{\Omega^1}|\mathscr{D}_\mathscr{C}T_j|^2\,dx < \infty, \quad \sup_j\frac{1}{\varepsilon_j^2}\int_{\Omega^1}|\partial_t T_j|^2\,dx < \infty. \tag{3.20}
$$

Note, that if T_j are the scaled solutions to problem (3.3), then from the Euler-Lagrange equation, associated with the functional (see (3.12)), follows that $E_{\varepsilon_j}(T_j) = 0$ and therefore conditions (3.20) are satisfied.

Due to (3.20) the sequence $\{T_j\}_{j=1}^{\infty}$ is uniformly bounded in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C}\times(-1,1))$ and a weakly converging subsequence (say $\{T_j\}_{j=1}^{\infty}$ itself) to a function T in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C}\times(-1,1))$ can be extracted.

The functional

$$H(T) = \int\limits_{\Omega^1} |\partial_t T|^2\, dx$$

is convex and continuous in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C}\times(-1,1))$; then it is weakly lower semi-continuous and $\partial_t T = 0$ a.e., because

$$\int_{\Omega^1} |\partial_t T|^2\, dx = H(T) \le \liminf_j H(T_j) = \liminf_j \int_{\Omega^1} |\partial_t T_j|^2\, dx = 0.$$

(see the last inequality in (3.20)). Hence $T(\mathscr{x}, t)$ is independent of t, i.e.

$$T(\mathscr{x}, t) = T(\mathscr{x}), \qquad \mathscr{x}\in\mathscr{C}, \quad -1\le t\le 1. \tag{3.21}$$

Let the following conditions are fulfilled

$$f_\varepsilon(\mathscr{x}, t) := f(\mathscr{x}, \varepsilon t) \underset{\varepsilon\to 0}{\to} f^0(\mathscr{x}) \quad in \quad \mathbb{L}_2(\Omega^1), \tag{3.22}$$

$q_\varepsilon^{\pm} \in \mathbb{H}^2(\mathscr{C})$ are uniformly bounded (with respect to ε) in $\mathbb{H}^2(\mathscr{C})$, and

$$\lim_{\varepsilon\to 0} q_\varepsilon^{+} = \lim_{\varepsilon\to 0} q_\varepsilon^{-} = q_0, \quad in \quad \mathbb{L}_2(\mathscr{C}), \tag{3.23}$$

and

$$\frac{1}{2\varepsilon}(q_\varepsilon^{+} - q_\varepsilon^{-}) \underset{\varepsilon\to 0}{\to} q_1 \quad in \quad \mathbb{L}_2(\mathscr{C}). \tag{3.24}$$

From (3.22)- (3.24) follows in particular, that

$$F_j(\mathscr{x}, t) \to F(\mathscr{x}, 0) \quad in \quad \mathbb{L}_2(\Omega^1). \tag{3.25}$$

Set

$$E^{(0)}(T) = \begin{cases} E^{(1)}(T) + E^{(2)}(T) & \text{for} \quad T\in\mathscr{P}(\mathscr{C}); \\ +\infty, & \text{for} \quad T\notin\mathscr{P}(\mathscr{C}). \end{cases} \tag{3.26}$$

where $\mathscr{P}(\mathscr{C})$ is defined in (3.12), and

$$\begin{aligned} E^{(1)}(T) &:= \frac{1}{2}\int_{\Omega^1} \langle(\mathscr{D}_{\Omega^1}T)(x,t), (\mathscr{D}_{\Omega^1}T)(x,t)\rangle d\sigma\, dt \\ &= \int_{\mathscr{C}} \langle(\mathscr{D}_{\mathscr{C}}T_{\mathscr{C}})(\mathscr{x}), (\mathscr{D}_{\mathscr{C}}T_{\mathscr{C}})(\mathscr{x})\rangle d\sigma, \end{aligned} \tag{3.27}$$

$$\begin{aligned} E^{(2)}(T) &:= \int_{\Omega^1} F(\mathscr{x}, 0) T(\mathscr{x}, t) d\sigma\, dt \\ &= 2\int_{\mathscr{C}} \left(f^0(\mathscr{x}) - \mathscr{H}_{\mathscr{C}}^0 q_0(\mathscr{x}) - q_1(\mathscr{x})\right) T_{\mathscr{C}}(\mathscr{x}) d\sigma. \end{aligned} \tag{3.28}$$

Let us check that the E_{ε_j} sequence Γ-convergs to $E^{(0)}$ in $\widetilde{\mathbb{H}}^1(\Omega^\varepsilon, \partial\mathscr{C} \times (-1,1))$. Indeed, we have

$$E_{\varepsilon_j}(T_j) = E^{(1)}_{\varepsilon_j}(T_j) + E^{(2)}_{\varepsilon_j}(T_j),$$

where

$$E^{(1)}_{\varepsilon_j}(T_j) = \int\limits_{\Omega^1} \left(\frac{1}{2} \langle \mathscr{D}_{\mathscr{C}} T_j, \mathscr{D}_{\mathscr{C}} T_j \rangle + \frac{1}{2\varepsilon_j^2} |\partial_t T_j|^2 \right) dx, \quad E^{(2)}_{\varepsilon_j}(T_j) = \int\limits_{\Omega^1} F_j T_j dx.$$

The functional $E^{(1)}(T)$ is convex and continuous and so it is weakly lower semicontinuous in $\widetilde{\mathbb{H}}^1(\Omega^\varepsilon, \partial\mathscr{C} \times (-1,1))$, therefore

$$\liminf_j E^{(1)}_{\varepsilon_j}(T_j) \geqslant \liminf_j E^{(1)}(T_j) \geqslant E^{(1)}(T).$$

Sequence $E^{(2)}_{\varepsilon_j}(T_j)$ converges to $E^{(2)}(T)$, because $F_j(\mathscr{x}, t) \to F(\mathscr{x}, 0)$ and $T_j \rightharpoonup T$ in $\mathbb{L}_2(\Omega^1)$. Consequently

$$\liminf_j E_{\varepsilon_j}(T_j) \geqslant E^{(0)}(T).$$

This proves $\liminf$ inequality for the sequence E_{ε_j}.

Note, that

$$E^{(2)}(T) = \int\limits_{\mathscr{C}} \int\limits_{-1}^{1} F(\mathscr{x}, 0) T(\mathscr{x}, t) dt\, d\sigma = 2 \int\limits_{\mathscr{C}} F(\mathscr{x}, 0) T_{\mathscr{C}}(\mathscr{x}) d\sigma.$$

To show that the lower bound is reached i.e. to build a recovery sequence T_j we fix $T_{\mathscr{C}} \in \mathbb{H}^1(\mathscr{C})$ and set $T(\mathscr{x}, t) = T_{\mathscr{C}}(\mathscr{x}),\ \mathscr{x} \in \mathscr{C},\ t \in (-1,1)$. Define recovery sequence as $T_j(x,t) = T(x,t) = T_{\mathscr{C}}(x)$ Then $\partial_t T_j = \partial_t T = 0$ and

$$\lim_{j\to\infty} E_{\varepsilon_j}(T_j) = \lim_{j\to\infty} E^{(1)}_{\varepsilon_j}(T) + \lim_{j\to\infty} E^{(2)}_{\varepsilon_j}(T) = E^{(1)}(T) + E^{(2)}(T) = E^{(0)}(T).$$

We have proved the following result.

Theorem 3.11 *If conditions* (3.22)- (3.24) *are fulfilled then the functional in* (3.8) Γ*-converges to the functional* $E^{(0)}(T)$ *defined in* (3.13) *as* $\varepsilon \to 0$.

Now we are able to prove the main Theorem 3.1 formulated in the introduction.

Proof of Theorem 3.1: The first part of the Theorem i.e. Γ-convergence of the functional (3.13) to the functional $E^{(0)}$ defined by (3.13), is proved in Theorem 3.11.

The concluding assertion, that the BVP (3.14) is an equivalent reformulation of the minimization problem with the energy functional (3.13), is explained in Theorem 3.6. □

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