THIN SHELLS WITH LIPSCHITZ BOUNDARY[(1)]

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In [Du11] we have revised an asymptotic model of a shell (Koiter, Sanchez-Palencia, Ciarlet etc.), based on the the calculus of tangent Günter's derivatives, developed in the papers of R. Duduchava, D. Mitrea and M. Mitrea [Du11, **?**, Du02b, DMM06]. As a result the 2-dimensional shell equation on a mid-surface $\mathscr{S}$ was written in terms of Günter's derivatives, unit normal vector field and the lamé constant, which coincides with the Lamé equation on the Hypersurface $\mathscr{S}$, investigated in [Du11, **?**, Du02b, DMM06].

The present investigation is inspired by the paper of G. Friesecke, R. D. James & S. Mller [FJM1], where a hierarchy of Plate Models are derived from nonlinear elasticity by Γ-Convergence. The final goal of the present investigation is to derive 2D shell equations in terms of Günter's derivatives by Γ-Convergence.

As a first step to the final goal in the paper of T. Buchukuri, R. Duduchava & G. Tephnadze [BDT1] was studied a mixed boundary value problem for the stationary heat transferA mixed boundary value problem for the stationary heat transfer equation in a thin layer around a surface $\mathscr{C}$ with the boundary. It was established what happens in Γ-limit when the thickness of the layer converges to zero. In particular, was shown that the Γ-limit of a mixed type Dirichlet-Neumann boundary value problem (BVP) for the Laplace equation in the initial thin layer is a Dirichlet BVP for the Laplace-Beltrami equation on the surface. For this was applied the variational formulation and the calculus of Günter's tangential differential operators on a hypersurface and layers. This approach allow global representation of basic differential operators and of corresponding BVPs in terms of the standard cartesian coordinates of the ambient Euclidean space $\mathbb{R}^n$.

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INTRODUCTION

Modern interest in shell theories has blossomed with the ubiquitous presence of thin films in science and technology. Thin structures encounter in engineering applications more and more often and there emerged numerous approaches proposed for modeling linearly elastic flexural shells. Started by the Cosserats pioneering work (1909), Goldenveiser (1961), Naghdi (1963), Vekua (1965), Novozhilov (1970), Koiter (1970) and many others contributed essentially the development of the shell theory. Ellipticity of the corresponding partial differential equations was proved later by Roug'e (1969) for cylindrical shells, by Coutris (1973) for the shell model proposed by Naghdi, Gordeziani (1974) for the shell model proposed by Vekua, Shoikhet (1974) for the shell model proposed by Novozhilov, Ciarlet & Miara (1992) for the model proposed by Koiter (cf. [Ci1], [Ci3]-[Ci6], [De1] for survey and further references).

Inspired by the books and papers of Sanchez-Palencia [Sa90, Sa92], Miara & Sanchez-Palencia [MS96], Ciarlet & Lods [CL1, CL2, CL3], Ciarlet, Lods & Miara [CLM1] and exposed in details in Ciarlet [Ci3, Ci5] we have developed in [Du11] asymptotic analysis of a linearly elastic shell based on the formal calculus of tangential Günter's derivatives, developed in the papers of the author with D. Mitrea and M. Mitrea [Du11, **?**, Du02b, DMM06]. As a result the 2-dimensional shell equation on a middle surface $\mathscr{S}$ is derived written in terms of Gunter's derivatives, unit normal vector field and the lamé constant, which coincides with the Lamé equation on the Hypersurface $\mathscr{S}$, investigated in [Du11, **?**, Du02b, DMM06].

The present investigation is inspired by the paper of G. Friesecke, R. D. James & S. Mller [FJM1], where a hierarchy of Plate Models are derived from nonlinear elasticity by Γ-Convergence. The final goal of the investigation is to derive 2D shell equations written in terms of Günter's derivatives by Γ-Convergence

Let us consider an example: a surface $\mathscr{S}$ be given by a local immersion

$$\Theta\,:\,\omega\to\mathscr{S}\,,\quad \omega\subset\mathbb{R}^{n-1}\,,\tag{0.1}$$

which means that the derivatives $\big\{\boldsymbol{g}_k:=\partial_k\Theta\big\}_{k=1}^{n-1}$ are linearly independent, i.e., the Jakobi matrix $\nabla_x\Theta_x$ has the maximal rank $n-1$. Thus, $\big\{\boldsymbol{g}_k\big\}_{k=1}^{n-1}$ is a **basis** (or a **covariant frame** if the basis is enriched with 0) in the space $\omega(\mathscr{S})$ of all tangential vector fields on $\mathscr{S}$. The system $\big\{\boldsymbol{g}^k\big\}_{k=1}^{n-1}$ which is biorthogonal $\langle\boldsymbol{g}_j,\boldsymbol{g}^k\rangle=\delta_{jk}$ forms the **contravariant basis** (the **contravariant frame**) in the same space $\omega(\mathscr{S})$ of all tangential vector fields on $\mathscr{S}$. Let $\boldsymbol{\nu}(\mathscr{x})=(\nu_1(\mathscr{x}),\dots,\nu_j(\mathscr{x}))^\top$ be the outer unit normal vector (the Gauß mapping) to $\mathscr{S}$ at $\mathscr{x}\in\mathscr{S}$ (see § 4 for details).

The Gram matrix $G_{\mathscr{S}}(\mathscr{x})=[g_{jk}(\mathscr{x})]_{n-1\times n-1}$, $g_{jk}:=\langle g_j,g_k\rangle$, is then positive definite, responsible for the Riemann metric on $\mathscr{S}$ and is called the **covariant metric tensor**. Moreover, it has the inverse matrix $G^{-1}_{\mathscr{S}}(\mathscr{x})=[g^{jk}(\mathscr{x})]_{n-1\times n-1}$, , $g^{jk}:=\langle\boldsymbol{g}^j,\boldsymbol{g}^k\rangle$ (cf. (2.4), (0.2)), which is called the **contravariant metric tensor**.

The Gram determinant

$$\mathscr{G}\big(\partial_1\Theta(x),\dots,\partial_{n-1}\Theta)(x)\big)=\det G_{\mathscr{S}}(x)\,,\qquad x\in\omega\subset\mathbb{R}^{n-1}\tag{0.2}$$

is responsible for the volume element $d\sigma$ of the surface, which is the vector product of the tangential vectors

$$d\sigma:=|\partial_1\Theta\wedge\ldots\wedge\partial_{n-1}\Theta|=\sqrt{\det G_{\mathscr{S}}}\,dx\,,\qquad dx=dx_1\cdots dx_{n-1}\,.\tag{0.3}$$

The surface divergence and the surface gradient are defined in the intrinsic coordinates by the equalities

$$\begin{aligned} \operatorname{div}_{\mathscr{S}} \boldsymbol{U} &:= \left[\det G_{\mathscr{S}}\right]^{-1/2} \sum_{j=1}^{n} \partial_j \left\{ \left[\det G_{\mathscr{S}}\right]^{1/2} U^j \right\}, \\ \nabla_{\mathscr{S}} f &= \sum_{j,k=1}^{n-1} (g^{jk} \partial_j f)\, \partial_k \end{aligned} \tag{0.4}$$

(see § 6 and [Ta96, Ch. 2, § 3]). Their composition is the **Laplace-Beltrami operator**

$$\boldsymbol{\Delta}_{\mathscr{S}} f := \operatorname{div}_{\mathscr{S}} \nabla_{\mathscr{S}} f = \left[\det G_{\mathscr{S}}\right]^{-1/2} \sum_{j,k=1}^{n-1} \partial_j \left\{ g^{jk} \left[\det G_{\mathscr{S}}\right]^{1/2} \partial_k f \right\}, \quad f \in C^2(\mathscr{S}), \tag{0.5}$$

which is self-adjoint

$$\Delta^*_{\mathscr{S}} = \left(\nabla_{\mathscr{S}} \operatorname{div}_{\mathscr{S}}\right)^* = \left(\operatorname{div}_{\mathscr{S}}\right)^* \left(\nabla_{\mathscr{S}}\right)^* = \nabla_{\mathscr{S}} \operatorname{div}_{\mathscr{S}} = \boldsymbol{\Delta}_{\mathscr{S}}. \tag{0.6}$$

The intrinsic parameters enable generalization to arbitrary manifolds, not necessarily immersed in the Euclidean space $\mathbb{R}^n$.

On the other hand sometimes it is more convenient to record these operators in Cartesian coordinates. To set the conditions for precise formulations let us consider the **natural basis**

$$\boldsymbol{e}^1 := (1, 0, \ldots, 0)^\top, \ldots, \boldsymbol{e}^n := (0, \ldots, 0, 1)^\top \tag{0.7}$$

in the Euclidean space $\mathbb{R}^n$ ($\left\{e^j\right\}_{j=1}^n$ is also called the **Cartesian basis** since it is ordered). Each point $x = (x_1, \ldots, x_n)^\top$ in the Euclidean space $\mathbb{R}^n$ is represented in the Cartesian basis $x = \sum_{j=1}^n x^j e_j$ in a unique way.

Let the operator (the matrix)

$$\pi_{\mathscr{S}} : \mathbb{R}^n \to \omega(\mathscr{S}), \quad \pi_{\mathscr{S}}(t) = I - \boldsymbol{\nu}(t)\boldsymbol{\nu}^\top(t) = \left[\delta_{jk} - \nu_j(t)\nu_k(t)\right]_{n\times n}, \quad t \in \mathscr{S} \tag{0.8}$$

denote the canonical orthogonal projection $\pi^2_{\mathscr{S}} = \pi_{\mathscr{S}}$ onto the space of tangential vector fields to $\mathscr{S}$ at the point $t \in \mathscr{S}$:

$$(\boldsymbol{\nu}, \pi_{\mathscr{S}} v) = \sum_j \nu_j v_j - \sum_{j,k} \nu_j^2 \nu_k v_k = 0 \qquad \text{for all} \quad v = (v_1, \ldots, v_n)^\top \in \mathbb{R}^n.$$

It turns out that the surface gradient is nothing but the collection of the weakly tangential **Günter's derivatives** (cf. [Gu94], [KGBB76], [Du02a])

$$\nabla_{\mathscr{S}} = \mathscr{D}_{\mathscr{S}} := (\mathscr{D}_1, \ldots, \mathscr{D}_n)^\top, \quad \mathscr{D}_j := \partial_j - \nu_j(\mathscr{x})\partial_{\boldsymbol{\nu}} = \partial_{\boldsymbol{d}^j}, \tag{0.9}$$

where $\partial_{\boldsymbol{\nu}} := \sum_{j=1}^n \nu_j \partial_j$ denotes the normal derivative. The first-order differential operators

$$\mathscr{D}_j = \partial_{\boldsymbol{d}^j}, \qquad 1 \leqslant j \leqslant n, \tag{0.10}$$

are the directional derivative along the vector fields $\boldsymbol{d}^j := \pi_{\mathscr{S}} \boldsymbol{e}^j$, $j = 1, \ldots, n$.

Moreover, the surface divergence coincides with the operator

$$\mathrm{div}_{\mathscr{S}}\boldsymbol{U} = \sum_{j=1}^{n} \mathscr{D}_j U_j^0\,, \qquad \text{for} \quad \boldsymbol{U} = \sum_{j=1}^{n} U_j^0 \partial_j \in \omega(\mathscr{S}) \tag{0.11}$$

and the Laplace-Beltrami operator coincides with (see also [MM84, pp. 2ff and p. 8.])

$$\boldsymbol{\Delta}_{\mathscr{S}}\,\varphi := \mathrm{div}_{\mathscr{S}}\nabla_{\mathscr{S}}\varphi = \sum_{j=1}^{n} \mathscr{D}_j^2\varphi\,, \quad \varphi \in C^2(\mathscr{S})\,. \tag{0.12}$$

Relatively simple form of recorded operators enables simplified treatment of corresponding boundary value problems, which require proofs of Korn's inequalities or similar.

The Laplace-Beltrami operator (0.12) is the natural operator associated with the Euler-Lagrange equations for a variational integral

$$\mathscr{E}[u] = -\frac{1}{2}\int_{\mathscr{S}} \|\mathscr{D}u\|^2\,dS. \tag{0.13}$$

A similar approach, based on the principle that, at equilibrium, the displacement minimizes the potential energy (Koiter's model), leads to the following form of the Lamé operator $\mathscr{L}_{\mathscr{S}}$ on $\mathscr{S}$ (cf. [DMM06])

$$\mathscr{L}_{\mathscr{S}}\boldsymbol{U} = \mu\pi_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\,\nabla_{\mathscr{S}}\boldsymbol{U} + (\lambda+\mu)\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U} + \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U}\,, \tag{0.14}$$

(cf. (0.8) for the projection $\pi_{\mathscr{S}}$). Here $\boldsymbol{U}$ is an arbitrary (tangential) vector fields on $\mathscr{S}$, $\lambda, \mu \in \mathbb{R}$ are the Lamé moduli, whereas

$$\mathscr{H}^0_{\mathscr{S}} = -\mathrm{div}_{\mathscr{S}}\boldsymbol{\nu} := -\sum_{j=1}^{n} \mathscr{D}_j\nu_j = \mathrm{Tr}\,\mathscr{W}_{\mathscr{S}}\,, \qquad \mathscr{W}_{\mathscr{S}} = -\big[\mathscr{D}_j\nu_k\big]_{n\times n}\,. \tag{0.15}$$

Note, that $\mathscr{H}_{\mathscr{S}} := (n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}$ and $\mathscr{W}_{\mathscr{S}}$ represent, respectively, the **mean curvature** and the **Weingarten mapping** (cf. (2.19)) of $\mathscr{S}$. This identification ensures that the boundary-value problem

$$\begin{cases} \mathscr{L}_{\mathscr{S}}\boldsymbol{U} = 0 \quad \text{in} \quad \mathscr{S}\,, \\ \boldsymbol{U}\big|_{\Gamma} = f \in \mathbb{H}^s(\partial\mathscr{S})\,, \quad f\cdot\boldsymbol{\nu} = f\cdot\boldsymbol{\nu}_{\Gamma} = 0 \quad \text{on} \quad \Gamma := \partial\mathscr{S}, \end{cases} \tag{0.16}$$

where $\boldsymbol{U} = \sum_{j=1}^{n} U_j^0\boldsymbol{d}^j \in \omega(\mathscr{S})\cap\mathbb{H}^{s+1/2}(\partial\mathscr{S})$ is the (tangential) generalized displacement vector field of the elastic hypersurface $\mathscr{S}$, is well-posed, whenever $\mu > 0$, $2\mu+\lambda > 0$, and $0 \leqslant s \leqslant 1$. Here $\mathbb{H}^s$ stands for the usual L^2-based Sobolev scale, $\boldsymbol{\nu}$ is the normal vector to $\mathscr{S}$ and $\boldsymbol{\nu}_{\Gamma}(t)$ is the unit tangential vector to $\mathscr{S}$ at the boundary point $t \in \Gamma := \partial\mathscr{S}$ and outer normal vector to the boundary $\Gamma = \partial\mathscr{S}$.

Chapter 1

AUXILIARY

In the present chapter we have collected, for the readers convenience, some auxiliary information, mostly from [Ci1, Ci2, Ci3, Ci4, DMM06, FJM1, Ta92].

1 Differentiation and implicit function theorem

In the present section we expose implicit and inverse function theorems, which are applied later.

Let us recall some standard notation: $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, \ldots\}$. For a natural number $n \in \mathbb{N}$ let $\mathbb{R}^n$ and C^n denote the n-dimensional spaces of vectors $x = (x_1, \ldots, x_n)^\top$ with real $x_j \in \mathbb{R}$ and complex $x_j \in \mathbb{C}$ entries and standard metrics, based on the scalar product

$$\langle x, y\rangle := x_1\overline{y_1} + \cdots + x_n\overline{y_n} \quad \text{for} \quad x, y \in C^n$$
$$\langle x, y\rangle := x_1 y_1 + \cdots + x_n y_n \quad \text{for} \quad x, y \in \mathbb{R}^n .$$

$\mathbb{N}_n$ and $\mathbb{N}_0^n$ denote the sets of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ with components from the corresponding sets

$$\partial^\alpha u(x) = \partial^\alpha_x u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \qquad \partial_j := \frac{\partial}{\partial x_j}, , \quad j = 1, 2, \ldots, n \tag{1.1}$$
$$\alpha \in \mathbb{N}_0^n, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n .$$

Let $\Omega \subset \mathbb{R}^n$ be an open domain. A continuous function $\Phi : \Omega \to \mathbb{R}^m$ is called **differentiable** at a point $x \in \Omega$ with **derivative** $D\Phi(x) : \mathbb{R}^n \to \mathbb{R}^m$, if $D\Phi(x)$ is a linear mapping (i.e., a matrix) and

$$\Phi(x + y) = \Phi(x) + D\Phi(x)y + R(x, y), \quad R(x, y) = \mathscr{o}(|y|) \tag{1.2}$$

for small $y \in \mathbb{R}^n$, $|y| \to 0$.

With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ the derivative $D\Phi(x)$ is the matrix of partial derivatives

$$D\Phi(x) = [\partial_j \Phi_k(x)]_{n\times m} \tag{1.3}$$

and transforms a column vector $u = (u_1, \dots, u_n)^\top$ into a new column vector

$$D\Phi(x)u = \left(\sum_{j=1}^{n} \partial_j \Phi_k(x) u_j\right)^\top .$$

The matrix $D\Phi$ in (1.3) is called the **Jacobi matrix**. If $n = m$ the corresponding determinant is called **Jacoby determinant** or **Jacobian**.

Φ is differentiable whenever all the partial derivatives exist.

Let $\Omega \subset \mathbb{R}^n$ be an open domain (Ω can be non-compact, e.g. $\Omega = \mathbb{R}^n$). For $r, m \in \mathbb{N}_0$ by $C^r(\Omega, \mathbb{R}^m)$ (or by $C^r(\Omega)$) is denoted the r-times continuously diferentiable mappings $\Phi : \Omega \to \mathbb{R}^m$ and $C^\infty(\Omega, \mathbb{R}^m) := \bigcap_{r=1}^\infty C^r(\Omega, \mathbb{R}^m)$.

The set of complex valued mappings will be denoted by $C^r(\Omega, C^m)$ (or by $C^r(\Omega)$).

The subspace $\mathbb{C}_0^\infty(\Omega)$ consists of infinitely differentiable functions on Ω with compact supports.

A composition of functions

$$F = \Psi \circ \Phi \,:\, \Omega \to \mathbb{R}^k, \quad \Phi \,:\, \Omega \to \mathscr{M} \subset \mathbb{R}^m, \quad \Phi \,:\, \mathscr{M} \to \mathbb{R}^k,$$

where Φ is differentiable at a point $x \in \Omega$ and Ψ is differentiable at a point $z = \Phi(x) \in \mathscr{M}$, is differentiable at a point x and the **chain rule** holds:

$$D(\Psi \circ \Phi)(x) = (D\Psi)(\Phi(x)) D\Phi(x) . \tag{1.4}$$

Let us recall that $\Omega \subset \mathbb{R}^n$ is called a **star-like domain** with respect to the point $x_0 \in \Omega$ if $y \in \Omega$ implies $x_0 + t(y - x_0) \in \Omega$ for all $0 \leqslant t \leqslant 1$.

The fundamental theorem of calculus, applied to $\varphi(t) = \Phi(x + ty)$ in a star-like domain with respect to $x \in \Omega$, gives the **Lagrange formula**

$$\Phi(x + y) = \Phi(x) + \int_0^1 D\Phi(x + ty) y \, dt = \Phi(x) + D\Phi(x + t_0 y) y \tag{1.5}$$

for $\Phi \in C^1(\Omega)$, all $y \in \Omega$ and some $0 \leq t_0 < 1$.

Let us consider a function

$$\Phi \,:\, \Omega \to \mathbb{R}^n , \qquad \Phi \in C^k , \tag{1.6}$$

which maps a domain $\Omega \subset \mathbb{R}^n$ to the same Euclidean space and $\Phi(x_0) = y_0$. It is important to know conditions ensuring the existence of the **inverse mapping**

$$\Phi^{-1} \,:\, \boldsymbol{V} \to \boldsymbol{U} \subset \Omega , \qquad \Phi(\Phi^{-1}(y)) \equiv y , \quad y \in \boldsymbol{V} \tag{1.7}$$

and its smoothness properties, at least locally, in a neighborhood of some y_0. The next inverse function theorem provides such conditions and, together with the Implicit function theorem (cf. Theorem 1.2), represent most fundamental results of multivariable analysis.

Theorem 1.1 (Inverse function theorem). *Let Ω be a domain in $\mathbb{R}^n$, $k \in \mathbb{N}$ and $\Phi \in C^k(\Omega, \mathbb{R}^n)$. Let the differential $D\Phi(x)$ be an invertible matrix at $x_0 \in \Omega$ and $\Phi(x_0) = y_0 \in \mathbb{R}^n$.*

*There exist neighborhoods $\boldsymbol{U} \subset \Omega$ of x_0 and $\boldsymbol{V} \subset \mathbb{R}^n$ of y_0 such that the mapping $\Phi : \boldsymbol{U} \to \boldsymbol{V}$ is one-to one and the inverse mapping $\Phi^{-1} : \boldsymbol{V} \to \boldsymbol{U}$ is C^k-smooth (i.e., Φ^{-1} is a C^k-***diffeomorphism***).*

Proof: Let

$$\Psi(x) := (D\Phi)(x_0)]^{-1}\big[\Phi(x_0 + x) - y_0\big] . \tag{1.8}$$

Then, obviously,

$$\Psi(0) = 0 \qquad \text{and} \quad (D\Psi)(0) = I .$$

Thus, the case reduces to $\Phi(0) = 0$, $(D\Phi)(0) = I$, $0 \in \Omega$, which we suppose fulfilled. Then we have to solve the equation $\Phi(u) = v$ for small v. Due to formula (1.2) this can be written as an equation

$$u + R(u) = v\,, \qquad R(0) = 0\,, \quad (DR)(0) = 0 \tag{1.9}$$

$$\text{where} \qquad R(u) = \mathscr{o}(|u|) .$$

with the mapping $R \in C^{k-1}(\Omega, \mathbb{R}^n)$. Solving (1.9) is equivalent to solving

$$T_v(u) = u\,, \qquad T_v(u) = v - R(u) . \tag{1.10}$$

Thus, we look for a fixed point $u = K(v) = \Phi^{-1}(v)$ and will show that $(DK)(0) = I$ or, equivalently, $K(v) = v + \mathscr{o}(|v|)$. The latter implies that for all x close to the origin (small enough)

$$(DK)(x) = \big(D\Psi(K(x))\big)^{-1} \tag{1.11}$$

and taking further derivatives it follows by induction that $K \in C^k$. To implement this idea we consider a metric space

$$\mathfrak{M}_v := \{u \in \Omega \ : \ |u - v| \leqslant \mathscr{A}_v\} \ ,$$

where (cf. (1.2) and (1.9))

$$\mathscr{A}_v := \sup_{|w|\leqslant 2|v|} |R(w)| = \mathscr{o}(|w|) = \mathscr{o}(|v|) . \tag{1.12}$$

Let us check that $\mathfrak{M}_v$ is invariant under the mapping

$$T_v \ : \ \mathfrak{M}_v \to \mathfrak{M}_v \tag{1.13}$$

provided that v is small enough. Indeed, since $T_v(u) - v = -R(u)$ we only need to check that $|R(u)| \leqslant \mathscr{A}_v$ for all $u \in \mathfrak{M}_v$ provided that v is small enough. Indeed, if $u \in \mathfrak{M}_v$ than, due to (1.12), $|u| \leqslant |v| + \mathscr{A}_v \leqslant 2|v|$ for a v small enough and

$$|R(u)| \leqslant \sup_{|w|\leqslant 2|v|} |R(w)| = \mathscr{A}_v .$$

This completes the proof of the mapping property (1.13).

Due to the Lagrange formulae (1.5) and the property $(DR)(0) = 0$ (see (1.5), by taking v sufficiently small, the mapping (1.13) becomes a contraction

$$|T_v(u) - T_v(w)| = |R(u) - R(w)| = |(DR)(u + t_0(w - u))(u - w)| \\ \leqslant r|u - w|\,, \qquad 0 < r < 1\,.$$

Then, by virtue of the fixed point theorem there exists a unique fixed point $u = K(v) \in \mathfrak{M}_v$. Moreover, from $u \in \mathfrak{M}_v$ we conclude that

$$|K(v) - v| = |u - v| \leqslant \mathscr{A}_v = \mathscr{o}\big(|v|\big)\,.$$

This completes the proof. ■

Theorem 1.2 (Implicit function theorem). *Let $\Omega \subset \mathbb{R}^m$, $\mathscr{E} \subset \mathbb{R}^n$ be domains and $k = 1, 2 \ldots$. Let $\Psi(x, y)\ :\ \Omega \times \mathscr{E} \to \mathbb{R}^n$ be a C^k-mapping, $\Psi(x_0, y_0) = 0$ and the partial $n \times n$ Jacoby matrix $D_y\Psi(x, y)$ be invertible at $(x_0, y_0) \in \Omega \times \mathscr{E}$.*

There exists a neighborhood $U_0 \subset \Omega$ of x_0 and a C^k-smooth mapping $y = \psi(x)$, $\psi\ :\ U_0 \to \mathscr{E}$ (called the **implicit function***) such that $\Psi(x, \psi(x)) \equiv 0$.*

The function $\psi(x)$ is unique: If there exists another continuous implicit function $\psi_1\ :\ U^1 \to \mathscr{E}$, the functions coincide $\psi_1(x) = \psi(x)$ in the common neighborhood $x \in U^0 \cap U^1$ of x_0.

Proof: Consider the mapping $\Phi\ :\ \Omega \times \mathscr{E} \to \mathbb{R}^m \times \mathbb{R}^n$ defined by

$$\Phi(x, y) := \big(x, \Psi(x, y)\big)\,. \tag{1.14}$$

The corresponding differential (the Jacobi matrix)

$$\big(D_{(x,y))}\Phi\big) = \begin{pmatrix} I & D_x\Psi \\ 0 & D_y\Psi \end{pmatrix} \tag{1.15}$$

is, obviously, invertible. Therefore, by virtue of the foregoing Theorem 1.2, there exists the inverse function $\Phi^{-1}\ :\ V^0 \times U_0 \to \mathbb{R}^m \times \mathbb{R}^n$ and at the point (x, y_0) acquires the form

$$\Phi^{-1}(x, y_0) = \big(x, \psi(x, y_0)\big)\,.$$

The function $\psi(x) = \psi(x, y_0)$ is the desired implicit function.

The uniqueness of the implicit function follows since, according to Theorem 1.1, there exists only the unique inverse function to $\Phi(x, y) = (x, \Psi(x, y))$. ■

2 Calculus of tangential differential operators

The content of the present section follows [DMM06, § 4] with a slight modification.

Throughout the present section we keep the convention similar to that in § 6: $\mathscr{S}$ is a hypersurface in $\mathbb{R}^n$, given by an immersion

$$\Theta\ :\ \Omega \to \mathscr{S}\,, \quad \Omega \subset \mathbb{R}^{n-1} \tag{2.1}$$

with a boundary $\Gamma = \partial\mathscr{S}$, given by the corresponding immersion

$$\Theta_\Gamma \,:\, \omega \to \Gamma := \partial\mathscr{S}\,, \qquad \omega \subset \mathbb{R}^{n-2}\,, \tag{2.2}$$

such that the corresponding differentials

$$D\Theta_j(p) := \mathrm{matr}\,[\partial_1\Theta_j(p), \ldots, \partial_{n-1}\Theta_j(p)]\,, \tag{2.3}$$

have the full rank

$$\mathrm{rank}\, D\Theta_j(p) = n-1\,, \qquad \forall p \in Y_j\,, \quad k = 1, \ldots, n\,, \quad j = 1, \ldots, M\,,$$

i.e. , all points of ω_j are regular for Θ_j for all $j = 1, \ldots, M$.

Let $\mathscr{S}$ be a hypersurface given by a collection of charts $\left\{(\mathscr{S}_j, \Theta_j)\right\}_{j=1}^M$, where

$$\Theta_j \,:\, \omega_j \to \mathscr{S}_j\,, \qquad \mathscr{S} = \bigcup_{j=1}^{M} \mathscr{S}_j\,, \qquad \omega_j \subset \mathbb{R}^{n-1}\,, \quad j = 1, \ldots, M \tag{2.4}$$

(cf. (2.2)). The derivatives

$$\boldsymbol{g}_k = \partial_k\Theta_j, \qquad k = 1, \ldots, n-1, \tag{2.5}$$

are then tangential vector fields on $\mathscr{S}$ and this system is a basis in the space of tangential vector fields $\omega(\mathscr{S})$. The symmetric **Gram matrix**

$$G_{\mathscr{S}}(x) := [\langle \boldsymbol{g}_k(\mathscr{x}), \boldsymbol{g}_m(\mathscr{x})\rangle]_{n-1\times n-1} = [\langle \partial_k\Theta_j(x), \partial_m\Theta_j(x)\rangle]_{n-1\times n-1}, \qquad x \in \omega_j \subset \mathbb{R}^{n-1} \tag{2.6}$$

defines the natural metric on the space of tangential vector fields $\omega(\mathscr{S})$, which is inherited from the ambient space $\mathbb{R}^n$. Namely, for arbitrary tangential vectors

$$u_k(x) = \alpha_k^1\partial_1\Theta_j(x) + \cdots + \alpha_k^{n-1}\partial_{n-1}\Theta_j(x) \in \omega(\mathscr{S})\,, \qquad \alpha_k^m \in \mathbb{R}\,, \quad k = 1, 2\,,$$

the inner product is defined by the bilinear **first fundamental form**

$$\langle u_1, u_2\rangle = \langle G_{\mathscr{S}} a_1, a_2\rangle\,, \qquad a_k = (\alpha_k^1, \ldots, \alpha_k^{n-1})^\top\,, \quad k = 1, 2\,. \tag{2.7}$$

$\boldsymbol{\nu}_\Gamma(t)$ is the outer normal vector field to the boundary Γ, which is tangential to $\mathscr{S}$ and $\boldsymbol{\nu}(\mathscr{x})$ is the outer unit normal vector field to $\mathscr{S}$, which has the most important role in the calculus of tangential differential operators we are going to apply. The **unit normal vector field** to the surface $\mathscr{S}$, also known as the **Gauß mapping**, is defined by the vector product of the covariant basis

$$\boldsymbol{\nu}(\mathscr{x}) := \pm\frac{\boldsymbol{g}_1(\mathscr{x}) \wedge \ldots \wedge \boldsymbol{g}_{n-1}(\mathscr{x})}{|\boldsymbol{g}_1(\mathscr{x}) \wedge \ldots \wedge \boldsymbol{g}_{n-1}(\mathscr{x})|}, \qquad \mathscr{x} \in \mathscr{S}. \tag{2.8}$$

The system of tangential vectors $\{\boldsymbol{g}_k\}_{k=1}^{n-1}$ to $\mathscr{S}$ (cf. (2.5)) is, known as the **covariant basis**. There exists the unique system $\left\{\boldsymbol{g}^k\right\}_{k=1}^{n-1}$ biorthogonal to it-the **contravariant basis**:

$$\langle \boldsymbol{g}_j, \boldsymbol{g}^k\rangle = \delta_{jk} \qquad j, k = 1, \ldots, n-1.$$

The contravariant basis is defined by the formula:

$$\boldsymbol{g}^k = \frac{1}{\det G_{\mathscr{S}}}\boldsymbol{g}_1 \wedge \cdots \wedge \boldsymbol{g}_{k-1} \wedge \boldsymbol{\nu} \wedge \boldsymbol{g}_{k+1} \wedge \cdots \wedge \boldsymbol{g}_{n-1}, \quad k = 1, \ldots, n-1, \tag{2.9}$$

where $G_{\mathscr{S}}(\mathscr{X})$ is the Gram matrix (see (2.6)).

Next we expose yet another definition of a hypersurface-an **implicit** one.

Definition 2.1 *Let $k \geq 1$ an $\omega \subset \mathbb{R}^n$ be a compact domain. An implicit C^k-smooth (an implicit Lipschitz) hypersurface in $\mathbb{R}^n$ is defined as the set*

$$\mathscr{S} = \Big\{\mathscr{X} \in \omega \ : \ \Psi_{\mathscr{S}}(\mathscr{X}) = 0\Big\}, \tag{2.10}$$

where $\Psi_{\mathscr{S}} : \omega \to \mathbb{R}$ is a C^k-mapping (or is a Lipschitz mapping) which is regular $\nabla\, \Psi(\mathscr{X}) \neq 0$.

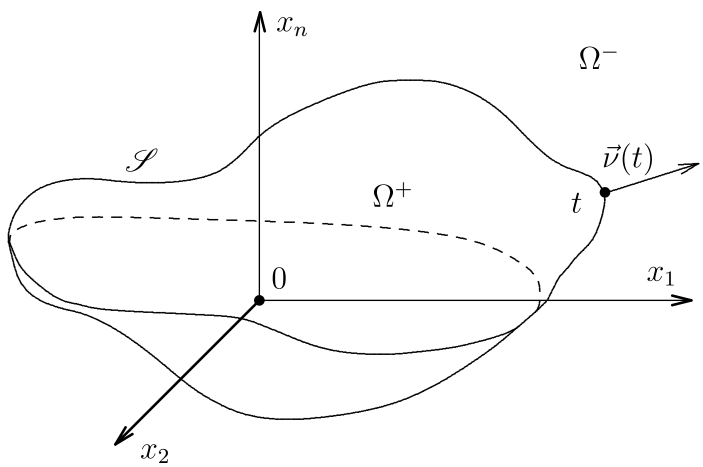

Fig. 1

Note, that Definition (2.1) and Definition 2.1 of a hypersurface $\mathscr{S}$ are equivalent and by taking a single function $\Psi_{\mathscr{S}}$ for the implicit definition of a hypersurface $\mathscr{S}$ we does not restrict the generality (see e.g., [Du02b]).

It is well known that using implicit surface functions gradient (see (5.2)) we can write an alternative definition of the unit normal vector field on the surface (see (2.8)):

$$\boldsymbol{\nu}(t) := \lim_{x \to t} \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|}, \qquad t \in \mathscr{S}. \tag{2.11}$$

In applications it is necessary to extend the vector field $\boldsymbol{\nu}t$ in a neighborhood of $\mathscr{S}$, preserving some important features. Here is the precise definition of extension.

Definition 2.2 *Let $\mathscr{S}$ be a surface in $\mathbb{R}^n$ with unit normal $\boldsymbol{\nu}$. A vector filed $\mathscr{N} \in C^1(\Omega_{\mathscr{S}})$ in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, will be referred to as a* **proper extension** *if $\mathscr{N}\Big|_{\mathscr{S}} = \boldsymbol{\nu}$, it is unitary $|\mathscr{N}| = 1$ in $\Omega_{\mathscr{S}}$ and if $\mathscr{N}$ satisfies the condition in the neighborhood*

$$\partial_j \mathscr{N}_k(x) = \partial_k \mathscr{N}_j(x) \qquad \text{for all} \quad x \in \Omega_{\mathscr{S}}, \quad j,k = 1, \ldots, n. \tag{2.12}$$

Such extension is needed, for example, to define correctly the normal derivative (the derivative along normal vector fields, outer or inner). It turned out that the "naive" extension (cf. (5.4))

$$\boldsymbol{\nu}(t) := \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|}, \qquad x \in \Omega_{\mathscr{S}} \tag{2.13}$$

is not proper. Indeed (see [DST15]), let $n = 2$ and $\mathscr{S}$ be the ellipse

$$\left\{x = (x_1, x_2) \in \mathbb{R}^2 | \Psi_{\mathscr{S}}(x_1, x_2) := x_1^2 + 2x_2^2 - 1 = 0\right\}.$$

Then

$$\begin{aligned}\mathscr{N}(x) := \frac{(\nabla\Psi_{\mathscr{S}})(x)}{|(\nabla\Psi_{\mathscr{S}})(x)|} &= \frac{(x_1, 2x_2)}{\sqrt{x_1^2 + 4x_2^2}},\\ \partial_1\mathscr{N}_2(x) &= -\frac{2x_2\, x_1}{(x_1^2 + 4x_2^2)^{3/2}},\\ \partial_2\mathscr{N}_1(x) &= -\frac{4x_1 x_2}{(x_1^2 + 4x_2^2)^{3/2}}.\end{aligned}$$

Hence $\partial_1\mathscr{N}_2(x) \neq \partial_2\mathscr{N}_1(x)$ unless $x_1 = 0$ or $x_2 = 0$.

For the proof of the next Proposition 2.3 and Corollary 2.4 on extension of the normal vector field we refer to [DST15].

Proposition 2.3 *Let $\mathscr{S} \subset \mathbb{R}^n$ be a hypersurface given by an implicit function*

$$\mathscr{S} = \left\{\mathscr{X} \in \mathbb{R}^n \ : \ \Psi_{\mathscr{S}}(\mathscr{X}) = 0\right\}.$$

Then the gradient $\nabla\Phi_{\mathscr{S}}(x)$ of the function

$$\Phi_{\mathscr{S}}(\mathscr{X} + t\boldsymbol{\nu}(\mathscr{X})) := t, \qquad \mathscr{X} + t\boldsymbol{\nu}(\mathscr{X}) \in \Omega_{\mathscr{S}}, \tag{2.14}$$

defined in the parameterized neighborhood

$$\Omega_{\mathscr{S}} := \{x = \mathscr{X} + t\boldsymbol{\nu}(\mathscr{X}) \ : \ \mathscr{X} \in \mathscr{S}, \quad -\varepsilon < t < \varepsilon\}$$

represents a unique proper extension of the unit normal vector field on the surface

$$\boldsymbol{\nu}(\mathscr{X}) = \lim_{x \to \mathscr{X}} \nabla\Phi_{\mathscr{S}}(x), \qquad \mathscr{X} \in \mathscr{S}.$$

Corollary 2.4 *For any proper extension $\mathscr{N}(x)$, $x \in \Omega_{\mathscr{S}} \subset \mathbb{R}^n$ of the unit normal vector field $\boldsymbol{\nu}$ to the surface $\mathscr{S} \subset \Omega_{\mathscr{S}}$ the equality*

$$\partial_{\mathscr{N}}\mathscr{N}(x) = 0 \qquad \textit{holds for all} \quad x \in \Omega_{\mathscr{S}}. \tag{2.15}$$

In particular, for the derivatives

$$\mathscr{D}_k = \partial_k - \mathscr{N}_k\partial_{\mathscr{N}}, \qquad k = 1, \ldots, n, \tag{2.16}$$

which are extension into the domain $\Omega_{\mathscr{S}}$ of Günter's derivatives $\mathscr{D}_k = \partial_k - \nu_k\partial_{\boldsymbol{\nu}}$ on the surface $\mathscr{S}$, we have the equality:

$$\mathscr{D}_k\mathscr{N}_j = \partial_k\mathscr{N}_j - \mathscr{N}_k\partial_{\mathscr{N}} = \partial_k\mathscr{N}_j, \qquad \mathscr{D}_k\mathscr{N}_j = \mathscr{D}_j\mathscr{N}_k, \quad \textit{for all} \quad j, k = 1, \ldots, n. \tag{2.17}$$

In the sequel we will dwell on a proper extension and apply the above properties of $\mathscr{N}$.

Lemma 2.5 (see [DMM06]) *For an arbitrary unitary extension $\mathscr{N}(x)\in C^1(\Omega_{\mathscr{S}})$, $|\mathscr{N}(x)|\equiv 1$, of $\boldsymbol{\nu}(\mathscr{x})$, in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, the following conditions are equivalent:*

i. $\partial_{\mathscr{N}}\mathscr{N}\big|_{\mathscr{S}}=0$, *i.e.,* $\partial_{\mathscr{N}}\mathscr{N}_j(x)\to 0$ *for* $x\to\mathscr{x}\in\mathscr{S}$ *and* $j=1,2,...,n$*;*

ii. $[\partial_k\mathscr{N}_j-\partial_j\mathscr{N}_k]\big|_{\mathscr{S}}=0$ *for* $k,\ j=1,\ 2,\dots,n$.

The **second fundamental form** of $\mathscr{S}$ has the form

$$\begin{aligned} II(\boldsymbol{U}(\mathscr{x}),\boldsymbol{V}(\mathscr{x}))\boldsymbol{\nu}(\mathscr{x}) &:= \partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x})-\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x})=(I-\pi_{\mathscr{S}})\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x})\\ &=\langle\boldsymbol{\nu}(\mathscr{x})\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x})\rangle\boldsymbol{\nu}(\mathscr{x}),\quad \forall\,\mathscr{x}\in\mathscr{S},\quad \boldsymbol{U},\boldsymbol{V}\in\omega(\mathscr{S}) \end{aligned}\tag{2.18}$$

and the **Weingarten matrix** (or the Weingarten mapping)

$$\mathscr{W}_{\mathscr{S}}:\omega(\mathscr{S})\longrightarrow\omega(\mathscr{S}),\tag{2.19}$$

is defined uniquely by the requirement that

$$\begin{aligned} \langle\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle=II(\boldsymbol{U},\boldsymbol{V})=\langle\boldsymbol{\nu},\partial_{\boldsymbol{U}}\boldsymbol{V}\rangle=-\langle\partial_{\boldsymbol{U}}\boldsymbol{\nu},\boldsymbol{V}\rangle=-\langle\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{\nu},\boldsymbol{V}\rangle\\ \forall\,\boldsymbol{U},\boldsymbol{V}\in\omega(\mathscr{S})\,. \end{aligned}\tag{2.20}$$

In the last equality in (2.20) we have applied the following: for a tangential vector field $\boldsymbol{V}\in\omega(\mathscr{S})$ holds $\langle\boldsymbol{\nu}(\mathscr{x}),\boldsymbol{V}(\mathscr{x})\rangle\equiv 0$, $\mathscr{x}\in\mathscr{S}$ and, by differentiating,

$$\langle\partial_{\boldsymbol{U}}\boldsymbol{\nu}(\mathscr{x}),\boldsymbol{V}(\mathscr{x})\rangle+\langle\boldsymbol{\nu}(\mathscr{x}),\partial_{\boldsymbol{U}}\boldsymbol{V}(\mathscr{x})\rangle\equiv 0\,,\qquad \mathscr{x}\in\mathscr{S}\,,\quad j=1,\dots,n\,,\tag{2.21}$$

$$\text{for all}\quad \boldsymbol{U}=\sum_{j=1}^n U_j d_j\,,\quad \boldsymbol{V}=\sum_{j=1}^n V_j d_j\,,\quad \boldsymbol{d}^j=\pi_{\mathscr{S}}\boldsymbol{e}^j\,,\quad \partial^{\mathscr{S}}_{\boldsymbol{U}}:=\sum_{j=1}^n U_j\mathscr{D}_j\,.$$

We can extend the Weingarten matrix $\mathscr{W}_{\mathscr{S}}(x)$ from the surface $\mathscr{S}$ to a neighbourhood as follows:

$$\mathscr{W}_{\mathscr{S}}(x):=-\nabla\mathscr{N}(x)=-\big[\partial_j\mathscr{N}_k(x)\big]_{n\times n}\,,\qquad x\in\Omega_{\mathscr{S}}\,.\tag{2.22}$$

Lemma 2.6 *The extended Weingarten matrix $\mathscr{W}_{\mathscr{S}}(x)$ in (2.22) has the following properties:*

i. $\mathscr{W}_{\mathscr{S}}(x)\mathscr{N}(x)=0$ *for all* $x\in\Omega_{\mathscr{S}}$*;*

ii. even if extension $\mathscr{N}(x)$ is not proper, the restriction to the hypersurface $\mathscr{W}_{\mathscr{S}}\big|_{\mathscr{S}}$ coincides with the Weingarten mapping of $\mathscr{S}$ and only depends on $\mathscr{S}$ (is independent of the choice of the extension $\mathscr{N}$);

iii. even if extension $\mathscr{N}(x)$ is not proper, $\mathrm{Tr}\,(\mathscr{W}_{\mathscr{S}})\big|_{\mathscr{S}}=\mathscr{H}^0_{\mathscr{S}}$, *where $\mathscr{H}^0_{\mathscr{S}}$ is the mean curvature of $\mathscr{S}$;*

iv. $\mathscr{W}_{\mathscr{S}}(x)\boldsymbol{V}(x)$, $x\in\mathscr{S}_C$, *is tangential to the level surface*

$$\mathscr{S}_C:=\{y\in\mathbb{R}^n\ :\ \Psi_{\mathscr{S}}(y)=C:=\Psi_{\mathscr{S}}(x)\}\tag{2.23}$$

for arbitrary vector field $\boldsymbol{V}:\mathscr{S}\to\mathbb{R}^n$.

Proof: First, $\mathscr{W}_{\mathscr{S}}\mathscr{N} = \nabla\|\mathscr{N}\|^2 = \nabla 1 = 0$ in $\Omega_{\mathscr{S}}$, justifying (i). Assertions (ii) and (iii) follow from Lemma 2.5.

Next, (iv) is proved as follows:

$$\langle\mathscr{N}(x), \mathscr{W}_{\mathscr{S}}\boldsymbol{V}(x)\rangle = -\sum_{j,k=1}^{n}\mathscr{N}_j(\partial_j\mathscr{N}_k)V_k = -\sum_{k=1}^{n}(\partial_{\mathscr{N}}\mathscr{N}_k)\omega = 0$$

due to (2.15), proved below. ■

We remind that

$$G_{\mathscr{S}}(\mathscr{X}) = G(\mathscr{X}) = [g_{jk}(\mathscr{X})]_{n-1\times n-1}\,, \qquad g_{jk} := \langle\boldsymbol{g}_j, \boldsymbol{g}_k\rangle$$

is the positive definite Gram matrix, which is known as the **covariant Riemannian metric tensor** and defines the metric on the surface $\mathscr{S}$ (cf. § 4).

Let $d\sigma = \sqrt{\det G_{\mathscr{S}}}dx$ and $d\mathfrak{s} = \sqrt{\det G_\Gamma}dx'$ stand for the volume elements on $\mathscr{S}$ and $\Gamma := \partial\mathscr{S}$, respectively ($x \in \mathbb{R}^{n-1}$, $x' \in \mathbb{R}^{n-2}$; cf. § 4).

Let

$$P(\nabla)u = \sum_{j=1}^{n}a_j\partial_j u + bu\,, \qquad a_j, b \in C^1(\mathbb{R}^{m\times m}) \tag{2.24}$$

be a first-order differential operator with real valued (variable) matrix coefficients, acting on vector-valued functions $u = (u_\beta)_\beta$ in $\mathbb{R}^n$ and its **principal symbol** is given by the matrix-valued function

$$\sigma(P;\xi) := \sum_{j=1}^{n}a_j\xi_j \qquad \xi = \left\{\xi_j\right\}_{j=1}^{n} \in \mathbb{R}^n\,. \tag{2.25}$$

Definition 2.7 *We say that P is a* **weakly tangential operator** *to the hypersurface $\mathscr{S}$, with unit normal $\boldsymbol{\nu}$, provided that*

$$\sigma(P;\boldsymbol{\nu}) = 0 \quad \textit{on the hypersurface } \mathscr{S}. \tag{2.26}$$

Next, call P a **strongly tangential operator** *to $\mathscr{S}$ provided that there exists an extended unit field $\mathscr{N}$ such that*

$$\sigma(P;\mathscr{N}) = 0 \quad \textit{in an open neighborhood of } \mathscr{S} \textit{ in } \mathbb{R}^n\,. \tag{2.27}$$

Note that in a strongly tangential operator the coordinate derivatives ∂_j can be replaced by the Günter's derivatives $\mathscr{D}_j$:

$$P(\nabla)u = \sum_{j=1}^{n}a_j\partial_j u + bu = \sum_{j=1}^{n}a_j\mathscr{D}_j u + bu = P(\mathscr{D})u\,, \qquad a_j, b \in C^1(\mathbb{R}^{m\times m}) \tag{2.28}$$

Most important tangential differential operators to the hypersurface are for us:

A. The weakly tangential Günter's derivatives (see (0.9))

$$\mathscr{D}_j := \partial_j - \nu_j\partial_{\boldsymbol{\nu}} = \partial_j - \nu_j\sum_{k=1}^{n}\nu_k\partial_k\,, \qquad j = 1,\dots,n.$$

B. The weakly tangential Stoke's derivatives $\mathscr{M}_{jk}=\nu_j\partial_k-\nu_k\partial_j$, introduced in § 5.

The Günter's and Stoke's derivatives are tangent since the corresponding vector fields are tangent

$$\begin{aligned}
&\mathscr{D}_j:=\partial_{\boldsymbol{d}^j}=\boldsymbol{d}^j\cdot\nabla\,,\qquad \mathscr{M}_{jk}:=\partial_{\mathfrak{m}_{jk}}=\mathfrak{m}_{jk}\cdot\nabla\,,\\
&\boldsymbol{d}^j:=\pi_{\mathscr{S}}\boldsymbol{e}^j=\boldsymbol{e}^j-\nu_j\boldsymbol{\nu}=\boldsymbol{\nu}\wedge\big(\boldsymbol{\nu}\wedge\boldsymbol{e}^j\big)=\sum_{k=1}^n(\delta_{jk}-\nu_j\nu_k)\boldsymbol{e}^k\,,\\
&\mathfrak{m}_{jk}:=\nu_je_k-\nu_ke_j\,,\quad \langle\boldsymbol{d}^j,\boldsymbol{\nu}\rangle=0\,,\qquad \langle\mathfrak{m}_{jk},\boldsymbol{\nu}\rangle=0\,,\quad j,k=1,\dots,n\,,
\end{aligned}\tag{2.29}$$

where $\pi_{\mathscr{S}}$ is the projection on the tangential space to the surface (see (0.8)). Therefore $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ can be applied to functions which are defined on the surface $\mathscr{S}$ only.

The generating vector fields $\{\boldsymbol{d}^j\}_{j=1}^n$ $\{\mathfrak{m}_{jk}\}_{j,k=1}^n$ are not frame since they are linearly dependent

$$\sum_{j=1}^n\nu_j(\mathscr{X})\boldsymbol{d}^j(\mathscr{X})\equiv0\,,\qquad \mathfrak{m}_{jj}=0\,,\tag{2.30}$$

but both systems $\{\boldsymbol{d}^j\}_{j=1}^n$ and $\{\mathfrak{m}_{jk}\}_{j,k=1}^n$ are full in the space of all tangential vector fields: any vector field $\boldsymbol{U}\in\omega(\mathscr{S})$ is represented as follows

$$\boldsymbol{U}(\mathscr{X})=\sum_{j=1}^nU^j(\mathscr{X})\boldsymbol{d}^j(\mathscr{X})=\sum_{0\leqslant j<k\leqslant1}^n c_{jk}(\mathscr{X})\mathfrak{m}_{jk}(\mathscr{X})\,.\tag{2.31}$$

For example, the covariant vector fields $\boldsymbol{g}_1(\mathscr{X}):=\partial_1\Theta_k(\mathscr{X}),\dots,\boldsymbol{g}_{n-1}(\mathscr{X}):=\partial_{n-1}\Theta_k(\mathscr{X})$, $\mathscr{X}\in\mathscr{S}_k$, $k=1,\dots,N$ on $\mathscr{S}$, which generate the derivatives $\partial_j=\partial_{dx_j}$, are represented as follows

$$\boldsymbol{g}_j(\mathscr{X})=\sum_{m=1}^n g_j^m(\mathscr{X})\boldsymbol{e}^m=\sum_{m=1}^n g_j^m(\mathscr{X})\boldsymbol{d}^m(\mathscr{X})\tag{2.32}$$

and $\{\boldsymbol{e}^m\}_{m=1}^n$ is a Cartesian frame in $\mathbb{R}^n$. Indeed, by applying the derivative to Θ_k we get

$$\begin{aligned}
\boldsymbol{g}_j=\sum_{m=1}^n g_j^m\boldsymbol{e}^m=\sum_{m=1}^n g_j^m\boldsymbol{d}^m\quad\text{since}\quad &\sum_{m=1}^n g_j^m[\boldsymbol{e}^m-\boldsymbol{d}^m]\\
=\sum_{m=1}^n g_j^m\nu_m\boldsymbol{\nu}=\langle\boldsymbol{g}_j,\boldsymbol{\nu}\rangle\boldsymbol{\nu}=0\,,&\quad j=1,\dots,n-1.
\end{aligned}$$

Let us recall the following result about surface divergence $\mathrm{div}_{\mathscr{S}}$, the surface gradient $\nabla_{\mathscr{S}}$ and the surface Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$.

Theorem 2.8 ([DMM06]) *For any function $\varphi\in C^1(\mathscr{S})$ we have*

$$\nabla_{\mathscr{S}}\varphi=\Big\{\mathscr{D}_1\varphi,\mathscr{D}_2\varphi,...,\mathscr{D}_n\varphi\Big\}^\top.\tag{2.33}$$

Also, for a 1-smooth tangential vector field $\boldsymbol{V} = \sum_{j=1}^{n} V^j e_j \in \omega(\mathscr{S})$,

$$\operatorname{div}_{\mathscr{S}} \boldsymbol{V} = -\nabla^*_{\mathscr{S}} \boldsymbol{V} := \sum_{j=1}^{n} \mathscr{D}_j V^j. \tag{2.34}$$

The Laplace-Beltrami operator $\Delta_{\mathscr{S}}$ *on* $\mathscr{S}$ *takes the form*

$$\boldsymbol{\Delta}_{\mathscr{S}}\, \psi = \operatorname{div}_{\mathscr{S}} \nabla_{\mathscr{S}}\, \psi = -\nabla^*_{\mathscr{S}} \Big(\nabla_{\mathscr{S}} \psi\Big) = \sum_{j=1}^{n} \mathscr{D}_j^2 \psi \tag{2.35}$$

$$= \sum_{j<k} \mathscr{M}_{jk}^2 \psi = \frac{1}{2} \sum_{j,k=1}^{n} \mathscr{M}_{jk}^2 \psi \qquad \forall\, \psi \in C^2(\mathscr{S})\,. \tag{2.36}$$

An important operator on forms is the **exterior derivative**. The **derivative of a 0-form**, i.e., of a scalar function

$$f\ :\ \mathscr{S} \to \mathbb{R}\,, \qquad f \in C^1(\mathscr{S})\,, \tag{2.37}$$

is a 1-form and maps

$$df(w)\ :\ \mathbb{T}_w\mathscr{S} \to \mathbb{R}\,. \tag{2.38}$$

Thus, $df(w)$ is a linear functional $df(w) \in \mathbb{T}^*_w\mathscr{S}$ over $\mathbb{T}_w\mathscr{S}$ for all $w \in \mathscr{S}$: being a vector $df(w) = Df(w) = (\partial_1 f(w), \ldots, \partial_{n-1} f(w))^\top$ the differential assigns to a vector $\xi \in \mathbb{T}_w\mathscr{S}$ the number

$$df(x)\xi = \sum_{j=1}^{n-1} \partial_j f(x)\xi_j\,, \qquad \partial_j f(x) := \partial_{dx_j} f(x), \quad x \in \mathscr{S}_k, \tag{2.39}$$

where $\big\{dx_j = \partial_j \Theta_k\big\}_{j=1}^{n-1}$ is the covariant basis on $\mathscr{S}$ and $\Theta_k\ :\ \Omega_k \to \mathscr{S}_k$, $k = 1, \ldots, N$ is the surface immersion.

From (2.32) and the definition of the derivative $\partial_j f(x) := \partial_{dx_j} f(x)$ in (2.39) follows that (see for the differential matrix $D\Theta_k$):

$$\begin{aligned} &\partial_{\mathscr{S}} := \big(\partial_1, \ldots, \partial_{n-1}\big)^\top := \big(\partial_{dx_1}, \ldots, \partial_{dx_{n-1}}\big)^\top = \big(D\Theta_k\big)^\top \nabla_{\mathscr{S}}, \\ &\nabla_{\mathscr{S}} := \big(\mathscr{D}_1, \ldots, \mathscr{D}_n\big)^\top \quad \text{or} \quad \partial_{dx_j} = \sum_{m=1}^{n} (\partial_j \Theta_k^m) \mathscr{D}_m\,, \quad m = 1, \ldots, n-1. \end{aligned} \tag{2.40}$$

Let $\mathscr{N}$ be a proper extension of the unit normal vector field $\boldsymbol{\nu}$ to $\mathscr{S}$ (cf. Definition 2.2). Then each operator $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ extends accordingly by setting

$$\mathscr{D}_j = \partial_j - \mathscr{N}_j \partial_{\mathscr{N}}\,, \qquad \mathscr{M}_{jk} := \mathscr{N}_j \partial_k - \mathscr{N}_k \partial_j\,, \qquad 1 \leqslant j,k \leqslant n \tag{2.41}$$

In the sequel, we shall make no distinction between the operator $\mathscr{D}_j$ or $\mathscr{M}_{jk}$ on $\mathscr{S}$ and the extended one in $\mathbb{R}^n$ given by (2.41). Note, that the extended operators $\mathscr{D}_j$ and $\mathscr{M}_{jk}$ becomes even *strongly tangent*.

For further reference, below we collect some of the most basic properties of this system of differential operators.

Lemma 2.9 *Let $\mathscr{N}$ be a proper extension of the unit vector field of normal vectors $\boldsymbol{\nu}$ to $\mathscr{S}$. The following relations are valid for $j,k=1,\dots,n$:*

i. $\mathscr{M}_{jj}=0$, $\mathscr{M}_{jk}=-\mathscr{M}_{kj}$;

ii. $\partial_k=\sum_{j=1}^n \mathscr{N}_j\mathscr{M}_{jk}+\mathscr{N}_k\partial_{\mathscr{N}}=-\sum_{k=1}^n \mathscr{N}_k\mathscr{M}_{jk}+\mathscr{N}_j\partial_{\mathscr{N}}$;

iii. $\sum_{k=1}^n \mathscr{M}_{jk}\mathscr{N}_k=-\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}}$, *where* $\mathscr{H}^0_{\mathscr{S}}(\mathscr{x})=-\operatorname{div}_{\mathscr{S}}\boldsymbol{\nu}(\mathscr{x})$ *and* $\mathscr{H}_{\mathscr{S}}(\mathscr{x}):=(n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}(\mathscr{x})$ *is the mean curvature at* $\mathscr{x}\in\mathscr{S}$ *(see* (0.15)*);*

iv. $\mathscr{D}_j=\sum_{k=1}^n \mathscr{N}_k\mathscr{M}_{kj}$;

v. $\mathscr{M}_{jk}=\mathscr{N}_j\mathscr{D}_k-\mathscr{N}_k\mathscr{D}_j$;

vi. $\sum_{j=1}^n \mathscr{N}_j\mathscr{D}_j=0$;

vii. $\sum_{r,j,k=m-1}^{m+1}\sigma(r,j,k)\mathscr{N}_i\mathscr{M}_{jk}=2\sum_{\{r,j,k\}\subset\{(m-1),m,(m+1)\}}\sigma(r,j,k)\mathscr{N}_i\mathscr{M}_{jk}=0$ *for* $m=2,\dots,n-1$, *where* $\sigma(r,j,k)$ *is the permutation sign:*

$$\sigma(j_1,\dots,j_k)=\begin{cases}+1 & \text{if } (j_1,\dots,j_k)\text{is an even permutation of the strongly}\\ & \text{ordered row } (m_1,\dots,m_k)\,,\quad m_1<\ldots<m_k\,,\\ 0 & \text{if } j_r=j_s \text{ for some } r,s=1\dots,k \text{ and } r\neq s\,,\\ -1 & \text{if } (j_1,\dots,j_k) \text{ is an odd permutation of the strongly}\\ & \text{ordered row } (m_1,\dots,m_k)\,,\quad m_1<\ldots<m_k\,;\end{cases}\tag{2.42}$$

viii. $[\mathscr{D}_j,\mathscr{D}_k]=\sum_{r=1}^n(\mathscr{M}_{jk}\mathscr{N}_r)\mathscr{D}_r+\big[\mathscr{N}_j\partial_{\mathscr{N}}\mathscr{N}_k-\mathscr{N}_k\partial_{\mathscr{N}}\mathscr{N}_j\big]\partial_{\mathscr{N}}$;

ix. $[\mathscr{D}_j,\mathscr{D}_k]=\sum_{r=1}^n(\mathscr{M}_{jk}\mathscr{N}_r)\mathscr{D}_r=\mathscr{N}_k[\mathscr{D}_{\mathscr{N}},\partial_j]-\mathscr{N}_j[\mathscr{D}_{\mathscr{N}},\partial_k]$;

x. $\partial_j\mathscr{N}_k=\mathscr{D}_j\mathscr{N}_k=\mathscr{D}_k\mathscr{N}_j$.

Proof: The identities (i)-(ii) and (iv)-(vii) are simple consequences of the definitions. For the equality (iii) we have

$$\begin{aligned}\sum_{k=1}^n\mathscr{M}_{jk}\mathscr{N}_k&=\sum_{k=1}^n\mathscr{M}_{jk}\mathscr{N}_k=\sum_{k=1}^n(\mathscr{N}_j\partial_k-\mathscr{N}_k\partial_j)\mathscr{N}_k\\&=\mathscr{N}_j\operatorname{div}\mathscr{N}-\frac{1}{2}\partial_j(\|\mathscr{N}\|^2)=-\mathscr{N}_j\mathscr{H}^0_{\mathscr{S}},\end{aligned}$$

as claimed.

To prove (viii) we calculate

$$\begin{aligned}
\mathscr{D}_j\mathscr{D}_k =& (\partial_j - \mathscr{N}_j\partial_{\mathscr{N}})(\partial_k - \mathscr{N}_k\partial_{\mathscr{N}}) = \partial_j\partial_k - (\partial_j\mathscr{N}_k)\partial_{\mathscr{N}}\\
&- \sum_{r=1}^{n}\Big[\mathscr{N}_k(\partial_j\mathscr{N}_r)\partial_r + \mathscr{N}_k\mathscr{N}_r\partial_r\partial_j + \mathscr{N}_j\mathscr{N}_r\partial_r\partial_k\Big] + \mathscr{N}_j\big(\partial_{\mathscr{N}}\mathscr{N}_k\big)\partial_{\mathscr{N}} + \mathscr{N}_j\mathscr{N}_k\partial^2_{\mathscr{N}}\\
=& -\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\partial_r + \mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}} + B_{jk}\\
=& -\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\mathscr{D}_r + \mathscr{N}_j(\partial_{\mathscr{N}}\mathscr{N}_k)\partial_{\mathscr{N}} + B_{jk}\,,
\end{aligned}\tag{2.43}$$

since

$$\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r)\mathscr{N}_r\partial_{\mathscr{N}} = \frac{1}{2}\sum_{r=1}^{n}\mathscr{N}_k(\partial_j\mathscr{N}_r^2)\partial_{\mathscr{N}} = \frac{1}{2}\mathscr{N}_k(\partial_j 1)\partial_{\mathscr{N}} = 0\,.$$

The operator

$$B_{jk} = \partial_j\partial_k - (\partial_j\mathscr{N}_k)\partial_{\mathscr{N}} - \sum_{r=1}^{n}\Big[\mathscr{N}_k\mathscr{N}_r\partial_r\partial_j + \mathscr{N}_j\mathscr{N}_r\partial_r\partial_k\Big] + \mathscr{N}_j\mathscr{N}_k\partial^2_{\mathscr{N}}$$

is symmetric $B_{jk} = B_{kj}$ and the desired commutator identity in (viii) follows from (2.43).

The first commutator identity in (ix) utilizes the facts that $\partial_{\mathscr{N}}\mathscr{N}_k = 0$ (cf. Lemma (2.12)) and follows from the identity in (viii). The second commutator identity in (ix) applies the same identity $\partial_{\mathscr{N}}\mathscr{N}_k = 0$, the identity $\partial_j\mathscr{N}_k = \partial_k\mathscr{N}_j$ (cf. (2.15)), and follows by a routine calculations.

The identities in (x) are already proved in (2.12) and (2.17). ∎

The next Lemma 2.10 provides an useful and interesting example of restriction of the differential form to hypersurface and to it's boundary.

Lemma 2.10 *Let $\Theta : \Omega \to \mathscr{S}$ be a smooth hypersurface in $\mathbb{R}^n$ with a smooth boundary $\Gamma := \partial\mathscr{S}$, while $d\sigma$ and $d\mathfrak{s}$ designate the respective volume elements on $\mathscr{S}$ and on Γ. Let $\boldsymbol{\nu}(\mathscr{x}) = \big(\nu_1(\mathscr{x}),\dots,\nu_n(\mathscr{x})\big)^\top$ be the outer unit normal vector to $\mathscr{S}$ at $\mathscr{x}\in\mathscr{S}$ and $\boldsymbol{\nu}_\Gamma(s) = \big(\nu^1_\Gamma(s),\dots,\nu^n_\Gamma(s)\big)^\top$-the unit tangential vector to $\mathscr{S}$ at the boundary point $s\in\Gamma$, which is outward (unit) normal vector to the boundary $\mathscr{S}$. Then*

$$\nu_j dS = \beta_j\big|_{\mathscr{S}}\,, \tag{2.44}$$

$$\big[\nu_j\nu^k_\Gamma - \nu_k\nu^j_\Gamma\big]\,d\mathfrak{s} = \beta_{jk}\big|_\Gamma\,, \tag{2.45}$$

where

$$\begin{aligned}
\beta_j :=& \big|dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge dx_n\big| = (-1)^{j-1}dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge dx_n\,,\\
\beta_{jk} :=& \big|dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge\widehat{dx_k}\wedge\ldots\wedge dx_n\big|\\
&= (-1)^{j+k-1}dx_1\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge\widehat{dx_k}\wedge\ldots\wedge dx_n
\end{aligned}$$

and $\widehat{dx_m}$ denotes that the factor dx_m is dropped.

The next Theorem generalizes Stoke's formulae (see [Ta96, § 2.2, Theorem 2.1] for the version on compact Riemannian manifolds).

Theorem 2.11 *For any real-valued function $\varphi \in C^1(\mathscr{S})$ and any $1 \leqslant j < k \leqslant n$, there hold:*

$$\int_{\mathscr{S}} \mathscr{M}_{jk}\varphi \, d\sigma = \oint_{\Gamma} \big[\nu_j \nu_\Gamma^k - \nu_k \nu_\Gamma^j\big]\varphi \, d\mathfrak{s}\,, \tag{2.46}$$

where $\boldsymbol{\nu}_\Gamma(\xi) = \big(\nu_\Gamma^1(\xi), \ldots, \nu_\Gamma^n(\xi)\big)^\top$ is the unit tangential vector to $\mathscr{S}$ at the boundary point $\xi \in \Gamma := \partial\mathscr{S}$ and outward (unit) normal vector to the boundary $\Gamma = \partial\mathscr{S}$.

Proof: With formula (2.44) at hand the integrand in (2.46) can be represented as a total differential

$$(\mathscr{M}_{jk}\varphi)\, d\sigma = (\partial_k\varphi)\omega_j\big|_{\mathscr{S}} - (\partial_j\varphi)\omega_k\big|_{\mathscr{S}} = d\big[\varphi\omega_{jk}\big]\big|_{\mathscr{S}}\,.$$

Applying the well known Stoke's formula

$$\int_{\mathscr{S}} d\beta := \int_{\Gamma} \beta \tag{2.47}$$

(see, e.g., [Du11]) and formula (2.45) we get:

$$\int_{\mathscr{S}} \mathscr{M}_{jk}\varphi \, d\sigma = \int_{\mathscr{S}} d\big[\varphi\omega_{jk}\big]\big|_{\mathscr{S}} = \int_{\Gamma} \varphi\omega_{jk}\big|_{\Gamma} = \int_{\Gamma} \big[\nu_j\nu_\Gamma^k - \nu_k\nu_\Gamma^j\big]\varphi\, d\mathfrak{s}$$

and (2.46) is proved. ■

The formal adjoint (in $\mathbb{R}^n$) to P is defined by

$$P^*u = -\sum_j \partial_j a_j^\top u + b^\top u$$

If $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain, and if P is a first-order operator, weakly tangent to $\partial\Omega$, then, applying (2.54) (cf. § 5), P can be integrated by parts over Ω without boundary terms, i.e.

$$(Pu, v)_\Omega := \int_\Omega \langle Pu, v\rangle\, dx = \int_\Omega \langle u, P^*v\rangle\, dx =: (u, P^*v)_\Omega \tag{2.48}$$

for all vector-valued sections of vector fields $u, v \in C^1(\bar\Omega)$.

For a weakly tangential differential operator Q on a closed hypersurface $\mathscr{S}$ let $Q^*_{\mathscr{S}}$ denote the "surface" adjoint:

$$(Q_{\mathscr{S}}\varphi, \psi)_{\mathscr{S}} := \oint_{\mathscr{S}} \langle Q_{\mathscr{S}}\varphi, \psi\rangle\, d\sigma = \oint_{\mathscr{S}} \langle \varphi, Q^*_{\mathscr{S}}\psi\rangle\, d\sigma = (\varphi, Q^*_{\mathscr{S}}\psi)_{\mathscr{S}} \tag{2.49}$$

for all vector-valued sections of vector fields $\varphi,\ \psi \in C^1(\bar\Omega)$.

Corollary 2.12 *The surface-adjoint operator $P^*_{\mathscr{S}}$ to the weakly tangential differential operator P in* (2.24) *is equal to the formally adjoint one*

$$P^*_{\mathscr{S}}\varphi = P^*\varphi = -\sum_{j=1}^{n}\partial_j a_j^\top\varphi + b^\top\varphi\,. \tag{2.50}$$

In particular, the Stoke's derivatives are skew-symmetric

$$\big(\mathscr{M}^*_{jk}\big)_{\mathscr{S}} = \mathscr{M}^*_{jk} = -\mathscr{M}_{jk} = \mathscr{M}_{kj} \qquad \forall\, j,k = 1,\ldots,n\,. \tag{2.51}$$

The adjoint operator to the operator $\mathscr{D}_j$ is

$$\big(\mathscr{D}_j\big)^*_{\mathscr{S}}\varphi = \mathscr{D}^*_j\varphi = -\mathscr{D}_j\varphi + \nu_j\mathscr{H}^0_{\mathscr{S}}\varphi\,, \qquad \varphi\in C^1(\mathscr{S})\,, \tag{2.52}$$

where $(n-1)^{-1}\mathscr{H}^0_{\mathscr{S}}(\mathscr{x}) = \mathscr{H}_{\mathscr{S}}(\mathscr{x})$ is the mean curvature of the surface $\mathscr{S}$ (cf. (0.15)*).*

For any real-valued function $\varphi\in C^1(\mathscr{S})$, any $1\leqslant j<k\leqslant n$ and for $\boldsymbol{\nu}_\Gamma = \big(\nu^1_\Gamma,\ldots,\nu^n_\Gamma\big)^\top$ being the the same as in Theorem 2.11 the following integration by parts formula

$$\int_{\mathscr{S}}\Big[(\mathscr{D}_j\varphi)\psi - \varphi\big(\mathscr{D}^*_j\psi\big)\Big]\,d\sigma = \oint_\Gamma \nu^j_\Gamma\varphi\psi\,d\mathfrak{s}\,, \tag{2.53}$$

holds. It is an analogue of the classical Gaußian **integration by parts** *formula*

$$\int_\Omega \partial_j f(y)g(y)\,dy = \oint_{\mathscr{S}}\nu_j(\tau)f(\tau)g(\tau)\,d\sigma - \int_\Omega f(y)\partial_j g(y)\,dy, \tag{2.54}$$

which holds for arbitrary $f,\ g\in\mathbb{W}^1(\mathscr{S})$.

In particular, the following **Gauß formulae for open surfaces** *is valid:*

$$\int_{\mathscr{S}}\mathscr{D}_j\varphi\,d\sigma = \oint_\Gamma\nu^j_\Gamma\varphi\,d\mathfrak{s} + \int_{\mathscr{S}}\nu_j\mathscr{H}^0_{\mathscr{S}}\varphi\,d\sigma\,. \tag{2.55}$$

Proof: We start by proving (2.51): by applying the Stoke's formulae

$$\oint_{\mathscr{S}}(\mathscr{M}_{jk}f)(\tau)\,d\sigma = 0, \qquad j,k=1,\ldots,n, \quad f\in\mathbb{W}^1_1(\mathscr{S}), \tag{2.56}$$

we get

$$\oint_{\mathscr{S}}(\mathscr{M}_{jk}\varphi)\psi\,d\sigma = \oint_{\mathscr{S}}(\mathscr{M}_{jk}\varphi\psi)\,d\sigma - \oint_{\mathscr{S}}\varphi(\mathscr{M}_{jk}\psi)\,d\sigma = -\oint_{\mathscr{S}}\varphi(\mathscr{M}_{jk}\psi)\,d\sigma$$

and the equality

$$\big(\mathscr{M}^*_{jk}\big)_{\mathscr{S}} = -\mathscr{M}_{jk} = \mathscr{M}_{kj} \tag{2.57}$$

follows. Moreover, note that the formal adjoint to $\mathscr{M}_{jk} = \mathcal{N}_j \mathscr{D}_k - \mathcal{N}_k \mathscr{D}_j$ is

$$\begin{aligned}\mathscr{M}^*_{jk}\varphi &= \big(\mathcal{N}_j\partial_k - \mathcal{N}_k\partial_j\big)^*\varphi = -\partial_j(\mathcal{N}_k\varphi) + \partial_k(\mathcal{N}_j\varphi)\\ &= \mathcal{N}_k\partial_j\varphi - \mathcal{N}_j\partial_k\varphi + (\partial_j\mathcal{N}_k)\varphi - (\partial_k\mathcal{N}_j)\varphi = -\mathscr{M}_{jk}\varphi\end{aligned}$$

(cf. (2.12)), where $\varphi \in C^1(\Omega_{\mathscr{S}})$ is defined in a neighborhood of $\mathscr{S}$. (2.51) is proved.

To prove (2.50) we note that, on $\mathscr{S}$,

$$\begin{aligned}P\varphi &= \sum_{j=1}^n a_j\partial_j\varphi + b\varphi = \sum_j a_j\big[\mathscr{D}_j + \nu_j\partial_{\boldsymbol{\nu}}\big]\varphi\\ &= \sum_{j=1}^n a_j\mathscr{D}_j\varphi + b\varphi + \sigma(P;\boldsymbol{\nu})\partial_{\boldsymbol{\nu}}\varphi = \sum_{j=1}^n a_j\mathscr{D}_j\varphi \qquad (2.58)\\ &= \sum_{j,k=1}^n a_j\nu_k\mathscr{M}_{kj}\varphi \qquad (2.59)\end{aligned}$$

due to Lemma 2.9.iv and the weak tangentiality of P. The property postulated in (2.50) follows from (2.59) and (2.51):

$$P^*_{\mathscr{S}}\varphi = \sum_{j,k=1}^n (\mathscr{M}_{kj})^*_{\mathscr{S}} a_j^\top \nu_k\varphi + b^\top\varphi = \sum_{j,k=1}^n (\mathscr{M}_{kj})^* a_j^\top\nu_k\varphi + b^\top\varphi = P^*\varphi\,.$$

With (2.50) and with (2.12) we get

$$\begin{aligned}(\mathscr{D}_j)^*_{\mathscr{S}}\varphi &= \mathscr{D}^*_j\varphi = -\partial_j\varphi + \sum_{k=1}^n \partial_k\big(\mathcal{N}_j\mathcal{N}_k\varphi\big)\\ &= -\partial_j\varphi + \sum_{k=1}^n\Big[\mathcal{N}_j\mathcal{N}_k\partial_k\varphi + \big(\mathcal{N}_k\partial_k\mathcal{N}_j\big)\varphi + \mathcal{N}_j\big(\partial_k\mathcal{N}_k\big)\varphi\Big]\\ &= -\mathscr{D}_j\varphi - \mathcal{N}_j\mathscr{H}^0_{\mathscr{S}}\varphi + (\partial_{\mathcal{N}}\mathcal{N}_j)\varphi\,, \qquad (2.60)\end{aligned}$$

where $\varphi \in C^1(\Omega_{\mathscr{S}})$ is defined in a neighborhood of $\mathscr{S}$ and

$$\mathscr{H}^0_{\mathscr{S}} := -\sum_{k=1}^n \mathscr{D}_k\mathcal{N}_k\,, \qquad \mathscr{H}^0_{\mathscr{S}}(\mathscr{x}) = -\sum_{k=1}^n \mathscr{D}_k\nu_k(\mathscr{x}) \qquad \text{for} \quad \mathscr{x}\in\mathscr{S}\,. \qquad (2.61)$$

(2.52) follows since (cf. (2.15)) $\partial_{\mathcal{N}}\mathcal{N}_j = 0$.

To prove (2.61) we apply

$$\begin{aligned}\partial_{\mathcal{N}}\mathcal{N}\big|_{\mathscr{S}} &= \left\{\sum_{j=1}^n \mathcal{N}_j\partial_j\mathcal{N}_k\right\}_{k=1}^n\Bigg|_{\mathscr{S}} = \left\{\sum_{j=1}^n \mathcal{N}_j\partial_k\mathcal{N}_j\right\}_{k=1}^n\Bigg|_{\mathscr{S}}\\ &= \frac12\nabla_x|\mathcal{N}|^2\Big|_{\mathscr{S}} = \frac12\nabla_x 1 = 0\,. \qquad (2.62)\end{aligned}$$

and proceed as follows

$$\sum_{k=1}^n \mathscr{D}_k\nu_k = \sum_{k=1}^n \left(\partial_k\nu_k - \nu_k\sum_{j=1}^n \nu_j\partial_j\nu_k\right) = -\mathscr{H}^0_{\mathscr{S}} - \sum_{j=1}^n \frac{\nu_j}{2}\partial_j 1 = -\mathscr{H}^0_{\mathscr{S}}\,.$$

For the proof of the last formula (2.53) we apply Lemma 2.9.iv, (2.51), the equalities $\sum\limits_{k=1}^n \nu_k^2 = 1$, $\sum\limits_{k=1}^n \nu_k\nu_\Gamma^k = 0$ and proceed as follows:

$$\oint_{\mathscr{S}} (\mathscr{D}_j\varphi)\psi\, d\sigma = \sum_{k=1}^n \oint_{\mathscr{S}} \nu_k(\mathscr{M}_{jk}\varphi)\psi\, d\sigma - \sum_{k=1}^n \oint_{\mathscr{S}} \psi(\mathscr{M}_{jk}\nu_k\psi)\, d\sigma$$
$$+ \sum_{k=1}^n \oint_\Gamma (\nu_k^2\nu_\Gamma^j - \nu_k\nu_j\nu_\Gamma^k)\varphi\psi\, d\mathfrak{s} = \oint_{\mathscr{S}} \psi(\mathscr{D}_j^*\psi)\, d\sigma + \oint_\Gamma \nu_\Gamma^j\varphi\psi\, d\mathfrak{s}\,.$$

Concerning the formula (2.55): it follows from formulae (2.53) and (2.52), if we insert $\psi(t) \equiv 1$ in (2.53) and note, that $\mathscr{D}_j 1 = 0$. ■

Lemma 2.13 *Let P be, as in ((2.24)), a first-order differential operator with C^1-smooth coefficients. P is weakly/strongly tangent if and only if the formally adjoint P^* is so.*

If P is weakly tangent to $\mathscr{S}$ and P is defined in a neighborhood of $\mathscr{S}$, then

$$(P\varphi)\Big|_{\mathscr{S}} = P\Big(\varphi|_{\mathscr{S}}\Big) \tag{2.63}$$

for every C^1 function φ defined in a neighborhood of $\mathscr{S}$. In particular,

$$\mathscr{D}_j\varphi\big|_{\mathscr{S}} = \mathscr{D}_j\big(\varphi\big|_{\mathscr{S}}\big)\,, \qquad \mathscr{M}_{jk}\varphi\big|_{\mathscr{S}} = \mathscr{M}_{jk}\big(\varphi\big|_{\mathscr{S}}\big)\,, \qquad j,k = 1,\dots,n\,. \tag{2.64}$$

Furthermore, (2.63) *is true for the adjoint P^*, and*

$$\int_{\mathscr{S}} \langle P\varphi, \psi\rangle\, d\sigma = \int_{\mathscr{S}} \langle\varphi, P^*\psi\rangle\, d\sigma + \oint_\Gamma \langle\sigma(P;\boldsymbol{\nu}_\Gamma)\varphi, \psi\rangle\, d\mathfrak{s} \tag{2.65}$$

for any vector-valued functions $\varphi, \psi \in \mathscr{S}$.

Proof: The first assertion follows since $\sigma(P^*;\xi) = -\sigma(P;\xi)^\top$, for each $\xi \in \mathbb{R}^n$.

Due to the representation (2.58) it suffices to prove (2.63) for only the operator $\mathscr{D}_j = \boldsymbol{d}^j\cdot\nabla$, where $\boldsymbol{d}^j = \pi_{\mathscr{S}}\boldsymbol{e}^j = \mathscr{N}\wedge\big(\mathscr{N}\wedge\boldsymbol{e}^j\big)$ is at least C^1-smooth vector field in a neighborhood $\Omega_{\mathscr{S}}$ of $\mathscr{S}$, tangent to the surface $\mathscr{S}$ at surface points (cf. (2.29)). Thus, we have to justify the following equality:

$$\mathscr{D}_j\varphi\big|_{\mathscr{S}} = \big(\boldsymbol{d}^j\cdot\nabla\big)\varphi\Big|_{\mathscr{S}} = \boldsymbol{d}^j\cdot\nabla\left(\varphi\Big|_{\mathscr{S}}\right) = \mathscr{D}_j\left(\varphi\Big|_{\mathscr{S}}\right)\,. \tag{2.66}$$

The vector field $\boldsymbol{d}^j(x) = \boldsymbol{d}^j(\theta, \mathscr{x})$ depends on the signed distance $\theta = \operatorname{dist}(x, \mathscr{S}) = \pm|x - \mathscr{x}|$ continuously ($\theta > 0$ for the outer domain and $\theta > 0$ for the inner one). Let $\mathscr{F}^t_{\boldsymbol{d}^j(\cdot)}$ be the integral curve of the vector field $\boldsymbol{d}^j$ and

$$\mathscr{F}^t_{\boldsymbol{d}^j(\cdot)} \,:\, \Omega_{\mathscr{S}} \to \Omega_{\mathscr{S}}\,, \qquad \mathscr{F}^t_{\boldsymbol{d}^j(0,\cdot)} = \mathscr{F}^t_{\boldsymbol{d}^j}(\cdot) \,:\, \mathscr{S} \to \mathscr{S} \tag{2.67}$$

be the flow generated by this vector field ℓ_θ in the neighborhood $\Omega_{\mathscr{S}}$ (cf. § 2). Since the flow depends continuously on the parameter θ, we get

$$\big(\boldsymbol{d}^j(\theta, \mathscr{x}) \cdot \nabla\big)\varphi\Big|_{\mathscr{S}} = \lim_{\theta\to 0} \frac{d}{dt}\, \varphi\left(\mathscr{F}^t_{\boldsymbol{d}^j(\theta,\mathscr{x})}\right)\Big|_{t=0} = \frac{d}{dt}\, \varphi\left(\mathscr{F}^t_{\boldsymbol{d}^j}\right)\big|_{t=0}$$
$$= \boldsymbol{d}^j \cdot \nabla\big(\varphi\big|_{\mathscr{S}}\big) = \mathscr{D}_j\left(\varphi\Big|_{\mathscr{S}}\right)$$

and (2.66) is proved.

Next, using (2.58), (2.53) and integrating by parts we get

$$\begin{aligned}
\int_{\mathscr{S}} \langle P\varphi, \psi\rangle\, d\sigma &= \sum_{j=1}^n \int_{\mathscr{S}} \langle a_j \mathscr{D}_j\varphi, \psi\rangle\, d\sigma + \int_{\mathscr{S}} \langle b\varphi, \psi\rangle\, d\sigma \\
&= \sum_{j=1}^n \int_{\mathscr{S}} \langle \varphi, \mathscr{D}^*_j a^\top_j \psi\rangle\, d\sigma + \int_{\mathscr{S}} \langle \varphi, b^\top\psi\rangle\, d\sigma + \sum_{j=1}^n \oint_\Gamma \langle \varphi, \nu^j_\Gamma a^\top_j\psi\rangle\, d\sigma \\
&= \int_{\mathscr{S}} \langle \varphi, P^*\psi\rangle\, d\sigma + \oint_\Gamma \langle \sigma(P; \boldsymbol{\nu}_\Gamma)\varphi, \psi\rangle\, d\mathfrak{s}
\end{aligned}$$

and this completes the proof. ∎

Remark 2.14 *By iteration, an identity similar in spirit to* (2.65) *holds for higher order weakly tangential differential operators which are higher order polynomials of Günter's or Stoke's derivatives (cf. Lemma 2.15).*

In this connection, let us also point out that the strongly tangential operator-the Stoke's gradient

$$\mathscr{M}_{\mathscr{S}} := \mathscr{N} \wedge \nabla_{\mathscr{S}} = \mathscr{N} \wedge \nabla = \big\{\mathscr{M}_{23}, -\mathscr{M}_{13}, \mathscr{M}_{12}\big\}\,, \qquad \mathscr{M}_{\mathscr{S}}\big|_{\mathscr{S}} = \boldsymbol{\nu} \wedge \nabla_{\mathscr{S}} \tag{2.68}$$

in $\mathbb{R}^3$ acting on scalar functions on $\mathscr{S}$, is naturally identified with the skew-symmetric matrix whose entries are the Stoke's derivatives, in the sense that

$$\boldsymbol{\nu} \wedge d = \frac{1}{2} \sum_{j,k=1}^3 \mathscr{M}_{jk}\, dx_j \wedge dx_k = \sum_{1\leqslant j<k\leqslant 3} \mathscr{M}_{jk}\, dx_j \wedge dx_k\,. \tag{2.69}$$

Further important examples of strongly tangential, first-order differential operators are offered by

$$\begin{aligned}
P_1\boldsymbol{U} &:= \operatorname{div}\boldsymbol{U} - \partial_{\boldsymbol{\nu}}\boldsymbol{U}\,\langle \boldsymbol{U}, \boldsymbol{\nu}\rangle, &&\text{with} \quad P^*_1\varphi = -\nabla\varphi + (\partial_{\boldsymbol{\nu}}\varphi + \mathscr{H}^0_{\mathscr{S}}\varphi)\boldsymbol{\nu}\,, \\
P_2\boldsymbol{U} &:= \operatorname{div}_{\mathscr{S}} \pi_{\mathscr{S}}\boldsymbol{U}, &&\text{with} \quad P^*_2\varphi = -\pi_{\mathscr{S}}\nabla_{\mathscr{S}}\varphi\,, \\
P_3\boldsymbol{U} &:= \partial_{\boldsymbol{\nu}}\pi_{\mathscr{S}}\boldsymbol{U} - \boldsymbol{\nu} \vee dU, &&\text{with} \quad P^*_3\varphi = -\pi_{\mathscr{S}}\partial_{\boldsymbol{\nu}}\varphi - \mathscr{H}^0_{\mathscr{S}}\pi_{\mathscr{S}}\varphi - \delta(\boldsymbol{\nu} \wedge \varphi)\,.
\end{aligned} \tag{2.70}$$

Indeed,

$$\sigma(P_1;\xi)=\langle\xi,\cdot\rangle-\langle\boldsymbol{\nu},\xi\rangle\langle\boldsymbol{\nu},\cdot\rangle\,,\quad \sigma(P_2;\xi)=\langle\xi,\pi_{\mathscr{S}}(\cdot)\rangle\,,\quad \sigma(P_3;\xi)=\langle\xi,\boldsymbol{\nu}\rangle\,\pi_{\mathscr{S}}-\boldsymbol{\nu}\vee(\xi\wedge\cdot)\,,$$

so that (2.27) is easily verified in each case.

In the sequel we use the following standard notation

$$\begin{aligned}\nabla^{\alpha}_{\mathscr{S}}:=\mathscr{D}_1^{\alpha_1}\ldots\mathscr{D}_n^{\alpha_n}\,,\qquad \alpha\in\mathbb{N}_0^n\,,\\ \mathscr{M}^{\beta}_{\mathscr{S}}:=\mathscr{M}_1^{\beta_1}\ldots\mathscr{M}_m^{\beta_m}\,,\quad \beta\in\mathbb{N}_0^m\,,\quad m=\frac{n(n-1)}{2}\,,\end{aligned}\tag{2.71}$$

where

$$\nabla_{\mathscr{S}}:=(\mathscr{D}_1,\ldots,\mathscr{D}_n)^{\top}\,,\qquad \mathscr{M}_{\mathscr{S}}:=(\mathscr{M}_{12},\ldots,\mathscr{M}_{n-1,n})^{\top}\tag{2.72}$$

and the selected Stoke's derivatives $\mathscr{M}_1:=\mathscr{M}_{1,2},\ldots,\mathscr{M}_m:=\mathscr{M}_{n-1,n}$ are non–vanishing, while the remaining non-vanishing Stoke's derivatives differ from the selected ones only by the sign. In contrast to the case of the usual derivatives ∂^{α} it does really matters in which sequence we apply the derivatives $\mathscr{D}_j^{\alpha_j}$ and $\mathscr{M}_k^{\beta_k}$ in (2.71), because they have variable coefficients. In this connection let us write precisely what is meant under the dual operators:

$$\begin{aligned}(\mathscr{D}_x^*)^{\alpha}:=(\mathscr{D}_n^*)^{\alpha_n}\ldots(\mathscr{D}_1^*)^{\alpha_1}\,,\quad \alpha\in\mathbb{N}_0^n\,,\\ (\mathscr{M}_x^*)^{\beta}:=(-1)^{|\beta|}(\mathscr{M}_m)^{\beta_m}\ldots(\mathscr{M}_1)^{\beta_1}\,,\quad \beta\in\mathbb{N}_0^m\,,\end{aligned}\tag{2.73}$$

Note, that we use the same operators $\mathscr{M}_1^*=-\mathscr{M}_1=-\mathscr{M}_{1,2},\ldots,\mathscr{M}_m^*=-\mathscr{M}_m:=-\mathscr{M}_{n-1,n}$ for the adjoint operators to the Stokes derivatives, because these operators are skew-symmetric $(\mathscr{M}_{j,k})^*=-\mathscr{M}_{j,k}$ (cf. (2.51)).

Lemma 2.15 *Let $\mathbf{G}(\mathscr{D})$ be a tangential differential operator of the form*

$$\mathbf{G}(\mathscr{D})=\sum_{|\alpha|\leqslant k}\boldsymbol{g}_{\alpha}(t)\mathscr{D}_t^{\alpha}=\sum_{|\beta|\leqslant k}f_{\beta}(t)\mathscr{M}_t^{\beta},\qquad t\in\mathscr{S}\,.\tag{2.74}$$

Then

$$\oint_{\mathscr{S}}\langle\mathbf{G}(\mathscr{D})\varphi,\psi\rangle\,d\sigma=\oint_{\mathscr{S}}\langle\varphi,\mathbf{G}^*(\mathscr{D})\psi\rangle\,d\sigma\,,\tag{2.75}$$

where

$$\mathbf{G}^*(\mathscr{D})=\sum_{|\alpha|\leqslant k}(\mathscr{D}^*)^{\alpha}g_{\alpha}^{\top}I=\sum_{|\beta|\leqslant k}(-1)^{|\beta|}\mathscr{M}^{\beta}f_{\beta}^{\top}I\tag{2.76}$$

and $\mathscr{D}^*$ and $\mathscr{M}^*$ are the adjoint operators (cf. (2.52) *and* (2.51)*).*

Remark 2.16 *Note that the operators* $i\mathscr{M}_j,\ j=1,\ldots,m$ *with variable coefficients*

$$\mathbf{A}(x,\mathscr{M}_x)u=\sum_{j=1}^{M}b_j(x)(i\mathscr{M}_j)^{m_j}\overline{b_j^\top(x)}u\,,\qquad b_j\in[C^\infty(\mathscr{S})]^{N\times N}\tag{2.77}$$

and polynomials with constant self adjoint $N\times N$ *matrix coefficients*

$$\mathbf{B}(\mathscr{M}_x)u=\sum_{j=1}^{M}a_j\mathscr{M}_j^{m_j}u\,,\qquad \overline{a_j^\top}=a_j=\mathrm{const}\quad\forall j=1,\ldots,M\,,\quad\forall m_j\in\mathbb{N}_0\,,\tag{2.78}$$

are all self adjoint on the hypersurface $\mathbf{A}^*_{\mathscr{S}}(\mathscr{M}_x)=\mathbf{A}(\mathscr{M}_x)$.

3 Equation of elastic hypersurface

One way of understanding the genesis of the Laplace-Beltrami operator $\boldsymbol{\Delta}_{\mathscr{S}}$ on the surface $\mathscr{S}$ (see (2.35)) is to consider the energy functional

$$\mathscr{E}[u]:=\int_{\mathscr{S}}\|\nabla\,u\|^2\,d\sigma,\qquad u\in C^\infty(\mathscr{S}).\tag{3.1}$$

Then any minimizer u of the functional (3.1) should satisfy

$$\begin{aligned}0=&\frac{d}{dt}\mathscr{E}[u+tv]\Big|_{t=0}=\int_{\mathscr{S}}\left[\langle\nabla\,u,\nabla\,v\rangle+\langle\nabla\,v,\nabla\,u\rangle\right]\,d\sigma\\ =&2\mathrm{Re}\int_{\mathscr{S}}\langle\nabla\,u,\nabla\,v\rangle\,d\sigma\qquad u\in C^\infty(\mathscr{S}),\qquad\forall\,v\in C_0^\infty(\mathscr{S}),\end{aligned}\tag{3.2}$$

which implies

$$\Delta u=0\quad\text{ on }\quad\mathscr{S}.\tag{3.3}$$

In other words, (3.3) is the Euler-Lagrange equation associated with the integral functional (3.1).

Similarly, minimizers of the energy functional

$$\mathscr{E}[\boldsymbol{U}]:=\int_{\mathscr{S}}\left[\|dU\|^2+\|\delta\boldsymbol{U}\|^2\right]d\sigma,\qquad\boldsymbol{U}\in\Lambda^\ell\omega(\mathscr{S}),\tag{3.4}$$

are null-solutions to the Hodge-Laplacian (cf. later (4.16)), while minimizers of the energy functional

$$\mathscr{E}[\boldsymbol{U}]:=\int_{\mathscr{S}}\|\nabla\boldsymbol{U}\|^2\,d\sigma,\qquad\boldsymbol{U}\in\omega(\mathscr{S}),\tag{3.5}$$

are null-solutions to the Bochner-Laplacian (cf. later (4.17)).

Our aim is to adopt a similar point of view in the case of anisotropic and isotropic (Lamé) system of elasticity on $\mathscr{S}$.

The Günter's derivatives $\{\mathscr{D}_j\}_{j=1}^n$ are tangent and represent a full system (cf. (2.29)-(2.31)). But the derivative $\mathscr{D}_j\boldsymbol{V}$ is not covariant and maps the tangential vectors to non-tangential ones $\mathscr{D}_j : \omega(\mathscr{S}) \not\to \omega(\mathscr{S})$. To improve this we just eliminate the normal component of the vector by applying the canonical orthogonal projection $\pi_{\mathscr{S}}$ onto $\omega(\mathscr{S})$ (cf. (0.8))

$$\mathscr{D}_j^{\mathscr{S}}\boldsymbol{V} := \pi_{\mathscr{S}}\mathscr{D}_j\boldsymbol{V} = \mathscr{D}_j\boldsymbol{V} - \langle\boldsymbol{\nu}, \mathscr{D}_j\boldsymbol{V}\rangle\boldsymbol{\nu} = \mathscr{D}_j\boldsymbol{V} + (\partial_{\boldsymbol{V}}\nu_j)\boldsymbol{\nu}\,, \tag{3.6}$$
$$\text{where}\quad \partial_{\boldsymbol{V}}\varphi := \sum_{k=1}^n V_k^0\partial_k\varphi = \sum_{k=1}^n V_k^0\mathscr{D}_k\varphi$$

and obtain an automorphisms of the space of tangential vector fields

$$\mathscr{D}_j^{\mathscr{S}} \;:\; \omega(\mathscr{S}) \to \omega(\mathscr{S})\,. \tag{3.7}$$

The starting point is to consider the total free (elastic) energy

$$\mathscr{E}[\boldsymbol{U}] := \int_{\mathscr{S}} E(y, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y))\,d\sigma, \quad \mathscr{D}^{\mathscr{S}}\boldsymbol{U} := \left[(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U})_k^0\right]_{n\times n}, \quad \boldsymbol{U}\in\omega(\mathscr{S}) \tag{3.8}$$

(cf. (3.6), (3.7)), ignoring at the moment the displacement boundary conditions (Koiter's model). As before, equilibria states correspond to minimizers of the above variational integral (see [NH80, § 5.2]). First we should identify the correct form of the stored energy density $E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x))$. We shall restrict attention to the case of linear elasticity. In this scenario, $E = (\mathfrak{S}_{\mathscr{S}}, \mathrm{Def}_{\mathscr{S}})$ depends bi-linearly on the stress tensor $\mathfrak{S}_{\mathscr{S}} = \left[\mathfrak{S}^{jk}\right]_{n\times n}$ and the deformation (strain) tensor

$$\mathrm{Def}_{\mathscr{S}} = \left[\mathfrak{D}_{jk}\right]_{n\times n}, \qquad \mathfrak{D}_{jk}\boldsymbol{U} := \frac{1}{2}\left[\left(\mathscr{D}_k^{\mathscr{S}}\boldsymbol{U}\right)_j + \left(\mathscr{D}_j^{\mathscr{S}}\boldsymbol{U}\right)_k\right], \quad j,k = 1,\dots,n \tag{3.9}$$

which, according to Hooke's law, satisfy $\mathfrak{S}_{\mathscr{S}} = \mathbb{T}\mathrm{Def}_{\mathscr{S}}$, for some linear, fourth-order tensor $\mathbb{T}$. If the medium is also homogeneous (i.e. the density and elastic parameters are position-independent), it follows that *E depends quadratically on the covariant derivative $\mathscr{D}^{\mathscr{S}}\boldsymbol{U}$*, i.e.

$$E(x, \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)) = \langle\mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x), \mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\rangle \tag{3.10}$$

for a linear operator

$$\mathbb{T} \;:\; \mathbb{M}^{n\times n}(\mathbb{R}) \longrightarrow \mathbb{M}^{n\times n}(\mathbb{R}), \tag{3.11}$$

where $\mathbb{M}^{n\times n}(\mathbb{R})$ stands for the vector space of all $n\times n$ matrices with real entries. Hereafter, we organize $\mathbb{M}^{n\times n}(\mathbb{R})$ as a real Hilbert space with respect to the inner product

$$\langle A, B\rangle := \mathrm{Tr}(AB^\top) = \sum_{i,j} a_{ij}b_{ij}, \qquad \forall\, A = [a_{ij}]_{i,j}\,,\; B = [b_{ij}]_{i,j} \in \mathbb{M}^{n\times n}(\mathbb{R}), \tag{3.12}$$

where $B^\top$ denotes transposed matrix, and Tr is the usual trace operator for square matrices.

A linear operator (3.11) is a tensor of order 4, i.e., $\mathbb{T} = \left[c_{ijk\ell}\right]_{ijk\ell}$, and

$$\mathbb{T}A = \left[\sum_{k,\ell} c_{ijk\ell}a_{k\ell}\right]_{ij}, \qquad \text{for}\;\; A = [a_{k\ell}]_{k\ell} \in \mathbb{M}^{n\times n}(\mathbb{R})\,. \tag{3.13}$$

$\mathbb{T}$ will be referred to in the sequel as the **elasticity tensor**. It is customary to assume that the elasticity tensor (3.11) is self-adjoint

$$\langle \mathbb{T}A, B\rangle = \langle A, \mathbb{T}B\rangle\,, \qquad A, B \in \mathbb{M}^{n\times n}(\mathbb{R})\,. \tag{3.14}$$

The condition rescaling (3.14), written in coordinate notation, is equivalent to the following equality

$$c_{ijk\ell} = c_{k\ell ij}\,, \qquad \forall\, i, j, k, \ell\,. \tag{3.15}$$

Indeed, the equality

$$\mathrm{Tr}((\mathbb{T}A)B^\top) = \sum_{i,j,k,\ell} c_{ijk\ell}a_{k\ell}b_{ij} = \sum_{i,j,k,\ell} c_{k\ell ij}a_{k\ell}b_{ij} = \mathrm{Tr}(A(\mathbb{T}B)^\top)$$

holds, for arbitrary $A = [a_{k\ell}]_{k\ell}$ and $B = [b_{k\ell}]_{k\ell}$, if and only if (3.15) holds: by inserting the delta functions $a_{k\ell} = \delta_{k\ell}$, $b_{ij} = \delta_{ij}$ we get the equality (3.15).

It is also customary to impose a symmetry condition, presented with two natural options:

$$\mathbb{T}(A^\top) = \mathbb{T}A \quad \text{and} \quad (\mathbb{T}A)^\top = \mathbb{T}A \qquad \forall\, A \in \mathbb{M}^{n\times n}(\mathbb{R})\,. \tag{3.16}$$

Then (3.16) amounts to the following symmetry in the indices of the elastic tensor:

$$c_{ijk\ell} = c_{ij\ell k} \quad \text{and} \quad c_{ijk\ell} = c_{jik\ell} \qquad \forall\, i, j, k, \ell\,. \tag{3.17}$$

Remark 3.1 *The conditions* (3.14) *and the first equality in* (3.16) *imply the second equality in* (3.16) *as well as the conditions* (3.14) *and the second equality in* (3.16) *imply the first equality in* (3.16)*. This is evident if we apply an equivalent formulation for corresponding tensors and matrices:* (3.15) *and* (3.17)*.*

A linear operator $\mathbb{T}$ in the energy functional of anisotropic elasticity (3.10) *satisfies the symmetry conditions* (3.14)*, and* (3.16)*. Equivalently, the corresponding elasticity tensor $\mathbb{T} = \left[c_{ijk\ell}\right]_{ijk\ell}$ has the symmetries* (3.15)*,* (3.17) *and, therefore, might have $n+n^2(n-1)^2/2$ different entries only.* ■

Remark 3.2 *It is rather natural to introduce the* **deformation tensor** *as the symmetrized covariant derivative (cf., e.g., [Ta96, V. I, Ch. 5, § 12]).*

$$\begin{aligned}(\mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U})(\boldsymbol{V},\boldsymbol{W}) &= \frac{1}{2}\Big\{\langle\partial_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle + \langle\partial_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\}\\ &= \frac{1}{2}\Big\{\langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle + \langle\partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\}, \qquad \forall\,\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}).\end{aligned} \tag{3.18}$$

It is also worth of mentioning that the antisymmetric part of the covariant derivative $\partial^{\mathscr{S}}_{\boldsymbol{U}}$

$$d\boldsymbol{U}(\boldsymbol{V},\boldsymbol{W}) = \langle d\boldsymbol{U}, \boldsymbol{V}\wedge\boldsymbol{W}\rangle = \frac{1}{2}\Big\{\langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle - \langle\partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\Big\}, \qquad \forall\,\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}), \tag{3.19}$$

is the exterior differential.

By inserting the value (3.9) of deformation tensor $\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}$ and applying the symmetry properties (3.17), we obtain

$$4\langle\mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(x),\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(x)\rangle=\langle\mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x),\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)\rangle=E(x,\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(x)) \tag{3.20}$$

(cf. (3.10)) which means that *the density of the elastic energy functional depends quadratically also on the deformation tensor*.

The density of the potential energy of an elastic medium should be strictly positive for the non-vanishing deformation tensor $\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}\neq 0$ (the energy conservation law). This leads to the following.

Lemma 3.3 *There exists a constant $C_0>0$ such that*

$$\langle\mathbb{T}\zeta,\zeta\rangle:=\sum_{i,j,k,\ell}c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell}\geqslant C_0\sum_{i,j}|\zeta_{i,j}|^2:=C_0|\zeta|^2 \tag{3.21}$$

for all symmetric and complex valued $\zeta_{ij}=\zeta_{ji}\in\mathbb{C}$ tensors $\zeta:=\big[\zeta_{ij}\big]_{n\times n}$.

Proof: The sum in the left hand side of (3.21) is real $\langle\mathbb{T}\zeta,\zeta\rangle=\overline{\langle\mathbb{T}\zeta,\zeta\rangle}$ (easy to check applying the symmetry properties (3.17) of the real valued coefficients). Dividing equality in (3.21) by $|\zeta|^2=\sum_{lm}|\zeta_{lm}|^2$ we find that it suffices to prove

$$\inf_{|\zeta|=1}\sum_{i,j,k,\ell}c_{ijk\ell}\zeta_{ij}\overline{\zeta}_{k\ell}\geqslant C_0>0\,. \tag{3.22}$$

If otherwise $C_0=0$, we select a sequence $\zeta^{(q)}_{jk}=\zeta^{(q)}_{kj}\in\mathbb{C}$, $q=1,2,\dots$ such that

$$\lim_{m\to\infty}\sum_{i,j,k,\ell}c_{ijk\ell}\zeta^{(q)}_{ij}\overline{\zeta^{(q)}_{k\ell}}=0\,,\qquad|\zeta^{(q)}|=1\,.$$

Since the space of tensors $[\zeta^{(q)}_{jk}]_{n\times n}$ is finite dimensional, there exists a convergent subsequence $\zeta^{(q_r)}_{k\ell}\to\zeta^{(0)}_{k\ell}$ as $r\to\infty$. Then we get an obvious contradiction

$$\sum_{i,j,k,\ell}c_{ijk\ell}\zeta^{(0)}_{ij}\overline{\zeta^{(0)}_{k\ell}}=0\,,\qquad|\zeta^{(0)}|=1\,.$$

which proves that $C_0>0$. ∎

Theorem 3.4 *The total free (elastic) energy functional (cf.* (3.8)*) acquires the form*

$$\mathscr{E}[\boldsymbol{U}]:=\int_{\mathscr{S}}\langle\mathbb{T}\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y),\mathscr{D}^{\mathscr{S}}\boldsymbol{U}(y)\rangle\,d\sigma=4\int_{\mathscr{S}}\langle\mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y),\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y)\rangle\,d\sigma\,,\quad\boldsymbol{U}\in\omega(\mathscr{S}) \tag{3.23}$$

and the Euler-Lagrange equation associated with the energy functional (3.23) *for a linear anisotropic elastic medium, reads*

$$\mathscr{L}_{\mathscr{S}}\boldsymbol{U}=\mathrm{Def}^*_{\mathscr{S}}\mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}\,,\qquad\boldsymbol{U}\in\omega(\mathscr{S})\,. \tag{3.24}$$

Here again $\mathbb{T}=\big[c_{ijk\ell}\big]_{ijk\ell}$ is the elasticity tensor which is positive definite (cf. (3.21)*) and has the symmetry properties* (3.15)*,* (3.17)*.*

Proof: The representation (3.23) follows from (3.8) and (3.20).

The Euler-Lagrange equation (3.24) is derived from (3.23) as a similar equation e3.3 is derived from (3.1):

$$\begin{aligned}\mathscr{E}[\boldsymbol{U}] &:= 4\int_{\mathscr{S}} \langle \mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y), \mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y)\rangle\, d\sigma \\ &= 4\int_{\mathscr{S}} \langle \mathrm{Def}^*_{\mathscr{S}}\mathbb{T}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}(y), \boldsymbol{U}(y)\rangle\, d\sigma = 0\end{aligned}$$

if and only if $\boldsymbol{U} \in \omega(\mathscr{S})$ is a solution of equation (3.24) due to the positive definiteness of the elasticity tensor $\mathbb{T}$ (cf. (3.21)). ■

Next we will find the Euler-Lagrange equation associated with the energy functional (3.8) for a linear isotropic elastic medium (Lamé equation) which is simpler. Such energy functional should be invariant with respect to any rotation. For the elasticity tensor $\mathbb{T}$ this results into the requirement that

$$\mathbb{T}(BAB^{-1}) = B(\mathbb{T}A)B^{-1}, \qquad \forall\, A, B \in \mathbb{M}^{n\times n}(\mathbb{R}) \text{ and unitary } \; B^\top = B^{-1}. \tag{3.25}$$

Examples of linear operators (3.11) satisfying (3.16) and (3.25) include

$$\mathbb{T} = \mathbb{T}A := (\mathrm{Tr}\, A)I \quad \text{and} \quad \mathbb{T}A := A + A^\top, \tag{3.26}$$

where I denotes the identity. The incisive step in the direction of identifying all such operators is the observation that any other operator of the type is a linear combination of these two. Namely, we have the following.

Lemma 3.5 *Let a linear operator* $\mathbb{T}$ *in* (3.11) *be frame indifferent (cf.* (3.25)*)*

$$\mathbb{T}(BAB^\top) = B(\mathbb{T}A)B^\top, \quad \textit{for all} \quad A \in \mathbb{M}^{3\times 3} \quad \textit{and for all orthogonal} \quad B \in \mathbb{SO}(3)$$

and have the symmetry property: one of conditions in (3.16) *holds.*

Then $\mathbb{T}$ *has the form*

$$\mathbb{T}A = \lambda\,(\mathrm{Tr}\, A)I + \mu\,(A + A^\top), \qquad A \in \mathbb{M}_{n,n}(\mathbb{R}), \tag{3.27}$$

where $\lambda, \mu \in \mathbb{R}$ *are some constants and it has both symmetry properties from* (3.16).

Proof: Let us first show that any linear operator (3.11) satisfying (3.16), (3.25) is represented in the form (3.27). By the previous discussion (cf. (3.26)), it suffices to prove that the space of linear operators (3.11) satisfying (3.16), (3.25) has dimension two.

It suffices to show that

$$\mathbb{T}D = aD + b(I - D) \quad \text{where} \quad D := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{3.28}$$

for the identity matrix I and two numbers $a, b \in \mathbb{R}$. Indeed, consider the following types of unitary matrices:

$$U_{j,k} := \begin{bmatrix} 1 & 0 & ... & ... & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & ... & 0 \\ 0 & ... & 0 & ... & 1 & ... & 0 \\ & & ... & ... & ... & & \\ 0 & ... & 1 & ... & 0 & ... & 0 \\ & & ... & ... & ... & & \\ 0 & 0 & 0 & ... & ... & 1 & 0 \\ 0 & 0 & 0 & ... & ... & ... & 1 \end{bmatrix}, \quad W_{j,k} := \begin{bmatrix} 1 & 0 & ... & ... & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & ... & 0 \\ 0 & ... & 0 & ... & 1 & ... & 0 \\ & & ... & ... & ... & & \\ 0 & ... & -1 & ... & 0 & ... & 0 \\ & & ... & ... & ... & & \\ 0 & 0 & 0 & ... & ... & 1 & 0 \\ 0 & 0 & 0 & ... & ... & ... & 1 \end{bmatrix}$$

where the only non-zero, off the diagonal entries are at (j, k) and (k, j). By multiplication $U_{j,k}A$ exchanges j-th with k-th rows in A, while $W_{j,k}A$, $j < k$, makes the same but changes the sign of j-th row before shifting it to k-th row.

By applying the unitary operator $U_{1,k}$, we get

$$\begin{aligned} \mathbb{T}E = & \sum_{j=1}^{n} \boldsymbol{e}^k \mathbb{T} U_{1,k} D U_{1,k}^{-1} = \sum_{j=1}^{n} \boldsymbol{e}^k U_{1,k} (\mathbb{T}D) U_{1,k}^{-1} \\ = & \sum_{j=1}^{n} \boldsymbol{e}^k U_{1,k} [aD - b(I - D)] U_{1,k}^{-1} = aE + b(I - E) \end{aligned} \tag{3.29}$$

for arbitrary diagonal matrix $E = [\delta_{jk}\boldsymbol{e}^k] = \sum_{j=1}^{n} \boldsymbol{e}^k U_{1,k} D U_{1,k}^{-1}$. Since for any $A \in \mathbb{M}^{n\times n}(\mathbb{R})$ we have $\mathbb{T}A = \frac{1}{2}\mathbb{T}(A + A^\top)$, thanks to (3.16), and since a self adjoint matrix can be diagonalized $\frac{1}{2}(A + A^\top) = UEU^{-1}$ with a suitable unitary matrix $\boldsymbol{U}$, the equality (3.29) holds for arbitrary A:

$$\mathbb{T}A = \mathbb{T}UEU^{-1} = \boldsymbol{U}(\mathbb{T}E)U^{-1} = \boldsymbol{U}[aE + b(I - E)]U^{-1} = aA + b(I - A)\,.$$

To check (3.28) we again apply the unitary matrices U_{i_o,j_o} and W_{i_o,j_o}. Set

$$A := \mathbb{T}D, \qquad A = \big[a_{ij}\big]_{1\leqslant i,j\leqslant n}$$

and observe that D is invariant under conjugation by W_{i_o,j_o}, i.e. $W_{i_o,j_o} D W_{i_o,j_o}^\top = D$, as long as $i_o \neq 1$ and $j_o \neq 1$. Thus, by (3.25), the same is true for $A = \mathbb{T}D$ which, in turn, eventually implies that

$$a_{i_o i_o} = a_{j_o j_o}, \qquad \forall\, i_o, j_o \neq 1. \tag{3.30}$$

The next observation is that D is invariant under conjugation by the product $U_{i_o j_o} W_{i_o,j_o}$, i.e. $U_{i_o j_o} W_{i_o,j_o} D W_{i_o,j_o}^\top U_{i_o j_o}^\top = D$, this time for every $1 \leqslant i_o \neq j_o \leqslant n$. Hence, by (3.25), the same holds for $A = \mathbb{T}D$, which ultimately implies that $a_{i_o j_o} = -a_{j_o i_o}$ for every pair of indices $1 \leqslant i_o \neq j_o \leqslant n$. Consequently,

$$a_{i_o j_o} = 0, \qquad \text{for every } \; 1 \leqslant i_o \neq j_o \leqslant n. \tag{3.31}$$

Under the current assumptions, i.e. (3.25), the first condition in (3.16), the desired conclusion, i.e. that $\mathbb{T}D$ has the two-parameter diagonal form indicated above, now follows readily from (3.30) and (3.31).

Let us analyze the case when the linear operator $\mathbb{T}$ satisfies (3.25) along with the second condition in (3.16). In this situation, let us consider the adjoint $\mathbb{T}^*$ to the tensor $\mathbb{T}$ with respect to the inner product (3.12) $\langle \mathbb{T}A, B\rangle = \langle A, \mathbb{T}^* B\rangle$. It can be readily checked that the adjoint $\mathbb{T}^*$ satisfies (3.25) and the first condition in (3.16), so the previous reasoning applies. Consequently, $\mathbb{T}^*$ can be represented in the form (3.27), which is invariant under the adjunction. Hence $\mathbb{T}$ can be written in the form (3.27) also. In particular, (3.27) holds in this case as well.

Concerning the equivalence of the first and the second condition in (3.16). Each of two conditions in (3.16) along with the condition (3.25) imply that the linear operator (3.11) has the form (3.27). Then, in particular, $\mathbb{T}$ is self adjoint. Since conditions in (3.16) are obtained by taking the adjoint, they are equivalent and the proof is completed. ■

Remark 3.6 *A posteriori, the conditions* (3.16) *and* (3.25) *imply that the linear operator* (3.11) *has the form* (3.27) *and, in particular, is self adjoint, i.e., imply the condition* (3.14).

Remark 3.7 *The above proof can be modified to hold in the case when* (3.25) *is (seemingly) weakened to allow only orientation preserving unitary matrices $\boldsymbol{U}$. All one has to do in this later case is to employ the invariance of D under conjugation by $U_{k_o\ell_o}U_{i_oj_o}W_{i_o,j_o}$ (with $k_o, \ell_o \neq 1$), in place of conjugation by (the inversion) $U_{i_oj_o}W_{i_o,j_o}$ as in the original proof.*

We are now ready to derive the Lamé equations of elasticity on a hypersurface.

Theorem 3.8 *On a smooth hypersurface $\mathscr{S}$ in $\mathbb{R}^n$, modeling a homogeneous, linear, isotropic, elastic medium, the Lamé operator $\mathscr{L}_\mathscr{S}$ is given by*

$$\mathscr{L}_\mathscr{S} = -\lambda\, \nabla_\mathscr{S} \operatorname{div}_\mathscr{S} + 2\mu \operatorname{Def}^*_\mathscr{S}\operatorname{Def}_\mathscr{S} = \lambda \operatorname{div}^*_\mathscr{S} \operatorname{div}_\mathscr{S} + 2\mu \operatorname{Def}^*_\mathscr{S}\operatorname{Def}_\mathscr{S}. \tag{3.32}$$

In particular, $\mathscr{L}_\mathscr{S}$ is a formally self-adjoint differential operator of second order.

Proof: According to the discussion in the first part of this section, the elasticity tensor in the case of linear, isotropic, elastic medium is given by (3.27), where λ, μ are the Lamé moduli. Applying the following properties of the trace

$$\operatorname{Tr}(A+B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)\,, \qquad \operatorname{Tr}(A^\top) = \operatorname{Tr}(A)\,,$$
$$\langle A + A^\top, A\rangle = Tr\big[(A+A^\top)A^\top\big] = \frac{1}{2}Tr\big[A^2 + 2AA^\top + (A^\top)^2\big]^2 = \frac{1}{2}Tr(A+A^\top)^2,$$

which are easy to verify directly, due to (3.10) the stored energy density is of the form

$$\begin{aligned} E(A) &= \langle \mathbb{T}A, A\rangle = \langle \lambda \operatorname{Tr}(A) I + \mu(A+A^\top), A\rangle = \lambda \operatorname{Tr}(A)\langle I, A\rangle + \mu\langle A + A^\top, A\rangle \\ &= \lambda\,(\operatorname{Tr} A)^2 + \frac{\mu}{2}\operatorname{Tr}\left((A+A^\top)^2\right). \end{aligned} \tag{3.33}$$

Further, by inserting $A := \mathscr{D}^\mathscr{S}\boldsymbol{U}$ in (3.33) and recalling (2.34), we get

$$E(x, \mathscr{D}^\mathscr{S}\boldsymbol{U}(x)) = \lambda\,(\operatorname{div}_\mathscr{S}\boldsymbol{U})^2(x) + 2\mu\,\langle(\operatorname{Def}_\mathscr{S}\boldsymbol{U})(x), (\operatorname{Def}_\mathscr{S}\boldsymbol{U})(x)\rangle \tag{3.34}$$

by (3.18) and since the trace

$$\operatorname{Tr}(\nabla_\mathscr{S}\boldsymbol{U}) = \sum_{j=1}^n \langle \partial_{h_j}\boldsymbol{U}, h_j\rangle = \operatorname{Div}_\mathscr{S}\boldsymbol{U} \tag{3.35}$$

is the divergence and is independent of a basis $\{h_j\}_{j=1}^n$. Thus, we are led to considering the variational integral

$$\mathscr{E}[\boldsymbol{U}] = \int_{\mathscr{S}} \Big[\lambda\,(\mathrm{div}_{\mathscr{S}}\,\boldsymbol{U})^2 + 2\mu\,\langle \mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U}, \mathrm{Def}_{\mathscr{S}}\,\boldsymbol{U}\rangle\Big]\,d\sigma, \qquad \boldsymbol{U}\in\omega(\mathscr{S}). \tag{3.36}$$

To determine the associated Euler-Lagrange equation, for a smooth and compactly supported vector field $\boldsymbol{V}\in\omega(\mathscr{S})\bigcap C_0^1(\mathscr{S})$ we compute

$$\frac{d}{dt}\mathscr{E}[\boldsymbol{U}+tV]\Big|_{t=0} = 2\int_{\mathscr{S}}\Big[\lambda\,\mathrm{div}_{\mathscr{S}}\boldsymbol{U}\mathrm{div}_{\mathscr{S}}\boldsymbol{V} + 2\mu\,\langle\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}, \mathrm{Def}_{\mathscr{S}}\boldsymbol{V}\rangle\Big]\,d\sigma$$

By applying the Gaußtheorem on the divergence $\mathrm{div}_{\mathscr{S}}$

$$\int_{\Omega}\mathrm{div}\,F(y)\,dy = \oint_{\mathscr{S}}\langle\boldsymbol{\nu}(\tau), F(\tau)\rangle\,d\sigma \tag{3.37}$$

and taking into account that $\boldsymbol{V}$ vanishes on the boundary $\Gamma=\partial\mathscr{S}$ we get

$$\begin{aligned}\frac{d}{dt}\mathscr{E}[\boldsymbol{U}+tV]\Big|_{t=0} &= 2\int_{\mathscr{S}}\langle(-\lambda\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}} + 2\mu\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}})\boldsymbol{U}, \boldsymbol{V}\rangle\,d\sigma\\ &= 2\int_{\mathscr{S}}\langle\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V}\rangle\,d\sigma = 0\,.\end{aligned} \tag{3.38}$$

Since the vector field $\boldsymbol{V}\in\omega(\mathscr{S})\bigcap C_0^1(\mathscr{S})$ is arbitrary, from (3.38) follows that the displacement vector field $\boldsymbol{U}$ satisfies the equality $\mathscr{L}_{\mathscr{S}}\boldsymbol{U}=0$.

That the operator $\mathscr{L}_{\mathscr{S}} = \lambda\,\mathrm{div}^*_{\mathscr{S}}\,\mathrm{div}_{\mathscr{S}} + 2\mu\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}$ is formally self adjoint, is obvious from its structure:

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V})_{\mathscr{S}} = \lambda(\mathrm{div}^*_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V})_{\mathscr{S}} + \mu(\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}\boldsymbol{U}, \boldsymbol{V})_{\mathscr{S}} = (\boldsymbol{U}, \mathscr{L}_{\mathscr{S}}\boldsymbol{V})_{\mathscr{S}}.$$ ■

4 THE SURFACE LAMÉ OPERATOR AND RELATED PDO'S

The present section deals mostly with the identification of the deformation tensor

$$\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})(\boldsymbol{V},\boldsymbol{W}) := \frac{1}{2}\{\langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle + \langle\partial^{\mathscr{S}}_{\boldsymbol{W}}\boldsymbol{U},\boldsymbol{V}\rangle\}, \qquad \forall\,\boldsymbol{U},\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}), \tag{4.1}$$

and the Lamé operator (3.32).

Theorem 4.1 *For the deformation tensor and the Lamé operator on $\mathscr{S}$ the following identities are valid:*

$$\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := \big[\mathfrak{D}_{jk}(\boldsymbol{U})\big]_{n\times n}\,, \tag{4.2}$$

$$\mathfrak{D}_{jk}(\boldsymbol{U}) = \frac{1}{2}\Big[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k + (\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})_j\Big] = \frac{1}{2}\Big[\mathscr{D}_jU_k + \mathscr{D}_kU_j + \partial_{\boldsymbol{U}}\big(\nu_j\nu_k\big)\Big]\,, \tag{4.3}$$

$$[\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})]^\top = \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \quad \textit{and} \quad \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{\nu} = 0\,, \tag{4.4}$$

$$\begin{aligned}\mathscr{L}_{\mathscr{S}} &= \mu\,\pi_{\mathscr{S}}\nabla^*_{\mathscr{S}}\,\nabla_{\mathscr{S}} + (\lambda+\mu)\,\nabla_{\mathscr{S}}\,\nabla^*_{\mathscr{S}} - \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\\ &= -\mu\,\boldsymbol{\Delta}_{\mathscr{S}} - (\lambda+\mu)\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}} - \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\,.\end{aligned} \tag{4.5}$$

Proof: Given the local nature of the identities we seek to prove, it suffices to work locally, in a small open subset $\mathscr{O}$ of $\mathscr{S}$, where an orthonormal basis $T_1, \dots, T_{n-1}$ to $\omega(\mathscr{S})$ has been fixed. We extend the basis by the outer unit normal vector field $T_n := \boldsymbol{\nu}$ so that $\{T_j\}_{1\leqslant j\leqslant n}$ becomes an orthonormal basis for $\mathbb{R}^n$, at points in $\mathscr{O}$.

Since $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})$ is a linear operator (see (4.1)) it is represented by an $n\times n$ matrix in the fixed basis $\{T_j\}_{1\leqslant j\leqslant n}$ and the first equality in (4.2) follows. The symmetry property of the matrix, recorded as the first equality in (4.4), follows from (4.1) since by interchanging vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ does not affect the definition (4.1).

For a tangential field $\boldsymbol{U}$ to $\mathscr{S}$ with $\operatorname{supp}\boldsymbol{U}\subset\mathscr{O}$ and arbitrary $\boldsymbol{V},\boldsymbol{W}\in\mathbb{R}^n$ we have

$$\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U}=\partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}}\boldsymbol{U}\,,\qquad \langle\partial^{\mathscr{S}}_{\boldsymbol{V}}\boldsymbol{U},\boldsymbol{W}\rangle=\langle\partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}}\boldsymbol{U},\pi_{\mathscr{S}}\boldsymbol{W}\rangle$$

and, by the definition of the deformation tensor (cf. (4.1)) obtain

$$\langle\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{V},\boldsymbol{W}\rangle:=\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})(\pi_{\mathscr{S}}\boldsymbol{V},\pi_{\mathscr{S}}\boldsymbol{W}),\qquad\forall\,\boldsymbol{V},\boldsymbol{W}\in\mathbb{R}^n.\tag{4.6}$$

Equality (4.6) implies the second equality in (4.4). Applying (3.18) and (3.6) we eventually obtain the second equality in (4.2):

$$\begin{aligned}\mathfrak{D}_{jk}(\boldsymbol{U})&=\frac{1}{2}\Big[\big(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U}\big)_j+\big(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U}\big)_k\Big]=\frac{1}{2}\Big[\mathscr{D}_kU_j+\mathscr{D}_jU_k+\partial_{\boldsymbol{U}}\big(\nu_j\nu_k\big)\Big]\\&=\frac{1}{2}\Big[\mathscr{D}_kU_j+\mathscr{D}_jU_k+\sum_{r=1}^{n}U_r(\mathscr{D}_r\nu_k)\nu_j+\sum_{r=1}^{n}U_r(\mathscr{D}_r\nu_j)\nu_k\Big]\,.\end{aligned}$$

We proceed with the proof of the last remaining equality (4.5). If $\boldsymbol{V}$ is also a smooth vector field, tangential to $\mathscr{S}$, applying (4.2) we get

$$\begin{aligned}&\int_{\mathscr{S}}\langle\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\boldsymbol{V}\rangle\,d\sigma=\int_{\mathscr{S}}\langle\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\mathrm{Def}_{\mathscr{S}}(\boldsymbol{V})\rangle\,d\sigma\\&\quad=\sum_{j,k=1}^{n}\frac{1}{4}\int_{\mathscr{S}}\Big[\mathscr{D}_kU_j+\mathscr{D}_jU_k+\partial_{\boldsymbol{U}}(\nu_j\nu_k)\Big]\Big[\mathscr{D}_kV_j+\mathscr{D}_jV_k+\partial_{\boldsymbol{V}}(\nu_j\nu_k)\Big]\,d\sigma\,.\end{aligned}\tag{4.7}$$

Next consider

$$\begin{aligned}&\sum_{j,k=1}^{n}\int_{\mathscr{S}}(\mathscr{D}_jU_k+\mathscr{D}_kU_j)(\mathscr{D}_jV_k+\mathscr{D}_kV_j)\,d\sigma=2\sum_{j,k=1}^{n}\int_{\mathscr{S}}\mathscr{D}^*_j(\mathscr{D}_jU_k+\mathscr{D}_kU_j)V_k\,d\sigma\\&\quad=2\sum_{j,k=1}^{n}\int_{\mathscr{S}}\Big[-V_k\mathscr{D}^2_jU_k-V_k\mathscr{D}_j\mathscr{D}_kU_j-\mathscr{H}^0_{\mathscr{S}}\nu_j(\mathscr{D}_jU_k)V_k-\mathscr{H}^0_{\mathscr{S}}\nu_j(\mathscr{D}_kU_j)V_k\Big]\,d\sigma\\&\quad=-2\int_{\mathscr{S}}\langle\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma-2\sum_{j,k=1}^{n}\int_{\mathscr{S}}\Big[V_k\mathscr{D}_j\mathscr{D}_kU_j+\mathscr{H}^0_{\mathscr{S}}\nu_j(\mathscr{D}_kU_j)V_k\Big]\,d\sigma\,,\end{aligned}\tag{4.8}$$

since $\sum\limits_{j=1}^{n}\nu_j\mathscr{D}_j=0$ on $\mathscr{S}$.

To proceed in the second integrand in (4.8): we employ the commutator identity from Lemma 2.9.ix and recall that the fields $\boldsymbol{U}$ and $\boldsymbol{V}$ are tangential to write

$$\sum_{j,k=1}^{n}\int_{\mathscr{S}} V_k\mathscr{D}_j\mathscr{D}_kU_j\,d\sigma=\sum_{j,k=1}^{n}\int_{\mathscr{S}}\Big[V_k\mathscr{D}_k\mathscr{D}_jU_j+V_k[\mathscr{D}_j,\mathscr{D}_k]U_j\Big]d\sigma$$

$$=\int_{\mathscr{S}}\langle\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle d\sigma+\sum_{j,k,l=1}^{n}\int_{\mathscr{S}}\big[V_k\nu_j\mathscr{D}_k\nu_l-\nu_kV_k\mathscr{D}_j\nu_l\big]\mathscr{D}_lU_j dS$$

$$=\int_{\mathscr{S}}\langle\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle d\sigma+\sum_{j,k,l=1}^{n}\int_{\mathscr{S}}V_k(\mathscr{D}_k\nu_l)\big[\mathscr{D}_l(\nu_jU_j)-(\mathscr{D}_l\nu_j)U_j\big]d\sigma$$

$$=\int_{\mathscr{S}}\langle\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle d\sigma-\sum_{j,k,l=1}^{n}\int_{\mathscr{S}}(\partial_k\nu_l)(\partial_l\nu_j)U_jV_k\,d\sigma$$

$$=\int_{\mathscr{S}}\langle\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma-\int_{\mathscr{S}}\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma \tag{4.9}$$

on $\mathscr{S}$, because $\sum_{j=1}^{n}\nu_jU_j=\sum_{k=1}^{n}\nu_kV_k=0$ and, due to (2.21)

$$\sum_{j,l,k=1}^{n}(\partial_k\nu_l)(\partial_l\nu_j)U_jV_k=\sum_{j,l,k=1}^{n}(\partial_l\nu_j)U_j(\partial_j\nu_k)V_k=\langle\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\mathscr{W}_{\mathscr{S}}\boldsymbol{V}\rangle=\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle.$$

For the third integrand in (4.8) we use Lemma 2.5.i and that the field $\boldsymbol{U}$ is tangential:

$$\sum_{j,k=1}^{n}\int_{\mathscr{S}}\mathscr{H}_{\mathscr{S}}^0\nu_j(\mathscr{D}_kU_j)V_k\,d\sigma=\mathscr{H}_{\mathscr{S}}^0\sum_{j,k=1}^{n}\int_{\mathscr{S}}V_k\left[\mathscr{D}_k\big(\nu_jU_j\big)-\big(\mathscr{D}_k\nu_j\big)U_j\right]\,d\sigma$$

$$=\int_{\mathscr{S}}\mathscr{H}_{\mathscr{S}}^0\langle\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma. \tag{4.10}$$

At this point, we may therefore conclude that

$$\sum_{j,k=1}^{n}\int_{\mathscr{S}}(\mathscr{D}_jU_k+\mathscr{D}_kU_j)(\mathscr{D}_jV_k+\mathscr{D}_kV_j)\,d\sigma$$

$$=2\int_{\mathscr{S}}\langle-\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}+\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U}-\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma. \tag{4.11}$$

We now proceed to analyze the remaining terms in (4.7). More precisely, we still have to take into account the terms containing either $\partial_{\boldsymbol{U}}(\nu_j\nu_k)$ or $\partial_{\boldsymbol{V}}(\nu_j\nu_k)$. We start with the identity

$$\sum_{j,k=1}^{n}(\mathscr{D}_kU_j)\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)=\sum_{j,k=1}^{n}\nu_k(\mathscr{D}_kU_j)\mathscr{D}_{\boldsymbol{V}}\nu_j+\sum_{j,k=1}^{n}(\mathscr{D}_{\boldsymbol{V}}\nu_k)\big[\mathscr{D}_k\big(\nu_jU_j\big)-U_j\mathscr{D}_k\nu_j\big]$$
$$=-\sum_{k,j=1}^{n}(\mathscr{D}_{\boldsymbol{V}}\nu_k)(\mathscr{D}_{\boldsymbol{U}}\nu_k)=-\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle, \tag{4.12}$$

valid at points on $\mathscr{S}$, because $\sum_{k}\nu_k\mathscr{D}_k=0$, $\sum_{j}\nu_jU_j=0$ and $\mathscr{D}_k\nu_j=\mathscr{D}_j\nu_k$. There are four such terms in (4.7), i.e. containing either $\mathscr{D}_{\boldsymbol{U}}(\nu_j\nu_k)$ or $\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)$. An inspection of the above calculation shows that, on $\mathscr{S}$, they are all equal to $-\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle$. We still have to compute the last integrand in (4.7):

$$\sum_{j,k=1}^{n}\mathscr{D}_{\boldsymbol{U}}(\nu_j\nu_k)\mathscr{D}_{\boldsymbol{V}}(\nu_j\nu_k)$$
$$=\sum_{j,k,r,l=1}^{n}\Big[U_r(\mathscr{D}_r\nu_j)\nu_k+U_r(\mathscr{D}_r\nu_k)\nu_j\Big]\Big[V_l(\mathscr{D}_l\nu_j)\nu_k+V_l(\mathscr{D}_l\nu_k)\nu_j\Big]$$
$$=2\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle+2\sum_{k,r,l=1}^{n}U_r(\mathscr{D}_r\nu_k)V_l\nu_k\frac{1}{2}\mathscr{D}_l\Big(\sum_{j=1}^{n}(\nu_j)^2\Big)=2\langle\mathscr{W}_{\mathscr{S}}^2\boldsymbol{U},\boldsymbol{V}\rangle,$$

on $\mathscr{S}$. At this point we combine all the above to get

$$4\sum_{j,k=1}^{n}\int_{\mathscr{S}}\mathfrak{D}_{jk}(\boldsymbol{U})\mathfrak{D}_{jk}(\boldsymbol{V})\,d\sigma=2\int_{\mathscr{S}}\langle-\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}-\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle\,d\sigma\,. \tag{4.13}$$

Having deduced (4.13), we may now compute

$$4\int_{\mathscr{S}}\langle\mathrm{Def}_{\mathscr{S}}^*\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\boldsymbol{V}\rangle\,d\sigma=\int_{\mathscr{S}}\langle\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}),\mathrm{Def}_{\mathscr{S}}(\boldsymbol{V})\rangle\,d\sigma$$
$$=4\sum_{j,k=1}^{n}\int_{\mathscr{S}}\mathfrak{D}_{jk}(\boldsymbol{U})\mathfrak{D}_{jk}(\boldsymbol{V})\,d\sigma=2\int_{\mathscr{S}}\langle-\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}-\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}\boldsymbol{U}\,,\,\boldsymbol{V}\rangle\,d\sigma$$
$$=2\int_{\mathscr{S}}\langle-\pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U}-\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U}-\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}\boldsymbol{U}\,,\,\boldsymbol{V}\rangle\,d\sigma, \tag{4.14}$$

since $\langle\boldsymbol{W},\boldsymbol{V}\rangle=\langle\pi_{\mathscr{S}}\boldsymbol{W},\boldsymbol{V}\rangle$ for a tangential vector field $\boldsymbol{V}$ and an arbitrary vector field $\boldsymbol{W}$ (also note that the original operator $\mathrm{Def}_{\mathscr{S}}^*\mathrm{Def}_{\mathscr{S}}\,:\,\omega(\mathscr{S})\to\omega(\mathscr{S})$ is tangential). We have also applied that the vector $\mathscr{W}_{\mathscr{S}}\boldsymbol{W}\in\omega(\mathscr{S})$ is tangential or an arbitrary vector field $\boldsymbol{W}$. Thus,

$$4\,\mathrm{Def}_{\mathscr{S}}^*\mathrm{Def}_{\mathscr{S}}=-2\pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}}-2\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}-2\,\mathscr{H}_{\mathscr{S}}^0\mathscr{W}_{\mathscr{S}}, \tag{4.15}$$

since the tangential vectors fields $\boldsymbol{U}$, $\boldsymbol{V}$ are arbitrary.

The first identity in (4.5) now follows easily from (4.15) and (3.32). The remaining identity in (4.5) then follows from what we have just proved and from Theorem 2.8. ■

Next recall the definition of the Hodge-Laplacian acting on 1-forms, i.e.

$$\Delta_{HL} := -\boldsymbol{d}^{\mathscr{S}}\boldsymbol{d}^{*}_{\mathscr{S}} - \boldsymbol{d}^{*}_{\mathscr{S}}\boldsymbol{d}^{\mathscr{S}} : \Lambda^1\omega(\mathscr{S}) \longrightarrow \Lambda^1\omega(\mathscr{S}) \tag{4.16}$$

where $\boldsymbol{d}^{\mathscr{S}}$ is the exterior derivative operator on $\mathscr{S}$, and $\boldsymbol{d}^{*}_{\mathscr{S}}$ its formal adjoint. As explained in §2, 1-forms on $\mathscr{S}$ are naturally identified with tangential fields to $\mathscr{S}$ so, from now on, we shall think of Δ_{HL} as mapping $\omega(\mathscr{S})$ into itself.

As pointed out in §2, the Hodge-Laplacian (4.16) is related to the Bochner-Laplacian on $\mathscr{S}$

$$\Delta_{BL} := -(\nabla^{\mathscr{S}})^{*}\nabla^{\mathscr{S}}, \tag{4.17}$$

via the Weitzenbock identity

$$\Delta_{BL} = \Delta_{HL} + \mathrm{Ric}_{\mathscr{S}}. \tag{4.18}$$

Our aim is to find alternative expressions for all these objects, starting with the Ricci tensor.

The Ricci curvature $\mathrm{Ric}_{\mathscr{S}}$ on $\mathscr{S}$ is a $(0,2)$-tensor defined as a contraction of $\boldsymbol{R}_{\mathscr{S}}$:

$$\mathrm{Ric}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) := \sum_{j=1}^{n}\langle \boldsymbol{R}_{\mathscr{S}}(h_j,\boldsymbol{V})\boldsymbol{U}, h_j\rangle = \sum_{j=1}^{n}\langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{V},h_j)h_j,\boldsymbol{U}\rangle, \tag{4.19}$$

$$\forall\, \boldsymbol{U},\boldsymbol{V}\in\omega(\mathscr{S}),$$

where $h_1,\dots,h_n$ is an orthonormal basis (of unit vectors) in $\omega(\mathscr{S})$. Thus, $\mathrm{Ric}_{\mathscr{S}}$ is a symmetric bilinear form.

Theorem 4.2 *For the Ricci tensor* $\mathrm{Ric}_{\mathscr{S}}$ *(cf.* (4.19)*) on* $\mathscr{S}$ *there holds*

$$\mathrm{Ric}_{\mathscr{S}} = -\mathscr{W}^2_{\mathscr{S}} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}. \tag{4.20}$$

In particular, when $n=3$ *–i.e. for a two dimensional hypersurface* $\mathscr{S}$ *in* $\mathbb{R}^3$*– the above identity reduces to*

$$\mathrm{Ric}_{\mathscr{S}} = -\det\mathscr{W}_{\mathscr{S}} = -\mathscr{K}_{\mathscr{S}}, \tag{4.21}$$

where $\mathscr{K}_{\mathscr{S}}$ *is the Gaußian curvature of the hypersurface* $\mathscr{S}$*.*

Proof: The Riemannian curvature tensor $\boldsymbol{R}_{\mathscr{S}}$ of $\mathscr{S}$ is given by

$$\boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W} = [\partial^{\mathscr{S}}_{\boldsymbol{U}},\partial^{\mathscr{S}}_{\boldsymbol{V}}]\boldsymbol{W} - \partial^{\mathscr{S}}_{[\boldsymbol{U},\boldsymbol{V}]}\boldsymbol{W}, \qquad \boldsymbol{U},\boldsymbol{V},\boldsymbol{W}\in\omega(\mathscr{S}), \tag{4.22}$$

where $[\boldsymbol{U},\boldsymbol{V}] := \partial_{\boldsymbol{U}}\boldsymbol{Y} - \partial_{\boldsymbol{Y}}\boldsymbol{U}$ is the usual commutator bracket. It is convenient to change this into a $(0,4)$-tensor by setting

$$\boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z}) := \langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle, \qquad \boldsymbol{U},\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z}\in\omega(\mathscr{S}). \tag{4.23}$$

Since $\mathbb{R}^n$ has zero curvature, it follows from Gauß's Theorema Egregium that, if $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$, $\boldsymbol{W}$ are tangential vector fields to $\mathscr{S}$, then

$$\langle \boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle = \langle II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{Z}), II_{\mathscr{S}}(\boldsymbol{V},\boldsymbol{W})\rangle - \langle II_{\mathscr{S}}(\boldsymbol{V},\boldsymbol{Z}), II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{W})\rangle \tag{4.24}$$

(see, e.g., [Ta96], Vol. II, p. 481). By inserting the second fundamental form $II_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) = \langle\partial_{\boldsymbol{U}}\boldsymbol{V} - \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V},\boldsymbol{\nu}\rangle = \langle\partial_{\boldsymbol{U}}\boldsymbol{V},\boldsymbol{\nu}\rangle$ (cf. (2.18)), we obtain:

$$\begin{aligned}\langle\boldsymbol{R}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V})\boldsymbol{W},\boldsymbol{Z}\rangle &= \langle\partial_{\boldsymbol{U}}\boldsymbol{Z},\boldsymbol{\nu}\rangle\langle\partial_{\boldsymbol{V}}\boldsymbol{W},\boldsymbol{\nu}\rangle - \langle\partial_{\boldsymbol{V}}\boldsymbol{Z},\boldsymbol{\nu}\rangle\langle\partial_{\boldsymbol{U}}\boldsymbol{W},\boldsymbol{\nu}\rangle\\ &= \langle\boldsymbol{Z},\partial_{\boldsymbol{U}}\boldsymbol{\nu}\rangle\langle\boldsymbol{W},\partial_{\boldsymbol{V}}\boldsymbol{\nu}\rangle - \langle\boldsymbol{Z},\partial_{\boldsymbol{V}}\boldsymbol{\nu}\rangle\langle\boldsymbol{W},\partial_{\boldsymbol{U}}\boldsymbol{\nu}\rangle\\ &= \langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{Z},\boldsymbol{U}\rangle\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{W},\boldsymbol{V}\rangle - \langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{Z},\boldsymbol{V}\rangle\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{W},\boldsymbol{U}\rangle. \end{aligned}\tag{4.25}$$

For the second equality in (4.25) we have used the fact that $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$, and $\boldsymbol{Z}$ are tangential, so in particular, $\partial_{\boldsymbol{U}}\langle\boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, $\partial_{\boldsymbol{V}}\langle\boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, $\partial_{\boldsymbol{V}}\langle\boldsymbol{W},\boldsymbol{\nu}\rangle = 0$, and $\partial_{\boldsymbol{U}}\langle\boldsymbol{W},\boldsymbol{\nu}\rangle = 0$ on $\mathscr{S}$.

Next, recall from (4.19) the definition of the Ricci tensor, i.e.

$$\mathrm{Ric}_{\mathscr{S}}(\boldsymbol{U},\boldsymbol{V}) = \sum_{j=1}^{n-1}\langle\boldsymbol{R}_{\mathscr{S}}(h_j,\boldsymbol{V})\boldsymbol{U},h_j\rangle,$$

where $h_1,\dots,h_{n-1}$ is, locally, an orthonormal basis in $\omega(\mathscr{S})$, and $\boldsymbol{U}$, $\boldsymbol{V}$ are arbitrary tangential vector fields to $\mathscr{S}$. If we set $h_n := \boldsymbol{\nu}$, and employ (4.25) together with $\mathscr{W}_{\mathscr{S}}\boldsymbol{\nu} = 0$, we obtain

$$\sum_{j=1}^{n-1}\langle\boldsymbol{R}_{\mathscr{S}}(T_j,\boldsymbol{V})\boldsymbol{U},T_j\rangle = \sum_{j=1}^{n}[\langle\boldsymbol{R}_{\mathscr{S}}T_j,T_j\rangle\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle - \langle\boldsymbol{R}_{\mathscr{S}}T_j,\boldsymbol{V}\rangle\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},T_j\rangle]$$

$$= -\mathscr{H}^0_{\mathscr{S}}\langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{V}\rangle - \langle\boldsymbol{R}_{\mathscr{S}}\boldsymbol{V},\sum_{j=1}^{n}\langle T_j,\boldsymbol{R}_{\mathscr{S}}\boldsymbol{U}\rangle T_j\rangle - \langle(\mathscr{W}^2_{\mathscr{S}} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}})\boldsymbol{U},\boldsymbol{V}\rangle, \tag{4.26}$$

which takes care of (4.20).

Finally, (4.21) is a consequence of what we have proved so far, in Lemma 2.6.ii, and the elementary identity $A^2 - (\mathrm{Tr}\,A)A = -(\det A)I$, valid for any 2×2 matrix A. ■

Lemma 4.3 *Let $H := \{h_j\}_{j=1}^n$, $|h_j| = 1$, be a basis in n-dimensional Banach space $\mathfrak{B}$. Consider the hyperspace $\mathfrak{B}_{\boldsymbol{\nu}} := \{u\in\mathfrak{B} : \langle u,\boldsymbol{\nu}\rangle = 0\}$ orthogonal to some vector $\boldsymbol{\nu}\in\mathfrak{B}$, $|\boldsymbol{\nu}|\neq 0$. Consider the system*

$$\hat{h}_j := h_j - \nu_j\boldsymbol{\nu}\,,\quad \nu_j := \langle\boldsymbol{\nu},h_j\rangle \qquad j = 1,\dots,n\,, \tag{4.27}$$

which is full in $\mathfrak{B}_{\boldsymbol{\nu}}$ but **linearly dependent** *and thus can not be a basis. Nevertheless, for a linear operator $A = [a_{jk}]_{n\times n} : \mathfrak{B}\to\mathfrak{B}$ with $A\boldsymbol{\nu} = 0$ and $A\mathfrak{B}_{\boldsymbol{\nu}}\subset\mathfrak{B}_{\boldsymbol{\nu}}$ (i.e., it maps $A : \mathfrak{B}_{\boldsymbol{\nu}}\to\mathfrak{B}$) we have*

$$\widehat{A} := [\widehat{a}_{jk}]_{n\times n} = [a_{jk}]_{n\times n} := A\,, \tag{4.28}$$

where $\widehat{A} := [\widehat{a}_{jk}]_{n\times n}$ is the matrix representations of A in the systems $\widehat{H} := \{\hat{h}_j\}_{j=1}^n\subset\mathfrak{B}_{\boldsymbol{\nu}}$.

Proof: Let us note that

$$\sum_{k=1}^{n} a_{jk}\nu_k = \sum_{k=1}^{n} a_{kj}\nu_k = 0 \qquad \text{for all}\quad j = 1,\dots,n\,,$$

where the first equality is equivalent to $A\boldsymbol{\nu} = 0$ and the second one-to $\langle \boldsymbol{\nu}, A\xi \rangle = 0$ for all $\xi \in \mathfrak{B}$. Applying the obtained equalities we find that

$$A\hat{h}_j = Ah_j - \nu_j A\boldsymbol{\nu} = \sum_{k=1}^{n} a_{kj} h_k = \sum_{k=1}^{n} a_{kj}\hat{h}_k + \sum_{k=1}^{n} a_{kj}\nu_k \boldsymbol{\nu} = \sum_{k=1}^{n} a_{kj}\hat{h}_k$$

which entails $\widetilde{a}_{kj} = a_{kj}$. ■

Theorem 4.4 *The following identities are valid:*

$$\Delta_{BL} = \pi_{\mathscr{S}} \boldsymbol{\Delta}_{\mathscr{S}} + \mathscr{W}_{\mathscr{S}}^2, \tag{4.29}$$

$$\Delta_{HL} = \pi_{\mathscr{S}} \boldsymbol{\Delta}_{\mathscr{S}} + 2\mathscr{W}_{\mathscr{S}}^2 - \mathscr{H}_{\mathscr{S}}^0 \mathscr{W}_{\mathscr{S}}. \tag{4.30}$$

Proof: In order to identify the Bochner-Laplacian operator Δ_{BL} on $\mathscr{S}$ we observe that, with tangential field $\boldsymbol{U}$ fixed, if the matrix $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})$ satisfies $\langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{V}, \boldsymbol{W} \rangle = \langle \partial^{\mathscr{S}}_{\pi_{\mathscr{S}}\boldsymbol{V}} \boldsymbol{U}, \pi_{\mathscr{S}} \boldsymbol{W} \rangle$, for each $\boldsymbol{V}, \boldsymbol{W} \in \mathbb{R}^n$ then, much as in the proof of Theorem 2.8

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\boldsymbol{e}^k, \boldsymbol{e}^j \rangle = \langle \partial_{\bar{e}_k} \boldsymbol{U}, \bar{e}_j \rangle = \mathscr{D}_k U_j - \sum_{r=1}^{n} \nu_j \nu_r \mathscr{D}_k(U_r). \tag{4.31}$$

On account of this we can now write

$$\begin{aligned}
\int_{\mathscr{S}} \langle (\nabla^{\mathscr{S}})^* \nabla^{\mathscr{S}} \boldsymbol{U}, \boldsymbol{V} \rangle \, d\sigma &= \int_{\mathscr{S}} \langle \nabla^{\mathscr{S}} \boldsymbol{U}, \nabla^{\mathscr{S}} \boldsymbol{V} \rangle \, d\sigma = \sum_{j,k=1}^{n-1} \int_{\mathscr{S}} \langle \nabla^{\mathscr{S}}_{T_j} \boldsymbol{U}, T_k \rangle \langle \nabla^{\mathscr{S}}_{T_j} \boldsymbol{V}, T_k \rangle \, d\sigma \\
&= \sum_{j,k=1}^{n} \int_{\mathscr{S}} \langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) T_j, T_k \rangle \langle \mathrm{Def}_{\mathscr{S}}(\boldsymbol{V}) T_j, T_k \rangle \, d\sigma = \sum_{j,k=1}^{n} \int_{\mathscr{S}} \mathfrak{D}_{jk}(\boldsymbol{U}) \mathfrak{D}_{jk}(\boldsymbol{V}) \, d\sigma \\
&= \sum_{j,k=1}^{n} \int_{\mathscr{S}} \Big[\mathscr{D}_k U_j \mathscr{D}_k V_j - \sum_{r=1}^{n} \nu_j \nu_r \mathscr{D}_j U_r \mathscr{D}_k V_j - \sum_{l=1}^{n} \nu_j \nu_l \mathscr{D}_k U_j \mathscr{D}_k V_l \\
&+ \sum_{r,l=1}^{n} \nu_r \nu_l \mathscr{D}_k U_r \mathscr{D}_k V_l \Big] \, d\sigma = \sum_{j,k=1}^{n} \int_{\mathscr{S}} \Big[(\mathscr{D}_k^* \mathscr{D}_k U_j) V_j - \sum_{r=1}^{n} U_r V_j (\partial_k \nu_r)(\partial_k \nu_j) \Big] \, d\sigma \\
&= \int_{\mathscr{S}} \langle -\boldsymbol{\Delta}_{\mathscr{S}} \boldsymbol{U} - \mathscr{W}_{\mathscr{S}}^2 \boldsymbol{U}, \boldsymbol{V} \rangle \, d\sigma.
\end{aligned} \tag{4.32}$$

In the next-to-the-last equality, we have applied the following identity to the terms under the integral sign in the fourth line above:

$$\sum_{r=1}^{n} \nu_r \mathscr{D}_s W_r = \mathscr{D}_s \Big(\sum_{r=1}^{n} \nu_r W_r \Big) - \sum_{r=1}^{n} W_r \mathscr{D}_s \nu_r = -\sum_{r=1}^{n} W_r \partial_s \nu_r, \quad \text{on} \quad \mathscr{S}, \tag{4.33}$$

valid for any tangential vector field $\boldsymbol{W}$, and any index $s \in \{1, \ldots, n\}$. In turn, the identity (4.33) can be seen from a direct computation (recall that $\partial_{\boldsymbol{\nu}} \nu_r = 0$ on $\mathscr{S}$). Finally, to justify the last equality in (4.32), it suffices to recall (2.35), (2.52) and the fact that $\sum\limits_{k=1}^{n} \nu_k \mathscr{D}_k = 0$.

The conclusion is that (4.29) holds. Finally, the identity (4.29) in concert with (4.18) and (4.20) implies (4.30). ■

Recall now from [EM70, *Note Added in Proof*, pp.161-162], [Ta92] (cf. also the remark at the end of this paper), and [Ta96, Vol. III], that the Navier-Stokes system for a velocity field $\boldsymbol{U}$, tangential to $\mathscr{S}$, and a (scalar-valued) pressure function p on $\mathscr{S}$ reads

$$\begin{aligned}\frac{\partial \boldsymbol{U}}{\partial t} - 2\,\mathrm{Def}^*_{\mathscr{S}}\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) + \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{U} - \nabla_{\mathscr{S}}p = f \quad \text{in} \quad \mathscr{S}\times(0,\infty),\\ \mathrm{div}_{\mathscr{S}}\boldsymbol{U} = 0, \quad \text{in} \quad \mathscr{S}.\end{aligned} \tag{4.34}$$

If $\mathscr{S}$ is embedded in $\mathbb{R}^n$ and the Riemannian metric is inherited from $\mathbb{R}^n$, a directional derivative $\partial_{\boldsymbol{U}}$ along a tangential vector field $\boldsymbol{U}\in\omega(\mathscr{S})$ maps the space of tangential vector fields to the space of possibly non-tangential vector fields

$$\partial_{\boldsymbol{U}}\,:\,\omega(\mathscr{S})\longrightarrow\omega(\mathscr{S})\,.$$

If composed with the projection

$$\partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{V} := \pi_{\mathscr{S}}\partial_{\boldsymbol{U}}\boldsymbol{V} = \partial_{\boldsymbol{U}}\boldsymbol{V} - \langle\boldsymbol{\nu},\partial_{\boldsymbol{U}}\boldsymbol{V}\rangle\boldsymbol{\nu} \tag{4.35}$$

(cf. (0.8)), it becomes an automorphism of the space of tangential vector fields. Such derivatives are compatible with the Riemannian metric on $\mathscr{S}$ and are torsion free as well. Therefore, they represent the natural Levi-Civita connection on $\mathscr{S}$.

Theorem 4.5 *The Navier-Stokes system* (4.34) *is equivalent to*

$$\begin{aligned}\frac{\partial \boldsymbol{U}}{\partial t} + \partial^{\mathscr{S}}_{\boldsymbol{U}}\boldsymbol{U} + \pi_{\mathscr{S}}\boldsymbol{\Delta}_{\mathscr{S}}\boldsymbol{U} + \mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U} - \nabla_{\mathscr{S}}p = f \quad \textit{in} \quad \mathscr{S}\times(0,\infty)\,,\\ \mathrm{div}_{\mathscr{S}}\boldsymbol{U} = 0 \quad \textit{in} \quad \mathscr{S}\,.\end{aligned} \tag{4.36}$$

Proof: This is a direct consequence of (4.15) and (4.35). ■

5 Lions' lemma and Korn's inequalities

For $1\leqslant p<\infty$, an integer $m=1,2,\dots$ and a closed C^{m+1}-smooth hypersurface $\mathscr{S}$ by $\mathbb{W}^m_p(\mathscr{S})$, $\mathbb{W}^m(\mathscr{S}) := \mathbb{W}^m_2(\mathscr{S})$ we denote the Sobolev spaces. The space $\mathbb{W}^{-m}_p(\mathscr{S})$ is defined as the dual to $\mathbb{W}^m_{p'}(\mathscr{S})$, $p' := \dfrac{p}{p-1}$, with respect to the sesquilinear form $(\varphi,\psi)_{\mathscr{S}}$ (cf. (2.49)) on functions $\varphi,\psi\in C^m(\mathscr{S})$ and extended by continuity to pairs $\varphi\in\mathbb{W}^m_{p'}(\mathscr{S})$ and $\psi\in\mathbb{W}^{-m}_p(\mathscr{S})$.

The embeddings $\mathbb{W}^m_p(\mathscr{S})\subset\mathbb{L}_p(\mathscr{S})\subset\mathbb{W}^{-m}_p(\mathscr{S})$ are continuous, even compact, and

$$\mathbb{W}^{-m}_p(\mathscr{S}) := \{\mathscr{D}^\alpha\varphi\,:\,\varphi\in\mathbb{L}_p(\mathscr{S})\text{ for all }\mathscr{D}^\alpha=\mathscr{D}^{\alpha_1}_1\cdots\mathscr{D}^{\alpha_n}_n\,,\quad|\alpha|=m\}\,.$$

If $\mathscr{S}$ is an open surface with the Lipschitz boundary $\Gamma=\partial\mathscr{S}\neq\emptyset$, $\widetilde{\mathbb{W}}^m_p(\mathscr{S})$ denotes the space of functions obtained by closing the space $C^\infty_0(\mathscr{S})$ of smooth functions with compact support in the norm of $\mathbb{W}^m_p(\widetilde{\mathscr{S}})$, where $\widetilde{\mathscr{S}}\supset\mathscr{S}$ is a closed surface which extends the surface $\mathscr{S}$. The notation $\mathbb{W}^m_p(\mathscr{S})$ is used for the factor space $\mathbb{W}^m_p(\widetilde{\mathscr{S}})/\widetilde{\mathbb{W}}^m_p(\widetilde{\mathscr{S}}\setminus\mathscr{S})$; the space $\mathbb{W}^m_p(\mathscr{S})$ can also be viewed as the restriction of all functions $\varphi\big|_{\mathscr{S}}$ of the space $\mathbb{W}^m_p(\widetilde{\mathscr{S}})$ to the subsurface $\mathscr{S}$ (cf. [Tr95] and [DS93] for details about these spaces spaces).

The following generalizes essentially J. L. Lions' Lemma (cf. [**?**, p.111], [Ta92], [AG1, Proposition 2.10], [Ci3, § 1.7], [Mc96]).

Lemma 5.1 *Let $\mathscr{S}$ be a 2-smooth closed hypersurface in $\mathbb{R}^n$. Then the inclusions $\varphi\in\mathbb{W}^{-1}_p(\mathscr{S})$, $\mathscr{D}_j\varphi\in\mathbb{W}^{-1}_p(\mathscr{S})$, for all $j=1,\dots,n$ imply $\varphi\in\mathbb{L}_p(\mathscr{S})$.*

Moreover, the assertion holds for a hypersurface $\mathscr{S}$ with the Lipschitz boundary $\Gamma:=\partial\mathscr{S}$ and the spaces $\mathbb{W}^{-1}_p(\mathscr{S})$ and $\widetilde{\mathbb{W}}^{-1}_p(\mathscr{S})$.

Proof: First we assume that $\mathscr{S}$ is a closed surface. The proof is based on the following facts from [DS93, Hr83, Ta96], which we recall without proofs.

A. There exists a "lifting operator" (a Bessel potential operator) $\Lambda(\mathscr{x},D)$, which has the inverse $\Lambda^{-1}(\mathscr{x},D)$ and they mapping isometrically the spaces

$$\Lambda^{-1}(\mathscr{x},D)\ :\ \mathbb{W}^{m-1}_p(\mathscr{S})\to\mathbb{W}^m_p(\mathscr{S}),\qquad \Lambda(\mathscr{x},D)\ :\ \mathbb{W}^m_p(\mathscr{S})\to\mathbb{W}^{m-1}_p(\mathscr{S})\tag{5.1}$$

for arbitrary $m=0,\pm1,\dots$.

B. $\Lambda^{-1}(\mathscr{x},D)$ is a pseudodifferential operator of order -1 and the commutant

$$[\mathscr{D}_j,\Lambda^{-1}(\mathscr{x},D)]:=\mathscr{D}_j\Lambda^{-1}(\mathscr{x},D)-\Lambda^{-1}(\mathscr{x},D)\mathscr{D}_j\tag{5.2}$$

with the pseudodifferential operator $\mathscr{D}_j$ has order -1, i.e., maps continuously the spaces

$$[\mathscr{D}_j,\Lambda^{-1}(\mathscr{x},D)]\ :\ \mathbb{W}^{-1}_p(\mathscr{S})\to\mathbb{L}_p(\mathscr{S}).$$

Let $\varphi\in\mathbb{W}^{-1}_p(\mathscr{S})$, $\mathscr{D}_j\varphi\in\mathbb{W}^{-1}_p(\mathscr{S})$, for all $j=1,\dots,n$. Then, due to (5.1), $\psi:=\Lambda^{-1}(\mathscr{x},D)\varphi\in\mathbb{L}_p(\mathscr{S})$ and, due to (5.2), $\mathscr{D}_j\psi=[\mathscr{D}_j,\Lambda^{-1}(\mathscr{x},D)]\varphi+\Lambda^{-1}(\mathscr{x},D)\mathscr{D}_j\varphi\in\mathbb{L}_p(\mathscr{S})$ for all $j=1,\dots,n$. From the definition of the space $\mathbb{W}^1_p(\mathscr{S})$ follows that $\psi\in\mathbb{W}^1_p(\mathscr{S})$. Due to (5.2) we get finally $\varphi=\Lambda(\mathscr{x},D)\psi\in\mathbb{L}_p(\mathscr{S})$.

If $\mathscr{S}$ has non-empty Lipschitz boundary $\Gamma\neq\emptyset$, there exist pseudodifferential operators

$$\begin{aligned}\Lambda^{-1}_-(\mathscr{x},D)\ &:\ \mathbb{W}^m_p(\mathscr{S})\to\mathbb{W}^{m+1}_p(\mathscr{S}),\\ \Lambda^{-1}_+(\mathscr{x},D)\ &:\ \widetilde{\mathbb{W}}^m_p(\mathscr{S})\to\widetilde{\mathbb{W}}^{m+1}_p(\mathscr{S}),\end{aligned}\tag{5.3}$$

arranging isomorphisms between the indicated spaces, and having the inverses $\Lambda^{-r}_-(\mathscr{x},D)$, $\Lambda^{-r}_+(\mathscr{x},D)$ (cf. [DS93]).

Moreover, pseudodifferential operators $\Lambda^{-1}_\pm(\mathscr{x},D)$ have order -1 and the commutants $[\mathscr{D}_j,\Lambda^{-1}_\pm(\mathscr{x},D)]:=\mathscr{D}_j\Lambda^{-1}_\pm(\mathscr{x},D)-\Lambda^{-1}_\pm(\mathscr{x},D)\mathscr{D}_j$ have order -1, i.e., map continuously the spaces $\mathbb{W}^{-1}_p(\mathscr{S})\to\mathbb{L}_p(\mathscr{S})$.

By using the formulated assertions the proof is completed as in the case of a closed surface $\mathscr{S}$. ■

The foregoing Lemma 5.1 has the following generalization for the Bessel potential spaces $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$ and $\mathbb{H}^s_p(\mathscr{S})$ (see [Tr95] and [DS93] for details about these spaces spaces).

Lemma 5.2 *If $\mathscr{S}$ is closed, sufficiently smooth,* $1 < p < \infty$*,* $s \in \mathbb{R}$*,* $m = 1, 2, \ldots$ *and*

$$\varphi \in \mathbb{H}^{s-m}_p(\mathscr{S}), \quad \mathscr{D}^\alpha \varphi = \mathscr{D}^{\alpha_1}_1 \cdots \mathscr{D}^{\alpha_n}_n \varphi \in \mathbb{H}^{s-m}_p(\mathscr{S}) \quad \textit{for all} \quad |\alpha| \leqslant m,$$

then $\varphi \in \mathbb{H}^s_p(\mathscr{S})$*.*

Moreover, the assertion holds for a hypersurface $\mathscr{S}$ *with the Lipschitz boundary* $\Gamma := \partial\mathscr{S}$ *and the spaces* $\mathbb{H}^s_p(\mathscr{S})$ *and* $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$*.*

Proof: Assume first $\mathscr{S}$ has no boundary. The proof is based, as in the foregoing case, on the following facts from [Hr83, Ta96, Tr95], which we recall without proofs.

A. There exists a "lifting operator" (the Bessel potential operator),

$$\Lambda^r(\mathscr{x}, D) \ : \ \mathbb{H}^s_p(\mathscr{S}) \to \mathbb{H}^{s-r}_p(\mathscr{S}), \quad r \in \mathbb{R} \tag{5.4}$$

arranging isomorphism between the indicated spaces, and having the inverse $\Lambda^{-r}(\mathscr{x}, D)$.

B. $\Lambda^r(\mathscr{x}, D)$ is a pseudodifferential operator of order $-r$ and the commutant

$$[\mathscr{D}^\alpha, \Lambda^r(\mathscr{x}, D)] := \mathscr{D}^\alpha \Lambda^r(\mathscr{x}, D) - \Lambda^r(\mathscr{x}, D)\mathscr{D}^\alpha \tag{5.5}$$

with the pseudodifferential operator $\mathscr{D}^\alpha = \mathscr{D}^{\alpha_1}_1 \cdots \mathscr{D}^{\alpha_n}_n$ has order $|\alpha| + r - 1$, i.e., maps continuously the spaces $\mathbb{H}^\gamma_p(\mathscr{S}) \to \mathbb{H}^{\gamma-|\alpha|-r+1}_p(\mathscr{S})$, $\forall\, \gamma \in \mathbb{R}$.

Assume that $m = 1$. Then $\varphi \in \mathbb{H}^{s-1}_p(\mathscr{S})$ and, due to (5.4), (5.5), it follows that $\psi := \Lambda^{s-1}_{\mathscr{S}}(\mathscr{x}, D)\varphi \in \mathbb{L}_p(\mathscr{S})$, $\mathscr{D}_j\psi = [\mathscr{D}_j, \Lambda^{s-1}_{\mathscr{S}}(\mathscr{x}, D)]\varphi + \Lambda^{s-1}_{\mathscr{S}}(\mathscr{x}, D)\mathscr{D}_j\varphi \in \mathbb{L}_p(\mathscr{S})$ for all $j = 1, \ldots, n$. By the definition of the space $\mathbb{W}^1_p(\mathscr{S})$ we conclude that $\psi \in \mathbb{W}^1_p(\mathscr{S})$. Due to (5.2) we get finally $\varphi = \Lambda^{1-s}(\mathscr{x}, D)\psi \in \mathbb{H}^s_p(\mathscr{S})$.

Now assume: $m = 2, 3, \ldots$ and the assertion is valid for $m - 1$. Then, due to the hypothesis, $\psi_j := \mathscr{D}_j\varphi \in \mathbb{H}^{s-m}_p(\mathscr{S})$ for $j = 1, \ldots, n$. Moreover, due to the same hypothesis,

$$\mathscr{D}^\alpha\psi_j := \mathscr{D}^\alpha\mathscr{D}_j\varphi \in \mathbb{H}^{s-m}_p(\mathscr{S}) \quad \text{for all} \quad |\alpha| \leqslant m-1 \quad \text{and all} \quad j = 1, \ldots, n.$$

Hence the induction hypothesis implies that $\psi_j := \mathscr{D}_j\varphi \in \mathbb{H}^{s-1}_p(\mathscr{S})$ for $j = 1, \ldots, n$. Now it follows from the already considered case $m = 1$ that $\varphi \in \mathbb{H}^s_p(\mathscr{S})$.

If $\mathscr{S}$ has non-empty Lipschitz boundary $\Gamma \neq \emptyset$, there exist pseudodifferential operators

$$\Lambda^r_-(\mathscr{x}, D) \ : \ \mathbb{H}^s_p(\mathscr{S}) \to \mathbb{H}^{s-r}_p(\mathscr{S}), \qquad \Lambda^r_+(\mathscr{x}, D) \ : \ \widetilde{\mathbb{H}}^s_p(\mathscr{S}) \to \widetilde{\mathbb{H}}^{s-r}_p(\mathscr{S}), \tag{5.6}$$

arranging isomorphisms between the indicated spaces, and having the inverses $\Lambda^{-r}_-(\mathscr{x}, D)$, $\Lambda^{-r}_+(\mathscr{x}, D)$ (cf. [DS93]).

Moreover, the pseudodifferential operators $\Lambda^{-r}_\pm(\mathscr{x}, D)$ have order $-r$ and the commutants $[\mathscr{D}^\alpha, \Lambda^{-r}_\pm(\mathscr{x}, D)] := \mathscr{D}^\alpha\Lambda^{-1}_\pm(\mathscr{x}, D) - \Lambda^{-r}_\pm(\mathscr{x}, D)\mathscr{D}^\alpha$ have order $|\alpha| - r - 1$, i.e., map continuously the spaces $\mathbb{H}^\gamma_p(\mathscr{S}) \to \mathbb{H}^{\gamma+r+1-|\alpha|}_p(\mathscr{S})$.

By using the formulated assertions the proof is completed as in the foregoing cases. ∎

Theorem 5.3 (Korn's I inequality "without boundary condition"). *Let $\mathscr{S} \subset \mathbb{R}^n$ be a Lipshitz hypersurface without boundary,* $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := [\mathfrak{D}_{jk}(\boldsymbol{U})]_{n\times n}$ *be the deformation tensor*

$$\mathfrak{D}_{jk}(\boldsymbol{U}) = \frac{1}{2}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \partial_{\boldsymbol{U}}\big(\nu_j\nu_k\big)\Big] = \frac{1}{2}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k + \sum_{m=1}^{n} U_m\mathscr{D}_m\big(\nu_j\nu_k\big)\Big]$$

(cf. (4.2)*) and*

$$\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\| := \left[\sum_{j,k=1}^{n}\big\|\mathfrak{D}_{jk}\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p\right]^{1/p}, \qquad \boldsymbol{U}\in\mathbb{W}^1_p(\mathscr{S}) \tag{5.7}$$

for $1 < p < \infty$. Then

$$\big\|\boldsymbol{U}\big|\mathbb{W}^1_p(\mathscr{S})\big\| \leqslant M\,\Big[\big\|\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p} \tag{5.8}$$

for some constant $M > 0$ or, equivalently, the mapping

$$\boldsymbol{U} \mapsto \Big[\big\|\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p} \tag{5.9}$$

is an equivalent norm on the space $\mathbb{W}^1_p(\mathscr{S})$.

Proof: Consider the space

$$\widehat{\mathbb{W}}^1_p(\mathscr{S}) := \Big\{\boldsymbol{U} = \big(U^0_1\dots,U^0_n\big)^\top\;:\;U^0_j\,,\,\mathfrak{D}_{jk}(\boldsymbol{U})\in\mathbb{L}_p(\mathscr{S})\ \textit{for all}\ j,k=1,\dots n\Big\} \tag{5.10}$$

endowed with the norm (cf. (5.8) and (5.9)):

$$\big\|\boldsymbol{U}\big|\widehat{\mathbb{W}}^1_p(\mathscr{S})\big\| := \Big[\big\|\boldsymbol{U}\big|\mathbb{L}_p(\mathscr{S})\big\|^p + \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_p(\mathscr{S})\big\|^p\Big]^{1/p}. \tag{5.11}$$

The derivatives here are understood in the sense of distributions: $\mathfrak{D}_{jk}(\boldsymbol{U})\in\mathbb{L}_p(\mathscr{S})$ means that there exists a function in $\mathbb{L}_p(\mathscr{S})$ denoted by $\mathfrak{D}_{jk}(\boldsymbol{U})$ such that

$$\begin{aligned}(\mathfrak{D}_{jk}(\boldsymbol{U}),\boldsymbol{V})_{\mathscr{S}} := \int_{\mathscr{S}}\Big[&U_j(\mathscr{x})\overline{\mathscr{D}^*_k V(\mathscr{x})} + U_k(\mathscr{x})\overline{\mathscr{D}^*_j V(\mathscr{x})}\\ &+\sum_{m=1}^{n} V(\mathscr{x})U_m(\mathscr{x})\mathscr{D}^*_m\big(\nu_j(\mathscr{x})\nu_k(\mathscr{x})\big)\Big]\,d\sigma \qquad \forall\,\boldsymbol{V}\in\mathbb{L}_p(\mathscr{S}),\end{aligned}$$

(cf. (2.52) for the formal dual $\mathscr{D}^*_m$).

It is obviously sufficient to prove, that the spaces $\mathbb{W}^1_p(\mathscr{S})$ and $\widehat{\mathbb{W}}^1_p(\mathscr{S})$ are identical. The inclusion $\mathbb{W}^1_p(\mathscr{S}) \subset \widehat{\mathbb{W}}^1_p(\mathscr{S})$ is trivial and we concentrate on the proof of the inverse inclusion $\widehat{\mathbb{W}}^1_p(\mathscr{S}) \subset \mathbb{W}^1_p(\mathscr{S})$.

To this end take $\boldsymbol{U} \in \widehat{\mathbb{W}}^1_p(\mathscr{S})$ and note that the inclusions $\boldsymbol{U} \in \mathbb{L}_p(\mathscr{S})$, $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) \in \mathbb{L}_p(\mathscr{S})$ (i.e., $\mathfrak{D}_{jk}\boldsymbol{U} \in \mathbb{L}_p(\mathscr{S})$ for all $j,k = 1,\dots,n$) imply

$$\mathfrak{D}^0_{jk}(\boldsymbol{U}) = \frac{1}{2}\Big[\mathscr{D}_k U_j + \mathscr{D}_j U_k\Big] = \mathfrak{D}_{jk}(\boldsymbol{U}) - \frac{1}{2}\sum_{r=1}^n \partial_r\big(\nu_j\nu_k\big)U_r \in \mathbb{L}_p(\mathscr{S}) \qquad (5.12)$$
$$\text{for all} \quad j,k = 1,\dots,n\,.$$

Then (cf. [DMM06, Proposition 4.4.iv] for the commutator $[\mathscr{D}_j, \mathscr{D}_k]$):

$$\mathscr{D}_j U_k \in \mathbb{H}^{-1}_p(\mathscr{S}) \qquad \big[\mathscr{D}_j,\mathscr{D}_k\big]U_m = \sum_{r=1}^n \big[\nu_j\mathscr{D}_k\nu_r - \nu_k\mathscr{D}_j\nu_r\big]\mathscr{D}_r U_m \in \mathbb{H}^{-1}_p(\mathscr{S}),$$
$$\mathscr{D}_k\mathscr{D}_j U_m = \mathscr{D}_j\widetilde{\mathfrak{D}}_{km}(\boldsymbol{U}) + \mathscr{D}_k\widetilde{\mathfrak{D}}_{jm}(\boldsymbol{U}) - \mathscr{D}_m\widetilde{\mathfrak{D}}_{jk}(\boldsymbol{U}) - \frac{1}{2}\big[\mathscr{D}_j,\mathscr{D}_k\big]U_m$$
$$-\frac{1}{2}\big[\mathscr{D}_j,\mathscr{D}_m\big]U_k - \frac{1}{2}\big[\mathscr{D}_k,\mathscr{D}_m\big]U_j \in \mathbb{H}^{-1}_p(\mathscr{S}) \quad \text{for} \quad j,k,m = 1,\dots,n\,,$$

Due to Lemma 5.1 of J. L. Lions this implies $\mathscr{D}_j U_m \in \mathbb{L}_p(\mathscr{S})$ for all $j,m = 1,\dots,n$ and the claimed result $\boldsymbol{U} \in \mathbb{W}^1_p(\mathscr{S})$ follows. ■

Remark 5.4 *The foregoing Theorem 5.3 is proved by P. Ciarlet in [Ci3] for the case $p = 2$, for curvilinear coordinates and covariant derivatives.*

A remarkable consequence of Korn's inequality (5.8) *is that the space*

$$\mathbb{W}^1_p(\mathscr{S}) := \Big\{\boldsymbol{U} = \big(U_1\dots,U_n\big)^\top \;:\; U_j\,,\ \mathscr{D}_k U_j \in \mathbb{L}_p(\mathscr{S}) \ \text{ for all } j,k = 1,\dots n\Big\}$$

and the space $\widehat{\mathbb{W}}^1_p(\mathscr{S})$ (cf. (5.10)*) are isomorphic (i.e. can be identified), although only $\dfrac{n(n+1)}{2} < n^2$ linear combinations of the n^2 derivatives $\mathscr{D}_j U_k$, $j,k = 1,\dots n$ participate in the definition of the space $\widehat{\mathbb{H}}^1_p(\mathscr{S})$.*

6 KILLING'S VECTOR FIELDS AND FURTHER KORN'S INEQUALITIES

Definition 6.1 *Let $\mathscr{S}$ be a hypersurface in the Euclidean space $\mathbb{R}^n$. The space $\mathscr{R}(\mathscr{S})$ of solutions to the deformation equations*

$$\mathfrak{D}_{jk}(\boldsymbol{U}) := \frac{1}{2}\big[(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})^0_k + (\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})^0_j\big]$$
$$= \frac{1}{2}\Big[\mathscr{D}_k U^0_j + \mathscr{D}_j U^0_k + \sum_{m=1}^n U^0_m\mathscr{D}_m\big(\nu_j\nu_k\big)\Big] = 0\,, \qquad (6.1)$$
$$\boldsymbol{U} = \sum_{j=1}^n U^0_j\boldsymbol{d}^j \in \omega(\mathscr{S}), \qquad j,k = 1,\dots,n$$

is called the space of **Killing's vector fields***.*

Killing's vector fields on a domain in the Euclidean space $\Omega \subset \mathbb{R}^n$ are known as the **rigid motions** and we start with this simplest class.

The space of rigid motions $\mathscr{R}(\Omega)$ extends naturally to the entire $\mathbb{R}^n$ and consists of linear vector-functions

$$\boldsymbol{V}(x) = a + \mathscr{B}x\,, \qquad \mathscr{B} = \big[b_{jk}\big]_{n\times n}, \qquad a \in \mathbb{R}^n\,, \quad x \in \mathbb{R}^n\,, \tag{6.2}$$

where the matrix $\mathscr{B}$ is skew symmetric

$$\mathscr{B} := \begin{bmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1(n-2)} & b_{1(n-1)} \\ -b_{12} & 0 & b_{21} & \cdots & b_{1(n-3)} & b_{2(n-2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -b_{1(n-2)} & -b_{2(n-3)} & -b_{3(n-4)} & \cdots & 0 & b_{(n-1)1} \\ -b_{1(n-1)} & -b_{2(n-2)} & -b_{3(n-3)} & \cdots & -b_{(n-1)1} & 0 \end{bmatrix} = -\mathscr{B}^\top \tag{6.3}$$

with real valued entries $b_{jk} \in \mathbb{R}$. For $n = 3, 4, \dots$ the space $\mathscr{R}(\mathbb{R}^n)$ is finite dimensional and $\dim \mathscr{R}(\mathbb{R}^n) = n + \dfrac{n(n-1)}{2} = \dfrac{n(n+1)}{2}$.

Note that for $n = 3$ the vector field $\boldsymbol{V} \in \mathscr{R}(\Omega)$, $\Omega \subset \mathbb{R}^3$, is the classical rigid displacement

$$\begin{aligned} &\boldsymbol{V}(x) = a + \mathscr{B}x = a + b \wedge x\,, \\ &b := (b_1, b_2, b_3)^\top \in \mathbb{R}^3\,, \quad x \in \Omega\,, \end{aligned} \qquad \mathscr{B} := \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}. \tag{6.4}$$

Definition 6.2 *We call a subset $\mathscr{M} \subset \mathbb{R}^n$* **essentially** m**-dimensional** *and write* $\operatorname{ess\,dim} \mathscr{M} = m$*, if there exist $m+1$ points $\mathscr{X}^0, \mathscr{X}^1, \cdots, \mathscr{X}^m \in \mathscr{M}$ such that the vectors $\big\{\mathscr{X}^j - \mathscr{X}^0\big\}_{j=1}^m$ are linearly independent.*

Note, that any m-dimensional subset $\mathscr{M} \subset \mathbb{R}^m$ is essentially m-dimensional, because contains m linearly independent vectors. Moreover, any collection of $m+1$ points in $\mathbb{R}^m$ (a 0-dimensional subset) is essentially m-dimensional, provided these points does not belong to any $m-1$ dimensional hyperplane.

Lemma 6.3 *Let*

$$\operatorname{Def}(\boldsymbol{U}) := \big[\mathfrak{D}^0_{jk}(\boldsymbol{U})\big]_{n\times n}, \tag{6.5}$$

$$\mathfrak{D}^0_{jk}(\boldsymbol{U}) = \frac{1}{2}\Big[\partial_k U^0_j + \partial_j U^0_k\Big], \qquad \boldsymbol{U} = \sum_{j=1}^n U^0_j \boldsymbol{e}^j.$$

be the deformation tensor in Cartesian coordinates.

The linear space $\mathscr{R}(\mathbb{R}^n)$ of rigid motions (of Killing's vector fields) in $\mathbb{R}^n$ consists of vector fields $\mathbf{K} = (K^0_1, \dots, K^0_n)^\top$ which are solutions to the system

$$2\mathfrak{D}^0_{jk}(\mathbf{K})(x) = \partial_k K^0_j(x) + \partial_j K^0_k(x) = 0 \qquad x \in \mathscr{S} \quad \text{for all} \quad j, k = 1, \dots, n. \tag{6.6}$$

If a rigid motion vanishes on an essentially $(n-1)$-dimensional subset $\mathbf{K}(\mathscr{X}) = 0$ for all $\mathscr{X} \in \mathscr{M}$, $\operatorname{ess\,dim} \mathscr{M} = n-1$, or at infinity $\mathbf{K}(x) = \mathscr{o}(1)$ as $|x| \to \infty$, then $\mathbf{K}$ vanishes identically $\mathbf{K}(x) \equiv 0$ on $\mathbb{R}^n$.

Proof: By differentiating (6.6) and recalling that $\partial_k\partial_l K^0_j = \partial_l\partial_k K^0_j$, we get

$$\partial_j\partial_k K^0_m = \partial_j\mathfrak{D}^0_{km}(\mathbf{K}) + \partial_k\mathfrak{D}^0_{jm}(\mathbf{K}) - \partial_m\mathfrak{D}^0_{jk}(\mathbf{K}) = 0 \quad \text{for all } j,k,m = 1,2,\ldots,n-1\,.$$

Therefore,

$$K^0_j(x) = a_j + b_{j1}x_1 + \cdots + b_{jn}x_n \qquad j = 1,2,\ldots,n$$

or

$$\mathbf{K}(x) = a + \mathscr{B}\,x \qquad \text{with} \quad \mathscr{B} = [b_{jk}]_{n\times n}\,. \tag{6.7}$$

From (6.6) we derive that $\mathscr{B}$ is a skew symmetric matrix (cf. (6.3)):

$$\partial_j K^0_k(x) = -\partial_k K^0_j(x) \equiv 0 \implies b_{jk} = -b_{kj} \quad j,k = 1,2,\ldots,n \implies \mathscr{B} = -\mathscr{B}^\top.$$

The inclusion $\mathbf{K} \in \mathscr{R}(\mathbb{R}^n)$ is proved.

The inverse statement, that any vector field $\mathbf{K} \in \mathscr{R}(\mathbb{R}^n)$ (of the form (6.2)) is a solution of the system (6.6), is easy to verify.

Let us prove the second assertion: for any linearly independent vectors $x^0,\ldots,x^{n-1}$ the condition

$$\mathbf{K}(x^k) = 0 \quad \implies \quad a + \mathscr{B}x^k = \mathbf{K}(x^k) = 0 \tag{6.8}$$

implies $a = 0$ and $\mathscr{B} = 0$, i.e., $\mathbf{K}(x) = 0$ for all $x \in \mathbb{R}^n$. Indeed, if $\mathscr{B} = 0$ then, obviously, $a = 0$. Accepting $\mathscr{B} \neq 0$, for rank of $\mathscr{B}$ we have the estimate $2 \leqslant \operatorname{rank}\mathscr{B} < n$ (if $\mathscr{B} \neq 0$ then, due to the symmetry $\mathscr{B} = -\mathscr{B}^\top$, there exists a non-degenerate minor of order at least 2). On the other hand, from (6.8) follows

$$\mathscr{B}(x^k - x^0) = 0 \qquad \forall\, k = 1,\ldots,n-1\,,$$

which contradicts the estimate $2 \leqslant \operatorname{rank}\mathscr{B} < n$ since $\left\{x^1 - x^0,\ldots,x^{n-1} - x^0\right\}$ are linearly independent.

If a rigid motion $\mathbf{K}(x)$ in (6.7) vanishes at infinity $\mathbf{K}(x) = \mathscr{O}(1)$ as $|x| \to \infty$, then obviously $a = 0$, $\mathscr{B} = 0$ and, therefore, $\mathbf{K}(x) = 0$ for all $x \in \mathbb{R}^n$. ■

Remark 6.4 *For the deformation tensor in Cartesian coordinates* $\operatorname{Def}(\boldsymbol{U})$ *(cf.* (6.5)*) in a domain* $\Omega \subset \mathbb{R}^n$ *Korn's inequality*

$$\left\|\boldsymbol{U}\middle|\mathbb{H}^1_p(\Omega)\right\| \leqslant M\left[\left\|\boldsymbol{U}\middle|\mathbb{L}_p(\Omega)\right\|^P + \left\|\operatorname{Def}(\boldsymbol{U})\middle|\mathbb{L}_p(\Omega)\right\|^p\right]^{1/p}, \quad 1 < p < \infty \tag{6.9}$$

with some constant $M > 0$ *is well known and is proved e.g. in [Ci2] (cf.* (5.7) *for a similar norm).*

In contrast to the rigid motions in $\mathbb{R}^n$ nobody can identify Killing's vector fields on hypersurfaces explicitly so far. The next Theorem 6.5 underlines importance of Killing's vector fields for the Lamé equation on hypersurfaces. Later we investigate properties of Killing's vector fields to prepare tools for investigations of boundary value problems for the Lamé equation.

Theorem 6.5 *Let $\mathscr{S}$ be an ℓ-smooth closed hypersurface in $\mathbb{R}^n$ and $\ell \geqslant 2$. The Lamé operator $\mathscr{L}_{\mathscr{S}}$ for an isotropic media*

$$\mathscr{L}_{\mathscr{S}} \ : \ \mathbb{H}^{s+1}_p(\mathscr{S}) \to \mathbb{H}^{s-1}_p(\mathscr{S}), \tag{6.10}$$
$$\mathscr{L}_{\mathscr{S}}\boldsymbol{U} = \mu\pi_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\,\nabla_{\mathscr{S}}\boldsymbol{U} + (\lambda+\mu)\,\nabla_{\mathscr{S}}\mathrm{div}_{\mathscr{S}}\boldsymbol{U} + \mu\,\mathscr{H}^0_{\mathscr{S}}\mathscr{W}_{\mathscr{S}}\boldsymbol{U}\ ,$$

is self adjoint $\mathscr{L}^*_{\mathscr{S}} = \mathscr{L}_{\mathscr{S}}$, elliptic, Fredholm and index $\mathrm{Ind}\,\mathscr{L}_{\mathscr{S}} = 0$ for all $1 < p < \infty$ and all $s \in \mathbb{R}$, provided that $|s| \leqslant \ell$.

The kernel of the operator $\mathrm{Ker}\mathscr{L}_{\mathscr{S}} \subset \mathbb{H}^s_p(\mathscr{S})$ is independent of the parameters p and s, coincides with the space of Killing's vector fields

$$\mathrm{Ker}\,\mathscr{L}_{\mathscr{S}} = \{\boldsymbol{U} \in \omega(\mathscr{S})\ :\ \mathscr{L}_{\mathscr{S}}\boldsymbol{U} = 0\} = \mathscr{R}(\mathscr{S}), \tag{6.11}$$

is finite dimensional and $\dim\mathscr{R}(\mathscr{S}) = \dim\mathrm{Ker}\,\mathscr{L}_{\mathscr{S}} < \infty$.

If $\mathscr{S}$ is C^∞ smooth, then the Killing's vector fields are smooth as well $\mathscr{R}(\mathscr{S}) \subset C^\infty(\mathscr{S})$.

$\mathscr{L}_{\mathscr{S}}$ is non-negative on the space $\mathbb{H}^1(\mathscr{S})$ and positive definite on the orthogonal complement $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ to the kernel

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geqslant 0 \qquad \text{for all} \quad \boldsymbol{U} \in \mathbb{H}^1(\mathscr{S})\,, \tag{6.12}$$

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geqslant C\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\|^2 \qquad \text{for all} \quad \boldsymbol{U} \in \mathbb{H}^1_{\mathscr{R}}(\mathscr{S})\,, \qquad C>0\,, \tag{6.13}$$

where $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ is the orthogonally complemented subspace to $\mathscr{R}(\mathscr{S})$ in $\mathbb{H}^1(\mathscr{S})$.

Moreover, the following Gåarding's inequality

$$(\mathscr{L}_{\mathscr{S}}\boldsymbol{U},\boldsymbol{U})_{\mathscr{S}} \geqslant C_1\|\boldsymbol{U}|\mathbb{H}^1(\mathscr{S})\|^2 - C_0\|\boldsymbol{U}|\mathbb{H}^{-r}(\mathscr{S})\|^2 \tag{6.14}$$

holds for all $\boldsymbol{U} \in \mathbb{H}^1(\mathscr{S})$, with any $-1 < r \leqslant \ell$ and some positive constants $C_0 > 0$, $C_1 > 0$.

Proof: The proof is exposed in [Du11]. Here we draw the following consequence.

Corollary 6.6 *Let $\mathscr{S} \subset \mathbb{R}^n$ be a Lipshitz hypersurface without boundary, $\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}) := \big[\mathfrak{D}_{jk}(\boldsymbol{U})\big]_{n\times n}$ be the deformation tensor*

$$\begin{aligned}\mathfrak{D}^0_{jk}(\boldsymbol{U}) = (\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U}))_{jk} &= \frac{1}{2}\big[(\mathscr{D}^{\mathscr{S}}_k\boldsymbol{U})_j + (\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k\big]\\ &= \frac{1}{2}\big[\mathscr{D}_j U^0_k + \mathscr{D}_k U^0_j + \partial_{\boldsymbol{U}}(\nu_j\nu_k)\big]\,, \quad \forall\, j,k = 1,\ldots,n.\end{aligned} \tag{6.15}$$

where $(\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U})_k$ denotes the $k-th$ component of the covariant derivative $\mathscr{D}^{\mathscr{S}}_j\boldsymbol{U}$. The norm $\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\|$ is defined by (5.7).

Then the following Korn's inequality

$$\big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\| \geqslant c\big\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\big\| \qquad \forall\,\boldsymbol{U} \in \mathbb{H}^1_{\mathscr{R}}(\mathscr{S}) \tag{6.16}$$

holds for some constant $c > 0$ or, equivalently, the mapping $\boldsymbol{U} \mapsto \big\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\big\|$ is an equivalent norm on the orthogonal complement $\mathbb{H}^1_{\mathscr{R}}(\mathscr{S})$ to the space of Killing's vector fields.

Proof: Due to Korn's inequality (5.8) for $p = 2$

$$\left\|\boldsymbol{U}\big|\mathbb{L}_2(\mathscr{S})\right\|^2 \geqslant M_1\Big[\left\|\boldsymbol{U}\big|\mathbb{H}^1(\mathscr{S})\right\|^2 - \left\|\mathrm{Def}_{\mathscr{S}}(\boldsymbol{U})\big|\mathbb{L}_2(\mathscr{S})\right\|^2\Big]$$

the mapping $\mathrm{Def}_{\mathscr{S}} : \mathbb{H}^1_{\mathscr{R}}(\mathscr{S}) \to \mathbb{L}_2(\mathscr{S})$ is Fredholm and has index 0. The inequality (6.16) follows since the mapping is injective (has an empty kernel). ■

Let us recall some results related to the uniqueness of solutions to arbitrary elliptic equation.

Definition 6.7 *Let Ω be an open subset with the Lipschitz boundary $\partial\Omega \neq \emptyset$ either on a Lipschitz hypersurface $\mathscr{S} \subset \mathbb{R}^n$ or in the Euclidean space $\mathbb{R}^{n-1}$.*

A class of functions $\mathscr{U}(\Omega)$ defined in a domain Ω in $\mathbb{R}^n$, is said to have the **strong unique continuation property***, if every $u \in \mathscr{U}(\Omega)$ in this class which vanishes to infinite order at one point must vanish identically.*

If a surface $\mathscr{S}$ is C^∞-smooth, any elliptic operator on $\mathscr{S}$ has the strong unique continuation property due to Holmgren's theorem. But we can have more.

Lemma 6.8 *Let $\mathscr{S}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$. The class of solutions to a second order elliptic equation $\mathbb{A}(\mathscr{X}, \mathscr{D})u = 0$, with Lipschitz continuous top order coefficients on a surface $\mathscr{S}$ has the strong unique continuation property.*

In particular, if the solution $u(\mathscr{X}) = 0$ vanishes in any open subset of $\mathscr{S}$ it vanishes identically on entire $\mathscr{S}$.

Proof: The result was proved in [AKS1] for a domain $\Omega \subset \mathbb{R}^n$ by the method of "Carleman estimates" (also see [Hr83, Volume 3, Theorem 17.2.6]). Another proof, involving monotonicity of the frequency function was discovered by N. Garofalo and F. Lin (see [GL1, GL2]). A differential equation $\mathbb{A}(\mathscr{X}, \mathscr{D})u(\mathscr{X}) = 0$ with Lipschitz continuous top order coefficients on a $\mathbb{W}^2_\infty$-smooth surface $\mathscr{S}$ is locally equivalent to a differential equation with Lipschitz continuous top order coefficients on a domain $\Omega \subset \mathbb{R}^{n-1}$. Therefore a solution $u(\mathscr{X})$ has the strong unique continuation property locally (on each coordinate chart) on $\mathscr{S}$.

Since $\mathscr{S}$ is covered by a finite number of local coordinate charts which intersect on open neighborhoods, a solution $u(\mathscr{X})$ has the strong unique continuation property globally on $\mathscr{S}$. ■

Remark 6.9 *If the top order coefficients of a second order elliptic equation $\mathbb{A}(\mathscr{X}, \mathscr{D})u = 0$ in open subsets $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, are merely Hölder continuous, with exponent less than 1, examples due to A. Plis [Pl63] and K. Miller [Mi03] show that a solution $u(x)$ does not have the strong unique continuation property.*

Lemma 6.10 *Let $\mathscr{C}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$ with the Lipschitz boundary $\Gamma := \partial\mathscr{C}$ and $\gamma \subset \Gamma$ be an open part of the boundary Γ. Let $\mathbb{A}(\mathscr{X}, \mathscr{D})$ be a second order elliptic system with Lipschitz continuous top order matrix coefficients on a surface $\mathscr{S}$.*

The Cauchy problem

$$\begin{cases} \mathbb{A}(\mathscr{X}, \mathscr{D})u = 0 & \text{on} \quad \mathscr{C}, \quad u \in \mathbb{H}^1(\Omega), \\ u(\mathfrak{s}) = 0 & \textit{for all} \quad \mathfrak{s} \in \gamma, \\ (\partial_{\boldsymbol{V}} u)(\mathfrak{s}) = 0 & \textit{for all} \quad \mathfrak{s} \in \gamma, \end{cases} \tag{6.17}$$

where the vector $\boldsymbol{V}$ is a non-tangential to Γ, but tangential to $\mathscr{S}$, has only a trivial solution $u(\mathscr{x})=0$ on entire $\mathscr{S}$.

Proof: With a local diffeomorphism the Cauchy problem (6.17) is transformed into a similar problem on a domain $\Omega\subset\mathbb{R}^{n-1}$ with the Cauchy data vanishing on some open subset of the boundary $\gamma\subset\Gamma:=\partial\Omega$.

Let us, for simplicity, use the same notation $\gamma\subset\Gamma=\partial\Omega$, the non-tangential vector $\boldsymbol{V}$ to γ, the function u and the differential operator $\mathbb{A}(x,\mathscr{D})$ for the transformed Cauchy problem in the transformed domain Ω. Moreover, we will suppose that γ is a part of the hypersurface $x_1=0$ (otherwise we can transform the domain Ω again). We also use new variables $t=x_1$ and $x:=(x_2,\dots,x_{n-1})$. Then $(0,x)\in\gamma$ while $(t,x)\in\Omega$ for all small $0<t<\varepsilon$ and some $x\in\Omega'$.

Thus, the natural basis element $\boldsymbol{e}^1$ (cf. (0.7)) is orthogonal to γ and, therefore, $\boldsymbol{e}^1=c_1(x)\boldsymbol{V}(x)+c_2(x)\boldsymbol{g}(x)$ for some unit tangential vector $\boldsymbol{g}(x)$ to γ for some scalar functions $c_1(x)$, $c_2(x)$ and all $x\in\Omega'$. Then, due to the third line in (6.17),

$$(\partial_t u)(0,x)=\partial_{\boldsymbol{e}^j}u(0,x)=c_1(x)\partial_{\boldsymbol{V}}u(0,x)+c_2(x)\partial_{\boldsymbol{g}}u(0,x)=0$$

because any derivative along tangential vector to γ vanishes $\partial_{\boldsymbol{g}}u(0,x)=0$ due to the second line in (6.17).

The second order equation $\mathbb{A}(t,x;\mathscr{D})$ can be written in the form

$$\mathbb{A}(t,x,D)u=\mathbb{A}(t,x;\boldsymbol{e}^1)\partial_t^2u+\mathbb{A}_1(t,x;D)\partial_tu+\mathbb{A}_2(t,x;D)u,\qquad D:=-i\partial_x,$$

where $\mathbb{A}(t,x;\boldsymbol{e}^1)$ is the (invertible) matrix function, $\mathbb{A}_1(t,x;D)$ and $\mathbb{A}_2(t,x;D)$ are differential operators of orders 1 and 2 respectively, compiled of derivatives ∂_x, $x\in\Omega'$. Therefore, if $\mathbb{A}_j^0(t,x;D):=\mathbb{A}^{-1}(t,x;\boldsymbol{e}^1)\mathbb{A}_j(t,x;D)$, $j=1,2$, the Cauchy problem (6.17) transforms into

$$\begin{cases}\partial_t^2u(t,x)+\mathbb{A}_1^0(t,x;D)\partial_tu(t,x)+\mathbb{A}_2^0(t,x;D)u(t,x)=0 & \text{on}\quad (t,x)\in\Omega_\varepsilon,\\ u(0,x)=0 & \text{for all}\quad x\in\Omega',\\ (\partial_tu)(0,x)=0 & \text{for all}\quad x\in\Omega',\end{cases}\tag{6.18}$$

where $\Omega_\varepsilon:=(0,\varepsilon)\times\Omega'\subset\Omega$, $u\in\mathbb{H}^1(\Omega_\varepsilon)$ and $\gamma:=\big\{(0,x)\ :\ x\in\Omega'\big\}$.

Now let us recall the inequality (see [Miz73, § 4.3, Theorem 4.3, § 6.14], [Sc77, § 4-7, Lemma 4-21]): There is a constant C which depends on ε and $\mathbb{A}(t,x;D)$ only and such that the inequality

$$\int_{\Omega_\varepsilon}e^{-\lambda t}|v(t,x)|^2dt\,dx\leqslant C\int_{\Omega_\varepsilon}e^{-\lambda t}|(\mathbb{A}(t,x;D)v)(t,x)|^2dt\,dx,\tag{6.19}$$

holds for $\mathbb{A}(t,x;D)v\in\mathbb{L}_2(\Omega_\varepsilon)$, $v\in C^\infty(\Omega_\varepsilon)$; moreover, $v(t,x)$ should vanishe near $t=\varepsilon$ and should have vanishing Cauchy data $v(0,x)=(\partial_tv)(0,x)=0$ for all $x\in\Omega'$.

Let $\rho\in C^2(0,\varepsilon)$ be a cut-off function: $\rho(t)=1$ for $0\leqslant t<\varepsilon/2$ and $\rho(t)=0$ for $3\varepsilon/4\leqslant t<\varepsilon$. Then $v:=\rho u\in\mathbb{H}^1(\Omega_\varepsilon)$ and since $\mathbb{A}(t,x;D)u=0$ on Ω_ε, we get

$$\begin{aligned}\mathbb{A}(t,x;D)(\rho u)&=\rho\mathbb{A}(t,x;D)u+(\partial_t^2\rho)u+(\partial_t\rho)\partial_tu+(\partial_t\rho)\mathbb{A}_1^0(t,x;D)u\\&=(\partial_t^2\rho)u+(\partial_t\rho)\partial_tu+(\partial_t\rho)\mathbb{A}_1^0(t,x;D)u.\end{aligned}$$

We have asserted $u \in \mathbb{H}^1(\Omega_\varepsilon)$, $\rho \in C^2$ and this implies $(\partial_t^2\rho)u \in \mathbb{L}_2(\Omega_\varepsilon)$, $(\partial_t\rho)\partial_t u \in \mathbb{L}_2(\Omega_\varepsilon)$. Note, that $\partial_t\rho(t)$ vanishes for $0 < t < \varepsilon/2$. Therefore $(\partial_t\rho)\mathbb{A}_1^0(t,x;D)u$ vanishes in a neighborhood of the boundary $\gamma \subset \Gamma$. Due to a priori regularity result (cf. [LM72, Ch. 2, § 3.2, § 3.3]), a solution to an elliptic equation in (6.18) has additional regularity $u \in \mathbb{H}^2(\Omega^0_\varepsilon)$ for arbitrary Ω^0_ε properly imbedded into Ω_ε. This implies $(\partial_t\rho)\mathbb{A}_1^0(t,x;D)u \in \mathbb{L}_2(\Omega_\varepsilon)$ and we conclude

$$\mathbb{A}(t,x;D)(\rho u) \in \mathbb{L}_2(\Omega_\varepsilon). \tag{6.20}$$

Introducing $v = \rho u$ into the inequality (6.19) we get

$$\begin{aligned}\int_{\Omega'}\int_0^{\varepsilon/4} e^{-\lambda t}|\rho(t)u(t,x)|^2 dt\,dx &\leqslant \int_{\Omega_\varepsilon} e^{-\lambda t}|\rho(t)u(t,x)|^2 dt\,dx\\ &\leqslant C\int_{\Omega'}\int_{\varepsilon/2}^{3\varepsilon/4} e^{-\lambda t}|(\mathbb{A}(t,x;D))\rho(t)u(t,x)|^2 dt\,dx.\end{aligned}$$

This implies for $\lambda > 0$

$$\int_{\Omega'}\int_0^{\varepsilon/4}|\rho(t)u(t,x)|^2 dt\,dx \leqslant e^{-\lambda\varepsilon/4}\int_{\Omega_\varepsilon}\left|(\mathbb{A}(t,x;D))\rho(t)u(t,x)\right|^2 dt\,dx \leqslant C_1 e^{-\lambda\varepsilon/4}.$$

where, due to (6.17), $C_1 > 0$ is a finite constant. By sending $\lambda \to \infty$ we get the desired result $u(t,x) = 0$ for all $0 \leqslant t \leqslant \varepsilon/4$ and all $x \in \Omega'$. Since $u(x)$ vanishes in a subset of the domain Ω, bordering γ, due to Lemma 6.8 the solution vanishes on entire Ω (on entire $\mathscr{C}$). ■

Due to our specific interest (see the next Lemma 6.12) and many applications, for example to control theory, the following boundary unique continuation property is of a special interest.

Definition 6.11 *Let $\mathscr{S}$ be a Lipschitz hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be an open subsurface with the Lipschitz boundary $\Gamma = \partial\mathscr{C}$.*

We say that a class of functions $\mathscr{U}(\Omega)$ has the **strong unique continuation property from the boundary** *if a vector-function $\boldsymbol{U} \in \mathscr{U}(\Omega)$ which vanishes $\boldsymbol{U}(\mathfrak{s}) = 0$, $\forall\,\mathfrak{s} \in \gamma$ on an open subset of the boundary $\gamma \subset \Gamma$, vanishes on the entire $\mathscr{C}$.*

Lemma 6.12 *Let $\mathscr{S}$ be a $\mathbb{W}^2_\infty$-smooth hypersurface in $\mathbb{R}^n$ and $\mathscr{C} \subset \mathscr{S}$ be an open $\mathbb{W}^2_\infty$-smooth subsurface.*

The set of Killing's vector fields $\mathscr{R}(\mathscr{S})$ on the open surface $\mathscr{C}$ has the strong unique continuation property from the boundary.

Proof: Let $\gamma \subset \Gamma := \partial\mathscr{C}$, $\operatorname{mes}\gamma > 0$ and $\boldsymbol{U}(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in \gamma \subset \Gamma := \partial\mathscr{C}$. Then (cf. (2.24))

$$\left\{\begin{array}{l}(\mathscr{D}_j U^0_k)(\mathfrak{s}) + (\mathscr{D}_k U^0_j)(\mathfrak{s}) = -\sum\limits_{m=1}^n U^0_m(\mathfrak{s})\mathscr{D}_m\big(\nu_j(\mathfrak{s})\nu_k(\mathfrak{s})\big) = 0,\\ U^0_k(\mathfrak{s}) = 0 \qquad \forall\,\mathfrak{s} \in \gamma, \quad j,k = 1,\dots,n.\end{array}\right. \tag{6.21}$$

Among tangential vector fields generating the Gunter's derivatives $\left\{\boldsymbol{d}^j(\mathfrak{s})\right\}_{j=1}^{n-1}$ only $n-1$ are linearly independent. One of vectors might collapse at a point $\boldsymbol{d}^j(\mathfrak{s})=0$ if the corresponding basis vector $\boldsymbol{e}^j$ is orthogonal to the surface at $\mathfrak{s}\in\mathscr{S}$, while others might be tangential to the subsurface Γ, except at least one, say $\boldsymbol{d}^n(\mathfrak{s})$, which is non-tangential to γ. Then from (6.21) follows

$$2(\mathscr{D}_n U_n^0)(\mathfrak{s})=0 \quad \text{and implies} \quad (\mathscr{D}_j U_n^0)(\mathfrak{s})=0 \qquad (6.22)$$
$$\text{for all} \quad \mathfrak{s}\in\gamma \quad \text{and all} \quad j=1,\dots,n.$$

Indeed, the vector $\boldsymbol{d}^j$, $1\leqslant j\leqslant n-1$, is a linear combination $\boldsymbol{d}^j(\mathfrak{s})=c_1(\mathfrak{s})\boldsymbol{d}^n(\mathfrak{s})+c_2(\mathfrak{s})\boldsymbol{\tau}^j(\mathfrak{s})$ of the non-tangential vector $\boldsymbol{d}^n(\mathfrak{s})$ and of the projection $\boldsymbol{\tau}^j(\mathfrak{s}):=\pi_\gamma\boldsymbol{d}^j(\mathfrak{s})$ of $\boldsymbol{d}^j(\mathfrak{s})$ to the subsurface γ at the point $\mathfrak{s}\in\gamma$. Since U^n vanishes identically on γ, the derivative $(\partial_{\boldsymbol{\tau}^j}U_n^0)(\mathfrak{s})=0$ vanishes as well and (6.22) follows:

$$(\mathscr{D}_j U_n^0)(\mathfrak{s})=c_1(\mathfrak{s})(\partial_{\boldsymbol{d}^n}U_n^0)(\mathfrak{s})+c_2(\mathfrak{s})(\partial_{\boldsymbol{\tau}^j}U_n^0)(\mathfrak{s})=c_1(\mathfrak{s})(\mathscr{D}_n U_n^0)(\mathfrak{s})=0 \qquad \forall\,\mathfrak{s}\in\gamma.$$

Equalities (6.21) and (6.22) imply

$$(\mathscr{D}_n U_j^0(\mathfrak{s})=-(\mathscr{D}_j U_n^0)(\mathfrak{s})=0 \qquad \forall\,\mathfrak{s}\in\gamma, \quad \forall\,j=1,\dots,n. \qquad (6.23)$$

Thus, we have the following Cauchy problem

$$\begin{cases} \mathscr{L}_{\mathscr{C}}(\mathscr{X},\mathscr{D})\boldsymbol{U}(\mathscr{X})=0 & \text{on} \quad \mathscr{C},\\ \boldsymbol{U}(\mathfrak{s})=0 & \text{for all} \quad \mathfrak{s}\in\gamma,\\ (\mathscr{D}_n\boldsymbol{U})(\mathfrak{s})=(\partial_{\boldsymbol{d}^n}\boldsymbol{U})(\mathfrak{s})=0 & \text{for all} \quad \mathfrak{s}\in\gamma, \end{cases} \qquad (6.24)$$

where $\boldsymbol{d}^n$ is a vector filed non-tangential to Γ. Due to Lemma 46.10, $\boldsymbol{U}(\mathscr{X})=0$ for all $\mathscr{X}\in\mathscr{C}$. ■

Before we draw some consequences from the proved unique continuation property, we should make some comments. The finite dimensionality of the linear space $\mathscr{R}(\mathscr{C})$ when the surface $\mathscr{C}$ is 2-smooth, was proved in the papers [CLM1, GS1, Ge1].

The foregoing Lemma 6.12 generalizes essentially the "infinitesimal rigid displacement lemma" (see [Ci3, Theorem 2.7-2] the following conditions are imposed:

i. $\mathscr{C}\subset\mathscr{S}$ is C^3-smooth, **elliptic** in $\mathbb{R}^3$, i.e., if

$$\sum_{k=1}^{2}|\xi^k|^2\leqslant C\sum_{k,j=1}^{2}\left|b_{jk}(\mathscr{X})\xi^j\xi^k\right| \qquad \forall\,\mathscr{X}\in\mathscr{S}, \quad \forall(\xi^1,\xi^2)^\top\in\mathbb{R}^2, \qquad (6.25)$$

where $b_{jk}(\mathscr{X})\ :\ \mathscr{S}\to\mathbb{R}$ are the covariant components of the curvature tensor of $\mathscr{S}$; the equivalent condition is that the Gaußian curvature is positive on the entire surface $\mathscr{S}$ or that tho principal curvatures of the surface $\mathscr{S}$ have the same sign everywhere on $\mathscr{S}$.

ii. The Killing's vector field $\boldsymbol{U}$ vanishes on the entire boundary $\partial\mathscr{S}$, i.e.,

$$\mathscr{R}_0(\mathscr{C})=\left\{\boldsymbol{U}\in\mathscr{R}\ :\ \boldsymbol{U}\big|_{\partial\mathscr{C}}=0\right\}=\{0\}. \qquad (6.26)$$

A similar assertion is proved by Lods & Mardare in [LoM1], but for $C^{2,1}$-smooth hypersurface with the Lipschitz boundary $\partial\mathscr{S}$ and when a Killing's vector field expires on the entire boundary $\partial\mathscr{S}$. An earlier version of the "infinitesimal rigid displacement lemma" is due to I. Vekua [Ve82], who proved it using the theory of "generalized analytic functions".

Corollary 6.13 (Korn's I inequality "with boundary condition"). *Let $\mathscr{C} \subset \mathbb{R}^n$ be a C^ℓ-smooth hypersurface with the Lipschitz boundary $\Gamma := \partial\mathscr{C} \neq \emptyset$ and $\ell \geqslant 2$, $|s| \leqslant \ell$. Then*

$$\left\|\boldsymbol{U}\middle|\mathbb{H}^s_p(\mathscr{C})\right\| \leqslant M\left\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U})\middle|\mathbb{H}^{s-1}_p(\mathscr{C})\right\| \qquad \forall\, \boldsymbol{U} \in \widetilde{\mathbb{H}}^s_p(\mathscr{C})$$

for some constant $M > 0$. In other words: the mapping

$$\boldsymbol{U} \mapsto \left\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U})\middle|\mathbb{H}^{s-1}_p(\mathscr{C})\right\| \tag{6.27}$$

is an equivalent norm on the space $\widetilde{\mathbb{H}}^s_p(\mathscr{C})$.

Proof: If the claimed inequality (6.27) is false, there exists a sequence $\boldsymbol{U}^j \in \widetilde{\mathbb{H}}^s_p(\mathscr{C})$, $j = 1, 2, \ldots$ such that

$$\left\|\boldsymbol{U}^j\middle|\mathbb{H}^s_p(\mathscr{C})\right\| = 1 \quad \forall j = 1, 2, \ldots \quad \lim_{j\to\infty}\left\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^j)\middle|\mathbb{H}^{s-1}_p(\mathscr{C})\right\| = 0.$$

Due to the compact embedding $\widetilde{\mathbb{H}}^s_p(\mathscr{C}) \subset \mathbb{H}^s_p(\mathscr{C}) \subset \mathbb{H}^{s-1}_p(\mathscr{C})$, a convergent subsequence $\boldsymbol{U}^{j_1}, \boldsymbol{U}^{j_2}, \ldots$ in $\mathbb{H}^{s-1}_p(\mathscr{C})$ can be selected. Let $\boldsymbol{U}^0 = \lim_{k\to\infty}\boldsymbol{U}^{j_k}$. Then

$$\left\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^0)\middle|\mathbb{H}^{s-1}_p(\mathscr{C})\right\| = \lim_{k\to\infty}\left\|\mathrm{Def}_{\mathscr{C}}(\boldsymbol{U}^{j_k})\middle|\mathbb{H}^{s-1}_p(\mathscr{C})\right\| = 0$$

and $\boldsymbol{U}^0$ is a Killing's vector field. Since $\boldsymbol{U}(x) = 0$ on Γ, due to Lemma 6.12 $\boldsymbol{U}^0(x) = 0$ for all $x \in \mathscr{C}$ which contradicts to $\left\|\boldsymbol{U}^0\middle|\mathbb{H}^s_p(\mathscr{C})\right\| = \lim_{k\to\infty}\left\|\boldsymbol{U}^{j_k}\middle|\mathbb{H}^s_p(\mathscr{C})\right\| = 1$. ■

Let us check the following equalities for a later use:

$$\nabla_{\Omega^\varepsilon}\boldsymbol{U} = \left[\mathscr{D}_j U^0_k\right]_{n+1\times n+1} + \langle \mathscr{N}, \boldsymbol{U}\rangle \mathscr{W}_{\Omega^\varepsilon}, \tag{6.28}$$

where

$$\boldsymbol{U} := \sum_{m=1}^{n+1} U^0_m \boldsymbol{d}^m = \sum_{m=1}^{n} U_m \boldsymbol{e}^m, \quad U^0_{n+1} = \sum_{m=1}^{n}\mathscr{N}_m U_m, \quad \mathscr{D}_{n+1} := \partial_{\mathscr{N}}, \quad \boldsymbol{d}^{n+1} := \mathscr{N}.$$

$\mathscr{W}_{\Omega^\varepsilon}$ is the extended Weingarten matrix

$$\mathscr{W}_{\Omega^\varepsilon} := \left[\mathscr{D}_j\mathscr{N}_k\right]_{n+1\times n+1} \tag{6.29}$$

and its last column and last row are 0, because $\mathscr{D}_j\mathscr{N}_{n+1} = \mathscr{D}_{n+1}\mathscr{N}_j = \mathscr{D}_{n+1}\mathscr{N}_{n+1} = 0$ for $j = 1, \ldots, n$.

In fact (see (2.13) fore some further details of calculation):

$$
\begin{aligned}
\nabla_{\Omega^\varepsilon}\boldsymbol{U} := \big[\partial_j U_k\big]_{n\times n} = \sum_{j,ky=1}^{n} \partial_j U_k \boldsymbol{e}^j \otimes \boldsymbol{e}^k \\
:= \sum_{j,k=1}^{n} [\mathscr{D}_j + \mathscr{N}_j\partial_{\mathscr{N}}][U_k^0 + \mathscr{N}_k\langle\mathscr{N},\boldsymbol{U}\rangle][\boldsymbol{d}^j + \mathscr{N}_j\mathscr{N}] \otimes [\boldsymbol{d}^k + \mathscr{N}_k\mathscr{N}] \\
= \sum_{j,k=1}^{n} (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes [\boldsymbol{d}^k + \mathscr{N}_k\mathscr{N}] + \sum_{j,k=1}^{n} \mathscr{D}_j[\mathscr{N}_k\langle\mathscr{N},\boldsymbol{U}\rangle]\boldsymbol{d}^j \otimes [\boldsymbol{d}^k + \mathscr{N}_k\mathscr{N}] \\
+ \sum_{j,k=1}^{n} \mathscr{N}_j^2(\partial_{\mathscr{N}} U_k^0)\mathscr{N} \otimes [\boldsymbol{d}^k + \mathscr{N}_k\mathscr{N}] + \sum_{j,k=1}^{n} \mathscr{N}_j^2\mathscr{N}_k^2\partial_{\mathscr{N}}\langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{N}\otimes\mathscr{N} \\
= \sum_{j,k=1}^{n} (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes \boldsymbol{d}^k + \sum_{j,k=1}^{n} \mathscr{N}_k(\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
+ \sum_{j,k=1}^{n} \langle\mathscr{N},\boldsymbol{U}\rangle(\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j \otimes [\boldsymbol{d}^k + \mathscr{N}_k\mathscr{N}] + \sum_{j,k=1}^{n} \mathscr{N}_k^2\mathscr{D}_j\langle\mathscr{N},\boldsymbol{U}\rangle\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
+ \sum_{k=1}^{n} (\mathscr{D}_{n+1}U_k^0)\boldsymbol{d}^{n+1} \otimes \boldsymbol{d}^k + \sum_{k=1}^{n} \big[\mathscr{N}_k\mathscr{D}_{n+1}U_k^0 + \mathscr{D}_{n+1}U_{n+1}^0\big]\,\boldsymbol{d}^{n+1} \otimes \boldsymbol{d}^{n+1} \\
= \sum_{j,k=1}^{n} (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes \boldsymbol{d}^k + \sum_{j,k=1}^{n} \big[\mathscr{D}_j(\mathscr{N}_k U_k^0) - U_k^0\mathscr{D}_j\mathscr{N}_k\big]\,\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
+ \langle\mathscr{N},\boldsymbol{U}\rangle \sum_{j,k=1}^{n} (\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j \otimes \boldsymbol{d}^k + \sum_{j=1}^{n} \mathscr{D}_j\langle\mathscr{N},\boldsymbol{U}\rangle\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
+ \sum_{k=1}^{n} (\mathscr{D}_{n+1}U_k^0)\boldsymbol{d}^{n+1} \otimes \boldsymbol{d}^k + (\mathscr{D}_{n+1}U_{n+1}^0)\boldsymbol{d}^{n+1} \otimes \boldsymbol{d}^{n+1} \\
= \sum_{j,k=1}^{n+1} (\mathscr{D}_j U_k^0)\boldsymbol{d}^j \otimes \boldsymbol{d}^k - \sum_{j,k=1}^{n} U_k^0(\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} + \langle\mathscr{N},\boldsymbol{U}\rangle \sum_{j,k=1}^{n} (\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j \otimes \boldsymbol{d}^k \\
= \big[\mathscr{D}_j U_k\big]_{(n+1)\times(n+1)} + \langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{W}_{\Omega^\varepsilon} - \sum_{j,k=1}^{n} U_k^0(\mathscr{D}_j\mathscr{N}_k)\boldsymbol{d}^j \otimes \boldsymbol{d}^{n+1} \\
= \big[\mathscr{D}_j U_k\big]_{(n+1)\times(n+1)} + \langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{W}_{\Omega^\varepsilon} - \big[(\mathscr{W}_{\Omega^\varepsilon}\boldsymbol{U}^0)_j\delta_{j,n+1}\big]_{(n+1)\times(n+1)},
\end{aligned}
$$

since

$$
\partial_{\mathscr{N}}\mathscr{N}_j = 0, \quad \sum_{j,k=1}^{n} \mathscr{N}_j^2 = 1, \quad \sum_{j=1}^{n} \mathscr{N}_j\mathscr{D}_j = 0, \quad \sum_{j=1}^{n} \mathscr{N}_j\boldsymbol{d}^j = 0,
$$

$$
\sum_{k=1}^{n} \mathscr{N}_k U_k^0 = 0, \quad \sum_{k=1}^{n} \mathscr{N}_k\mathscr{D}_j\mathscr{N}_k = \frac{1}{2}\mathscr{D}_j\sum_{k=1}^{n}\mathscr{N}_k^2 = \frac{1}{2}\mathscr{D}_j 1 = 0, \quad j = 1, 2\ldots, n+1.
$$

For a domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\mathscr{M} := \partial\Omega$ and $\mathscr{M}_0 \subset \mathscr{M}$-a subsurface of non-zero measure let $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ denote a subspace of functions $\varphi \in \mathbb{W}^1(\Omega, \mathscr{M}_0)$ which is the closure of the set $C^\infty(\Omega, \mathscr{M}_0)$ of smooth functions $\varphi(x)$ which have vanishing trace on $\mathscr{M}_0$, i.e. $\varphi^+(\mathscr{x}) = 0$ for all $\mathscr{x} \in \mathscr{M}_0$. The space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ inherits the standard norm from $\mathbb{W}^1(\Omega)$:

$$\|\varphi \,|\, \mathbb{W}^1(\Omega) \,\| := \left[\|\varphi \,|\, \mathbb{L}_p(\Omega)\| + \sum_{j=1}^n \|\partial_j\varphi \,|\, \mathbb{L}_p(\Omega)\| \right]^{1/p}.$$

Since the space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$ does not contain constants, it is easy to prove the following.

Lemma 6.14 *The formula*

$$\|\varphi \,|\, \widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0) \,\| := \left[\sum_{j=1}^n \|\partial_j\varphi \,|\, \mathbb{L}_p(\Omega)\| \right]^{1/p}. \tag{6.30}$$

defines an equivalent norm in the space $\widetilde{\mathbb{W}}^1(\Omega, \mathscr{M}_0)$.

If ε is sufficiently small, the boundary $\mathscr{M}_\varepsilon := \partial\Omega_\varepsilon$ is represented as the union of three C^1-smooth surfaces $\mathscr{M}_\varepsilon = \mathscr{M}_{\varepsilon,D} \cup \mathscr{M}^-_{\varepsilon,N} \cup \mathscr{M}^+_{\varepsilon,N}$, where $\mathscr{M}_{\varepsilon,D} = \partial\mathscr{C} \times [-\varepsilon, \varepsilon]$ is the lateral surface, $\mathscr{M}^+_{\varepsilon,N} = \mathscr{C} \times \{+\varepsilon\}$ is the upper surface and $\mathscr{M}^-_{\varepsilon,N} = \mathscr{C} \times \{-\varepsilon\}$ is the lower surface of the of the boundary $\mathscr{M}_\varepsilon$ of the layer domain Ω_ε.

The next Lemma 6.15 is proved for a later use in § 3.

Lemma 6.15 $T \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M})$*, Let* $\mathscr{M}_0 := \gamma \times [-\varepsilon, \varepsilon]$*, where* $\gamma \subset \Gamma := \partial\mathscr{C}$ *is a subset of the boundary of the surface* $\mathscr{C}$ *of non-trivial measure. If* $g \in \mathbb{L}_2(\Omega_\varepsilon)$*, for the linear functional*

$$E_\varepsilon(u) = \int\limits_{\Omega_\varepsilon} g(x)u(x)\,dx, \qquad u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0) \tag{6.31}$$

we have the following estimate:

$$E_\varepsilon(u) \leqslant C\|g|\mathbb{L}_2(\Omega_\varepsilon)\| \; \|\mathscr{D}_\mathscr{C} u|\mathbb{L}_2(\Omega_\varepsilon)\| \tag{6.32}$$

for some constant $C > 0$ *independent of* $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$.

Proof: To prove (6.32) we recall that $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$ vanishes on the lateral subsurface $\mathscr{x} \in \mathscr{M}_0 \subset \mathscr{M}_D := \partial\mathscr{C} \times (-\varepsilon, \varepsilon)$.

Let $\mathscr{C}_t$ be the "parallel" surface to the mid-surface $\mathscr{C}$ on a distance $|t|$ and for negative $t < 0$ the surface $\mathscr{C}_t$ is "below" $\mathscr{C}$, while for positive $t > 0$ is "above" $\mathscr{C}$, i.e., in the direction of the normal vector filed $\boldsymbol{\nu}(\mathscr{x})$, $\mathscr{x} \in \mathscr{C}$. Note, that $\mathscr{C}_{\pm 1\varepsilon} = \mathscr{M}^\pm_D$. Taking $u(\mathscr{x}, t)$, $\mathscr{x} \in \mathscr{C}$, $-\varepsilon < t < \varepsilon$ from a dense subset of the space $\widetilde{\mathbb{W}}^1(\Omega_\varepsilon, \mathscr{M}_0)$ we can assume that

$u(\cdot,t) \in \widetilde{\mathbb{W}}^1(\mathscr{C}_t)$ for all fixed $-\varepsilon \leqslant t \leqslant \varepsilon$. Since $u(\mathscr{x},t)$ vanishes on the part of the boundary $\mathscr{M}_0 \cap \partial\mathscr{C}_t$, the Sobolev semi-norm

$$\|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\|_0 := \|\mathscr{D}_{\mathscr{C}}u(\cdot,t)|\mathbb{L}_2(\mathscr{C}_t)\| = \left[\sum_{j=1}^{3}\int_{\mathscr{C}_t}|\mathscr{D}_j u(\mathscr{x},t)|^2 d\sigma\right]^{1/2}$$

turns into the norm and is equivalent to the standard Sobolev norm

$$\|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\| := \left[\int_{\mathscr{C}_t}|u(\mathscr{x},t)|^2 d\sigma + \sum_{j=1}^{3}\int_{\mathscr{C}_t}|\mathscr{D}_j u(\mathscr{x},t)|^2 d\sigma\right]^{1/2}$$

for all $t \in [-\varepsilon,\varepsilon]$, which means

$$M\|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\| \leqslant \|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\|_0 \leqslant \|u(\cdot,t)|\mathbb{W}^1(\mathscr{C}_t)\|$$

for some constant $0 < M < 1$, independent of t and u. From this equivalence we get the estimate

$$\|u(\cdot,t)|\mathbb{L}_2(\mathscr{C}_t)\|^2 \leqslant \frac{1-M^2}{M^2}\|\mathscr{D}_{\mathscr{C}}u(\cdot,t)|\mathbb{L}_2(\mathscr{C}_t)\|^2. \tag{6.33}$$

By integrating the obtained inequality with respect to the variable t we get the following final estimate

$$\|u|\mathbb{L}_2(\Omega_\varepsilon)\| \leqslant \frac{\sqrt{1-M^2}}{M}\|\mathscr{D}_{\mathscr{C}}u|\mathbb{L}_2(\Omega_\varepsilon)\| \qquad \forall u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon,\mathscr{M}_0). \tag{6.34}$$

The estimate in (6.32) follows with the help of the Cauchy inequality and inequality (6.34):

$$\int_{\Omega_\varepsilon} g(x)u(x)dx \leqslant \|g|\mathbb{L}_2(\Omega_\varepsilon)\|\|u|\mathbb{L}_2(\Omega_\varepsilon)\| \leqslant \frac{\sqrt{1-M^2}}{M}\|g|\mathbb{L}_2(\Omega_\varepsilon)\|\|\mathscr{D}_{\mathscr{C}}u|\mathbb{L}_2(\Omega_\varepsilon)\|. \qquad \blacksquare$$

Remark 6.16 *Let us stress that in estimate* (6.32) *we only need the surface derivatives* $\mathscr{D}_1, \mathscr{D}_2$ *and* $\mathscr{D}_3$. *If we would have* $g \in \mathbb{W}^{-1}(\Omega_\varepsilon)$, *then we should assume* $u \in \widetilde{\mathbb{W}}^1(\Omega_\varepsilon)$. *These spaces are dual and, therefore if the integral in the functional* E_ε *in* (6.31), *is understood as the duality, the functional* E_ε *is bounded, but then estimate writes*

$$E_\varepsilon(u) \leqslant C\|g|\mathbb{L}_2(\Omega_\varepsilon)\| \, \|\mathscr{D}_{\Omega_\varepsilon}u|\mathbb{L}_2(\Omega_\varepsilon)\|, \qquad \mathscr{D}_{\Omega_\varepsilon} = (\mathscr{D}_1,\mathscr{D}_2,\mathscr{D}_3,\mathscr{D}_4)^\top. \tag{6.35}$$

In this estimate all derivatives, the surface and the transversal $\partial_t = \partial_{\boldsymbol{\nu}} = \mathscr{D}_4$ *(the normal to the surface* $\mathscr{C}$*) are participating.*

7 AUXILIARY FROM THE OPERATOR THEORY

The results exposed in the present section will be applied to complex valued matrices, which are identified with operators in the finite dimensional space $\mathbb{C}$. Nevertheless, we will formulate results in general setting of operators in a Hilbert space.

Throughout this section we assume that $\mathfrak{H}$ is a Hilbert space with respect to some continuous scalar product, a bilinear form $(\cdot,\cdot)\ :\ \mathfrak{H}\times\mathfrak{H}\to\mathbb{C}$, i.e.,

$$(\lambda\, u+\mu\, w, v)=\overline{\lambda}(u,v)+\overline{\mu}(w,v)\,,\qquad (u,\lambda\, v+\mu\, z)=\lambda(u,v)+\mu(u,z)\,,$$
$$\big|(u,v)\big|\leqslant C\|u\|_{\mathfrak{H}}\|v\|_{\mathfrak{H}}\,,\quad \forall\, u,w\in\mathfrak{H}\,,\quad \forall\, v,z\in\mathfrak{H}\,,$$
$$(\varphi,\psi)=\overline{(\psi,\varphi)}\qquad \forall\,\varphi,\psi\in\mathfrak{H}.$$

Recall, that the dual operator $(\boldsymbol{A}^*\varphi,\psi)=(\varphi,\boldsymbol{A}\psi)$ maps continuously the same space $\boldsymbol{A}^*\ :\ \mathfrak{H}\longrightarrow\mathfrak{H}$ and $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is **self-adjoint** operator if

$$(\boldsymbol{A}\varphi,\psi)=(\varphi,\boldsymbol{A}\psi)\qquad \forall\,\varphi,\ \psi\in\mathfrak{H}. \tag{7.1}$$

$\boldsymbol{A}\in\mathscr{L}(\mathfrak{H},\mathfrak{H})$ is **positive definite** (or **coercive**) if the inequality

$$(\boldsymbol{A}\varphi,\varphi)\geqslant C\|\varphi\big|\mathfrak{H}\|^2 \tag{7.2}$$

holds for some constant $C>0$ and all $\varphi\in\mathfrak{H}$.

Lemma 7.1 *Let $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$. The inequality*

$$\|\boldsymbol{A}\varphi\big|\mathfrak{H}\|\geqslant C\|\varphi\big|\mathfrak{H}\| \tag{7.3}$$

with some constant $C>0$ holds if and only if the operator $\boldsymbol{A}$ is normally solvable $\Im\, A=\overline{\Im\, A}$ and injective $\operatorname{Ker}\boldsymbol{A}=\{0\}$.

Proof. If the inequality (7.3) holds, then $\boldsymbol{A}\varphi=0$, $\varphi\in\mathfrak{H}$ implies $\varphi=0$ and $\operatorname{Ker}\boldsymbol{A}=\{0\}$. Now let $\psi_j=\boldsymbol{A}\varphi_j\to\psi_0$ (convergence in the norm). From (7.3) follows the convergence $\varphi_j\to\varphi_0$. Due to continuity of $\boldsymbol{A}$ this implies $\boldsymbol{A}\varphi_0=\psi_0\in\Im\,\boldsymbol{A}$ and $\Im\,\boldsymbol{A}$ is closed.

Vice versa, let $\boldsymbol{A}$ be normally solvable and $\operatorname{Ker}\boldsymbol{A}=\{0\}$. Then $\Im\,\boldsymbol{A}$ is a Hilbert space, subspace of $\mathfrak{H}$ and the operator $\boldsymbol{A}\ :\ \mathfrak{H}\to\Im\,\boldsymbol{A}$ is bijective. Due to the Banach's Inverse mapping theorem $\boldsymbol{A}$ is invertible: there exists $\mathbf{B}\in\mathscr{L}(\Im\,\boldsymbol{A})$ such that $\mathbf{AB}x=x\ \mathbf{BA}y=y$ for all $x\in\Im\,\boldsymbol{A}$ and all $y\in\mathfrak{H}$. Inserting in $\|\mathbf{B}\psi\big|\mathfrak{H}\|\leqslant C\|\psi\big|\Im\,\boldsymbol{A}\|:=\|\psi\big|\mathfrak{H}\|$ the equality $\psi=\boldsymbol{A}\varphi$, $\varphi\in\mathfrak{H}$, we get (7.3). ∎

Definition 7.2 *For an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ the closed set*

$$\Sigma(\boldsymbol{A}):=\overline{\{(\boldsymbol{A}\varphi,\varphi)\ :\ \varphi\in\mathfrak{H}\}}, \tag{7.4}$$

where the overbar denotes closing of the set, is called the **spectral set** *of $\boldsymbol{A}$.*

Lemma 7.3 *If the spectral set $\Sigma(\boldsymbol{A})$ of an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is real valued $\Sigma(\boldsymbol{A})\subset\mathbb{R}$, then $\boldsymbol{A}$ is self-adjoint.*

Proof. We proceed as follows:

$$\begin{aligned}(\boldsymbol{A}\varphi,\psi) &= \frac{1}{4}\Big\{(\boldsymbol{A}[\varphi+\psi],\varphi+\psi)-(\boldsymbol{A}[\varphi-\psi],\varphi-\psi)\\ &\quad +i(\boldsymbol{A}[\varphi+i\psi],\varphi+i\psi)-i(\boldsymbol{A}[\varphi-i\psi],\varphi-i\psi)\Big\}\\ &= \frac{1}{4}\Big\{\overline{(\boldsymbol{A}[\varphi+\psi],\varphi+\psi)}-\overline{(\boldsymbol{A}[\varphi-\psi],\varphi-\psi)}\\ &\quad +i\overline{(\boldsymbol{A}[\varphi+i\psi],\varphi+i\psi)}-i\overline{(\boldsymbol{A}[\varphi-i\psi],\varphi-i\psi)}\Big\}\\ &= \frac{1}{4}\Big\{(\varphi+\psi,\boldsymbol{A}[\varphi+\psi])-(\varphi-\psi,\boldsymbol{A}[\varphi-\psi])\\ &\quad +i(\varphi+i\psi,\boldsymbol{A}[\varphi+i\psi])-i(\varphi-i\psi,\boldsymbol{A}[\varphi-i\psi])\Big\}\\ &= (\varphi,\boldsymbol{A}\psi)\,,\qquad \varphi,\psi\in\mathfrak{H}\end{aligned}$$

since $(\boldsymbol{A}u,u)=\overline{(\boldsymbol{A}u,u)}$ by the condition $\Sigma(\boldsymbol{A})\subset\mathbb{R}$ and $\overline{(\boldsymbol{A}u,u)}=(u,\boldsymbol{A}u)$ by the definition. ■

Corollary 7.4 *If an operator $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ is positive definite, it is self-adjoint and invertible.*

Proof. If $\boldsymbol{A}$ is positive definite, its spectral set is real valued and $\boldsymbol{A}$ is self-adjoint.

From (7.2) we get

$$\|\boldsymbol{A}\varphi\big|\mathfrak{H}\|\|\varphi\big|\mathfrak{H}\|\geqslant(\boldsymbol{A}\varphi,\varphi)\geqslant C\|\varphi\big|\mathfrak{H}\|^2$$

and further

$$\|\boldsymbol{A}\varphi\big|\mathfrak{H}\|\|\geqslant C\|\varphi\big|\mathfrak{H}\|\,,\qquad \varphi\in\mathfrak{H}\,. \tag{7.5}$$

Due to Lemma 7.1 the inequality (7.2) implies that $\boldsymbol{A}$ is normally solvable and has a trivial kernel $\operatorname{Ker}\boldsymbol{A}=\{0\}$. Being self-adjoint $\boldsymbol{A}^*=\boldsymbol{A}$ the operator has the trivial cokernel $\dim\operatorname{Coker}\boldsymbol{A}=\dim Ker\,\boldsymbol{A}=0$ (due to (7.2) $\boldsymbol{A}\varphi=0$ implies that $\varphi=0$). Therefore, $\boldsymbol{A}$ is invertible. ■

Let $\boldsymbol{A}\in\mathscr{L}(\mathfrak{H})$ and $\boldsymbol{A}=\mathbf{R}\mathbf{H}_{\boldsymbol{A}}$ be its left polar decomposition, where $\mathbf{R}\in\mathbb{SO}(\mathfrak{H})$ is the orthogonal (unitary) operator $\mathbf{R}^*=\mathbf{R}^{-1}$ and $\mathbf{H}_{\boldsymbol{A}}$ is positive, self adjoint (Hermitian) operator

$$\langle\mathbf{H}_{\boldsymbol{A}}\varphi,\varphi\rangle\geqslant C_0\|\varphi\|^2,\qquad C_0>0,\qquad \mathbf{H}_{\boldsymbol{A}}^*=\mathbf{H}_{\boldsymbol{A}},\quad\forall\,\varphi\in\mathfrak{H}.$$

Let us check, that $\mathbf{H}_{\boldsymbol{A}}=\sqrt{\boldsymbol{A}^*\boldsymbol{A}}$. Indeed, if $\boldsymbol{A}=\mathbf{R}\mathbf{U}_{\boldsymbol{A}}$, then $\boldsymbol{A}^*=\mathbf{H}_{\boldsymbol{A}}^*\mathbf{R}^*=\mathbf{H}_{\boldsymbol{A}}\mathbf{R}^{-1}$ and $\sqrt{\boldsymbol{A}^*\boldsymbol{A}}=\sqrt{\mathbf{H}_{\boldsymbol{A}}\mathbf{R}^{-1}\mathbf{R}\mathbf{H}_{\boldsymbol{A}}}=\sqrt{\mathbf{H}_{\boldsymbol{A}}^2}=\mathbf{H}_{\boldsymbol{A}}$. ■

Similarly, for the right polar decomposition $\boldsymbol{A}=\mathbf{H}'_{\boldsymbol{A}}\mathbf{R}'$ we get $\mathbf{H}'_{\boldsymbol{A}}=\sqrt{\boldsymbol{A}\,\boldsymbol{A}^*}$.

Note, that if $\boldsymbol{A}$ is positive definite (or, at least, has a real valued spectral set), then $\boldsymbol{A}$ is self adjoint $\boldsymbol{A}^*=\boldsymbol{A}$ and the polar decomposition is trivial $\mathbf{H}_A=\mathbf{H}'_A=\sqrt{\boldsymbol{A}\boldsymbol{A}}=\boldsymbol{A}$, $\mathbf{R}=\mathbf{R}'=\mathbf{I}$.

Note, that the norm has the following property:

$$\|\mathbf{R}\mathbf{A}\mathbf{R}\| = \|\mathbf{R}\mathbf{A}\| = \|\mathbf{A}\mathbf{R}\| = \|\boldsymbol{A}\| \qquad \forall\, \boldsymbol{A} \in \mathscr{L}(\mathfrak{H}), \quad \text{and} \quad \forall\, \mathbf{R} \in \mathbb{SO}(\mathfrak{H}). \tag{7.6}$$

Indeed,

$$\|\mathbf{R}\mathbf{A}\| = \inf_{\varphi\in\mathfrak{H}} \sqrt{((\mathbf{R}\mathbf{A})^*\mathbf{R}\mathbf{A}\varphi, \varphi)} = \inf_{\varphi\in\mathfrak{H}} \sqrt{(\boldsymbol{A}^*\mathbf{R}^*\mathbf{R}\mathbf{A}\varphi, \varphi)} = \inf_{\varphi\in\mathfrak{H}} \sqrt{(\boldsymbol{A}^*\boldsymbol{A}\varphi, \varphi)} = \|\boldsymbol{A}\|\,.$$

By using the obtained equality and recalling that $\|\boldsymbol{A}\| = \|\boldsymbol{A}^*\|$ and $\mathbf{R} \in \mathbb{SO}(\mathfrak{H})$ implies $\mathbf{R}^* \in \mathbb{SO}(\mathfrak{H})$, we prove the following

$$\|\mathbf{A}\mathbf{R}\| = \|(\mathbf{A}\mathbf{R})^*\| = \|\mathbf{R}^*\boldsymbol{A}^*\| = \|\boldsymbol{A}\|\,.$$

As a consequence, $\|\mathbf{R}\mathbf{A}\mathbf{R}\| = \|\mathbf{R}\mathbf{A}\| = \|\mathbf{A}\mathbf{R}\| = \|\boldsymbol{A}\|$. ■

Next we will prove, that if $\boldsymbol{A} = \mathbf{R}\mathbf{H}_{\boldsymbol{A}}$ is the left polar decomposition, then

$$\begin{aligned} &\operatorname{dist}(\boldsymbol{A}, \mathbb{SO}(\mathfrak{H})) = \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\| && \text{if } \boldsymbol{A} \text{ is invertible,}\\ &\operatorname{dist}(\boldsymbol{A}, \mathbb{SO}(\mathfrak{H})) \leqslant \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\| && \text{otherwise.} \end{aligned} \tag{7.7}$$

Indeed, due to (7.6),

$$\begin{aligned} \operatorname{dist}(\boldsymbol{A}, \mathbb{SO}(\mathfrak{H}) &= \inf_{\boldsymbol{V}\in\mathbb{SO}(\mathfrak{H})} \|\mathbf{R}\mathbf{H}_{\boldsymbol{A}} - \boldsymbol{V}\| = \inf_{\boldsymbol{V}\in\mathbb{SO}(fH)} \|\mathbf{R}^*(\mathbf{R}\mathbf{H}_{\boldsymbol{A}} - \boldsymbol{V})\| \\ &= \inf_{\boldsymbol{V}\in\mathbb{SO}(\mathfrak{H})} \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{R}^*\boldsymbol{V}\| \leqslant \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\|\,, \end{aligned}$$

since $\mathbf{I}, \mathbf{R}^*\boldsymbol{V} \in \mathbb{SO}(\mathfrak{H})$.

The second inequality in (7.7) is proved. To prove the first equality in (7.7) we assume $\boldsymbol{A}$ is invertible and...

The subsequent proof has to be modified later

Let us show that a small perturbation $\mathbf{R}(t) := \mathbf{R} + t\mathbf{U}$, $|t| < \varepsilon$, of $\mathbf{R}$ by arbitrary matrix $\mathbf{U}$ with the constraint $\mathbf{R}(t) \in \mathbb{SO}(\mathfrak{H})$ (i.e., $\mathbf{R}(t)\mathbf{R}^*(t) = \mathbf{I}$ for all $|t| < \varepsilon$) gives

$$\begin{aligned} \inf_{t\in\mathbb{R}} \|\boldsymbol{A} - \mathbf{R}(t)\| &= \inf_{t\in\mathbb{R}} \inf_{\varphi\in\mathfrak{H}} \sqrt{((\boldsymbol{A} - \mathbf{R}(t))\varphi, (\boldsymbol{A} - \mathbf{R}(t))\varphi)} \\ &= \inf_{t\in\mathbb{R}} \inf_{\varphi\in\mathfrak{H}} \sqrt{((\boldsymbol{A} - \mathbf{R}(t))^*(\boldsymbol{A} - \mathbf{R}(t))\varphi, \varphi)} \\ &= \inf_{t\in\mathbb{R}} \inf_{\varphi\in\mathfrak{H}} \sqrt{(\left[(\mathbf{H}_{\boldsymbol{A}} - \mathbf{I})^2 - t(\mathbf{U}^*\boldsymbol{A} + \boldsymbol{A}^*\mathbf{U})\right]\varphi, \varphi)} \\ &= \inf_{t\in\mathbb{R}} \inf_{\varphi\in\mathfrak{H}} \sqrt{((\mathbf{H}_{\boldsymbol{A}} - \mathbf{I})^2\varphi, \varphi)} = \inf_{\varphi\in\mathfrak{H}} \sqrt{((\mathbf{H}_{\boldsymbol{A}} - \mathbf{I})\varphi, (\mathbf{H}_{\boldsymbol{A}} - \mathbf{I})\varphi)} \\ &= \|\mathbf{H}_{\boldsymbol{A}} - \mathbf{I}\| \end{aligned}$$

because the minimization by t shows, that the norm minimizes at $(\mathbf{U}^*\boldsymbol{A} + \boldsymbol{A}^*\mathbf{U}) = (\mathbf{U}^*\boldsymbol{A} + \boldsymbol{A}^*\mathbf{U})^* = 0$. ■

8 GEOMETRIC RIGIDITY

The basic rigidity result relevant to passage to the thin plate limit is the following.

Proposition 8.1 (see [FJM1]) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$ and $1 < p < \infty$. There exists a constant $C(\Omega)$ with the following property: For each $U \in \mathbb{W}^1(\Omega)$ there is an associated rotation $R_U \in \mathbb{SO}(n)$ such that,*

$$\left\|\nabla U - R_U \middle| \mathbb{L}_p(\Omega)\right\| \leqslant C(\Omega) \left\|\mathrm{dist}(\nabla U, \mathbb{SO}(n)) \middle| \mathbb{L}_p(\Omega)\right\|. \tag{8.1}$$

The result is sharp in the sense that neither the norm on the right hand side nor the power with which it appears can be improved.

By considering the special case when the right hand side in (8.1) is zero, Proposition 8.1 reduces to the following.

Corollary 8.2 (Liouville theorem) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$. If U is a $\mathbb{W}^1(\Omega)$ map which satisfies the partial differential equation*

$$\nabla U(x) \in \mathbb{SO}(n) \qquad \textit{a.e. in} \quad \Omega, \tag{8.2}$$

then it is affine $U(x) = Rx + c$, $R \in \mathbb{SO}(n)$, $c =$ const or, equivalently, $\nabla U = R \in \mathbb{SO}(n)$.

Proof: In the setting of Sobolev maps this was first proved by Reshetnyak [Re67]. A short modern proof belongs to G. Friesecke, R.D. James & S. Müller [FJM1] and consists of three observations.

First, for $n \times n$ matrix $A = [a_{jk}]_{n\times n}$ let cof A denote the matrix of cofactors of A, i.e.,

$$\mathrm{cof}\, A = \left[(-1)^{j+k} \mathrm{det} A_{jk}\right]_{n\times n}, \tag{8.3}$$

where A_{jk} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the j-th row and the k-th column. It is well-known that

$$\mathrm{div}\, \mathrm{cof} \nabla U = 0 \qquad \text{for all} \quad U \in \mathbb{W}^1(\Omega). \tag{8.4}$$

Note first that if the equality (8.4) is proved for $U \in C^2(\Omega)$, it can be extended to arbitrary $U \in \mathbb{W}^1(\Omega)$.

We have to prove

$$C_i := \sum_{k=1}^{n} \partial_k (\mathrm{cof} \nabla U)_{ki} = 0, \quad i = 1, \dots, n. \tag{8.5}$$

Note, that C_i can be formally written as

$$C_i = \det \begin{pmatrix} \partial_1 & \partial_2 & \cdots & \partial_n \\ \partial_1 v_1^{(i)} & \partial_2 v_1^{(i)} & \cdots & \partial_n v_1^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 v_{n-1}^{(i)} & \partial_2 v_{n-1}^{(i)} & \cdots & \partial_n v_{n-1}^{(i)} \end{pmatrix}, \tag{8.6}$$

where $v^{(i)} = (U_1, \cdots, U_{i-1}, U_{i+1}, \cdots, U_n)$. The equality (8.5) follows from the following assertion: For any $u = (u_1, \cdots u_{n-1}) \in C^2(\mathbb{R}^{n-1})$

$$\det \begin{pmatrix} \partial_1 & \partial_2 & \cdots & \partial_n \\ \partial_1 u_1 & \partial_2 u_1 & \cdots & \partial_n u_1 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 u_{n-1} & \partial_2 u_{n-1} & \cdots & \partial_n u_{n-1} \end{pmatrix} = 0, \tag{8.7}$$

which can be easily proved by induction, expanding the determinant with respect to the last row.

Second, (8.2) implies that U is harmonic, and in particular smooth. To prove this recall, that if $A \in \mathbb{GL}(n)$ is an invertible matrix, $A^{-1} = \det A(\operatorname{cof} A)^\top$. In particular, for $B \in \mathbb{SO}(n)$, which means $B^{-1} = B^\top$, $\det B = 1$, we get $\operatorname{cof} B = B$. Then from the asserted inclusion $\nabla U \in \mathbb{SO}(n)$ we get $\nabla(U)(x) = \operatorname{cof} \nabla U(x)$ and by taking the divergence we get the following:

$$\Delta U = \operatorname{div} \nabla \boldsymbol{U} = \operatorname{div} \operatorname{cof} \nabla U(x) = 0.$$

Third, the second gradient squared of any harmonic map can be expressed pointwise via derivatives of the inner products,

$$\frac{1}{2}\left(|\nabla U|^2 - n\right) = \langle \nabla U, \Delta \nabla U \rangle + |\nabla^2 U|^2 = |\nabla^2 U|^2; \tag{8.8}$$

but $|\nabla U|^2 - n = 0$ when U satisfies (8.2). ■

An estimate in terms of $\varepsilon + \sqrt{\varepsilon}$, where $\varepsilon := \left\|\operatorname{dist}(\nabla U, \mathbb{SO}(n))\middle|\mathbb{L}_p(\Omega)\right\|$, is much easier to prove, but is insufficient for the application to plate theory, where one needs to sum the estimate over many small cubes of size h.

Corollary 8.3 (see [Re67]) *If $U_j \to U$ in $\mathbb{W}^1(\Omega)$ and $\operatorname{dist}(\nabla U_j, \mathbb{SO}(n)) \to 0$ in measure, then $\nabla U_j \to R$ in $\mathbb{L}_2(\Omega)$ for some constant rotation matrix $R \in \mathbb{SO}(n)$).*

Let us remind, that the space of $n \times n$ matrices $\mathbb{M}^{n\times n}(\mathbb{R})$ is a real Hilbert space with respect to the inner product (3.12) and the norm

$$\|A\| := \sqrt{\operatorname{Tr}(AA^\top)} = \sqrt{\operatorname{Tr}(A^\top A)} = \sqrt{\sum_{i,j} a_{ij}^2}, \qquad \forall\, A = [a_{ij}]_{n\times n}. \tag{8.9}$$

Note, that the norm has the following property:

$$\|RAR\| = \|RA\| = \|AR\| = \|A\| \qquad \forall\, A \in \mathbb{M}^{n\times n}, \quad \text{and} \quad \forall\, R \in \mathbb{SO}(n). \tag{8.10}$$

Indeed,

$$\|RA\| = \sqrt{\operatorname{Tr}[(RA)^\top (RA)]} = \sqrt{\operatorname{Tr}(A^\top R^\top RA)} = \sqrt{\operatorname{Tr}(A^\top A)} = \|A\|,$$

$$\|AR\| = \sqrt{\operatorname{Tr}[(AR)(AR)^\top]} = \sqrt{\operatorname{Tr}(ARR^\top A^\top)} = \sqrt{\operatorname{Tr}(AA^\top)} = \|A\|$$

and, as a consequence, $\|RAR\| = \|RA\| = \|AR\| = \|A\|$. ■

Let $A = RH_A$ be the left polar decomposition of a matrix A, where $R \in \mathbb{SO}(n)$ is the orthogonal (unitary) matrix $R^\top = R^{-1}$ and H_A is positive, self adjoint (Hermitian) matrix

$$\langle H_A\xi, \xi\rangle \geqslant C_0\|\xi\|^2, \qquad C_0 > 0, \qquad H_A^* = H_A, \qquad \forall\, \xi \in \mathbb{C}.$$

Let us check, that $H_A = \sqrt{A^\top A}$. Indeed, if $A = RU_A$, then $A^\top = H_A^\top R^\top = H_A R^{-1}$ and $\sqrt{A^\top A} = \sqrt{H_A R^{-1} R H_A} = \sqrt{H_A^2} = H_A$. ■

Similarly, for the right polar decomposition $A = H'_A R'$ we get $H'_A = \sqrt{A\,A^\top}$.

By analogue with (7.7) is proved, that if $A = RH_A$ is the left polar decomposition of A, then

$$\begin{aligned} &\mathrm{dist}(A, \mathbb{SO}(n)) = \|H_A - I\| && \text{if } \det A \neq 0 \quad \text{(i.e., } A \text{ is invertible),} \\ &\mathrm{dist}(A, \mathbb{SO}(n)) \leqslant \|H_A - I\| && \text{otherwise.} \end{aligned} \tag{8.11}$$

Chapter 2

Γ-CONVERGENCE BY NUMBERS

3 SOME PRELIMINARIES

The main purpose of the present chapter is to introduce Γ-convergence, discuss it‘s properties and demonstrate its application on a simplest version of dimension-reduced problems. Γ-convergence was introduced by De Giorgi in [DF1] and represents powerful toolkit for the investigation of various dimension-reduction problems.

Our exposition follows mostly the book [Br1]. Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ and $\mathbb{R}$ the set of real numbers.

Definition 3.1 *Let $f : X \to \mathbb{R}$. We define the lower limit (*$\liminf$ *for short) of f at x as*

$$\begin{aligned} \liminf_{y\to x} f(y) &= \inf\left\{ \liminf_j f(x_j) : x_j \in X,\ x_j \to x \right\} \\ &= \inf\left\{ \lim_j f(x_j) : x_j \in X,\ x_j \to x,\ \exists \lim_j f(x_j) \right\}. \end{aligned}$$

and upper limit ($\limsup$ *for short) of f at x as*

$$\begin{aligned} \limsup_{y\to x} f(y) &= \sup\left\{ \limsup_j f(x_j) : x_j \in X,\ x_j \to x \right\} \\ &= \sup\left\{ \lim_j f(x_j) : x_j \in X,\ x_j \to x,\ \exists \lim_j f(x_j) \right\}. \end{aligned}$$

The lower limit is linked to our minimum problems much more then the upper limit. The first notion will be preferred in our statements, but many results will obviously hold for the limsup, with the due changes. Definition (3.1) can also be given if f is not defined in the whole X (in this case the x_j must be taken in the domain of f); in particular, we can have $X = \mathbb{N}$ and $x = \infty$ and recover the usual definition of $\liminf$ and $\limsup$ for sequences.

By taking $x_j = x$ we always get $\liminf_{y\to x} f(y) \leqslant f(x)$. Moreover, it can easily checked that

$$\liminf_{y\to x} (-f(y)) = -\liminf_{y\to x} f(y),$$

$$\liminf_{y\to x} \left(f\left(y\right)+g\left(y\right)\right) \geq \liminf_{y\to x} f\left(y\right)+\liminf_{y\to x} g\left(y\right), \tag{2.1}$$

$$\liminf_{y\to x} \left(f\left(y\right)+g\left(y\right)\right) \leqslant \limsup_{y\to x} f\left(y\right)+\liminf_{y\to x} g\left(y\right).$$

A way to interpret these limit operators is that they give the sharpest upper and lower bounds for the behaviour of f close to x : that is, for all $\varepsilon > 0$ we will have

$$\liminf_{y\to x} f\left(y\right)-\varepsilon < f\left(x'\right) < \liminf_{y\to x} f\left(y\right)+\varepsilon,$$

provided that $d\left(x, x'\right) < \delta = \delta\left(\varepsilon\right)$. With this observation in mind it can be easily checked that we have the equivalent topological definitions:

$$\liminf_{y\to x} f\left(y\right) = \sup_{U\in N(x)} \inf_{y\in U} f\left(y\right), \quad \limsup_{y\to x} f\left(y\right) \inf_{U\in N(x)} \sup_{y\in U} f\left(y\right), \tag{2.2}$$

where we have used the notation $N\left(x\right)$ for the family of all open sets containing a point $x \in X$.

4 Lower semicontinuity

Definition 4.1 *A function $f : X \to \overline{\mathbb{R}}$ will be said to be (sequentially) lower semicontinuous functions (l.s.c for short) at $x \in X$, if for every sequence (x_j) converging to x we have*

$$f\left(x\right) \leqslant \liminf_{j} f\left(x_j\right),$$

or, in other words,

$$f\left(x\right) = \min\left\{\liminf_{j} f\left(x_j\right) : \ x_j \to x\right\}.$$

We will say that f is lower semicontinuous functions (l.s.c for short) (on X) if it is l.s.c at all $x \in X$.

Remark 4.2 *The following conditions are equivalent:*

(i) *f is lower semicontinuous.*

(ii) *we have $f\left(x\right) = \liminf_{y\to x} f\left(y\right)$, for all $x \in X$.*

(iii) *for all $t \in \mathbb{R}$ the sublevel set $\{f \leqslant t\}$ is closed.*

Indeed the equivalence of (i) and (ii) is given by (2.2). Note that (i) implies that if $f\left(x_j\right) \leqslant t$ and $x_j \to x$ then $f\left(x\right) \leqslant t$, while if there exists x and $x_j \to x$ such that $f\left(x\right) > t > \liminf_j f\left(x_j\right)$ then (iii) is violated for such a t.

Remark 4.3 (i) *If f and g are l.s.c at x, then so is $f+g$ by (2.1).*

(ii) *Let* $\{f_j : i \in I\}$ *be a family of l.s.c functions (*I *an arbitrary set of indices, not necessarily countable). Then the function defined by* $f(x) = \sup_i f_i(x)$ is l.s.c. *In fact, for fixed* $x \in X$ *and* $x_j \to x$, *we have*

$$f_i(x) \leqslant \liminf_j f_i(x_j) \leqslant \liminf_j f(x_j).$$

By taking the supremum for $i \in I$ *we obtain* $f(x) \leqslant \liminf_j f(x_j)$. *In particular, the supremum of a family of continues functions is l.s.c.*

(iii) *If* $f = \chi_E$ *is the characteristic function of the set* E, *then* f *is l.s.c, if and only if* E *is open, by Remark (4.2)* (iii).

(iv) *A function* $f : X \to \overline{\mathbb{R}}$ *is called upper semicontinuous if* $-f$ *is l.s.c. All the results of this section have an obvious counterpart for upper semicontinuous functions. In particular* $f = \chi_E$ *is upper semicontinuous if and only if* E *closed.*

5 CONVEXITY

Definition 5.1 *We recall that a function* $f : \mathbb{R}^n \to (-\infty, +\infty]$ *is convex if we have*

$$f(tz_1 + (1-t) z_2) \leqslant tf(z_1) + (1-t) f(z_2)$$

for all $z_1, z_2 \in \mathbb{R}^n$ *and* $t \in (0,1)$.

Remark 5.2 *(a) The convexity of* f *is equivalent to requiring that Jensen's inequality holds:*

$$f\left(\int_X g d\mu\right) \leqslant \int_X f(g) d\mu \tag{2.3}$$

for all probability spaces (X, μ) *and measurable* $f : X \to \mathbb{R}^n$.

(b) If $f \in C^1(\mathbb{R}^n)$, *then it is convex if and only if*

$$f(z) \leqslant f(w) + \left\langle f'(z), z - w \right\rangle \tag{2.4}$$

for all $z, w \in \mathbb{R}^n$.

(c) The supremum of the family of convex functions is convex.

(d) If f is *a convex function and* f *is finite at every point of en open set* Ω, *then* f *is continuous on* Ω *and locally Lipschitz continuous on* Ω.

(e) If f *is convex and there exists* $1 \leqslant p < \infty$ *and* $c > 0$ *such that*

$$0 \leqslant f(z) \leqslant c(1 + |z|^p)$$

for all $z \in \mathbb{R}^n$, *then* f *satisfies the local Lipschitz condition*

$$|f(z) - f(w)| \leqslant c' \left(1 + |z|^{p-1} + |w|^{p-1}\right) |z - w| \tag{2.5}$$

for all $z, w \in \mathbb{R}^n$ *for some* c' *defending only* c *and* p.

(f) If $f_j : \mathbb{R}^n \to \mathbb{R}$ *is a sequence of locally equi-bounded convex functions then there exists a subsequence of* (f_j) *converging uniformly on all compact subsets of* $\mathbb{R}^n$.

We can now recall the definition of Γ-convergence and make some first remark.

6 Γ-CONVERGENCE

Definition 6.1 *(Γ-convergence) We say that a sequence* $f_j : X \to \overline{\mathbb{R}}$ *Γ-converges in* X *to* $f_\infty : X \to \overline{\mathbb{R}}$ *if for all* $x \in X$ *we have*

(i) ($\liminf$ inequality) For every sequence (x_j) converging to x,

$$f_\infty(x) \leqslant \liminf_j f_j(x_j); \tag{2.6}$$

(ii) ($\limsup$ inequality) There exists a sequence $\left(x_j^0\right)$ converging to x, such that

$$f_\infty(x) \geq \limsup_j f_j\left(x_j^0\right); \tag{6.1}$$

The function f_∞ is called the Γ-limit of (f_j) and we write $f_\infty = \Gamma\text{-}\lim_j f_j$.

Pointwise definition. The definition above can be also given a fixed point $x \in X$: we say that (f_j) Γ-convergence at x to the value $f_\infty(x)$ if (i), (ii) above hold. In this case we write

$$f_\infty(x) = \Gamma\text{-}\lim_j f_j(x).$$

In this notation, f_j Γ-convergence to f_∞ if and only if

$$f_\infty(x) = \Gamma\text{-}\lim_j f_j(x)$$

at all $x \in X$.

If we want to highlight the role of the metric, we can add the dependence on the distance d, and write $\Gamma(d)\text{-}\lim_j$ $\Gamma(d)$-convergence and so on.

Different ways of writing the $\limsup$ **inequality.** Note that if (x_j) satisfies the $\limsup$ inequality, then by (2.6) we have

$$f_\infty(x) \leqslant \liminf_j f_j(x) \leqslant \limsup_j f_j(x) \leqslant f_\infty(x),$$

so that indeed

$$f_\infty(x) = \lim_j f_j(x_j),$$

hence (ii) can be substituted by

$(ii)'$ (exitance of a recovery sequence) there exists a sequence (x_j) converging to x, such that

$$f_\infty(x) = \lim_j f_j(x_j); \tag{2.8}$$

On the other hand, sometimes it is more convenient to prove (ii) with a small error and then deduce its validity by an approximation argument: that is, (ii) can be replaced by

$(ii)''$ (approximate $\limsup$ inequality) for all $\varepsilon > 0$ there exists a sequence (x_j) converging to x, such that

$$f_\infty(x) \geq \limsup_j f_j(x_j) - \varepsilon. \tag{2.9}$$

In the following (and in the literature) all conditions (ii), $(ii)'$, $(ii)''$ are equally referred to as the $\limsup$ inequality or as the exitance of a recovery sequence.

Note that the $\liminf$ inequality (i) can be rewritten as

$$f_\infty(x) \leqslant \inf\left\{\liminf_j f_j(x_j):\ x_j \to x\right\}.$$

Trivially, we always have

$$\inf\left\{\liminf_j f_j(x_j):\ x_j \to x\right\} \leqslant \inf\left\{\limsup_j f_j(x_j):\ x_j \to x\right\}$$

and if $(\overline{x_j})$ is a recovery sequence for (ii) we have

$$\inf\left\{\limsup_j f_j(x_j):\ x_j \to x\right\} \leqslant \limsup_j f_j(\overline{x_j}) \leqslant f_\infty(x),$$

so that (i) and (ii) imply that we have

$$f_\infty(x) = \min\left\{\liminf_j f_j(x_j):\ x_j \to x\right\} = \min\left\{\limsup_j f_j(x_j):\ x_j \to x\right\}. \quad (2.10)$$

(and actually both minima are obtained as limits along a recovery sequence). It is important to keep in mind this characterization as many properties of the Γ-limit will be easily explained from it.

Remark 6.2 ***(Γ-convergence as an equality of upper and lower bounds)*** *It is sometimes convenient to state the equality in (2.10) as an equality of imfima*

$$f_\infty(x) = \inf\left\{\liminf_j f_j(x_j):\ x_j \to x\right\} = \inf\left\{\limsup_j f_j(x_j):\ x_j \to x\right\}. \quad (2.11)$$

This equality is indeed equivalent to the definition of Γ-limit, that is, the Γ-limit exists if and only if the two infima in (2.11) are equal. This characterization will be important in that in this way the exitance of the Γ-limit (which not always exists) is expressed as the equality of two quantities which are always defined and which can (and will) be studied separately. The first quantity can be thought as the lower bound for the Γ-limit, the second as an upper bound.

By (2.11) we obtain in particular that the Γ-limit, if it exists, is unique.

Remark 6.3 ***(stability under continuous perturbations).*** *An important property of Γ-converges is its stability under continuous perturbations: if (f_j) Γ-converges to f_∞ and $g: X \to [-\infty, +\infty]$ is a d-continuous function then $(f_j + g)$ Γ-converges to $f_\infty + g$.*

This is an immediate consequence of the definition, since if (i) holds then for all $x \in X$ and $x_j \to x$ we get

$$f_\infty(x) + g(x) \leqslant \liminf_j f_j(x_j) + \lim_j g(x_j) = \liminf_j (f_j(x_j) + g(x_j)),$$

while if $(ii)'$ above holds then we get

$$f_\infty(x) + g(x) = \lim_j f_j(x_j) + \lim_j g(x_j) = \lim_j (f_j(x_j) + g(x_j)).$$

Remark 6.4 ***(Γ-limit of a constant sequence).*** *Consider the simplest case $f_j = f$ for all $j \in \mathbb{N}$. In this case it will be easily seen that f_j Γ-convergence. By the* $\liminf$ *inequality, the limit f_∞ must satisfy*

$$f_\infty\left(x\right) \leqslant \liminf_j f\left(x_j\right)$$

for all x and $x_j \to x$. If f is not lower semicontinuous then there exists $\overline{x}$ and a sequence $\overline{x_j} \to \overline{x}$ such that

$$\liminf_j f_j\left(\overline{x_j}\right) < f\left(\overline{x}\right),$$

hence, in particular $f_\infty\left(\overline{x}\right) \neq f\left(\overline{x}\right)$. This shows that Γ-convergence does not satisfy the requirement that a constant sequence $f_j = f$ converges to f (if f is not lower semicontinuous). We will see however that this holds true in the family of lower semicontinuous functions (see Remark (6.6))

Remark 6.5 ***(dependence on the metric).*** *The choice of the metric on X is clearly a fundamental step in problems involving Γ-limit. In general, even when two distance d and d' are comfortable; That is*

$$\lim_j d'\left(x_j, x\right) = 0 \Rightarrow \lim_j d\left(x_j, x\right) = 0. \tag{2.12}$$

The exitance of Γ-limit in one metric does not imply the existence of the Γ-limit in the second (see examples in Section 1.3). However, in this situation, if both Γ-limits exist then we have

$$\Gamma\left(d\right) - \lim_j f_j \leqslant \Gamma\left(d^{'}\right) - \lim_j f_j.$$

This is clear, for example, from the characterization (2.10) *since the set of converging sequences for d is larger than that is for $\mathrm{d}^{'}$.*

Remark 6.6 ***(comparison with pointwise and uniform limits).*** *As a very particular case, we can consider the metric $\mathrm{d}^{'}$ of the discrete topology (where the only converging sequences are constant sequences). In this case the Γ-limit coincides with the pointwise limit (if it exists). If d is any other metric then (2.12) holds trivially, so that we obtain*

$$\Gamma\left(d\right) - \lim_j f_j \leqslant \lim_j f_j$$

as a particular case of the previous remark.

If f_j converges uniformly to a f on an open set U (in particular if $f_j = f$) and f is l.s.c. then we have also that f_j Γ-convergence to f. Indeed, the $\limsup$ inequality is obtained by the constant sequence, while the liminf inequality is immediately verified once we remark that if $x_j \to x \in U$, then $x_j \in U$, for j large enough, so that

$$\liminf_j f_j\left(x_j\right) = \lim_j\left(f_j\left(x_j\right) - f\left(x_j\right)\right) + \liminf_j f_j\left(x\right) \geq f\left(x\right).$$

7 SOME EXAMPLES ON THE REAL LINE

In this section we will compute some simple Γ-limit of functions defined on the real line (equipped with the usual Euclidean distance) and will also compare it with the point-wise convergence, which can be thought of as a Γ-limit with respect to the discrete metric, as explained in Remark 6.6.

We have seen that a constant sequence $f_j = f$ Γ-converges to f if and only if f is lower semicontinuous. Hence, if f is not l.s.c the point-wise limit and Γ-limit are different. Now we construct an example where these two limits differ even if the pointwise limit is lower semicontinuous.

Example 1. Let
$$f_j(t) = f_1(jt),$$
where
$$f_1(t) = \sqrt{2}t\exp\left(-\left(2t^2-1\right)/2\right)$$
or
$$f_1(t) = \begin{cases} \pm 1, & \text{if} \quad t = \pm 1, \\ 0, & otherwise. \end{cases}$$
Then $f_j \to 0$ pointwise, but $\Gamma - \lim_j f_j = f$, where
$$f(t) = \begin{cases} 0, & \text{if} \quad t \neq 0, \\ -1, & \text{if} \quad t = 0. \end{cases}$$
Indeed, the sequence f_j converges locally uniformly (and hence also Γ-converges) to 0 in $\mathbb{R}\setminus\{0\}$, while clearly the optimal sequence for $x = 0$ is $-1/j$, for which $f_j(x_j) = -1$. In this case the pointwise and Γ-limits both exist and are different at one point.

Example 2. Take
$$f_j(t) = -f_1(jt),$$
where f_1 is as in the previous example. Clearly, the Γ-limit remains unchanged. This shows that in general
$$\Gamma\text{-}\lim_j(-f_j) \neq -\Gamma\text{-}\lim_j f_j,$$
$$\Gamma\text{-}\lim_j(f_j+g_j) \neq \Gamma\text{-}\lim_j f_j + \Gamma\text{-}\lim_j g_j.$$
(taking in the example $g_j = -f_j$) even if all functions are continuous.

The pointwise and Γ-limits may exist and be different at every point. Take $g_j = f_j$, where
$$g_j(t) = \begin{cases} 0, & \text{if} \quad t \notin \mathbb{Q} \text{ or } t = \frac{k}{n}, \text{ with } k \in \mathbb{Z} \text{ and } n \in \{1,...,j\} \\ -1, & otherwise. \end{cases}$$
We then have $f_j \to -1$. The liminf inequality is trivial and limsup inequality is easily obtained by remarking that $\{g_j = -1\}$ is dance for all $j \in \mathbb{N}$.

Example 3. There may be no pointwise converging subsequence of (f_j) but the $\Gamma\text{-}\lim_j f_j$ may exist all the same. Take, for example, $f_j(t) = -\cos(jt)$. In this case $\Gamma\text{-}\lim_j f_j = -1$. Again, the liminf inequality is trivial, while the $\limsup$ inequality is easily obtained by taking, for example, $x_j = [jx/2\pi]\,2\pi/j$ ($[t]$ the integer part of t).

The sequence f_j may be converging pointwise, but may not Γ-converge. Take for example

$$f_j = (-1)^j g_j,$$

where g_j, is defined in Example 2. In this case $f_j \to 0$ pointwise, but the Γ-$\lim_j f_j$ does not exist at any point.

8 Function spaces and their properties

In all the follows (a,b) is bounded open interval of $\mathbb{R}$.

The norm (or quasi norm) of the space $L_p(a,b)$ $(0 < p \leqslant \infty)$ is defined by

$$\|f\|_{L_p(a,b)} := \left(\int_a^b |f(x)|^p \, d\mu(x) \right)^{1/p}, \qquad (0 < p < \infty) \tag{2.13}$$

and

$$\|f\|_{L_\infty(a,b)} = \inf_M \left(\mu \left\{ x : |f(x)| \geq M \right\} = 0 \right) \tag{2.14}$$

It is well known that

$$\|f\|_{L_\infty(a,b)} = \lim_{p \to \infty} \left(\int_{(a,b)} |f(x)|^p \, d\mu(x) \right)^{1/p}. \tag{2.15}$$

Definition 8.1 ***(Weak derivative).*** *We say that $u \in L_1(a,b)$ is weakly differentiable if a function $g \in L_1(a,b)$ exists such that the following integration by parts formula holds*

$$\int_a^b u\varphi' dt = -\int_a^b g\varphi dt$$

for all $\varphi \in C_0^1(a,b)$. If such g exists then it is called the weak derivative of u and is denoted u'.

Remark 8.2 *The notion of weak derivative is an extension of notion of classical derivative: If $u \in C^1(a,b)$ and its classical derivative belongs to $L_1(a,b)$ then the classical derivative coincides with its weak derivative. The function $x \to |x|$ is weakly differentiable in any (a,b), but $u \notin C^1(-1,1)$ and its weak derivative is the function $x \to x/|x|$, which in turn is not weakly differentiable in $(-1,1)$.*

Definition 8.3 ***(Sobolev spaces)*** *Let $p \in [0.+\infty]$. The Sobolev spaces $\mathbb{W}^{1,p}(a,b)$ is defined as the space of all weakly differentiable $u \in L_p(a,b)$ such that $u' \in L_p(a,b)$. The norm of u in $\mathbb{W}^{1,p}(a,b)$ is defined as*

$$\|u\|^p_{\mathbb{W}^{1,p}(a,b)} = \|u\|^p_{L_p(a,b)} + \left\| u' \right\|^p_{L_p(a,b)}.$$

The space $W^{1,p}_{loc}(\mathbb{R})$ consists of such functions u for which $u \in W^{1,p}_{loc}(I)$ for all bounded open intervals $I \subset \mathbb{R}$.

Remark 8.4 *The Sobolev spaces $\mathbb{W}^{1,p}(a,b)$ equipped with the natural norm, indicated in Definition 8.3, is a Banach space. This is easily checked upon identifying $\mathbb{W}^{1,p}(a,b)$ with the subspace of $L_p(a,b) \times L_p(a,b)$ of all pairs $\left(u,u^{'}\right)$ with $u \in \mathbb{W}^{1,p}(a,b)$. The same identification shows that $\mathbb{W}^{1,p}(a,b)$ is separable if $1 \leqslant p < \infty$.*

Theorem 8.5 ***(pointwise value of Sobolev spaces).*** *Let $u \in \mathbb{W}^{1,p}(a,b)$. Then there exists $\dot{u} \in C_0^1[(a,b)]$ such, that $\dot{u} = u$ a.e. on (a,b) and*

$$\dot{u}(y) - \dot{u}(x) = \int_x^y u^{'}(t)\,dt \tag{2.16}$$

for all $x, y \in [a,b]$. We commontly identify u with its continuous reprensentative $u^{'}$ whenever pointwise values are taken into account.

Remark 8.6 ***(boundary values).*** *If $\mathbb{W}^{1,p}(a,b)$ then the boundary values $u(a)$ and $u(b)$ are uniquely defined by the values $\dot{u}(a)$ and $\dot{u}(b)$, respectively. We may then extand a function $\mathbb{W}^{1,p}(a,b)$ to the function $W_{loc}^{1,p}(\mathbb{R})$ by simply setting $u(t) = u(a)$ for $t \leqslant a$ and $u(t) = b$, for $t \geq b$.*

Theorem 8.7 ***(equivalent definitions of Sobolev spaces).*** *Let $1 \leqslant p \leqslant \infty$. Then the following statements are equivalent:*

(i) $u \in \mathbb{W}^{1,p}(a,b)$.

(ii) There exists $C \geq 0$, such that

$$\left|\int_a^b u\varphi^{'}dt\right| \leqslant C\,\|\varphi\|_{L_{p^{'}}(a,b)},$$

for arbitrary $\varphi \in C_0^1(a,b)$.

(iii) There exists $C \geq 0$, such that for all for all $I \subset (a,b)$ and for all $h \in \mathbb{R}$ such that $|h| \leqslant dist\,(I,\{a,b\})$ we have

$$\|\tau_h u - u\|_{L_p(I)} \leqslant C\,|h|,$$

where $\tau_h u = u(t-h)$.

(iv) There exists a sequence (u_j) in $C^\infty([a,b])$ such that

$$\lim_j \|u_j - u\|_{\mathbb{W}^{1,p}(a,b)} = 0. \tag{2.17}$$

(v) There exists a sequence (u_j) in $C^\infty(\mathbb{R})$ such that (2.17) holds.

(vi) There exists a sequence (u_j) in $C^\infty(\mathbb{R})$ such that $\sup_j \|u_j\|_{\mathbb{W}^{1,p}(a,b)} < \infty$ and

$$\lim_j \|u_j - u\|_{L_p(a,b)} = 0.$$

Remark 8.8 *(a) The best constant C in (ii) and (iii) above is $\left\|u^{'}\right\|_{L_p(a,b)}$.*

(b) If $p=1$ then (i) ⇒(ii)⇔(iii). Note that the function $x \to x/|x|$ satisfies (ii)-(vi) with $p=1$ but does not belongs to $\mathbb{W}^{1,1}(-1,1)$.

(c) By (iii) we easily see that $\mathbb{W}^{1,\infty}(a,b)$ coincides with the space $lip(a,b)$ of all Lipschitz functions on (a,b) and $\left\|u^{'}\right\|_{L_\infty(a,b)}$ is the best Lipschitz constant for u.

Theorem 8.9 ***(embedding results).*** *There exists a constant $C=C(a,b)$ such that*

$$\|u\|_{L_\infty(a,b)} \leqslant C\|u\|_{\mathbb{W}^{1,p}(a,b)} \tag{2.18}$$

Moreover, we have compact embeddings

$$\mathbb{W}^{1,p}(a,b) \subset C^0([a,b]) \tag{2.19}$$

for $1<p\leqslant\infty$ and

$$\mathbb{W}^{1,1}(a,b) \subset L_q(a,b)$$

for all $q \geq 1$.

Definition 8.10 *The space $W_0^{1,p}(a,b)$ is defined as the closure of $C_0^\infty(a,b)$ in the $W_0^{1,p}$-norm, or equivalently, as the set of those $u \in \mathbb{W}^{1,p}(a,b)$ with boundary values $u(a)=u(b)=0$.*

Theorem 8.11 ***(Poincare's inequality).*** *There exists a constant $C=C(a,b)$ such that*

$$\|u\|_{\mathbb{W}^{1,p}(a,b)} \leqslant C\left\|u^{'}\right\|_{L_p(a,b)}, \tag{2.20}$$

for all $u \in \mathbb{W}^{1,p}(a,b)$ such that $\dot{u}(x)=0$, for some $x\in[a,b]$. In particular this holds for $u \in W_0^{1,p}(a,b)$.

Definition 8.12 *Let $u:(a,b)\to\mathbb{R}$ be measurable function. The total variation of u on (a,b) is defined as*

$$Var(u,(a,b)) := \\ = \inf_{v=u,\text{ a.e. on }(a,b)} \sup\left\{\sum_{i=1}^{N}|v(t_{i+1})-v(t_i)| : \ a<t_0<...<t_N<b, \ N\in\mathbb{N}\right\}$$

If $Var(u,(a,b))<\infty$ then we say that u is a function of bounded variation. We simply write $Var(u)$ if (a,b) is fixed.

Remark 8.13 *If $u \in \mathbb{W}^{1,1}(a,b)$, then*

$$Var(u,(a,b)) = \int_a^b \left|u^{'}\right| dt.$$

In particular, u is a function of bounded variation. Note that also

$$v(x) = x/|x|$$

is a function of bounded variation with $Var(v,(-1,1))=2$.

9 More properties of Γ-limits

From the definition of Γ-convergence we immediately obtain the following properties.

Remark 9.1 *If $\{f_{j_k}\}$ is a subsequence of $\{f_j\}$, then*

$$\Gamma - \lim\inf_j f_j \leqslant \Gamma - \lim\inf_k f_{j_k}, \qquad \Gamma - \lim\sup_k f_{j_k} \leqslant \Gamma - \lim\sup_j f_j.$$

In particular, if $f_\infty = \Gamma - \lim_j f_j$ exists then for every increasing sequence of integers j_k $f_\infty = \Gamma - \lim_k f_{j_k}$.

Remark 9.2 *If g is a continuous function then $f_\infty + g = \Gamma - \lim_j (f_j + g)$; more in general, if $g_j \to g$ uniformly then $f_\infty + g = \Gamma - \lim_j (f_j + g_j)$. In particular, if $f_j \to f$ uniformly on an open set U, then $\Gamma - \lim_j f_j = sc\, f$ On U.*

Remark 9.3 *If $f_j \to f$ pointwise, then $\Gamma - \lim\sup_j f_j \leqslant f$ and, hence, $\Gamma - \lim\sup_j f_j \leqslant sc\, f$.*

We can state some simple but important cases when Γ-limit does exist and is computed easily.

Proposition 9.4 (Γ-limit of monotone sequaences) *(i) (decreasing sequences) If $f_{j+1} \leqslant f_j$ for all $j \in \mathbb{N}$, then*

$$\Gamma - \lim_j f_j = sc(\inf_j f_j) = sc(\lim_j f_j). \tag{9.1}$$

(ii) (increasing sequences) If $f_j \leqslant f_{j+1}$ for all $j \in \mathbb{N}$, then

$$\Gamma - \lim_j f_j = sc(\sup_j sc\, f_j) = \lim_j scf_j. \tag{9.2}$$

In particular, if f_j is l.s.c. for every $j \in \mathbb{N}$, then

$$\Gamma - \lim_j f_j = \lim_j f_j. \tag{9.3}$$

In particular, if f_j is l.s.c. for every $j \in \mathbb{N}$, then

Proof: As $f_j \to \inf_k f_k$ pointwise, by Remark 9.3 we have $\Gamma - \lim\sup_j f_j \leqslant sc(\inf_k f_k)$, while the other inequality is trivially derived from the inequality $sc(\inf_k f_k) \leqslant \inf_k f_k \leqslant f_j$ and (i) is proved.

To prove (ii) note that since $sc\, f_j \to \sup_k sc\, f_k$ pointwise, by Remark 9.3

$$\Gamma - \lim\sup_j f_j = \Gamma - \lim\sup_j sc\, f_j \leqslant \sup_k sc\, f_k.$$

On the other hand $sc\, f_k \leqslant f_j$ for all $j \geqslant k$ so that the converse inequality follows easily. ■

Remark 9.5 *By Proposition 9.4.(ii), if* f_j *is a equi-mildly coercive non-decreasing sequence of l.s.c. functions, then* $\sup_j \min_X f_j = \min_X \sup_j f_j$.

Proposition 9.6 (Compactness of Γ-convergence) *Let* (X, d) *be a separable metric space. and for all* $j \in \mathbb{N}$ *let* $f_j : X \to \overline{\mathbb{R}}$ *be a function. Then there exists a subsequence* f_{j_k} *such that the* Γ*-limit* $\Gamma - \lim_k f_{j_k}$ *exists for all* $x \in X$.

Proof: Let $\{U_k\}$ be a countable base of open sets in the topology of X. Since $\overline{\mathbb{R}}$ is compact, there exists an increasing sequence of integers $\{\sigma_j^0\}_j$ along which the limit

$$\lim_j \inf_{y \in U_0} f_{\sigma^0 j}(y)$$

exists, and for all $k \geqslant 1$ we define $\{\sigma_j^k\}_j$ recurrently as a subsequence of $\{\sigma_j^{k-1}\}_j$ along which the limit

$$\lim_j \inf_{y \in U_0} f_{\sigma^k j}(y)$$

exists. The "duiagonal" sequence $j+k := \sigma_k^k$ being a subsequence of $\{\sigma_j^j\}_j$, has the property that the limit

$$\lim_j \inf_{y \in U_\ell} f_{j_k}(y)$$

exists for all $\ell \in \mathbb{N}$. In particular, we have

$$\lim \inf_k \inf_{y \in U_\ell} f_{j_k}(y) = \lim \sup_k \inf_{y \in U_\ell} f_{j_k}(y)$$

for all $\ell \in \mathbb{N}$, and the claimed convergence follows. ■

Remark 9.7 *If* (X, d) *is not a separable metric space, then Proposition 9.6 fails. As an example we can take* $X = \{-1, 1\}^{\mathbb{N}}$ *equipped with the discrete topology.* X *is metrizable and* Γ*-convergence on* X *is equivalent to pointwise convergence. We take the sequence* $f_j : X \to \{-1, 1\}$ *defined by* $f_j(x) = x_j$ *if* $x = (x_0, x_1, \ldots)$*. If* $\{f_{j_k}\}$ *is a subsequence of* $\{f_j\}$ *and we define* x *by* $x_{j_k} = (-1)^k$ *and* $x_j = 1$ *if* $j \notin \{j_k : k \in \mathbb{N}\}$*, then the limit* $\lim_k f_{j_k}(x)$ *does not exist. Hence no subsequence of* $\{f_j\}$ Γ*-converges.*

Γ-convergence enjoys the following useful property.

Proposition 9.8 (Urisohn property of Γ-convergence) *We have* $f_\infty = \Gamma - \lim_j f_j$ *if and only if for every subsequence* $\{f_{j_k}\}$ *there exists a further subsequence which* Γ*-converges to* f_∞.

Proof: Clearly, if f_j Γ-converges to f_∞, then every subsequence of f_j Γ-converges to the same limit (see Remark 9.1).

For an increasing sequence of integers $\{j_k\}$ we have

$$\Gamma - \lim \inf_j f_j \leqslant \Gamma - \lim \inf_k f_{j_k} \leqslant \Gamma - \lim \sup_k f_{j_k} \leqslant \Gamma - \lim \sup_j f_j.$$

Hence if $\Gamma - \lim\inf_k f_{j_k}(x) = f_\infty(x)$ but $\Gamma - \lim\, f_j(x)$ does not exist we have

$$\text{either}\quad f_\infty(x) \leqslant \Gamma - \lim\sup_j f_j(x) \quad \text{or} \quad f_\infty(x) \geqslant \Gamma - \lim\inf_j f_j(x).$$

In the first case we have

$$f_\infty(x) < \sup_{U\in\mathscr{N}(x)} \lim\sup_j \inf_{y\in U} f_j(y),$$

so that there exists $U \in \mathscr{N}(x)$ with the property

$$f_\infty(x) < \lim\sup_j \inf_{y\in U} f_j(y).$$

This means that there exists a subsequence $\{f_{j_k}\}$ of $\{f_j\}$ along which

$$f_\infty(x) < \lim\inf_k \inf_{y\in U} f_{j_k}(y),$$

so that $f_\infty(x) < \Gamma - \lim\inf_k \inf\limits_{y\in U} f_{j_k}(y)$ leads to a contradiction. In the second case a sequence x_j converging to x exists such that $\lim\inf_j f_j(x_j) < f_\infty(x)$. This means that $\Gamma - \lim\sup_k f_{j_k}, f_\infty(x)$, thus giving a contradiction. ■

Proposition 9.9 *Let X be a topological vector space. If $\{f_j\}$ is the sequence of convex functions, the Γ-limit $f := \Gamma - \lim\sup_j f_j$ is also a convex function.*

The statement fails in general case.

Proof: We leave the proof to the reader as an exercise (see [Br1], Exercise 1.6). ■

Chapter 3

Γ-CONVERGENCE OF HEAT TRANSFER EQUATION

A mixed boundary value problem for the stationary heat transfer equation in a thin layer around a surface $\mathscr{C}$ with the boundary is investigated. The main object is to trace what happens in Γ-limit when the thickness of the layer converges to zero. The limit Dirichlet BVP for the Laplace-Beltrami equation on the surface is described explicitly and we show how the Neumann boundary conditions in the initial BVP transform in the Γ-limit. For this we apply the variational formulation and the calculus of Günter's tangential differential operators on a hypersurface and layers, which allow global representation of basic differential operators and of corresponding boundary value problems in terms of the standard Euclidean coordinates of the ambient space $\mathbb{R}^n$.

The exposition follows the paper of T. Buchukuri, R. Duduchava & G. Tephnadze [BDT1].

1 INTRODUCTION

The main object of the paper is to demonstrate what happens with a boundary value problem for the Laplace equation in a thin layer Ω^ε around a surface $\mathscr{C}$ in $\mathbb{R}^3$ when the thickness of the layer ε diminishes to zero: $\varepsilon \to 0$. We impose the Neumann boundary conditions on the upper and lower faces of the layer $\mathscr{C} \times \{\pm\varepsilon\}$ and the Dirichlet boundary conditions on the lateral surface $\partial\mathscr{C} \times (-\varepsilon, \varepsilon)$.

The limit of the associated functionals is understood in the sense of Γ-convergence and the main tool is the representation of differential operators with the help of Gunter's derivatives-the system of tangential derivatives on the surface $\mathscr{D}_j := \partial_j - \nu_j\partial_{\boldsymbol{\nu}}$, $j = 1, 2, 3$ and the normal derivative $\partial_{\boldsymbol{\nu}} := \sum_{j=1}^{3} \nu_j\partial_j$, where $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^\top$ is the unit normal vector field on the mid surface $\mathscr{C}$. The first-order differential operator $\mathscr{D}_j$ is the directional derivative along $\pi\, e^j$, where $\pi : \mathbb{R}^3 \to T\mathscr{C}$ is the orthogonal projection onto the tangent plane to $\mathscr{C}$ and $e^1, \dots, e^n$ is the canonical basis in the Euclidean space $e^j = (\delta_{jk})_{1\leq k\leq 3} \in \mathbb{R}^3$, with δ_{jk} denoting the Kronecker symbol (cf. [Gu94], [KGBB76], [Du02a]).

Calculus of Gunter's derivatives on a hypersurface allows representation of the most basic partial differential operators (PDO's), as well as their associated boundary value problems, on a hypersurface $\mathscr{C}$ in global form, in terms of the standard spatial coordinates in $\mathbb{R}^n$. Such BVPs arise in a variety of situations and have many practical applications. See, for example, [**?**, §72] for the heat conduction by surfaces, [**?**, §10] for the equations of surface flow, [Ci1], [**?**] for the vacuum Einstein equations describing gravitational fields, [**?**] for the Navier-Stokes equations on spherical domains, as well as the references therein.

A hypersurface $\mathscr{C}$ in $\mathbb{R}^3$ has the natural structure of a 2-dimensional Riemannian manifold and the aforementioned PDE's are not the immediate analogues of the ones corresponding to the flat, Euclidean case, since they have to take into consideration geometric characteristics of $\mathscr{C}$ such as curvature. Inherently, these PDE's are originally written in local coordinates, intrinsic to the manifold structure of $\mathscr{C}$.

The surface gradient

$$\mathscr{D} := (\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3)^\top \tag{1.1}$$

is defined on $\mathscr{C}$, and has a relatively simple structure. In terms of (1.1), the Laplace-Beltrami operator on $\mathscr{C}$ simply becomes (see [MM84, pp. 2ff and p. 8.])

$$\Delta_{\mathscr{C}} = \mathscr{D}^*\mathscr{D} \quad \text{on} \quad \mathscr{C}.$$

Alternatively, this is the natural operator associated with the Euler-Lagrange equations for the variational integral

$$\mathscr{E}[u] = -\frac{1}{2}\int_{\mathscr{C}} \langle \mathscr{D}u, \mathscr{D}u\rangle\, dS, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$.

A similar approach, based on the principle that, at equilibrium, the displacement minimizes the potential energy, leads to the derivation of the equation for the elastic hypersurface (cf. [DMM06, **?**] for the isotropic case).

These results are useful in numerical and engineering applications (cf. [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]) and we plan to treat a number of special surfaces in greater detail in a subsequent publication.

We consider heat conduction by an "isotropic" medium, governed by the Laplace equations, with the classical mixed Dirichlet-Neumann boundary conditions on the boundary in the layer domain $\Omega^\varepsilon := \mathscr{C} \times (-\varepsilon, \varepsilon)$ of thickness 2ε, where $\mathscr{C} \subset \mathscr{S}$ is a smooth subsurface of a closed hypersurface $\mathscr{S}$ with smooth nonempty boundary $\partial\mathscr{C}$. In particular, we confine ourselves with zero Dirichlet and non-zero Neumann data (see Remark 4.1 for the case of non-zero Dirichlet data):

$$\begin{aligned}
&\Delta_{\Omega^\varepsilon}\widetilde{T}(\mathscr{x}, t) = f(\mathscr{x}, t), && (\mathscr{x}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon),\\
&\widetilde{T}^+(\mathscr{x}, t) = 0, && (\mathscr{x}, t) \in \partial\mathscr{C} \times (-\varepsilon, \varepsilon),\\
&(\partial_t\widetilde{T})^+(\mathscr{x}, \pm\varepsilon) = q_\varepsilon^\pm(\mathscr{x}), && \mathscr{x} \in \mathscr{C}.
\end{aligned} \tag{1.3}$$

In the investigation we apply that the Laplace operator $\Delta_{\Omega^\varepsilon} = \partial_1^2 + \partial_2^2 + \partial_3^2$ is represented as the sum of the Laplace-Beltrami operator on the mid-surface, the square of the transversal derivative and the lower order term

$$\Delta_{\Omega^\varepsilon}\widetilde{T} = \Delta_{\mathscr{C}}\widetilde{T} + \partial_t^2\widetilde{T} + 2\mathscr{H}_{\mathscr{C}}\partial_t\widetilde{T}, \tag{1.4}$$

where $\mathscr{D}_4 = \partial_t$. The Laplace-Beltrami operator $\Delta_{\mathscr{C}}$ defined in (0.12) and the mean curvature $\mathscr{H}_{\mathscr{C}}(\mathscr{x}) = \sum_{k=1}^{3} \mathscr{D}_k\mathscr{N}_k(\mathscr{x})$ of the surface are extended properly from $\mathscr{C}$ (see the forthcoming Lemma 2.2).

Introducing the function $G(\mathscr{x},t)$ which has the same Dirichlet and Neumann traces as T on the $\partial\mathscr{C}\times(-\varepsilon,\varepsilon)$ and on $\mathscr{C}\times\{\pm\varepsilon\}$ respectively

$$G(\mathscr{x},t)=\frac{1}{4\varepsilon}(t+\varepsilon)^2q^+_\varepsilon(\mathscr{x})-\frac{1}{4\varepsilon}(t-\varepsilon)^2q^-_\varepsilon(\mathscr{x}),\tag{1.5}$$

we can reduce the problem (1.3) to the following boundary value problem with respect to unknown function $T=\widetilde{T}-G$

$$\begin{aligned}\Delta_{\Omega^\varepsilon}T(\mathscr{x},t)&=F(\mathscr{x},t),\qquad(\mathscr{x},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon),&(1.6)\\ T^+(\mathscr{x},t)&=0,\qquad(\mathscr{x},t)\in\partial\mathscr{C}\times(-\varepsilon,\varepsilon),&(1.7)\\ (\partial_tT)^+(\mathscr{x},\pm\varepsilon)&=0,\qquad\mathscr{x}\in\mathscr{C}.&(1.8)\end{aligned}$$

where

$$\begin{aligned}F(\mathscr{x},t)&:=f(\mathscr{x},t)-\frac{1}{4\varepsilon}\big((t+\varepsilon)^2\Delta_{\mathscr{C}}q^+_\varepsilon(\mathscr{x})-(t-\varepsilon)^2\Delta_{\mathscr{C}}q^-_\varepsilon(\mathscr{x})\big)\\ &-\frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{x})}{2\varepsilon}\big((t+\varepsilon)q^+_\varepsilon(\mathscr{x})-(t-\varepsilon)q^-_\varepsilon(\mathscr{x})\big)-\frac{1}{2\varepsilon}\big(q^+_\varepsilon(\mathscr{x})-q^-_\varepsilon(\mathscr{x})\big),\quad(1.9)\\ &\qquad(\mathscr{x},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon).\end{aligned}$$

The BVP (1.6)-(1.8) is reformulated as the minimization problem for the functional which, after scaling (stretching the variable $t=\varepsilon\tau$ and dividing the entire functional by ε) has the following form

$$\begin{aligned}E_\varepsilon(T_\varepsilon):&=\int_{-1}^{1}\int_{\mathscr{C}}\left[\frac{1}{2}(\mathscr{D}_{\mathscr{C}}T_\varepsilon)^2(\mathscr{x},\tau)+\frac{1}{2\varepsilon^2}(\partial_\tau T_\varepsilon)^2(\mathscr{x},\tau)+F_\varepsilon(\mathscr{x},\tau)T_\varepsilon(\mathscr{x},\tau)\right]d\sigma d\tau\quad(1.10)\\ F_\varepsilon(\mathscr{x},t)&:=F(\mathscr{x},\varepsilon t)=f(\mathscr{x},\varepsilon t)-\frac{\varepsilon}{4}\big((t+1)^2\Delta_{\mathscr{C}}q^+_\varepsilon(\mathscr{x})-\frac{\varepsilon}{4}(t-1)^2\Delta_{\mathscr{C}}q^-_\varepsilon(\mathscr{x})\big)\\ &-\frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{x})}{2}\big((t+1)q^+_\varepsilon(\mathscr{x})-(t-1)q^-_\varepsilon(\mathscr{x})\big)-\frac{1}{2\varepsilon}\big(q^+_\varepsilon(\mathscr{x})-q^-_\varepsilon(\mathscr{x})\big),\qquad(1.11)\\ T_\varepsilon(\mathscr{x},\tau)&:=T\left(\mathscr{x},\varepsilon\tau\right),\quad T_\varepsilon\in\widetilde{\mathbb{H}}^1(\Omega^1,\partial\mathscr{C}\times(-1,1)),\quad F_\varepsilon\in\widetilde{\mathbb{H}}^{-1}(\Omega^1),\quad q^\pm_\varepsilon\in\widetilde{\mathbb{H}}^2(\mathscr{C}),\\ &\qquad(\mathscr{x},t)\in\mathscr{C}\times(-\varepsilon,\varepsilon).\end{aligned}$$

(For the definition of $\widetilde{\mathbb{H}}^1(\Omega^1,\partial\mathscr{C}\times(-1,1))$ see (4.9).)

Let

$$\mathscr{P}(\mathscr{C}):=\left\{T\in\mathbb{H}^1(\Omega^1):T(\mathscr{x},\tau)=T_{\mathscr{C}}(\mathscr{x}),\qquad T_{\mathscr{C}}\in\widetilde{\mathbb{H}}^1(\mathscr{C}),\quad\tau\in[-1,1]\right\}.\tag{1.12}$$

The main result of the present investigation is the following Theorem 1.1.

Theorem 1.1 *Let*

$$f_\varepsilon(\mathscr{x},t):=f(\mathscr{x},\varepsilon t)\underset{\varepsilon\to0}{\to}f^0(\mathscr{x})\quad in\quad\mathbb{L}_2(\Omega^1),$$

$q_\varepsilon^\pm \in \widetilde{\mathbb{H}}^2(\mathscr{C})$ be uniformly bounded (with respect to ε) in $\mathbb{H}^2(\mathscr{C})$, and

$$\lim_{\varepsilon\to 0} q_\varepsilon^+ = \lim_{\varepsilon\to 0} q_\varepsilon^- = q_0, \quad q_0 \in \mathbb{L}_2(\mathscr{C}),$$
$$\frac{1}{2\varepsilon}(q_\varepsilon^+ - q_\varepsilon^-) \underset{\varepsilon\to 0}{\to} q_1 \quad in \quad \mathbb{L}_2(\mathscr{C}).$$

Then the functional in (1.10) *Γ-converges to the functional*

$$E^{(0)}(T) = \begin{cases} \int\limits_{\mathscr{C}} \left[\langle \mathscr{D}_{\mathscr{C}} T_{\mathscr{C}}(\mathscr{x}), \mathscr{D}_{\mathscr{C}} T_{\mathscr{C}}(\mathscr{x})\rangle + 2\big(f^0(\mathscr{x}) - \mathscr{H}_{\mathscr{C}}^0 q_0(\mathscr{x}) - q_1(\mathscr{x})\big) T_{\mathscr{C}}(\mathscr{x})\right] d\sigma, & \\ & if \quad T \in \mathscr{P}(\mathscr{C}); \\ +\infty, & if \quad T \notin \mathscr{P}(\mathscr{C}). \end{cases} \tag{1.13}$$

The following Dirichlet boundary value problem for Laplace-Beltrami equation on the mid surface $\mathscr{C}$

$$\begin{aligned} &\Delta_{\mathscr{C}} T(\mathscr{x}) = f^0(\mathscr{x}) - \mathscr{H}_{\mathscr{C}}^0 q_0(\mathscr{x}) - q_1(\mathscr{x}), \quad \mathscr{x} \in \mathscr{C}, \\ &T^+(\mathscr{x}) = 0, \qquad \mathscr{x} \in \partial\mathscr{C}, \\ &T \in \mathbb{H}^1(\mathscr{C}), \quad f^0, q_0, q_1 \in \mathbb{L}_2(\mathscr{C}), \end{aligned} \tag{1.14}$$

is an equivalent reformulation of the minimization problem with the energy functional (1.13).

Remark 1.2 *The BVP* (1.14) *is the "Γ limit" of the initial BVP* (1.3) *in the following sense: The corresponding functional* (1.13) *is the Γ-limit of the functional* (1.10)*, corresponding to the BVP* (1.6)-(1.8).

It is remarkable to note that the weak derivative q^0 of the Neumann condition from the initial BVP (1.3) *migrated into the right hand side of the limit equation.*

Note as well that the Γ-limit $T_{\mathscr{C}}(\mathscr{x})$ of a solution $T(\mathscr{x}, \varepsilon\tau)$. $T \in \mathbb{H}^1(\Omega_\varepsilon)$ to the BVP (1.6)-(1.8) *has better smoothness $T_{\mathscr{C}} \in \mathbb{H}^1(\mathscr{C})$ than expected.*

Γ-limits of boundary value problems in thin structures, reformulated as a minimization problem for the associated energy functional, were studied by many authors (see, e.g., [FJM1, **?**, Ve82, Br1] and the literature cited therein). But mostly the Lamé equations for elastic plates $\mathscr{C} \subset \mathbb{R}^2$ and zero boundary conditions were treated (the Laplace equation for a plate is studied in [Br1]). In the papers [**?**, Ve82] the case of shells is treated, but with a different technique. Our approach is based on the calculus of Günter's derivatives, which we find more appropriate for such problems.

The layout of the paper is as follows. In § 1-§ 2 we review some basic differential-geometric concepts which are relevant for the work at hand (e.g., hypersurfaces and different methods of their identification). In § 3 we identify the most important partial differential operators on hypersurfaces, such as gradient, divergence, Laplace-Beltrami operator. In § 4 we consider the energy functional (1.2) and the associated Euler-Lagrange equation. In sections § 5, § 6 the aforementioned approach is applied and proved main theorems of the present paper, including Theorem 1.1.

2 LAPLACE OPERATOR IN A LAYER DOMAIN

We will keep the notation of § 1: Θ, ω, $\mathscr{S}$ and $\mathscr{C}$. We consider a **layer domain**

$$\Omega^\varepsilon := \Big\{\mathscr{x}_t \in \mathbb{R}^n\ :\ \mathscr{x}_t = \mathscr{x} + t\boldsymbol{\nu}(\mathscr{x}) = \Theta(x) + t\boldsymbol{\nu}\big(\Theta(x)\big)\,, \quad x\in\omega\,,\ -\varepsilon<t<\varepsilon\Big\}$$
$$= \mathscr{C}\times(-\varepsilon,\varepsilon), \tag{2.1}$$

where $\boldsymbol{\nu}(\mathscr{x}) = \boldsymbol{\nu}(\Theta(y))$ for $\mathscr{x} = \Theta(y)\in\mathscr{S}$, is the outer unit normal vector field (see (2.8) and (5.4)). The surface $\mathscr{C}$ is a mid-surface for the layer domain.

We will also use the notation $\boldsymbol{\nu}(y) := \boldsymbol{\nu}(\Theta(y))$ for brevity unless this leads to a confusion. The coordinate t will be referred to as the **transverse variable**.

Without going into detail let us remark only that if the hypersurface $\mathscr{S}$ is C^2-smooth and $1/\varepsilon$ is more than the maximum of modules of all principal curvatures of the surface $\mathscr{S}$ (i.e., of all eigenvalues $|\lambda_1(\mathscr{x})|,\dots,|\lambda_{n-1}(\mathscr{x})|, \lambda_n(\mathscr{x})\equiv 0$ of the Weingarten matrix $\mathscr{W}_{\mathscr{S}}(\mathscr{x})$, $\mathscr{x}\in\mathscr{S}$), then the mapping

$$\begin{aligned}&\Theta^\varepsilon\ :\ \omega^\varepsilon := \omega\times(-\varepsilon,\varepsilon)\to\Omega^\varepsilon\,, \quad \omega^\varepsilon\subset\mathbb{R}^n\,,\\ &\Theta^\varepsilon(y,t) := \Theta(y) + t\boldsymbol{\nu}(y)\,, \qquad (y,t)\in\omega^\varepsilon\end{aligned} \tag{2.2}$$

is a diffeomorphism.

We will also suppose that $\mathscr{N}$ is a proper extension of the outer unit normal vector field $\boldsymbol{\nu}$ into the layer neighborhood Ω^ε (cf. Definition 2.2).

The n-tuple $\boldsymbol{g}_1 := \partial_1\Theta,\dots,\boldsymbol{g}_{n-1} := \partial_{n-1}\Theta, \boldsymbol{g}_n := \mathscr{N}$, where $\mathscr{N}$ is the proper extension of $\boldsymbol{\nu}$ in the neighborhood Ω^ε, is a basis in Ω^ε and arbitrary vector field $\boldsymbol{U} = \sum_{j=1}^n U_j^0\boldsymbol{e}^j$ on Ω^ε is represented with this basis in "curvilinear coordinates".

Let us consider the system of $(n+1)$-vectors

$$\boldsymbol{d}^j := \boldsymbol{e}^j - \mathscr{N}_j\mathscr{N}\,, \qquad j=1,\dots,n \quad \text{and} \quad \boldsymbol{d}^{n+1} := \mathscr{N}, \tag{2.3}$$

where $\boldsymbol{e}^1,\dots,\boldsymbol{e}^n$ is the Cartesian basis in $\mathbb{R}^n$ (cf. (0.7)); the first n vectors $\boldsymbol{d}^1,\dots,\boldsymbol{d}^n$ are tangential to the surface $\mathscr{C}$, while the last one $\boldsymbol{d}^{n+1} = \mathscr{N}$ is orthogonal to all $\boldsymbol{d}^1,\dots,\boldsymbol{d}^n$. This system is, obviously, linearly dependent, but full and any vector field $\boldsymbol{U}\in\mathscr{W}(\Omega^\varepsilon)$ is written in the following form:

$$\boldsymbol{U} = \sum_{j=1}^n U_j\boldsymbol{e}^j = \sum_{j=1}^{n+1} U_j^0\boldsymbol{d}^j. \tag{2.4}$$

Since the system $\{\boldsymbol{d}^j\}_{j=1}^{n+1}$ is linearly dependent

$$\sum_{j=1}^n \mathscr{N}_j\boldsymbol{d}^j = 0, \qquad \langle\mathscr{N},\boldsymbol{d}^j\rangle = 0, \quad j=1,\dots,n, \tag{2.5}$$

the representation (2.4) is not unique. To fix the unique representation in (2.4) we will keep the following convention:

$$U_j^0 := U_j - \langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{N}_j, \quad j=1,\dots,n, \quad U_{n+1}^0 = \langle\mathscr{N},\boldsymbol{U}\rangle = \sum_{j=1}^n U_j\mathscr{N}_j. \tag{2.6}$$

The convention (2.6) is natural because if the vector $\boldsymbol{U}(\mathscr{x})$ is tangent to $\mathscr{C}$ for $\mathscr{x}\in\mathscr{C}$, then $U^0_j(\mathscr{x}):=U_j(\mathscr{x})$ for $j=1,\dots,n$ and $U^0_{n+1}(\mathscr{x})=0$.

Moreover, if the scalar product of vectors

$$\boldsymbol{U}:=\sum_{j=1}^{n}U_j\boldsymbol{e}^j=\sum_{j=1}^{n+1}U^0_j\boldsymbol{d}^j,\quad \boldsymbol{V}:=\sum_{j=1}^{n}V_j\boldsymbol{e}^j=\sum_{j=1}^{n+1}V^0_j\boldsymbol{d}^j \tag{2.7}$$

is defined by the equality

$$\langle\boldsymbol{U},\boldsymbol{V}\rangle^0:=\sum_{j=1}^{n+1}U^0_jV^0_j,$$

then the "new" and the "old" scalar products coincide:

$$\begin{aligned}\langle\boldsymbol{U},\boldsymbol{V}\rangle^0&=\sum_{j=1}^{n+1}U^0_jV^0_j=\sum_{j=1}^{n}(U_j-\mathscr{N}_j\langle\mathscr{N},\boldsymbol{U}\rangle)(V_j-\mathscr{N}_j\langle\mathscr{N},\boldsymbol{V}\rangle)+\langle\mathscr{N},\boldsymbol{U}\rangle\langle\mathscr{N},\boldsymbol{V}\rangle\\&=\sum_{j=1}^{n}U_jV_j=\langle\boldsymbol{U},\boldsymbol{V}\rangle.\end{aligned} \tag{2.8}$$

In particular,

$$\|\boldsymbol{U}\|^0:=\sum_{j=1}^{n+1}|U^0_j|^2=\sum_{j=1}^{n}|U_j|^2=\|\boldsymbol{U}\|. \tag{2.9}$$

Note for a later use, that due to the equalities (2.5) and the convention (2.6) we get

$$\begin{aligned}\partial_{\boldsymbol{U}}&=\sum_{j=1}^{n}U_j\partial_j=\sum_{j=1}^{n}[U^0_j\partial_j+\langle\mathscr{N},\boldsymbol{U}\rangle\mathscr{N}_j\partial_j]=\sum_{j=1}^{n}U^0_j(\partial_j-\mathscr{N}_j\partial_{\mathscr{N}})+\langle\mathscr{N},\boldsymbol{U}\rangle\partial_{\mathscr{N}}\\&=\sum_{j=1}^{n}U^0_j\mathscr{D}_j+U^0_{n+1}\mathscr{D}_{n+1}=\sum_{j=1}^{n+1}U^0_j\mathscr{D}_j=:\mathscr{D}_{\boldsymbol{U}}.\end{aligned}$$

Definition 2.1 *For a function $\varphi\in\mathbb{H}^1(\Omega^\varepsilon)$ the extended gradient is*

$$\mathscr{D}_{\Omega^\varepsilon}\varphi=\Big\{\mathscr{D}_1\varphi,...,\mathscr{D}_n\varphi,\mathscr{D}_{n+1}\varphi\Big\}^\top=\sum_{j=1}^{n+1}(\mathscr{D}_j\varphi)\boldsymbol{d}^j,\qquad \mathscr{D}_{n+1}\varphi:=\partial_{\mathscr{N}}\varphi \tag{2.10}$$

and for a smooth vector field $\boldsymbol{U}=\sum_{j=1}^{n+1}U^0_j\boldsymbol{d}^j\in\mathscr{W}(\Omega^\varepsilon)$ (see (2.4), (2.6)) the extended divergence is

$$\operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}:=\sum_{j=1}^{n+1}\mathscr{D}_jU^0_j+\mathscr{H}^0_{\mathscr{C}}\langle\mathscr{N},\boldsymbol{U}\rangle=-\nabla^*_{\Omega^\varepsilon}\boldsymbol{U}, \tag{2.11}$$

since

$$\mathscr{H}^0_{\Omega^\varepsilon}(x) := \sum_{j=1}^{n} \partial_j \mathscr{N}_j(x) = \sum_{j=1}^{n+1} \mathscr{D}_j \mathscr{N}_j(x) = \sum_{j=1}^{n} \mathscr{D}_j \nu_j(\mathscr{x}) = \mathscr{H}^0_{\mathscr{C}}(\mathscr{x}), \qquad x \in \Omega^\varepsilon, \qquad \mathscr{x} = \pi_{\mathscr{S}} x$$

and $\mathscr{H}^0_{\mathscr{C}}(\mathscr{x})$ *differs from the mean curvature* $\mathscr{H}_{\mathscr{C}}(\mathscr{x})$ *(see* (5.15)*) by the constant multiplier* $\mathscr{H}^0_{\mathscr{C}}(\mathscr{x}) = (n-1)\mathscr{H}_{\mathscr{C}}(\mathscr{x})$.

Lemma 2.2 *The classical gradient* $\nabla\varphi := \Big\{\partial_1\varphi, ..., \partial_n\varphi\Big\}^\top$*, written in the full system of vectors* $\big\{\boldsymbol{d}^j\big\}_{j=1}^{n+1}$ *in* (2.3) *coincides with the extended gradient* $\mathscr{D}_{\Omega^\varepsilon}\varphi$ *in* (2.10).

Similarly: the classical divergence $\operatorname{div}\boldsymbol{U} := \sum_{j=1}^{n}\partial_j U_j$ *of a vector field* $\boldsymbol{U} := \sum_{j=1}^{n} U_j \boldsymbol{e}^j$*, written in the full system* (2.3)*, coincides with the extended divergence* $\operatorname{div}\boldsymbol{U} = \operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}$ *in* (2.11).

The extended gradient and the negative extended divergence are dual $\mathscr{D}^*_{\Omega^\varepsilon} = -\operatorname{div}_{\Omega^\varepsilon}$ *and* $\operatorname{div}^*_{\Omega^\varepsilon} = -\mathscr{D}_{\Omega^\varepsilon}$.

The Laplace-Beltrami operator $\Delta_{\Omega^\varepsilon} := \operatorname{div}_{\Omega^\varepsilon}\mathscr{D}_{\Omega^\varepsilon}\varphi = -\mathscr{D}^*_{\Omega^\varepsilon}\Big(\mathscr{D}_{\Omega^\varepsilon}\varphi\Big)$ *on* Ω^ε*, written in the full system* (2.3)*, acquires the following form*

$$\Delta_{\Omega^\varepsilon}\varphi = \sum_{j=1}^{n}\mathscr{D}_j^2\varphi + \partial^2_{\mathscr{N}}\varphi + \mathscr{H}^0_{\mathscr{C}}\partial_{\mathscr{N}}\varphi = \sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi + \mathscr{H}^0_{\mathscr{C}}\mathscr{D}_{n+1}\varphi\,, \quad \varphi \in \mathbb{H}^2(\Omega^\varepsilon)\,. \tag{2.12}$$

Proof: A similar lemma is proved in [Du10, Lemma 4.3], but definition of the divergence $\operatorname{div}_{\Omega^\varepsilon}$ is different there. Therefore we expose the full proof below.

That the gradients coincide follows from the choice of the full system (2.3):

$$\begin{aligned}\nabla\varphi := \Big\{\partial_1\varphi, ..., \partial_n\varphi\Big\}^\top &= \sum_{j=1}^{n}(\partial_j\varphi)\boldsymbol{e}^j = \sum_{j=1}^{n}(\mathscr{D}_j\varphi + \mathscr{N}_j\mathscr{D}_{n+1}\varphi)\boldsymbol{e}^j \\ &= \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^j + (\mathscr{D}_{n+1}\varphi)\mathscr{N} = \sum_{j=1}^{n+1}(\mathscr{D}_j\varphi)\boldsymbol{d}^j = \mathscr{D}_{\Omega^\varepsilon}\varphi\end{aligned} \tag{2.13}$$

since

$$\begin{gathered}\boldsymbol{e}^j = \boldsymbol{d}^j + \mathscr{N}_j\mathscr{N}, \quad \partial_j = \mathscr{D}_j + \mathscr{N}_j\partial_{\mathscr{N}}, \\ \sum_{j=1}^{n}\mathscr{N}_j\mathscr{D}_j = 0, \quad \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{e}^j = \sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^j.\end{gathered} \tag{2.14}$$

By applying (2.6) and (2.14) we proceed as follows:

$$\begin{aligned}\operatorname{div}\boldsymbol{U}=&\sum_{j=1}^{n}\partial_j U_j=\sum_{j=1}^{n}\mathscr{D}_j U_j+\sum_{j=1}^{n}\mathscr{N}_j\partial_{\mathscr{N}}U_j=\sum_{j=1}^{n}\mathscr{D}_j\left[U_j^0+\mathscr{N}_j\langle\mathscr{N},\boldsymbol{U}\rangle\right]\\&+\sum_{j=1}^{n}\partial_{\mathscr{N}}\left(\mathscr{N}_j U_j\right)=\sum_{j=1}^{n}\mathscr{D}_j U_j^0+\sum_{j=1}^{n}(\mathscr{D}_j\mathscr{N}_j)\langle\mathscr{N},\boldsymbol{U}\rangle+\mathscr{D}_{n+1}U_{n+1}^0\\=&\sum_{j=1}^{n+1}\mathscr{D}_j U_j^0+\mathscr{H}_{\mathscr{C}}^0\langle\mathscr{N},\boldsymbol{U}\rangle=\operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}.\end{aligned}\tag{2.15}$$

The proved equality and the classical equality $\nabla^* = -\operatorname{div}$, ensure the both claimed equalities $\mathscr{D}^*_{\Omega^\varepsilon} = -\operatorname{div}_{\Omega^\varepsilon}$ and $\operatorname{div}^*_{\Omega^\varepsilon} = -\mathscr{D}_{\Omega^\varepsilon}$:

$$(\mathscr{D}_{\Omega^\varepsilon}\varphi,\boldsymbol{U})=(\nabla\varphi,\boldsymbol{U})=-(\varphi,\operatorname{div}\boldsymbol{U})=-(\varphi,\operatorname{div}_{\Omega^\varepsilon}\boldsymbol{U}).$$

Formula (2.12) for the Laplace-Beltrami operator is a direct consequence of equalities (2.13), (2.15) and definitions. Indeed, the first n components of the gradient

$$\nabla\varphi=\mathscr{D}_{\Omega^\varepsilon}\varphi=\sum_{j=1}^{n}(\mathscr{D}_j\varphi)\boldsymbol{d}^j+(\mathscr{D}_{n+1}\varphi)\mathscr{N}$$

have the property $(\mathscr{D}_j\varphi)^0=\mathscr{D}_j\varphi-\langle\mathscr{N},\mathscr{D}_{\Omega^\varepsilon}\varphi\rangle\mathscr{N}_j=\mathscr{D}_j\varphi$ because (see the third formula in (2.14)) $\langle\mathscr{N},\mathscr{D}_{\Omega^\varepsilon}\varphi\rangle=\sum_{j=1}^{n}\mathscr{N}_j\mathscr{D}_j\varphi=0$ and we can write

$$\begin{aligned}\boldsymbol{\Delta}\varphi=\operatorname{div}\nabla\varphi=\operatorname{div}_{\Omega^\varepsilon}\mathscr{D}_{\Omega^\varepsilon}\varphi=&\sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi+\mathscr{H}_{\mathscr{C}}^0\langle\mathscr{N},\nabla\varphi\rangle\\=&\sum_{j=1}^{n+1}\mathscr{D}_j^2\varphi+\mathscr{H}_{\mathscr{C}}^0\mathscr{D}_{n+1}\varphi=\Delta_{\Omega^\varepsilon}\varphi.\end{aligned}$$

□

3 CONVEX ENERGIES

Let again Ω^ε be a layer domain of width 2ε in the direction transversal to the mid-surface $\mathscr{C}$ (cf. § 3).

Any minimizer u of the energy functional

$$\mathscr{E}^\varepsilon(u):=\int_{\Omega^\varepsilon}\langle\nabla u,\nabla u\rangle\,dy,\qquad u\in\mathbb{H}^1(\Omega^\varepsilon)\tag{3.1}$$

should satisfy

$$\begin{aligned}0=&\frac{d}{dt}\mathscr{E}^\varepsilon(u+tv)\Big|_{t=0}=\int_{\Omega^\varepsilon}\left[\langle\nabla u,\nabla v\rangle+\langle\nabla v,\nabla u\rangle\right]\,dy\\=&2\mathrm{Re}\int_{\Omega^\varepsilon}\langle\nabla u,\nabla v\rangle\,dy=-2\mathrm{Re}\int_{\Omega^\varepsilon}\langle\operatorname{div}\nabla u,v\rangle\,dy=-2\mathrm{Re}\int_{\Omega^\varepsilon}\langle\Delta u,v\rangle\,dy\end{aligned}$$

for arbitrary $v \in \widetilde{\mathbb{H}}^1(\Omega^\varepsilon)$, which implies

$$\Delta\, u = 0 \qquad \text{on} \quad \Omega^\varepsilon. \tag{3.2}$$

In other words, (3.2) is the Euler-Lagrange equation associated with the energy functional (7.1).

Similarly, minimizers of the energy functional

$$\mathscr{E}_0(u) := \int_{\mathscr{C}} \langle \nabla_{\mathscr{C}} u, \nabla_{\mathscr{C}} u\rangle \, d\sigma, \qquad u \in H^1(\mathscr{C})$$

on the hypersurface $\mathscr{C}$ should satisfy the following Laplace-Beltrami equation

$$\Delta_{\mathscr{C}} u := \mathrm{div}_{\mathscr{C}} \nabla_{\mathscr{C}} u = 0 \qquad \text{on} \quad \mathscr{C}. \tag{3.3}$$

To treat the dimension reduction problem for the Laplace equation (see [Br1] for a similar consideration in case of a flat 3D body), we assume, without restricting generality, that Ω^1 (i.e., for $\varepsilon = 1$) is still a layer domain. Otherwise we can first change the variable $\mathscr{x}_n = \varepsilon_0 \bar{\mathscr{x}}_n$, $0 < \bar{\mathscr{x}}_n < 1$, where $0 < \varepsilon_0 < 1$ is such that Ω^{ε_0} is still a layer domain.

Next we introduce a new coordinate system (cf. (2.6))

$$\begin{gathered} x := \sum_{m=1}^n x_m \boldsymbol{e}^m = \sum_{m=1}^n \mathscr{x}_m \boldsymbol{d}^m + t\boldsymbol{d}^{n+1}, \\ \mathscr{x}_k := x_k - \mathscr{N}_k \langle \mathscr{N}, x\rangle, \quad k = 1, \dots, n, \quad t = \mathscr{x}_{n+1} := \langle x, \mathscr{N}\rangle = \sum_{m=1}^n x_m \mathscr{N}_m \end{gathered} \tag{3.4}$$

and define the scalar product of elements as follows (cf. similar in (2.7)):

$$\langle \mathscr{x}, \mathscr{y}\rangle := \sum_{j=1}^{n+1} \mathscr{x}_j \mathscr{y}_j \quad \text{for} \quad \mathscr{x} := \sum_{m=1}^{n+1} \mathscr{x}_m \boldsymbol{d}^m, \quad \mathscr{y} := \sum_{m=1}^{n+1} \mathscr{y}_m \boldsymbol{d}^m.$$

Then (cf. (2.8)-(2.9))

$$\begin{aligned} \langle \mathscr{x}, \mathscr{y}\rangle = \sum_{j=1}^{n+1} \mathscr{x}_j \mathscr{y}_j = \sum_{j=1}^n (x_j - \mathscr{N}_j \langle \mathscr{N}, x\rangle)((y_j - \mathscr{N}_j \langle \mathscr{N}, y\rangle)) + \langle \mathscr{N}, x\rangle \langle \mathscr{N}, y\rangle \\ = \sum_{j=1}^n x_j \overline{y_j} = \langle x, y\rangle. \end{aligned}$$

In particular,

$$\|\mathscr{x}\| := \sum_{j=1}^{n+1} |\mathscr{x}_j|^2 = \sum_{j=1}^n |x_j|^2 = \|x\|. \tag{3.5}$$

Due to Lemma 2.2 the classical gradient in the energy functional (7.1) can be replaced by the extended gradient

$$\mathscr{E}^\varepsilon(u) := \int_{\Omega^\varepsilon} \langle \mathscr{D}_{\Omega^\varepsilon} u(y), \mathscr{D}_{\Omega^\varepsilon} u(y)\rangle \, dy = \int_{-\varepsilon}^{\varepsilon} \int_{\mathscr{C}} \left[|\mathscr{D}_{\mathscr{C}} u(\mathscr{x}, t)|^2 + |\partial_t u(\mathscr{x}, t)|^2 \right] \, d\sigma \, dt, \tag{3.6}$$

where $\mathscr{D}_{\mathscr{C}} := (\mathscr{D}_1, \ldots, \mathscr{D}_n)^\top$ is the surface gradient and $u \in \mathbb{H}^1(\Omega^\varepsilon)$ is arbitrary, because $\mathscr{D}_{n+1} = \partial_{\mathscr{N}} = \partial_t$. Here $\mathscr{C}$ is the mid surface of the layer domain $\Omega^\varepsilon = \mathscr{C} \times (-\varepsilon, \varepsilon)$ and $d\sigma$ is the surface measure on $\mathscr{C}$.

Due to the representation (7.14) and the new coordinate system (3.4) we can apply the scaling with respect to the variable t and study the scaled energy. The approach is based on Γ-convergence (see [Br1, FJM1]) and can be applied to a general energy functional which is convex and has square growth. The problem we have in mind is the following: *Do these energies defined on thin n-dimensional domains Ω^ε converge (and in which sense) to an energy defend on the $n-1$ dimensional Hypersurface $\mathscr{C}$ (the mid-surface of Ω^ε) when the domain Ω^ε is "squeezed" infinitely in the transversal direction to $\mathscr{C}$?*

In the next two sections we apply the results developed in the present paper to boundary value problems for the heat conduction by a hypersurface. In particular we shall show, that if the thickness of the layer domain Ω^ε, with the mid-surface $\mathscr{C}$, tends to zero, the functionals in variational formulation of the linear heat conduction equation, Gamma-converge to the functional corresponding to some explicit boundary value problem for the Laplace-Beltrami equation on the mid-surface $\mathscr{C}$.

4 VARIATIONAL REFORMULATION OF HEAT TRANSFER PROBLEMS

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ with the piecewise smooth boundary $\partial\Omega = \overline{\mathscr{C}}_D \cup \overline{\mathscr{C}}_N$, where $\mathscr{C}_D$ and $\mathscr{C}_N$ are open non-intersecting surfaces $\mathscr{C}_D \cap \mathscr{C}_N = \varnothing$ and their common boundary is a smooth arc. Denote by $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)^\top$ the unit normal on $\mathscr{C}$, external with respect to Ω.

We consider the general steady-state, linear heat transfer problem for a medium occupying domain Ω. We assume that on the $\mathscr{C}_D$ part of the boundary $\partial\Omega$ the temperature g is prescribed, while on the $\mathscr{C}_N$ part of $\partial\Omega$ is prescribed the heat flux q.

We look for a temperature distribution $T(x)$ in Ω, which satisfies the linear heat conduction equation

$$\operatorname{div}(\mathscr{A}(x)\nabla T)(x) = f(x), \qquad x \in \Omega \tag{4.1}$$

and boundary conditions

$$T^+(y) = g(y) \qquad \text{on } \mathscr{C}_D, \tag{4.2}$$

$$-\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\rangle = q(y) \qquad \text{on } \mathscr{C}_N, \tag{4.3}$$

where $\mathscr{A}$ is the thermal conductivity, f is the heat source, g is the distribution of temperature and q is the heat flux. All these quantities are supposed known.

We assume, that $\mathscr{A}(x)$ is a bounded measurable and positive definite 3×3 matrix-function (cf. a similar condition (3.38))

$$\langle \mathscr{A}(x)\xi, \xi\rangle \geqslant C\|\xi\|^2, \qquad x \in \Omega, \quad \xi \in \mathbb{R}^3.$$

The following inequality is an obvious consequence of the positive definiteness of $\mathscr{A}$:

$$(\mathscr{A}\boldsymbol{U}, \boldsymbol{U}) \geqslant C\|\boldsymbol{U}|\mathbb{L}_2(\Omega)\|^2$$

for all 3-vectors $\boldsymbol{U} = (U_1, U_2, U_3)^\top \in \mathbb{L}_2(\Omega)$. Further we assume that the traces $\mathscr{A}^+(y)$ at the boundary $\mathscr{C}$ exist. Then $\mathscr{A}^+$ has the same properties as $\mathscr{A}$ on Ω, namely, is a bounded, measurable positive definite matrix function.

We impose the following natural constraints on the solution T and on the prescribed data f, g, q:

$$T \in \mathbb{H}^1(\Omega), \quad f \in \widetilde{\mathbb{H}}^{-1}(\Omega), \quad g \in \mathbb{H}^{1/2}(\mathscr{C}_D), \quad q \in \mathbb{H}^{-1/2}(\mathscr{C}_N). \tag{4.4}$$

The existence of the traces $\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+\rangle \in \mathbb{H}^{-1/2}(\mathscr{C}_3)$, which is not ensured by the trace theorem, follows from the Green formula

$$\int_\Omega (\operatorname{div} \mathscr{A}(x)\nabla T)(x)\psi(x)dx = \int_{\mathscr{C}} \langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\rangle \psi^+(y)\, d\sigma - \int_\Omega \langle \mathscr{A}(x)\nabla T(x), \nabla\psi(x)\rangle\, dx \tag{4.5}$$

by the duality between the spaces $\mathbb{H}^{1/2}(\mathscr{C})$ and $\widetilde{\mathbb{H}}^{-1/2}(\mathscr{C})$ due to the fact that T is a solution to the equation (4.1). For this we rewrite (4.5) in the form

$$\int_{\mathscr{C}} \langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T)^+(y)\rangle \psi^+(y)\, d\sigma = \int_\Omega f(x)\psi(x)dx + \int_\Omega \langle \mathscr{A}(x)\nabla T(x), \nabla\psi(x)\rangle\, dx,$$

and note that $\psi \in \mathbb{H}^1(\Omega)$ is arbitrary and, therefore, $\psi^+ \in \mathbb{H}^{1/2}(\mathscr{C})$ is arbitrary.

First we will reduce the BVP (4.1)–(4.3) to the equivalent BVP with vanishing Dirichlet data.

Remark 4.1 *Let us assume the subsurface $\mathscr{C}_D$ is smooth and $g \in \mathbb{H}^s(\mathscr{C}_D)$, $s \geqslant \frac{1}{2}$. There exists a domain Ω' with a smooth boundary $\mathscr{C}' := \partial\Omega'$, with the properties: $\Omega \subset \Omega'$ and $\mathscr{C}_D \subset \mathscr{C}'$. Let $g^0 \in \mathbb{H}^s(\mathscr{C}')$ be such extension of g which maintains the space.*

The Dirichlet BVP

$$\begin{array}{ll} \operatorname{div}(\mathscr{A}(x)\nabla G)(x) = 0, & x \in \Omega', \\ G^+(y) = g^0(y) & \text{on } \mathscr{C}' \end{array} \tag{4.6}$$

has a unique solution

$$G(x) = W\left(\frac{1}{2}I + W_0\right)^{-1} g^0(x), \qquad x \in \Omega', \qquad G \in \mathbb{H}^{s+1/2}(\Omega'),$$

where W is the double layer potential for the operator $\operatorname{div}\mathscr{A}(x)\nabla$ and W_0 is its direct value (a singular integral operator) on the surface $\mathscr{C}'$ $I : \mathbb{H}^s(\mathscr{C}') \to \mathbb{H}^s(\mathscr{C}')$ is a unit operator). Then the BVP

$$\begin{array}{ll} \operatorname{div}(\mathscr{A}(x)\nabla T_0)(x) = f(x), & x \in \Omega, \\ T_0^+(y) = 0 & \text{on } \mathscr{C}_D, \\ -\langle \boldsymbol{\nu}(y), \mathscr{A}^+(y)(\nabla T_0)^+(y)\rangle = q_0(y) & \text{on } \mathscr{C}_N \end{array} \tag{4.7}$$

is an equivalent reformulation of the BVP (4.1)–(4.3)*, now with vanishing Dirichlet traces. The solutions and Neumann datae are related as follows:*

$$\begin{array}{c} T_0(x) := T(x) - G(x), \quad x \in \Omega, \\ q_0(y) := q(y) - \left(\partial_{\boldsymbol{\nu}} W\left(\frac{1}{2}I + W_0\right)^{-1} g^0\right)^+(y), \quad x \in \mathscr{C}. \end{array} \tag{4.8}$$

Note, that if we require higher smoothness for the Neumann data $q \in \mathbb{H}^r(\mathscr{C}_N)$, $r > -1/2$ and take $g \in \mathbb{H}^{r+1}(\mathscr{C}_D)$ (i.e., $s = r + 1$ in (4.6)*), the Neumann data in the BVP* (4.7) *inherits the same smoothness $q_0 \in \mathbb{H}^r(\mathscr{C}_N)$.*

Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipshitz boundary $\mathscr{M} := \partial\Omega$ and $\mathscr{M}_0 \subset \partial\Omega$-be a subsurface of the boundary surface which has the non-zero measure. By $\widetilde{\mathbb{H}}^1(\Omega, \mathscr{M}_0)$ we denote a subspace of $\mathbb{H}^1(\Omega)$ of those functions which have vanishing traces on the part of the boundary

$$\widetilde{\mathbb{H}}^1(\Omega, \mathscr{M}_0) := \left\{\varphi \in \mathbb{H}^1(\Omega) \ : \ \varphi^+(y) = 0 \quad \forall\, y \in \mathscr{M}_0\right\}. \tag{4.9}$$

This space inherits the standard norm from $\mathbb{H}^1(\Omega)$:

$$\|\varphi \,|\, \mathbb{H}^1(\Omega)\,\| : \ = \left[\|\varphi \,|\, \mathbb{L}_2(\Omega)\,\|^2 + \sum_{j=1}^{n} \|\partial_j\varphi \,|\, \mathbb{L}_2(\Omega)\|^2\right]^{1/2}.$$

Consider the functional

$$\Phi(T) = \int\limits_{\Omega} \left[\frac{1}{2}\langle \mathscr{A}(x)\nabla T(x), \nabla T(x)\rangle + f(x)T(x)\right] dx + \int\limits_{\mathscr{C}_N} q(y)T^+(y)d\sigma \tag{4.10}$$

where f and q satisfy conditions (4.4) and $T \in \mathbb{H}^1(\Omega)$ has vanishing traces on $\mathscr{C}_D$, i.e., $T \in \widetilde{\mathbb{H}}^1(\Omega, \mathscr{C}_D)$ (see (4.9)).

The second summand in the integral on Ω is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{-1}(\Omega)$ and $\mathbb{H}^1(\Omega)$. Concerning the integral on $\mathscr{C}_N$: it is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{1/2}(\mathscr{C}_N)$ and $\mathbb{H}^{-1/2}(\mathscr{C}_N)$ because $q \in \mathbb{H}^{-1/2}(\mathscr{C}_N)$ and the conditions $T \in \widetilde{\mathbb{H}}^1(\Omega, \mathscr{C}_D)$, $\operatorname{supp} T^+ \subset \mathscr{C}_N$ imply the inclusion $T^+ \in \widetilde{\mathbb{H}}^{1/2}(\mathscr{C}_N)$.

Theorem 4.2 *The problem* (4.1)-(4.3) *with vanishing Dirichlet condition* $T^+(y) = g(y) = 0$ *for all* $y \in \mathscr{C}_D$ *is reformulated into the following equivalent variational problem: Let* f *and* q *satisfy conditions* (4.4) *and look for a temperature distribution* $T \in \widetilde{\mathbb{H}}^1(\Omega, \mathscr{C}_D)$ *(see* (4.9)*) which is a stationary point of the functional* (4.10)*.*

Proof: Let $T(x)$ be a stationary point of the functional (4.10). Consider the variation

$$\delta\Phi = \frac{d}{d\varepsilon}\Phi(T + \varepsilon\boldsymbol{V})|_{\varepsilon=0} = \int\limits_{\Omega} \left[\langle \mathscr{A}(x)\nabla T(x), \nabla \boldsymbol{V}(x)\rangle + f(x)\boldsymbol{V}(x)\right] dx + \int\limits_{\mathscr{C}_N} q(y)\boldsymbol{V}^+(y)d\sigma. \tag{4.11}$$

The trial function $\boldsymbol{V} \in \mathbb{H}^1(\Omega)$ is such that $T + \varepsilon\boldsymbol{V}$ satisfies the boundary conditions. Then from the equalities $T^+(y) + \boldsymbol{V}^+(y) = 0 = T^+(y)$ on $\mathscr{C}_D$ follows that $T^+(y) = \boldsymbol{V}^+(y) = 0$ on $\mathscr{C}_D$, i.e., T and $\boldsymbol{V}$ have the traces vanishing on the part $\mathscr{C}_D$ of the boundary:

It is clear, that for those $\boldsymbol{V}$ for which the functional $\Phi(T + \varepsilon\boldsymbol{V})$ has a stationary point, we have $\delta\Phi = 0$. By applying the Gauß theorem to the first summand under the integral on

Ω in (4.11), we obtain the associated Euler-Lagrange equation

$$\int\limits_{\Omega}\big[-\operatorname{div}\mathscr{A}(x)\nabla T(x)+f(x)\big]\boldsymbol{V}(x)\,dx+\int\limits_{\mathscr{C}_D}\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T)^+(y)\rangle\boldsymbol{V}^+(y)d\sigma$$

$$+\int\limits_{\mathscr{C}_N}\Big[q(y)+\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T)^+(y)\rangle\Big]\boldsymbol{V}^+(y)d\sigma=0. \quad (4.12)$$

Since the trial function $\boldsymbol{V}$ vanishes on $\mathscr{C}_D$ (see (4.9)), the integral on $\mathscr{C}_D$ in (4.12) vanishes. Now taking arbitrary function $\boldsymbol{V}\in C_0^\infty(\Omega)$ (vanishing in the vicinity of the boundary $\mathscr{C}$), all summands in (4.12) except the first one vanish and we obtain

$$\int\limits_{\Omega}\big[-\operatorname{div}\mathscr{A}(x)\nabla T(x)+f(x)\big]\boldsymbol{V}(x)\,dx=0,$$

which is equivalent to the basic differential equation in (4.1).

Therefore from (4.12) follows that

$$\int\limits_{\mathscr{C}_N}\Big[q(y)+\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T)^+(y)\rangle\Big]\boldsymbol{V}^+(y)\,d\sigma=0. \quad (4.13)$$

The trace $\boldsymbol{V}^+$ of a trial function in (4.13) is arbitrary, we derive, the boundary condition (4.3).

Vice versa: Let T be a solution to the mixed problem (4.1)-(4.3) with vanishing Dirichlet traces $T^+(y)=g(y)=0$ on $\mathscr{C}$, by taking the scalar product of the basic equation in (4.1) with the solution T, by applying the Green formulae and the boundary conditions (4.2) with $g=0$, we get the following equality:

$$0=\int\limits_{\Omega}\big[-\operatorname{div}\mathscr{A}(x)\nabla T(x)+f(x)\big]T(x)\,dx=\int\limits_{\Omega}\big[\mathscr{A}(x)\nabla T(x)+f(x)\big]\nabla T(x)\,dx$$

$$+\int\limits_{\mathscr{C}_D\cup\mathscr{C}_N}\langle\boldsymbol{\nu}(y),\mathscr{A}^+(y)(\nabla T)^+(y)\rangle T^+(y)d\sigma$$

$$=\int\limits_{\Omega}\big[\mathscr{A}(x)\nabla T(x)+f(x)\big]\nabla T(x)\,dx\int\limits_{\mathscr{C}_N}q(y)T^+(y)d\sigma.$$

Therefore, T is a stationary point of the functional Φ in (4.10). □

If $\mathscr{C}_D=\mathscr{C},\ \mathscr{C}_N=\emptyset$, the problem (4.1)-(4.3) reduces to the problem with a Dirichlet boundary condition

$$T^+(y)=0 \qquad \text{on} \quad \mathscr{C}$$

and the corresponding functional Φ in variational formulation (see (4.10)) takes the form

$$\Phi_D(T)=\frac{1}{2}\int\limits_{\Omega}\Big[\langle\mathscr{A}(x)\nabla T(x),\nabla T(x)\rangle+f(x)T(x)\Big]\,dx.$$

If $\mathscr{C}_D = \emptyset,\ \mathscr{C}_N = \mathscr{C}$, from (4.1)-(4.3) we get the problem with Neumann boundary condition

$$-\langle \mathscr{A}^+(y)\boldsymbol{\nu}(y), (\nabla T)^+(y) = q(y) \qquad \text{on} \quad \mathscr{C}$$

and the corresponding functional in variational formulation (see (4.10)) takes the form

$$\Phi_N(T) = \frac{1}{2}\int\limits_{\Omega} \Big[\langle \mathscr{A}(x)\nabla T(x), \nabla T(x)\rangle + f(x)T(x)\Big]\,dx + \int\limits_{\mathscr{C}} q(y)T^+(y)d\sigma.$$

We conclude the section with some auxiliary results on Lebesgue points of integrable functions which is important in the next section.

Let $B(x)$ be a ball in the Euclidean space $B \subset \mathbb{R}^n$ centered at x. *The derivative of the integral at* x is defined to be

$$\lim_{B(x)\to x} \frac{1}{|B(x)|}\int_{B(x)} f(y)\,\mathrm{d}y, \tag{4.14}$$

where $|B(x)|$ denotes the volume (i.e., the Lebesgue measure) of $B(x)$, and $B(x) \to x$ means that the diameter of $B(x)$ tends to 0. Note that

$$\begin{aligned}\left|\frac{1}{|B(x)|}\int_{B(x)} f(y)\,\mathrm{d}y - f(x)\right| &= \left|\frac{1}{|B(x)|}\int_{B(x)} [f(y) - f(x)]\,\mathrm{d}y\right| \\ &\leqslant \frac{1}{|B(x)|}\int_{B(x)} |f(y) - f(x)|\,\mathrm{d}y. \end{aligned} \tag{4.15}$$

The points x for which the right hand side tends to zero are called the *Lebesgue points of f*.

Theorem 4.3 (Lebesgue Differentiation Theorem, Lebesgue 1910.) *For an integrable function $f \in \mathbb{L}_1(\Omega)$ the derivative of the integral* (4.14) *exists and is equal to $f(x)$ at almost every point $x \in \Omega$.*

Moreover, almost every point $x \in \Omega$ is a Lebesgue point of f (see (4.15)*).*

Corollary 4.4 *If $g \in \mathbb{L}_2(\Omega)$, $f \in \mathbb{L}_2(\Omega \times (-1,1))$, then*

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} (g(\cdot), f(\cdot,\tau)))_\Omega d\tau = (g(\cdot), f(\cdot,t)))_\Omega \tag{4.16}$$

for almost all $t \in (-1,1)$.

Proof: It is clear, that $g \cdot f \in \mathbb{L}_1(\Omega \times (-1,1))$ and for the function $h(t) := (g(\cdot), f(\cdot,t))_\Omega$ the inclusion $h \in \mathbb{L}_1((-1,1))$ is true. Thence we can apply Theorem 4.3 to the function $h(t)$ and get (4.16). □

5 Heat transfer in thin Layers

Let $\mathscr{C}$ be a C^2 smooth orientable surface in $\mathbb{R}^3$ given by a single chart (immersion)

$$\theta \ : \ \omega \to \mathscr{C}, \qquad \omega \subset \mathbb{R}^2$$

and let $\boldsymbol{\nu}(\mathscr{x}),\ \mathscr{x} \in \mathscr{C}$ be the unit normal vector field on $\mathscr{C}$ with the fixed orientation. Chart is supposed to be single just for convenience and multi-chart case can be considered similarly. Denote by Ω^ε the layer domain i.e. the set of all points in $\mathbb{R}^3$ in the distance less then ε from $\mathscr{C}$. Then for sufficiently small ε the map $\Theta \ : \ \mathscr{C} \times (-\varepsilon, \varepsilon) \to \Omega^\varepsilon$

$$\Theta(\mathscr{x}, t) = \mathscr{x} + t\boldsymbol{\nu}(\mathscr{x}) = \theta(x) + t\boldsymbol{\nu}(\theta(x)), \qquad x \in \omega$$

is C^1 homeomorphism and $\Theta(\mathscr{C} \times \{0\}) = \mathscr{C}$.

As noted above we can extend unit normal vector field to the entire Ω^ε properly by assuming

$$\boldsymbol{\nu}(\mathscr{x} + t\boldsymbol{\nu}(\mathscr{x})) = \boldsymbol{\nu}(\mathscr{x}), \qquad \mathscr{x} \in \mathscr{C}, \qquad -\varepsilon < t < \varepsilon.$$

If ε is sufficiently small, the boundary $\mathscr{M}^\varepsilon := \partial\Omega^\varepsilon$ is represented as the union of three C^1-smooth surfaces $\mathscr{M}^\varepsilon = \mathscr{M}_{\varepsilon,D} \cup \mathscr{M}^-_{\varepsilon,N} \cup \mathscr{M}^+_{\varepsilon,N}$, where $\mathscr{M}_{\varepsilon,D} = \partial\mathscr{C} \times [-\varepsilon, \varepsilon]$ is the lateral surface, $\mathscr{M}^+_{\varepsilon,N} = \mathscr{C} \times \{+\varepsilon\}$ is the upper surface and $\mathscr{M}^-_{\varepsilon,N} = \mathscr{C} \times \{-\varepsilon\}$ is the lower surface of the of the boundary $\mathscr{M}^\varepsilon$ of layer domain Ω^ε.

In the present section will be considered the heat conduction problem by an "isotropic" medium, governed by the BVP (cf. (1.4) for $\Delta_{\Omega^\varepsilon}$)

$$\begin{aligned}
&\Delta_{\Omega^\varepsilon} T(\mathscr{x}, t) = f(\mathscr{x}, t), && (\mathscr{x}, t) \in \mathscr{C} \times (-\varepsilon, \varepsilon),\\
&T^+(\mathscr{x}, t) = 0, && (\mathscr{x}, t) \in \partial\mathscr{C} \times (-\varepsilon, \varepsilon),\\
&(\partial_t T)^+(\mathscr{x}, \pm\varepsilon) = q^\pm_\varepsilon(\mathscr{x}), && \mathscr{x} \in \mathscr{C}.
\end{aligned} \tag{5.1}$$

The case of an "anisotropic" medium will be treated in a forthcoming publication.

We impose the following constraints

$$\begin{gathered}
T \in \mathbb{H}^1(\Omega^\varepsilon), \qquad q^\pm_\varepsilon \in \widetilde{\mathbb{H}}^2(\mathscr{C}), \qquad f \in \mathbb{L}_2(\Omega^1),\\
0 \ \text{ is the Lebesgue point for the function } \ \widetilde{f}(t) := \int_{\mathscr{C}} |f(\mathscr{x}, t)|^2 d\sigma
\end{gathered} \tag{5.2}$$

(see (4.15) and note that $\|\widetilde{f}\big|\mathbb{L}_1(-1,1)\| \leqslant \|f\big|\mathbb{L}_2(\Omega^1)\|^2$). The latter constraint implies that $\widetilde{f}(0)$ exists and, due to Theorem 4.3,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \widetilde{f}(t) dt = \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_{\mathscr{C}} |f(\mathscr{x}, t)|^2 d\sigma dt = \widetilde{f}(0).$$

The formulated BVP (5.1) governs a heat transfer in the body Ω^ε when there are thermal sources or sinks in Ω^ε. The temperature on the lateral surface $\partial\mathscr{C} \times (-\varepsilon, \varepsilon)$ is zero, the heat fluxes are fixed on the upper and lower surfaces $\mathscr{C}^\pm := \mathscr{C} \times \{\pm\varepsilon)$. It is well known, that the boundary value problem (5.1) as well as it's equivalent problem (1.6)-(1.8) have the unique solution $T \in \mathbb{H}^1(\Omega^\varepsilon)$ (respectively, $T_0 \in \mathbb{H}^1(\Omega^\varepsilon)$; see, e.g., [**?**]).

The energy functional associated with the problem (5.1) reads (cf. Theorem 7.1)

$$E(T_\varepsilon) : = \int\limits_{-\varepsilon}^{\varepsilon}\int\limits_{\mathscr{C}}\left[\frac{1}{2}(\mathscr{D}_{\mathscr{C}}T^2(\mathscr{x},\tau) + \frac{1}{2\varepsilon^2}(\partial_\tau T^2(\mathscr{x},\tau) + F(\mathscr{x},\tau)T_\varepsilon(\mathscr{x},\tau)\right] d\sigma d\tau, \quad (5.3)$$

$$\begin{aligned} F(\mathscr{x},t) \; &:= \; f(\mathscr{x},t) - \frac{1}{4\varepsilon}\big((t+\varepsilon)^2\Delta_{\mathscr{C}}q_\varepsilon^+(\mathscr{x}) - (t-\varepsilon)^2\Delta_{\mathscr{C}}q_\varepsilon^-(\mathscr{x})\big) \\ &- \; \frac{\mathscr{H}^0_{\mathscr{C}}}{2\varepsilon}\big((t+\varepsilon)q_\varepsilon^+(\mathscr{x}) - (t-\varepsilon)q_\varepsilon^-(\mathscr{x})\big) - \frac{1}{2\varepsilon}\big(q_\varepsilon^+(\mathscr{x}) - q_\varepsilon^-(\mathscr{x})\big), \qquad (5.4) \\ &\qquad (\mathscr{x},t) \in \mathscr{C}\times(-\varepsilon,\varepsilon). \end{aligned}$$

More generally, we consider the non-linear functional

$$E_\varepsilon(T) \;=\; \int\limits_{\Omega^\varepsilon}[\mathscr{K}_0(\mathscr{D}_{\Omega^\varepsilon}T(x),\, T(x)) + F_\varepsilon(x)T(x)]\,dx, \qquad (5.5)$$

where $\mathscr{K}_0(\mathscr{D}_{\Omega^\varepsilon}T,\, T)$ is strictly convex and has quadratic estimate. In the case of the functional (5.3),

$$\mathscr{K}_0(\mathscr{D}_{\Omega^\varepsilon}T,\, T) = \frac{1}{2}\langle\mathscr{D}_{\Omega^\varepsilon}T, \mathscr{D}_{\Omega^\varepsilon}T\rangle = \frac{1}{2}(\mathscr{D}_{\Omega^\varepsilon}T)^2 = \frac{1}{2}(\mathscr{D}_{\mathscr{C}}T_\varepsilon)^2(\mathscr{x},\tau) + \frac{1}{2\varepsilon^2}(\partial_\tau T_\varepsilon)^2(\mathscr{x},\tau) \quad (5.6)$$

and it is clear that the kernel is strictly convex because the quadratic function $F(x) = x^2$ is strictly convex $[\theta x_1 + (1-\theta)x_2]^2 < \theta x_1^2 + (1-\theta)x_2^2$ for all $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$, $0 < \theta < 1$. The kernel has a trivial quadratic estimate, because is a quadratic function.

A nice proof of the next Lemma 6.14 is exposed in [**?**, Example 3.6]

Lemma 5.1 *Let Ω be a domain in $\mathbb{R}^n$ with the Lipshitz boundary $\mathscr{M} := \partial\Omega$ and $\mathscr{M}_0 \subset \mathscr{M}$ be a subsurface of non-zero measure. Then the inequality*

$$\|\varphi\,|\,\mathbb{L}_2(\Omega)\| \leqslant C\|\nabla\varphi\,|\,\mathbb{L}_2(\Omega)\| = C\left[\sum_{j=1}^n\|\partial_j\varphi\,|\,\mathbb{L}_2(\Omega)\|^2\right]^{1/2} \qquad (5.7)$$

holds for all functions $\varphi \in \widetilde{\mathbb{H}}^1(\Omega, \mathscr{M}_0)$ and the constant C is independent of φ.

Now we perform the scaling of the variable $t = \varepsilon\tau$, $-1 < \tau < 1$, and study the following functionals in the scaled domain $\Omega^1 = \mathscr{C}\times(-1,1)$

$$E_\varepsilon\left(T_\varepsilon\right) = \int_{-1}^{1}\int_{\mathscr{C}}\left[\mathscr{K}_0\left(\mathscr{D}_{\mathscr{C}}T_\varepsilon,\, \frac{1}{\varepsilon}\partial_t T_\varepsilon, T_\varepsilon\right) + F_\varepsilon T_\varepsilon\right] d\sigma d\tau \qquad (5.8)$$

where $\mathscr{D}_{\mathscr{C}} = (\mathscr{D}_1,\, \mathscr{D}_2,\, \mathscr{D}_3)$, $\mathscr{D}_4 = \partial_t$. The functionals $E_\varepsilon\,(T_\varepsilon)$ are related to the original functional $E\,(T)$ by the equality

$$E_\varepsilon\,(T_\varepsilon) = \frac{1}{\varepsilon}E\,(T)\,, \quad \text{where} \quad T_\varepsilon(\mathscr{x},t) = T\,(\mathscr{x}_1,\mathscr{x}_2,\mathscr{x}_3,\varepsilon t)\,, \quad \text{and} \tag{5.9}$$

$$\begin{aligned} F_\varepsilon(\mathscr{x},t) &= F(\mathscr{x},\varepsilon t) = f(\mathscr{x},\varepsilon t) - \frac{\varepsilon}{4}\big((t+1)^2\Delta_{\mathscr{C}} q^+_\varepsilon(\mathscr{x}) - \frac{\varepsilon}{4}(t-1)^2\Delta_{\mathscr{C}} q^-_\varepsilon(\mathscr{x})\big) \\ &\quad - \frac{\mathscr{H}^0_{\mathscr{C}}(\mathscr{x})}{2}\big((t+1)q^+_\varepsilon(\mathscr{x}) - (t-1)q^-_\varepsilon(\mathscr{x})\big) - \frac{1}{2\varepsilon}\big(q^+_\varepsilon(\mathscr{x}) - q^-_\varepsilon(\mathscr{x})\big), \\ &\qquad (\mathscr{x},t) \in \mathscr{C} \times (-\varepsilon,\varepsilon). \end{aligned} \tag{5.10}$$

Lemma 5.2 *Let F_ε be uniformly bounded in $\mathbb{L}_2(\Omega^1)$*

$$\sup_{\varepsilon<\varepsilon_0} \|F_\varepsilon|\mathbb{L}_2(\Omega^1)\| < \infty. \tag{5.11}$$

Then the energy functional $E_\varepsilon(T)$ in (5.8) *is correctly defined on the space $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1,1))$, is strictly convex and has the following quadratic estimate*

$$\begin{aligned} &E_\varepsilon(\theta T_1 + (1-\theta)T_2) < \theta E_\varepsilon(T_1) + (1-\theta)E_\varepsilon(T_2), \qquad 0<\theta<1 \\ &C_1\int\limits_{\Omega^1} \mathscr{K}_0\left(\mathscr{D}_{\mathscr{C}}T,\, \frac{1}{\varepsilon}\partial_t T, T\right) d\sigma dt - C_2 \leqslant E_\varepsilon(T) \\ &\leqslant C_3\left[1 + \int\limits_{\Omega^1} \mathscr{K}_0\left(\mathscr{D}_{\mathscr{C}}T,\, \frac{1}{\varepsilon}\partial_t T, T\right) d\sigma dt\right], \\ &\qquad\qquad \forall\, T_1,\ T_2, T \in \widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C} \times (-1,1)) \end{aligned} \tag{5.12}$$

for some positive constants C_1, C_2 and C_3 not depending on ε.

Proof: Let us decompose the functional $E_\varepsilon(T)$ in (5.8) into the sum of non-linear and linear parts

$$\begin{aligned} &E_\varepsilon(T) = E^{(1)}_\varepsilon(T) + E^{(2)}_\varepsilon(T) \\ &E^{(1)}_\varepsilon(T) := \int_{\Omega^1} \mathscr{K}_0\left(\mathscr{D}_{\mathscr{C}}T,\, \frac{1}{\varepsilon}\partial_t T, T\right) dx, \\ &E^{(2)}_\varepsilon(T) := \int_{\Omega^1} F_\varepsilon(x)T(x)dx. \end{aligned} \tag{5.13}$$

By the conditions imposed on $\mathscr{K}_0$ in (5.5), the first (non-linear) functional $E^{(1)}_\varepsilon(T)$ is strictly convex and has a quadratic estimate:

$$\begin{aligned} &C^0_1\int\limits_{\Omega^1} \left(\langle \mathscr{D}_{\mathscr{C}}T_j, \mathscr{D}_{\mathscr{C}}T_j\rangle + \frac{1}{\varepsilon_j^2}|\partial_t T_j|^2\right) dx - C^0_2 \leqslant E^{(1)}_\varepsilon(T) \\ &\qquad \leqslant C^0_3\left[1 + \int\limits_{\Omega^1} \left(\langle \mathscr{D}_{\mathscr{C}}T_j, \mathscr{D}_{\mathscr{C}}T_j\rangle + \frac{1}{\varepsilon_j^2}|\partial_t T_j|^2\right) dx\right]. \end{aligned} \tag{5.14}$$

On the other hand $E^{(2)}_\varepsilon(T)$ is linear and, therefore, strictly convex (see the first inequality in (5.12)). Thus, we only have to prove the two-sided quadratic estimate in (5.12) for the linear functional $E^{(2)}_\varepsilon(T)$. Due to Lemma 5.1 and the equality (2.13) we can write:

$$\begin{aligned} |E^{(2)}_\varepsilon(T)| \leqslant \left|\int_{\Omega^1} F_\varepsilon(x)T(x)dx\right| \leqslant \|F_\varepsilon|\mathbb{L}_2(\Omega^1)\|\,\|T|\mathbb{L}_2(\Omega^1)\| \leqslant M\|\nabla T|\mathbb{L}_2(\Omega^1)\| \\ \leqslant M\Big(\frac{1}{\eta}+\eta\|\nabla T|\mathbb{L}_2(\Omega^1)\|^2\Big) \leqslant M\Big(\frac{1}{\eta}+\eta\|\mathscr{D}_{\Omega^1}T|\mathbb{L}_2(\Omega^1)\|^2\Big). \quad (5.15)\end{aligned}$$

Choosing $\eta = 1$ in (5.15) and taking into account (5.14) we get the right inequality in the second line of (5.12), whereas taking η sufficiently small we obtain

$$E_\varepsilon(T) \geqslant |E^{(1)}_\varepsilon(T)| - |E^{(2)}_\varepsilon(T) \geqslant C_1\|\mathscr{D}_{\Omega^\varepsilon}T|\mathbb{L}_2(\Omega^\varepsilon)\|^2 - C_2. \qquad \square$$

Let $F_j = F_{\varepsilon_j}$, $0 < \varepsilon_j \leq 1$, $\lim_{j\to\infty}\varepsilon_j = 0$ and F_{ε_j} be uniformly bounded (see (5.11)). Further let $T_j = T_{\varepsilon_j} \in \widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C}\times(-1,1))$, $\quad j = 1,2,\dots$ be the sequence of functions with "finite energy":

$$\sup_j E_{\varepsilon_j}(T_j) < +\infty. \quad (5.16)$$

Then from (5.14)–(5.15) we get

$$\begin{aligned} C^0_1\|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^1)\|^2 &= \int_{\Omega^1}\left(\frac{1}{2}\langle\mathscr{D}_{\mathscr{C}}T_j, \mathscr{D}_{\mathscr{C}}T_j\rangle + \frac{1}{2\varepsilon_j^2}|\partial_t T_j|^2\right)dx \\ &= C^0_1 E_{\varepsilon_j}(T_j) - C^0_1\int_{\Omega^1} F_j(\mathscr{x},t)T_j(\mathscr{x},t)d\sigma dt \\ &\leqslant C^0_2\big(1 + \|F_j|\mathbb{L}_2(\Omega^1)\|\|T_j|\mathbb{L}_2(\Omega^1)\|\big) \\ &\leqslant C^0_3\Big(1 + \|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^\varepsilon)\|^2\Big)^{\frac{1}{2}}, \end{aligned} \quad (5.17)$$

since, due to Lemma 5.1,

$$\|T_j|\mathbb{L}_2(\Omega^1)\| \leqslant C_0\|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^1)\|. \quad (5.18)$$

Consequently,

$$\sup_j\|\mathscr{D}_{\Omega^1}T_j|\mathbb{L}_2(\Omega^1)\| = \sup_j\left(\int_{\Omega^1}\left(\frac{1}{2}\langle\mathscr{D}_{\mathscr{C}}T_j, \mathscr{D}_{\mathscr{C}}T_j\rangle + \frac{1}{2\varepsilon_j^2}|\partial_t T_j|^2\right)dx\right)^{1/2} < +\infty. \quad (5.19)$$

From (5.17)-(5.19) follows

$$\sup_j\int_{\Omega^1}|T_j|^2\,dx < \infty, \quad \sup_j\int_{\Omega^1}|\mathscr{D}_{\mathscr{C}}T_j|^2\,dx < \infty, \quad \sup_j\frac{1}{\varepsilon_j^2}\int_{\Omega^1}|\partial_t T_j|^2\,dx < \infty. \quad (5.20)$$

Note, that if T_j are the scaled solutions to problem (1.3), then from the Euler-Lagrange equation, associated with the functional (see (4.12)), follows that $E_{\varepsilon_j}(T_j) = 0$ and therefore conditions (5.20) are satisfied.

Due to (5.20) the sequence $\{T_j\}_{j=1}^{\infty}$ is uniformly bounded in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C}\times(-1,1))$ and a weakly converging subsequence (say $\{T_j\}_{j=1}^{\infty}$ itself) to a function T in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C}\times(-1,1))$ can be extracted.

The functional

$$H(T) = \int\limits_{\Omega^1} |\partial_t T|^2 \, dx$$

is convex and continuous in $\widetilde{\mathbb{H}}^1(\Omega^1, \partial\mathscr{C}\times(-1,1))$; then it is weakly lower semi-continuous and $\partial_t T = 0$ a.e., because

$$\int_{\Omega^1} |\partial_t T|^2 \, dx = H(T) \le \liminf_j H(T_j) = \liminf_j \int_{\Omega^1} |\partial_t T_j|^2 \, dx = 0.$$

(see the last inequality in (5.20)). Hence $T(\mathscr{x}, t)$ is independent of t, i.e.

$$T(\mathscr{x}, t) = T(\mathscr{x}), \qquad \mathscr{x} \in \mathscr{C}, \quad -1 \le t \le 1. \tag{5.21}$$

Let the following conditions are fulfilled

$$f_\varepsilon(\mathscr{x}, t) := f(\mathscr{x}, \varepsilon t) \underset{\varepsilon\to 0}{\rightarrow} f^0(\mathscr{x}) \quad in \quad \mathbb{L}_2(\Omega^1), \tag{5.22}$$

$q_\varepsilon^{\pm} \in \mathbb{H}^2(\mathscr{C})$ are uniformly bounded (with respect to ε) in $\mathbb{H}^2(\mathscr{C})$, and

$$\lim_{\varepsilon\to 0} q_\varepsilon^{+} = \lim_{\varepsilon\to 0} q_\varepsilon^{-} = q_0, \quad in \quad \mathbb{L}_2(\mathscr{C}), \tag{5.23}$$

and

$$\frac{1}{2\varepsilon}(q_\varepsilon^{+} - q_\varepsilon^{-}) \underset{\varepsilon\to 0}{\rightarrow} q_1 \quad in \quad \mathbb{L}_2(\mathscr{C}). \tag{5.24}$$

From (5.22)- (5.24) follows in particular, that

$$F_j(\mathscr{x}, t) \to F(\mathscr{x}, 0) \quad in \quad \mathbb{L}_2(\Omega^1). \tag{5.25}$$

Set

$$E^{(0)}(T) = \begin{cases} E^{(1)}(T) + E^{(2)}(T) & \text{for} \quad T \in \mathscr{P}(\mathscr{C}); \\ +\infty, & \text{for} \quad T \notin \mathscr{P}(\mathscr{C}). \end{cases} \tag{5.26}$$

where $\mathscr{P}(\mathscr{C})$ is defined in (1.12), and

$$\begin{aligned} E^{(1)}(T) &:= \frac{1}{2}\int_{\Omega^1} \langle (\mathscr{D}_{\Omega^1} T)(x,t), (\mathscr{D}_{\Omega^1} T)(x,t)\rangle d\sigma\, dt \\ &= \int_{\mathscr{C}} \langle (\mathscr{D}_{\mathscr{C}} T_{\mathscr{C}})(\mathscr{x}), (\mathscr{D}_{\mathscr{C}} T_{\mathscr{C}})(\mathscr{x})\rangle d\sigma, \end{aligned} \tag{5.27}$$

$$\begin{aligned} E^{(2)}(T) &:= \int_{\Omega^1} F(\mathscr{x}, 0) T(\mathscr{x}, t) d\sigma\, dt \\ &= 2\int_{\mathscr{C}} \left(f^0(\mathscr{x}) - \mathscr{H}_{\mathscr{C}}^0 q_0(\mathscr{x}) - q_1(\mathscr{x})\right) T_{\mathscr{C}}(\mathscr{x}) d\sigma. \end{aligned} \tag{5.28}$$

Let us check that the E_{ε_j} sequence Γ-convergs to $E^{(0)}$ in $\widetilde{\mathbb{H}}^1(\Omega^\varepsilon, \partial\mathscr{C} \times (-1,1))$. Indeed, we have

$$E_{\varepsilon_j}(T_j) = E^{(1)}_{\varepsilon_j}(T_j) + E^{(2)}_{\varepsilon_j}(T_j),$$

where

$$E^{(1)}_{\varepsilon_j}(T_j) = \int\limits_{\Omega^1} \left(\frac{1}{2} \langle \mathscr{D}_{\mathscr{C}} T_j, \mathscr{D}_{\mathscr{C}} T_j \rangle + \frac{1}{2\varepsilon_j^2} |\partial_t T_j|^2 \right) dx, \quad E^{(2)}_{\varepsilon_j}(T_j) = \int\limits_{\Omega^1} F_j T_j dx.$$

The functional $E^{(1)}(T)$ is convex and continuous and so it is weakly lower semicontinuous in $\widetilde{\mathbb{H}}^1(\Omega^\varepsilon, \partial\mathscr{C} \times (-1,1))$, therefore

$$\liminf_j E^{(1)}_{\varepsilon_j}(T_j) \geqslant \liminf_j E^{(1)}(T_j) \geqslant E^{(1)}(T).$$

Sequence $E^{(2)}_{\varepsilon_j}(T_j)$ converges to $E^{(2)}(T)$, because $F_j(\mathscr{x}, t) \to F(\mathscr{x}, 0)$ and $T_j \rightharpoonup T$ in $\mathbb{L}_2(\Omega^1)$. Consequently

$$\liminf_j E_{\varepsilon_j}(T_j) \geqslant E^{(0)}(T).$$

This proves $\liminf$ inequality for the sequence E_{ε_j}.

Note, that

$$E^{(2)}(T) = \int\limits_{\mathscr{C}} \int\limits_{-1}^{1} F(\mathscr{x}, 0) T(\mathscr{x}, t) dt\, d\sigma = 2 \int\limits_{\mathscr{C}} F(\mathscr{x}, 0) T_{\mathscr{C}}(\mathscr{x}) d\sigma.$$

To show that the lower bound is reached i.e. to build a recovery sequence T_j we fix $T_{\mathscr{C}} \in \mathbb{H}^1(\mathscr{C})$ and set $T(\mathscr{x}, t) = T_{\mathscr{C}}(\mathscr{x}),\ \mathscr{x} \in \mathscr{C},\ t \in (-1,1)$. Define recovery sequence as $T_j(x,t) = T(x,t) = T_{\mathscr{C}}(x)$ Then $\partial_t T_j = \partial_t T = 0$ and

$$\lim_{j\to\infty} E_{\varepsilon_j}(T_j) = \lim_{j\to\infty} E^{(1)}_{\varepsilon_j}(T) + \lim_{j\to\infty} E^{(2)}_{\varepsilon_j}(T) = E^{(1)}(T) + E^{(2)}(T) = E^{(0)}(T).$$

We have proved the following result.

Theorem 5.3 *If conditions* (5.22)- (5.24) *are fulfilled then the functional in* (5.8) Γ*-converges to the functional* $E^{(0)}(T)$ *defined in* (1.13) *as* $\varepsilon \to 0$.

Now we are able to prove the main Theorem 1.1 formulated in the introduction.

Proof of Theorem 1.1: The first part of the Theorem i.e. Γ-convergence of the functional (1.13) to the functional $E^{(0)}$ defined by (1.13), is proved in Theorem 5.3.

The concluding assertion, that the BVP (1.14) is an equivalent reformulation of the minimization problem with the energy functional (1.13), is explained in Theorem 7.1. □

Chapter 4

MELLIN CONVOLUTION EQUATIONS IN THE BESSEL POTENTIAL SPACES

In the present chapter we expose investigations of Mellin convolution equations in the Bessel potential spaces, published in the papers [DD16, Du13]. Such equations are important while investigating boundary value problems (BVPs) for elliptic equations on surfaces and domains with Lipschitz boundary and will be applied in the next chapter to the investigations of BVPs for the Laplace-Beltrami amd L'ame equations on surfaces.

1 Introduction

It is well-known that various boundary value problems for PDE in planar domains with angular points on the boundary, e.g. Lamé systems in elasticity (cracks in elastic media, reinforced plates), Maxwell's system and Helmholtz equation in electromagnetic scattering, Cauchy–Riemann systems, Carleman–Vekua systems in generalized analytic function theory etc. can be studied with the help of the Mellin convolution equations of the form

$$\mathbf{A}\varphi(t) := c_0\varphi(t) + \frac{c_1}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \int_0^\infty \mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\,\frac{d\tau}{\tau} = f(t), \tag{1.1}$$

with the kernel $\mathscr{K}$ satisfying the condition

$$\int_0^\infty t^{\beta-1}|\mathscr{K}(t)|\,dt < \infty, \quad 0 < \beta < 1, \tag{1.2}$$

which makes it a bounded operator in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$, provided $1 \leqslant p \leqslant \infty$, $-1 < \gamma < p-1$, $\beta := (1+\gamma)/p$ (cf. [Du79]).

In particular, integral equations with fixed singularities in the kernel

$$c_0(t)\varphi(t) + \frac{c_1(t)}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \sum_{k=0}^{n}\frac{c_{k+2}(t)t^{k-r}}{\pi i}\int_0^\infty \frac{\tau^r\varphi(\tau)\,d\tau}{(\tau+t)^{k+1}} = f(t), \quad 0 \leqslant t \leqslant 1, \tag{1.3}$$

where $0 \leqslant r \leqslant k$ are of type (1.1) after localization, i.e. after "freezing" the coefficients.

The Fredholm theory and the unique solvability of equations (1.1) in the weighted Lebesgue spaces were accomplished in [Du79]. This investigation was based on the following observation: if $1 < p < \infty$, $-1 < \gamma < p-1$, $\beta := (1+\gamma)/p$, the following mutually invertible exponential transformations

$$\begin{aligned} &Z_\beta : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+),\\ &Z_\beta\varphi(\xi) := e^{-\beta\xi}\varphi(e^{-\xi}), \quad \xi \in \mathbb{R} := (-\infty, \infty),\\ &Z_\beta^{-1} : \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma),\\ &Z_\beta^{-1}\psi(t) := t^{-\beta}\psi(-\ln t), \quad t \in \mathbb{R}^+ := (0, \infty), \end{aligned} \tag{1.4}$$

transform the equation (1.1) from the weighted Lebesgue space $f, \varphi \in \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ into the Fourier convolution equation $W^0_{\mathscr{A}_\beta}\psi = g$, $\psi = Z_\beta\varphi$, $g = Z_\beta f \in \mathbb{L}_p(\mathbb{R})$ of the form

$$\begin{aligned} W^0_{\mathscr{A}_\beta}\psi(x) &= c_0\psi(x) + \int_{-\infty}^{\infty} \mathscr{K}_1(x-y)\varphi(y)\,dy,\\ \mathscr{K}_1(x) &= e^{-\beta x}\Big[\frac{c_1}{1-e^{-x}} + \mathscr{K}(e^{-x})\Big]. \end{aligned}$$

Note that the symbol of the operator $W^0_{\mathscr{A}_\beta}$, viz. the Fourier transform of the kernel

$$\begin{aligned} \mathscr{A}_\beta(\xi) &:= c_0 + \int_{-\infty}^{\infty} e^{i\xi x}\mathscr{K}_1(x)\,dx\\ &:= c_0 - ic_1\cot\pi(\beta - i\xi) + \int_{-\infty}^{\infty} e^{(i\xi-\beta)x}\mathscr{K}(e^{-x})\,dx, \quad \xi \in \mathbb{R} \end{aligned} \tag{1.5}$$

is a piecewise continuous function. Let us recall that the theory of Fourier convolution operators with discontinuous symbols is well developed, cf. [Du75a, Du75b, Du77, Du78, Th85]. This allows one to investigate various properties of the operators (1.1), (1.3). In particular, Fredholm criteria, index formula and conditions of unique solvability of the equations (1.1) and (1.3) have been established in [Du79].

Similar integral operators with fixed singularities in kernel arise in the theory of singular integral equations with the complex conjugation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)\,dt}{\tau - t} + \frac{e(t)}{\pi i}\overline{\int_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau - t}} = f(t), \quad t \in \Gamma$$

and in more general R-linear equations

$$a(t)\varphi(t) + b(t)\overline{\varphi(t)} + \frac{c(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)\,dt}{\tau - t} + \frac{d(t)}{\pi i}\int_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau - t} +$$

$$+\frac{e(t)}{\pi i}\overline{\int_\Gamma \frac{\varphi(\tau)\,dt}{\tau-t}}+\frac{g(t)}{\pi i}\overline{\int_\Gamma \frac{\overline{\varphi(\tau)}\,dt}{\tau-t}}=f(t),\quad t\in\Gamma,$$

if the contour Γ possesses corner points. Note that a complete theory of such equations is presented in [DL85, DLS95].

Let $t_1,\ldots,t_n\in\Gamma$ be the corner points of a piecewise-smooth contour Γ, and let $\mathbb{L}_p(\Gamma,\rho)$ denote the weighted $\mathbb{L}_p$-space with a power weight $\rho(t):=\prod_{j=1}^n|t-t_j|^{\gamma_j}$. Assume that the parameters p and $\beta_j:=(1+\gamma_j)/p$ satisfy the conditions

$$1<p<\infty,\ \ 0<\beta_j<1,\ \ j=1,\ldots,n.$$

If the coefficients of the above equations are piecewise-continuous matrix functions, one can construct a function $\mathscr{A}_{\vec\beta}(t,\xi)$, $t\in\Gamma$, $\xi\in\mathbb{R}$, $\vec\beta:=(\beta_1,\ldots,\beta_n)$, called the symbol of the equation (of the related operator). It is possible to express various properties of the equation in terms of $\mathscr{A}_{\vec\beta}$:

- The equation is Fredholm in $\mathbb{L}_p(\Gamma,\rho)$ if and only if its symbol is elliptic., i.e. iff $\inf_{(t,\xi)\in\Gamma\times\mathbb{R}}|\,\mathscr{A}_{\vec\beta}(t,\xi)|>0$;
- To an elliptic symbol $\mathscr{A}_{\vec\beta}(t,\xi)$ there corresponds an integer valued index $\mathrm{ind}\mathscr{A}_{\vec\beta}(t,\xi)$, the winding number, which coincides with the Fredholm index of the corresponding operator modulo a constant multiplier.

For more detailed survey of the theory and various applications to the problems of elasticity we refer the reader to [Du75a, Du75b, Du77, Du79, Du82, Du84a, Du84b, Du86, Sc85].

Similar approach to boundary integral equations on curves with corner points based on Mellin transformation has been exploited by M. Costabel and E. Stephan [Co83, CS84].

However, one of the main problems in boundary integral equations for elliptic partial differential equations is the absence of appropriate results for Mellin convolution operators in the Bessel potential spaces, cf. [Du82, Du84b, Du86] and recent publications on nano-photonics [BCC12a, BCC12a, GB10]. Such results are needed to obtain an equivalent reformulation of boundary value problems into boundary integral equations in the Bessel potential spaces. Nevertheless, numerous works on Mellin convolution equations seem to pay almost no attention to the mentioned problem.

The first arising problem is the boundedness results for Mellin convolution operators in the Bessel potential spaces. The conditions on kernels known so far are very restrictive. The following boundedness result for the Mellin convolution operator can be proved

Proposition 1.1 *Let $1<p<\infty$ and let $m=1,2,\ldots$ be an integer. If a function $\mathscr{K}$ satisfies the condition*

$$\int_0^1 t^{\frac1p-m-1}|\mathscr{K}(t)|\,dt+\int_1^\infty t^{\frac1p-1}|\mathscr{K}(t)|\,dt<\infty, \tag{1.6}$$

then the Mellin convolution operator (see (1.1)*)*

$$\boldsymbol{A}=\mathfrak{M}^0_{\mathscr{A}_{1/p}}:\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)\longrightarrow\mathbb{H}^s_p(\mathbb{R}^+)$$

with the symbol (*see* (1.5))

$$\mathscr{A}_{1/p}(\xi) := c_0 + c_1 \coth \pi\Big(\frac{i}{p} + \xi\Big) + \int_0^\infty t^{\frac{1}{p}-i\xi} \mathscr{K}(t)\, \frac{dt}{t}\,, \quad \xi \in \mathbb{R}, \tag{1.7}$$

is bounded for any $0 \leqslant s \leqslant m$.

Note that the condition

$$K_\beta := \int_0^\infty t^{\beta-1} |\mathscr{K}(t)|\, dt < \infty \tag{1.8}$$

ensures that the operator

$$\mathfrak{M}^0_a : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$$

is bounded, while the norm of the Mellin convolution

$$\mathfrak{M}^0_{a_\beta}\varphi(t) := \int_0^\infty \mathscr{K}\Big(\frac{t}{\tau}\Big)\varphi(\tau)\, \frac{d\tau}{\tau} \tag{1.9}$$

admits the estimate $\|\mathfrak{M}^0_{a_\beta}\| \leqslant K_\beta$.

The above-formulated result has very restricted application. For example, the operators

$$\begin{aligned}
\boldsymbol{N}_\alpha\varphi(t) &= \frac{\sin\alpha}{\pi} \int_0^\infty \frac{t\,\varphi(\tau)\, d\tau}{t^2 + \tau^2 - 2t\tau\cos\alpha}\,,\\
\boldsymbol{N}^*_\alpha\varphi(t) &= \frac{\sin\alpha}{\pi} \int_0^\infty \frac{\tau\,\psi_j(\tau)\, d\tau}{t^2 + \tau^2 - 2t\tau\cos\alpha}\,,\\
\boldsymbol{M}_\alpha\varphi(t) &= \frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{[\tau\,\cos\alpha - t]\varphi(\tau)\, d\tau}{t^2 + \tau^2 - 2t\,\tau\cos\alpha}\,, \quad -\pi < \alpha < \pi,
\end{aligned} \tag{1.10}$$

which we encounter in boundary integral equations for elliptic boundary value problems (see [BDKT13]), as well as the operators

$$\boldsymbol{N}_{m,k}\varphi(t) := \frac{t^k}{\pi i} \int_0^\infty \frac{\tau^{m-k}\varphi(\tau)\, d\tau}{(\tau + t)^{m+1}}\,, \quad k = 0, \ldots, m,$$

represented in (1.3), do not satisfy the conditions (1.6). In particular, $\boldsymbol{N}_\alpha$ satisfies condition (1.6) only for $m = 1$ and $\boldsymbol{N}_{m,k}$ only for $m = k$. Although, as we will see below in Theorem 3.4, all operators $\boldsymbol{N}_\alpha$, $\boldsymbol{N}^*_\alpha$ and $\boldsymbol{N}_{m,k}$ are bounded in the Bessel potential spaces in the setting (**??**) for all $s \in \mathbb{R}$.

Here we introduce *admissible kernels*, which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at the infinity

$$\mathscr{K}(t) := \sum_{j=0}^{\ell} \frac{d_j}{t-c_j} + \sum_{j=\ell+1}^{\infty} \frac{d_j}{(t-c_j)^{m_j}}\,, \quad j=0,1,\dots, \tag{1.11}$$
$$c_0,\dots,c_\ell \in \mathbb{R}, \quad 0<\alpha_k := \arg c_k < 2\pi, \quad k=\ell+1,\ell+2,\dots.$$

$\mathscr{K}(t)$ have poles at $c_0, c_1, \ldots \in \mathbb{C}\setminus\{0\}$ and complex coefficients $d_j \in \mathbb{C}$. The Mellin convolution operator

$$\mathbf{K}^m_c\varphi(t) := \frac{1}{\pi}\int_0^\infty \frac{\tau^{m-1}\varphi(\tau)\,d\tau}{(t-c\,\tau)^m}. \tag{1.12}$$

with the kernel

$$\mathscr{K}^m_c(t) := \frac{1}{(t-c)^m}, \qquad 0<\arg c<2\pi$$

(see Definition **??**) turns out to be bounded in the Bessel potential spaces (see Theorem 3.4).

In order to study Mellin convolution operators in the Bessel potential spaces, we use the "lifting" procedure, performed with the help of the Bessel potential operators $\mathbf{\Lambda}^s_+$ and $\mathbf{\Lambda}^{s-r}_-$, which transform the initial operator $\mathfrak{M}^0_a$ into the lifted operator $\mathbf{\Lambda}^{s-r}_-\mathfrak{M}^0_a\mathbf{\Lambda}^{-s}_+$ acting already on a Lebesgue $\mathbb{L}_p$ spaces. However, the lifted operator is neither Mellin nor Fourier convolution and to describe its properties, one has to study the commutants of the Bessel potential operators and Mellin convolutions with meromorphic kernels. It turns out that the Bessel potentials alter after commutation with Mellin convolutions and the result depends essentially on poles of the meromorphic kernels. These results allows us to show that the lifted operator $\mathbf{\Lambda}^{s-r}_-\mathfrak{M}_a\mathbf{\Lambda}^{-s}_+$ belongs to the Banach algebra of operators generated by Mellin and Fourier convolution operators with discontinuous symbols. Since such algebras have been studied before [Du87], one can derive various information (Fredholm properties, index, the unique solvability) about the initial Mellin convolution equation $\mathfrak{M}^0_a\varphi = g$ in the Bessel potential spaces in the settings $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $g \in \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+)$ and in the settings $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $g \in \mathbb{H}^{s-r}_p(\mathbb{R}^+)$.

The results of the present work is already applied in [DTT14] to the investigation of some boundary value problems studied before by Lax–Milgram Lemma in [BCC12a, BCC12a]. Note that the present approach is more flexible and provides better tools for analyzing the solvability of the boundary value problems and the asymptotic behavior of their solutions.

It is worth noting that the obtained results can also be used to study Schrödinger operator on combinatorial and quantum graphs. Such a problem has attracted a lot of attention recently, since the operator mentioned above possesses interesting properties and has various applications, in particular, in nano-structures (see [Ku04, Ku05] and the references there). Another area for application of the present results are Mellin pseudodifferential operators on graphs. This problem has been studied in [1], but in the periodic case only. Moreover, some of the results can be applied in the study of stability of approximation methods for Mellin convolution equations in the Bessel potential spaces.

The present paper is organized as follows. In the first section we observe Mellin and Fourier convolution operators with discontinuous symbols acting on Lebesgue spaces. Most of these results are well known and we recall them for convenience. In the second section we define Mellin convolutions with admissible meromorphic kernels and prove their boundedness in the Bessel potential spaces. In Section **??** is proved the key result on commutants of the Mellin convolution operator (with admissible meromorphic kernel) and a Bessel potential. In Section **??** we enhance results on Banach algebra generated by Mellin and Fourier convolution operators by adding explicit definition of the symbol of a Hankel operator, which belong to this algebra. In Sections 5 the obtained results are applied to describe Fredholm properties and the index of Mellin convolution operators with admissible meromorphic kernels in the Bessel potential spaces.

2 Mellin Convolution and the Bessel Potential Operators

Let N be a positive integer. If there arises no confusion, we write $\mathfrak{A}$ for both scalar and matrix $N \times N$ algebras with entries from $\mathfrak{A}$. Similarly, the same notation $\mathfrak{B}$ is used for the set of N-dimensional vectors with entries from $\mathfrak{B}$. It will be usually clear from the context what kind of space or algebra is considered.

The integral operator (1.1) is called Mellin convolution. More generally, if $a \in \mathbb{L}_\infty(\mathbb{R})$ is an essentially bounded measurable $N \times N$ matrix function, the Mellin convolution operator $\mathfrak{M}^0_a$ is defined by

$$\mathfrak{M}^0_a\varphi(t) := \mathscr{M}^{-1}_\beta a \mathscr{M}_\beta \varphi(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} a(\xi) \int\limits_0^\infty \left(\frac{t}{\tau}\right)^{i\xi-\beta} \varphi(\tau)\,\frac{d\tau}{\tau}\,d\xi, \qquad \varphi \in \mathbb{S}(\mathbb{R}^+),$$

where $\mathbb{S}(\mathbb{R}^+)$ is the Schwartz space of fast decaying functions on $\mathbb{R}^+$, whereas $\mathscr{M}_\beta$ and $\mathscr{M}^{-1}_\beta$ are the Mellin transform and its inverse, i.e.

$$\mathscr{M}_\beta\psi(\xi) := \int\limits_0^\infty t^{\beta-i\xi}\psi(t)\,\frac{dt}{t}, \quad \xi \in \mathbb{R},$$

$$\mathscr{M}^{-1}_\beta\varphi(t) := \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} t^{i\xi-\beta}\varphi(\xi)\,d\xi, \quad t \in \mathbb{R}^+.$$

The function $a(\xi)$ is usually referred to as a symbol of the Mellin operator $\mathfrak{M}^0_a$. Further, if the corresponding Mellin convolution operator $\mathfrak{M}^0_a$ is bounded on the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ of N-vector functions endowed with the norm

$$\big\|\varphi \mid \mathbb{L}_p(\mathbb{R}^+, t^\gamma)\big\| := \left[\int\limits_0^\infty t^\gamma |\varphi(t)|^p\,dt\right]^{1/p},$$

then the symbol $a(\xi)$ is called a Mellin $\mathbb{L}_{p,\gamma}$ *multiplier*.

The transformations

$$\mathbf{Z}_\beta : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}), \quad \mathbf{Z}_\beta\varphi(\xi) := e^{-\beta t}\varphi(e^{-\xi}), \quad \xi \in \mathbb{R},$$
$$\mathbf{Z}^{-1}_\beta : \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma), \quad \mathbf{Z}^{-1}_\beta\psi(t) := t^{-\beta}\psi(-\ln t), \quad t \in \mathbb{R}^+,$$

arrange an isometrical isomorphism between the corresponding $\mathbb{L}_p$-spaces. Moreover, the relations

$$\begin{gathered}\mathscr{M}_\beta = \mathscr{F}\mathbf{Z}_\beta, \quad \mathscr{M}_\beta^{-1} = \mathbf{Z}_\beta^{-1}\mathscr{F}^{-1},\\ \mathfrak{M}_a^0 = \mathscr{M}_\beta^{-1} a \mathscr{M}_\beta = \mathbf{Z}_\beta^{-1}\mathscr{F}^{-1} a \mathscr{F}\mathbf{Z}_\beta = \mathbf{Z}_\beta^{-1} W_a^0 \mathbf{Z}_\beta,\\ -1 < \gamma < p-1, \qquad \beta := \frac{1+\gamma}{p}. \qquad 0, \beta < 1,\end{gathered} \tag{2.1}$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transform and its inverse,

$$\mathscr{F}\varphi(\xi) := \int\limits_{-\infty}^{\infty} e^{i\xi x}\varphi(x)\,dx, \quad \mathscr{F}^{-1}\psi(x) := \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-i\xi x}\psi(\xi)\,d\xi, \ \ x \in \mathbb{R}\,,$$

show a close connection between Mellin $\mathfrak{M}_a^0$ and Fourier

$$W_a^0\varphi := \mathscr{F}^{-1} a \mathscr{F}\varphi, \ \ \varphi \in \mathbb{S}(\mathbb{R}),$$

convolution operators, as well as between the corresponding transforms. Here $\mathbb{S}(\mathbb{R})$ denotes the Schwartz class of infinitely smooth functions, decaying fast at the infinity.

An $N \times N$ matrix function $a(\xi)$, $\xi \in \mathbb{R}$ is called a *Fourier $\mathbb{L}_p$-multiplier* if the operator $W_a^0 : \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R})$ is bounded. The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$.

From (2.1) immediately follows the following

Proposition 2.1 (See [Du79]) *Let $1 < p < \infty$. The class of Mellin $\mathbb{L}_{p,\gamma}$-multipliers coincides with the Banach algebra $\mathfrak{M}_p(\mathbb{R})$ of Fourier $\mathbb{L}_p$ -multipliers for arbitrary $-1 < \gamma < p-1$ and is independent of the parameter γ.*

Thus, a Mellin convolution operator $\mathfrak{M}_a^0$ in (2.1) is bounded in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ if and only if $a \in \mathfrak{M}_p(\mathbb{R})$.

It is known, see, e.g. [Du79], that the Banach algebra $\mathfrak{M}_p(\mathbb{R})$ contains the algebra $\mathbf{V}_1(\mathbb{R})$ of all functions with bounded variation provided that

$$\beta := \frac{1+\gamma}{p}, \ \ 1 < p < \infty, \ \ -1 < \gamma < p-1. \tag{2.2}$$

As it was already mentioned, the primary aim of the present paper is to study Mellin convolution operators $\mathfrak{M}_a^0$ acting in the Bessel potential spaces,

$$\mathfrak{M}_a^0 : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+). \tag{2.3}$$

The symbols of these operators are $N \times N$ matrix functions $a \in C\mathfrak{M}_p^0(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only one possible jump at infinity. We commence with the definition of the Besseel potential spaces and Bessel potentials, arranging isometrical isomorphisms between these spaces and enabling the lifting procedure, writing a Fredholm equivalent operator (equation) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ for the operator $\mathfrak{M}_a^0$ in (2.3).

For $s \in \mathbb{R}$ and $1 < p < \infty$, the Bessel potential space, known also as a fractional Sobolev space, is the subspace of the Schwartz space $\mathbb{S}'(\mathbb{R})$ of distributions having the finite norm

$$\big\|\varphi \mid \mathbb{H}^s_p(\mathbb{R})\big\| := \Bigg[\int\limits_{-\infty}^{\infty} \big|\mathscr{F}^{-1}\big(1+|\xi|^2\big)^{s/2}(\mathscr{F}\varphi)(t)\big|^p\, dt\Bigg]^{1/p} < \infty.$$

For an integer parameter $s = m = 1, 2, \dots$, the space $\mathbb{H}^s_p(\mathbb{R})$ coincides with the usual Sobolev space endowed with an equivalent norm

$$\big\|\varphi \mid \mathbb{W}^m_p(\mathbb{R})\big\| := \Bigg[\sum_{k=0}^{m}\int\limits_{-\infty}^{\infty} \Big|\frac{d^k\varphi(t)}{dt^k}\Big|^p\, dt\Bigg]^{1/p}.$$

If $s < 0$, one gets the space of distributions. Moreover, $\mathbb{H}^{-s}_{p'}(\mathbb{R})$ is the dual to the space $\mathbb{H}^s_p(\mathbb{R}^+)$, provided $p' := \frac{p}{p-1}$, $1 < p < \infty$. Note that $\mathbb{H}^s_2(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle\varphi,\psi\rangle_s = \int\limits_{\mathbb{R}} (\mathscr{F}\varphi)(\xi)\overline{(\mathscr{F}\psi)(\xi)}(1+\xi^2)^s\, d\xi, \quad \varphi,\psi \in \mathbb{H}^s(\mathbb{R}).$$

By r_Σ we denote the operator restricting functions or distributions defined on $\mathbb{R}$ to the subset $\Sigma \subset \mathbb{R}$. Thus $\mathbb{H}^s_p(\mathbb{R}^+) = r_+(\mathbb{H}^s_p(\mathbb{R}))$, and the norm in $\mathbb{H}^s_p(\mathbb{R}^+)$ is defined by

$$\big\|f \mid \mathbb{H}^s_p(\mathbb{R}^+)\big\| = \inf_{\ell}\big\|\ell f \mid \mathbb{H}^s_p(\mathbb{R})\big\|,$$

where ℓf stands for any extension of f to a distribution in $\mathbb{H}^s_p(\mathbb{R})$.

Further, we denote by $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ the (closed) subspace of $\mathbb{H}^s_p(\mathbb{R})$ which consists of all distributions supported in the closure of $\mathbb{R}^+$.

Notice that $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ is always continuously embedded in $\mathbb{H}^s_p(\mathbb{R}^+)$, and if $s \in (1/p - 1, 1/p)$, these two spaces coincide. Moreover, $\mathbb{H}^s_p(\mathbb{R}^+)$ may be viewed as the quotient-space $\mathbb{H}^s_p(\mathbb{R}^+) := \mathbb{H}^s_p(\mathbb{R})/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^-)$, $\mathbb{R}^- := (-\infty, 0)$.

Let $a \in \mathbb{L}_{\infty,loc}(\mathbb{R})$ be a locally bounded $m \times m$ matrix function. The Fourier convolution operator (FCO) with the symbol a is defined by

$$W^0_a := \mathscr{F}^{-1}a\mathscr{F}.$$

If the operator

$$W^0_a : \mathbb{H}^s_p(\mathbb{R}) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R})$$

is bounded, we say that a is an $\mathbb{L}_p$-multiplier of order r and use "$\mathbb{L}_p$-multiplier" if the order is 0. The set of all $\mathbb{L}_p$-multipliers of order r (of order 0) is denoted by $\mathfrak{M}^r_p(\mathbb{R})$ (by $\mathfrak{M}_p(\mathbb{R})$, respectively).

For an $\mathbb{L}_p$-multiplier of order r, $a \in \mathfrak{M}^r_p(\mathbb{R})$, the Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ is defined by the equality

$$W_a = r_+W^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+) \tag{2.4}$$

and the Hankel operator by the equality

$$H_a = r_+ \boldsymbol{V} W_a^0 : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+), \quad \boldsymbol{V}\psi(t) := \psi(-t), \tag{2.5}$$

where $r_+ := r_{\mathbb{R}^+} : \mathbb{H}^s_p(\mathbb{R}) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$ is the restriction operator to the semi-axes $\mathbb{R}^+$. Note that the generalized Hörmander's kernel of a FCO W_a depends on the difference of arguments $\mathscr{K}_a(t-\tau)$, while the Hörmander's kernel of a Hankel operator $r_+\boldsymbol{V}W^0_a$ depends of the sum of the arguments $\mathscr{K}_a(t+\tau)$.

We did not use in the definition of the class of multipliers $\mathfrak{M}^r_p(\mathbb{R})$ the parameter $s \in \mathbb{R}$. This is due to the fact that $\mathfrak{M}^r_p(\mathbb{R})$ is independent of s: if the operator W_a in (2.5) is bounded for some $s \in \mathbb{R}$, it is bounded for all other values of s. Another definition of the multiplier class $\mathfrak{M}^r_p(\mathbb{R})$ is written as follows: $a \in \mathfrak{M}^r_p(\mathbb{R})$ if and only if $\lambda^{-r}a \in \mathfrak{M}_p(\overline{\mathbb{R}}) = \mathfrak{M}^0_p(\overline{\mathbb{R}})$, where $\lambda^r(\xi) := (1+|\xi|^2)^{r/2}$. This assertion is one of the consequences of the following theorem.

Theorem 2.2 *Let $1 < p < \infty$. Then:*

1. *For any $r, s \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$ the convolution operators (ΨDOs)*

$$\begin{aligned} \boldsymbol{\Lambda}^r_\gamma = W_{\lambda^r_\gamma} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) &\longrightarrow \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+), \\ \boldsymbol{\Lambda}^r_{-\gamma} = r_+ W^0_{\lambda^r_{-\gamma}}\ell : \mathbb{H}^s_p(\mathbb{R}^+) &\longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+), \\ \lambda^r_{\pm\gamma}(\xi) := (\xi \pm \gamma)^r, \quad \xi \in \mathbb{R}, &\quad \operatorname{Im}\gamma > 0, \end{aligned} \tag{2.6}$$

where $\ell : \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R})$ is an extension operator and r_+ is the restriction from the axes $\mathbb{R}$ to the semi-axes $\mathbb{R}^+$, arrange isomorphisms of the corresponding spaces. The final result is independent of the choice of an extension ℓ.

2. *For arbitrary operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ of order r, the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) & \xrightarrow{\ \mathbf{A}\ } & \mathbb{H}^{s-r}_p(\mathbb{R}^+) \\ \boldsymbol{\Lambda}^{-s}_\gamma \Big\uparrow & & \Big\downarrow \boldsymbol{\Lambda}^{s-r}_{-\gamma} \\ \mathbb{L}_p(\mathbb{R}^+) & \xrightarrow[\boldsymbol{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\boldsymbol{\Lambda}^{-s}_\gamma]{} & \mathbb{L}_p(\mathbb{R}^+) \end{array} . \tag{2.7}$$

The diagram (2.6) provides an equivalent lifting of the operator $\mathbf{A}$ of order r to the operator $\boldsymbol{\Lambda}^{s-r}_{-\gamma}\mathbf{A}\boldsymbol{\Lambda}^{-s}_\gamma : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ of order 0.

3. *For any bounded convolution operator $W_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ of order r and for any pair of complex numbers γ_1, γ_2 such that $\operatorname{Im}\gamma_j > 0$, $j = 1, 2$, the lifted operator*

$$\begin{aligned} \boldsymbol{\Lambda}^\mu_{-\gamma_1} W_a \boldsymbol{\Lambda}^\nu_{\gamma_2} = W_{a_{\mu,\nu}} \ &: \ \widetilde{\mathbb{H}}^{s+\nu}_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-r-\mu}_p(\mathbb{R}^+), \\ a_{\mu,\nu}(\xi) &:= (\xi-\gamma_1)^\mu a(\xi)(\xi+\gamma_2)^\nu \end{aligned} \tag{2.8}$$

is again a Fourier convolution.
In particular, the lifted operator W_{a_0} in $\mathbb{L}_p$-spaces, $\boldsymbol{\Lambda}^{s-r}_{-\gamma}W_a\boldsymbol{\Lambda}^{-s}_\gamma : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ has the symbol

$$a_{s-r,-s}(\xi) = \lambda^{s-r}_{-\gamma}(\xi)a(\xi)\lambda^{-s}_\gamma(\xi) = \Big(\frac{\xi-\gamma}{\xi+\gamma}\Big)^{s-r}\frac{a(\xi)}{(\xi+i)^r}\,.$$

4. *The Hilbert transform* $S_{\mathbb{R}^+} = i\mathbf{K}_1^1 = W_{-\,\mathrm{sign}}$ *is a Fourier convolution operator and*

$$\boldsymbol{\Lambda}^s_{-\gamma_1}\boldsymbol{K}^1_1\boldsymbol{\Lambda}^{-s}_{\gamma_2} = W_{i\,g^s_{-\gamma_1,\gamma_2}\,\mathrm{sign}}, \tag{2.9}$$

where

$$g^s_{-\gamma_1,\gamma_2}(\xi) := \left(\frac{\xi-\gamma_1}{\xi+\gamma_2}\right)^s. \tag{2.10}$$

Proof: For the proof of items 1-3 we refer the reader to [Du79, Lemma 5.1] and [DS93, ES81]. The item 4 is a consequence of the proved items 2 and 3 (see [Du79, Du13]). ■

Remark 2.3 *The class of Fourier convolution operators is a subclass of pseudodifferential operators (ΨDOs). Moreover, for integer parameters $m = 1, 2, \dots$ the Bessel potentials $\boldsymbol{\Lambda}^m_{\pm} = W_{\lambda^m_{\pm\gamma}}$, which are Fourier convolutions of order m, are ordinary differential operators of the same order m:*

$$\boldsymbol{\Lambda}^m_{\pm\gamma} = W_{\lambda^m_{\pm\gamma}} = \left(i\,\frac{d}{dt}\pm\gamma\right)^m = \sum_{k=0}^m \binom{m}{k} i^k(\pm\gamma)^{m-k}\,\frac{d^k}{dt^k}\,. \tag{2.11}$$

These potentials map both spaces (cf. (2.6)*)*

$$\begin{aligned}\boldsymbol{\Lambda}^m_{\pm\gamma}\quad &: \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+),\\ &: \mathbb{H}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{s-m}_p(\mathbb{R}^+),\end{aligned} \tag{2.12}$$

but the mappings are not isomorphisms because the inverses $\boldsymbol{\Lambda}^{-m}_{\pm\gamma}$ are bounded only for one pair of spaces, indicated in (2.6).

Remark 2.4 *For any pair of multipliers $a \in \mathfrak{M}^r_p(\mathbb{R})$, $b \in \mathfrak{M}^s_p(\mathbb{R})$ the corresponding convolution operators on the half axes W^0_a and W^0_b have the property $W^0_aW^0_b = W^0_bW^0_a = W^0_{ab}$.*

For the corresponding Wiener-Hopf operators on the half axes a similar equality

$$W_aW_b = W_{ab} \tag{2.13}$$

holds if and only if either the function $a(\xi)$ has an analytic extension in the lower half plane, or the function $b(\xi)$ has an analytic extension in the upper half plane (see [Du79]).

Note that, actually (2.8) *is a consequence of* (2.13).

3 Mellin Convolutions with Admissible Meromorphic Kernels

Now we consider kernels $\mathscr{K}(t)$, exposed in (1.11), which are meromorphic functions on the complex plane $\mathbb{C}$, vanishing at infinity, having poles at $c_0, c_1, \dots \in \mathbb{C}\setminus\{0\}$ and complex coefficients $d_j \in \mathbb{C}$.

Definition 3.1 *We call a kernel $\mathscr{K}(t)$ in* (1.11) *is admissible iff:*

(i) $\mathscr{K}(t)$ *has only a finite number of poles* $c_0, \ldots, c_\ell$ *which belong to the positive semi-axes, i.e.,* $\arg c_0 = \cdots = \arg c_\ell = 0$*;*

(ii) *The corresponding multiplicities are one* $m_0 = \cdots = m_\ell = 1$*;*

(iii) *The remainder points* $c_{\ell+1}, c_{\ell+2}, \ldots$ *do not condense to the positive semi-axes and their real parts are bounded uniformly*

$$\varliminf_{j\longrightarrow\infty} c_j \notin [0, \infty), \qquad \sup_{j=\ell+1,\ell+2,\ldots} \operatorname{Re} c_j \leqslant K < \infty. \tag{3.1}$$

(iv) $\mathscr{K}(t)$ *is a kernel of an operator, which is a composition of finite number of operators with admissible kernels.*

Example 3.2 *The function*

$$\mathscr{K}(t) = \exp\left(\frac{1}{t-c}\right), \quad \operatorname{Re} c < 0 \ \ \textit{or} \ \ \operatorname{Im} c \neq 0$$

is an example of the admissible kernel which also satisfies the condition of the next Theorem 3.4. Other examples of operators with admissible kernels (which also satisfies the condition of the next Theorem 3.4) are operators which we encounter in (1.3)*, in* (1.10) *and in* (2.4) *and, in general, any finite sum in* (1.11)*.*

Example 3.3 *The function*

$$\mathscr{K}(t) = \frac{\ln\, t - c_1c_2}{t - c_1c_2}\,, \quad \operatorname{Im} c_1 \neq 0, \quad \operatorname{Im} c_2 \neq 0$$

is another example of the admissible kernel and represnts the composition of operators $c_2\mathbf{K}^1_{c_1}\mathbf{K}^1_{c_2}$ *(see* (2.10)*) with admissible kernels which also satisfies the condition of the next Theorem 3.4. More trivial examples of operators with admissible kernels (which also satisfies the condition of the next Theorem 3.4) are operators which we encounter in* (1.3)*, in* (1.10) *and in* (2.4) *and, in general, any finite sum in* (1.11)*.*

Theorem 3.4 *Let conditions*

$$\beta := \frac{1+\gamma}{p}, \quad 1 < p < \infty, \quad -1 < \gamma < p - 1. \tag{3.2}$$

hold, $\mathscr{K}(t)$ *in* (1.11) *be an admissible kernel and*

$$K_\beta := \sum_{j=0}^{\infty} 2^{m_j} |d_j|\, |c_j|^{\beta - m_j} < \infty. \tag{3.3}$$

Then the Mellin convolution $\mathfrak{M}^0_{a_\beta}$ *in* (1.9) *with the admissible meromorphic kernel* $\mathscr{K}(t)$ *in* (1.11) *is bounded in the Lebesgue space* $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ *and its norm has the estimate* $\|\mathfrak{M}^0_{a_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+, t^\gamma))\| \leqslant M K_\beta$ *with some* $M > 0$*.*

We can drop the constant M *and replace* 2^{m_j} *by* $2^{\frac{m_j}{2}}$ *in the estimate* (3.3) *provided* $\operatorname{Re} c_j < 0$ *for all* $j = 0, 1, \ldots$*.*

Proof: The first $\ell+1$ summands in the definition of the admissible kernel (1.11) correspond to the Cauchy operators

$$A_0\varphi(t)=\sum_{j=0}^{\ell}\frac{d_j}{\pi}\int_0^\infty\frac{\varphi(\tau)\,d\tau}{t-c_j\tau},\quad c_j>0,\quad j=0,1,\dots,\ell,$$

and their boundedness property in the weighted Lebesgue space

$$A_0:\mathbb{L}_p(\mathbb{R}^+,t^\gamma)\longrightarrow\mathbb{L}_p(\mathbb{R}^+,t^\gamma) \tag{3.4}$$

under constraints (2.2) is well known (see [Kh57] and also [GK79]). Therefore we can ignore the first ℓ summands in the expansion of the kernel $\mathscr{K}(t)$ in (1.11). To the boundedness of the operator $\mathfrak{M}^0_{a^\ell_\beta}$ with the remainder kernel

$$\mathscr{K}^\ell(t):=\sum_{j=\ell+1}^{\infty}\frac{d_j}{(t-c_j)^{m_j}}\,,\quad c_j\neq0,\quad j=0,1,\dots,$$
$$0<\alpha_k:=\arg c_k<2\pi,\quad k=\ell+1,\ell+2,\dots$$

(see (1.11)), we apply the estimate (1.8)

$$\left\|\mathfrak{M}^0_{a^\ell_\beta}\mid\mathscr{L}(\mathbb{L}_p(\mathbb{R}^+,t^\gamma))\right\|\leqslant\int_0^\infty t^{\beta-1}|\mathscr{K}^\ell(t)|\,dt\leqslant\sum_{j=\ell+1}^{\infty}|d_j|\int_0^\infty\frac{t^{\beta-1}dt}{|t-c_j|^{m_j}}\,. \tag{3.5}$$

Now note that

$$|t-c_j|^{-m_j}=\left(t^2+|c_j|^2-2\operatorname{Re}c_jt\right)^{-\frac{m_j}{2}}\leqslant\left(\frac{t^2+|c_j|^2}{2}\right)^{-\frac{m_j}{2}}$$
$$\leqslant2^{m_j}(t+|c_j|)^{-m_j}\qquad\text{for all }\ t\geqslant2K=2\sup|\operatorname{Re}c_j|>0.$$

due to the constraints (3.1). On the other hand,

$$|t-c_j|^{-m_j}\leqslant M(t+|c_j|)^{-m_j}\ \text{ for all }\ 0\leqslant t\leqslant2K$$

and a certain constant $M>0$. Therefore

$$|t-c_j|^{-m_j}\leqslant M2^{m_j}(t+|c_j|)^{-m_j}\ \text{ for all }\ 0<t<\infty. \tag{3.6}$$

Next we recall the formula from [GR94, Formula 3.194.4]

$$\int_0^\infty \tag{3.7}$$

$$\frac{t^{\beta-1}\,dt}{(t+c)^m}=(-1)^{m-1}\binom{\beta dir0o}{m dir0o}\frac{\pi c^{\beta-m}}{\sin\pi\beta},\quad-\pi<\arg c<\pi,\quad\operatorname{Re}\beta<1, \tag{3.8}$$
$$\binom{\beta-1}{m-1}:=\frac{(\beta-1)\cdots(\beta-m+1)}{(m-1)}\,,\quad\binom{\beta-1}{0}:=1$$

to calculate the integrals. By inserting the estimate (3.6) into (3.5) and applying (3.7), we get

$$\begin{aligned}\left\|\mathfrak{M}^0_{a^\ell_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+,t^\gamma))\right\| &\leqslant \sum_{j=\ell+1}^{\infty}|d_j|\int_0^\infty \frac{t^{\beta-1}dt}{|t-c_j|^{m_j}}\\ &\leqslant M_0\sum_{j=\ell+1}^{\infty}2^{m_j}|d_j|\int_0^\infty\frac{t^{\beta-1}dt}{(t+|c_j|)^{m_j}}\\ &\leqslant \frac{\pi M_0}{\sin\pi\beta}\sum_{j=\ell+1}^{\infty}2^{m_j}|d_j|\left|\binom{\beta-1}{m_j-1}\right|c_j^{\beta-m_j}\\ &\leqslant M\sum_{j=\ell+1}^{\infty}2^{m_j}|d_j|c_j^{\beta-m_j}=MK_\beta,\quad M:=\frac{\pi M_0}{\sin\pi\beta},\end{aligned}\tag{3.9}$$

since (see (3.7))

$$\left|\binom{\beta-1}{m_j-1}\right|\leqslant 1,$$

where K_β is from (3.3). The boundedness (3.4) and the estimate (3.9) imply the claimed estimate

$$\left\|\mathfrak{M}^0_{a_\beta} \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+,t^\gamma))\right\| \leqslant MK_\beta.$$

If $\operatorname{Re} c_j<0$ for all $j=0,1,\ldots$, we have

$$\frac{1}{|t-c_j|^{m_j}} = \left(t^2+|c|^2-2\operatorname{Re}c_jt\right)^{-\frac{m_j}{2}} \leqslant \left(t^2+|c|^2\right)^{-\frac{m_j}{2}} \leqslant 2^{\frac{m_j}{2}}\left(t+|c_j|\right)^{-m_j}$$

valid for all $t>0$ and a constant M does not emerge in the estimate. ■

Let us find the symbol (the Mellin transform of the kernel) of the operator (2.10) for $0<\arg\, c<2\pi$, $m=1,2,\ldots$ (see (2.9), (2.10)). For this we apply formula (3.7):

$$\begin{aligned}\mathscr{M}_\beta\mathscr{K}^m_c(\xi)=\int_0^\infty t^{\beta-i\xi-1}\mathscr{K}^m_c(t)\,dt &= \frac{1}{\pi}\int_0^\infty\frac{t^{\beta-i\xi-1}}{(t+(-c))^m}\,dt\\ &=\binom{\beta-i\xi-1}{m-1}\frac{(-1)^{m-1}(-c)^{\beta-i\xi-m}}{\sin\pi(\beta-i\xi)}\\ &=\binom{\beta-i\xi-1}{m-1}\frac{(-1)^{m-1}e^{-i\pi(\beta-i\xi-m)}c^{\beta-i\xi-m}}{\sin\pi(\beta-i\xi)},\end{aligned}$$

since if $-\pi<\arg(-c)<\pi$ and $0<\arg\, c<2\pi$, then $-c=e^{-\pi i}c$. In particular,

$$\mathscr{M}_\beta\mathscr{K}^1_c(\xi)=\frac{e^{-i\pi(\beta-i\xi-1)}c^{\beta-i\xi-1}}{\sin\pi(\beta-i\xi)},\quad 0<\arg c<2\pi,\tag{3.10a}$$

$$\mathscr{M}_\beta\mathscr{K}^1_{-d}(\xi)=\frac{d^{\beta-i\xi-1}}{\sin\pi(\beta-i\xi)},\quad -\pi<\arg d<\pi,\tag{3.10b}$$

$$\mathscr{M}_\beta\mathscr{K}^1_{-1}(\xi)=\frac{1}{\sin\pi(\beta-i\xi)},\quad \xi\in\mathbb{R}.\tag{3.10c}$$

Now let us find the symbol of the Cauchy singular integral operator $K^1_1 = -iS_{\mathbb{R}^+}$ (see (**??**), (**??**)). For this we apply Plemeli formula and formula (3.7):

$$\mathscr{M}_\beta\mathscr{K}^1_1(t) := \int_0^\infty t^{\beta-i\xi-1}\mathscr{K}^1_1(t)\,dt = -\frac{1}{\pi}\int_0^\infty \frac{t^{\beta-i\xi-1}\,dt}{t-1}$$
$$= \lim_{\varepsilon\longrightarrow 0}\frac{1}{2\pi}\int_0^\infty \Big[\frac{t^{\beta-i\xi-1}}{t+e^{i(\pi-\varepsilon)}} + \frac{t^{\beta-i\xi-1}}{t+e^{-i(\pi-\varepsilon)}}\Big]\,dt$$
$$= \lim_{\varepsilon\longrightarrow 0}\frac{e^{i(\pi-\varepsilon)(\beta-i\xi-1)} + e^{-i(\pi-\varepsilon)(\beta-i\xi-1)}}{2\sin\pi(\beta-i\xi)} = \cot\pi(\beta-i\xi).$$

For an admissible kernel with poles $\arg c_0 = \arg c_\ell = 0$ (and, therefore, $m_0 = \cdots = m_\ell = 1$) and $0 < \arg c_j < 2\pi$, $j = \ell+1, \ldots$ we get

$$\mathscr{M}_\beta\mathscr{K}(\xi) = \cot\pi(\beta-i\xi)\sum_{j=0}^{\ell} d_j c_j^{\beta-i\xi-1}$$
$$+\frac{1}{\sin\pi(\beta-i\xi)}\sum_{j=\ell+1}^{\infty} d_j\binom{\beta-i\xi-1}{m_j-1}(-1)^{m_j-1}e^{-i\pi(\beta-i\xi-m_j)}c_j^{\beta-i\xi-m_j}. \quad (3.11)$$

Theorem 3.5 *If $\mathscr{K}$ is an admissible kernel the corresponding Mellin convolution operator with the kernel $\mathscr{K}$*

$$\mathbf{K}\varphi(t) := \int_0^\infty \mathscr{K}\left(\frac{t}{\tau}\right)\varphi(\tau)\frac{d\tau}{\tau},$$
$$\mathbf{K}\ :\ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+), \quad (3.12)$$

is bounded for all $1 < p < \infty$ and $s \in \mathbb{R}$.

The condition on the parameter p can be relaxed to $1 \leqslant p \leqslant \infty$, provided the admissible kernel $\mathscr{K}$ in (1.11) *has no poles on positive semi-axes $\alpha_j = \arg c_j \neq 0$ for all $j = 0, 1, \ldots$.*

Proof: Due to the representation (1.11), we have to prove the theorem only for a model kernel

$$\mathscr{K}^m_c(t) := \frac{1}{\pi(t-c)^m}, \qquad c \neq 0, \quad 0 < \arg c < 2\pi, \quad m = 1, 2, \ldots. \quad (3.13)$$

The respective Mellin convolution operator $\mathbf{K}^m_c$ (see (2.10)) is bounded in $\mathbb{L}_p(\mathbb{R}^+)$ for all $1 \leqslant p \leqslant \infty$ for arbitrary $0 < |\arg c| < \pi$ (cf. (1.2)).

To accomplish the boundedness result of $\mathbf{K}^m_c$ in $\mathbb{L}_p(\mathbb{R}^+)$ we need to consider the case $\arg c = 0$ (i.e., $c \in (0,\infty)$) and, therefore, $m = 1$ (see Definition **??**). Then the operator $\mathbf{K}^1_c$ coincides with the "dilated" Cauchy singular integral operator with a constant multiplier

$$\mathbf{K}^1_c\varphi(t) := \frac{1}{\pi}\int_0^\infty \frac{\varphi(\tau)d\tau}{t-c\,\tau} = -\frac{i}{c}\,(S_{\mathbb{R}^+}\varphi)\Big(\frac{t}{c}\Big), \quad (3.14)$$

where

$$S_{\mathbb{R}^+}\varphi(t) := \frac{1}{\pi i}\int\limits_0^\infty \frac{\varphi(\tau)\,d\tau}{\tau - t}, \tag{3.15}$$

and is bounded in $\mathbb{L}_p(\mathbb{R}^+)$ for all $1 < p < \infty$ (cf., e.g., [Du79, GK79]).

Now let $0 \leqslant \arg c < 2\pi$ and $m = 1$. Then, if $\varphi \in C_0^\infty(\mathbb{R}^+)$ is a smooth function with compact support and $k = 1, 2, \ldots$, integrating by parts we get

$$\begin{aligned}\frac{d^k}{dt^k}\,\mathbf{K}_c^1\varphi(t) &= \frac{1}{\pi}\int\limits_0^\infty \frac{d^k}{dt^k}\,\frac{1}{t - c\,\tau}\varphi(\tau)\,d\tau = \frac{(-c)^{-k}}{\pi}\int\limits_0^\infty \frac{d^k}{d\tau^k}\,\frac{1}{t - c\,\tau}\varphi(\tau)\,d\tau\\ &= \frac{c^{-k}}{\pi}\int\limits_0^\infty \frac{1}{t - c\,\tau}\,\frac{d^k\varphi(\tau)}{d\tau^k}\,d\tau = c^{-k}\Big(\mathbf{K}_c^1\,\frac{d^k}{dt^k}\varphi\Big)(t).\end{aligned} \tag{3.16}$$

For $m = 2, 3, \ldots$ and $0 < \arg c < 2\pi$ we get similarly

$$\begin{aligned}\frac{d}{dt}\,\mathbf{K}_c^m\varphi(t) &= \frac{1}{\pi}\int\limits_0^\infty \frac{d}{dt}\,\frac{\tau^{m-1}}{(t - c\,\tau)^m}\,\varphi(\tau)\,d\tau\\ &= \sum_{j=0}^{m-1}\frac{(-c)^{-1-j}}{\pi}\int\limits_0^\infty \frac{d}{d\tau}\,\frac{\tau^{m-1-j}}{(t - c\,\tau)^{m-j}}\,\varphi(\tau)\,d\tau\\ &= -\sum_{j=0}^{m-1}\frac{(-c)^{-1-j}}{\pi}\int\limits_0^\infty \frac{\tau^{m-1-j}}{(t - c\,\tau)^{m-j}}\,\frac{d}{d\tau}\,\varphi(\tau)\,d\tau\\ &= -\sum_{j=0}^{m-1}(-c)^{-1-j}\Big(\mathbf{K}_c^{m-j}\,\frac{d}{dt}\,\varphi\Big)(t)\end{aligned}$$

and, recurrently,

$$\frac{d^k}{dt^k}\,\mathbf{K}_c^m\varphi(t) = (-1)^k\sum_{j=0}^{m-1}(-c)^{-k-j}\gamma_j^k\Big(\mathbf{K}_c^{m-j}\,\frac{d^k}{dt^k}\varphi\Big)(t), \quad k = 1, 2, \ldots, \tag{3.17}$$

$$\gamma_j^1 = j + 1, \quad \gamma_0^k = 1, \quad \gamma_j^k := \sum_{r=0}^{j}\gamma_r^{k-1}, \quad j = 0, 1, \ldots, m, \quad k = 1, 2, \ldots.$$

Recall now that for an integer $s = n$ the spaces $\mathbb{H}_p^n(\mathbb{R}^+)$, $\widetilde{\mathbb{H}}_p^n(\mathbb{R}^+)$ coincide with the Sobolev spaces $\mathbb{W}_p^n(\mathbb{R}^+)$, $\widetilde{\mathbb{W}}_p^n(\mathbb{R}^+)$, respectively (these spaces are isomorphic and the norms are equivalent) and $C_0^\infty(\mathbb{R}^+)$ is a dense subset in $\widetilde{\mathbb{W}}_p^n(\mathbb{R}^+) = \widetilde{\mathbb{H}}_p^n(\mathbb{R}^+)$. Then, using the equalities (3.15), (3.17) and the boundedness of the operators $\mathbf{K}_c^{m-j}$ (see (3.13)–(3.15)), we proceed as follows:

$$\Big\|\mathbf{K}_c^m\varphi \mid \mathbb{H}_p^n(\mathbb{R}^+)\Big\| = \sum_{k=0}^{n}\Big\|\frac{d^k}{dt^k}\,\mathbf{K}_c^m\varphi \mid \mathbb{L}_p(\mathbb{R}^+)\Big\| =$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{m-1} |c|^{-k-j} \gamma_j^k \left\| \mathbf{K}_c^{m-j} \frac{d^k}{dt^k} \varphi \mid \mathbb{L}_p(\mathbb{R}^+) \right\|$$
$$\leqslant M \sum_{k=0}^{n} \left\| \frac{d^k}{dt^k} \varphi \mid \mathbb{L}_p(\mathbb{R}^+) \right\| = M \big\| \varphi \mid \mathbb{H}_p^n(\mathbb{R}^+) \big\|,$$

where $M > 0$ is a constant, and the boundedness (3.12) follows for $s = 0, 1, 2, \dots$. The case of arbitrary $s > 0$ follows by the interpolation between the spaces $\mathbb{H}_p^m(\mathbb{R}^+)$ and $\mathbb{H}_p^0(\mathbb{R}^+) = \mathbb{L}_p(\mathbb{R}^+)$, also between the spaces $\widetilde{\mathbb{H}}_p^m(\mathbb{R}^+)$ and $\widetilde{\mathbb{H}}_p^0(\mathbb{R}^+) = \mathbb{L}_p(\mathbb{R}^+)$.

For $s < 0$ the boundedness (3.12) follows by duality: the adjoint operator to $\mathbf{K}_c^m$ is

$$\mathbf{K}_c^{m,*}\varphi(t) := \frac{1}{\pi} \int_0^\infty \frac{t^{m-1}\varphi(\tau)\, d\tau}{(\tau - c\,t)^m} = \sum_{j=1}^{m} \omega_j \mathbf{K}_{c^{-1}}^j \varphi(t),$$

for some constant coefficients $\omega_1, \dots, \omega_m$. The operator $\mathbf{K}_c^{m,*}$ has the admissible kernel and, due to the proved part of the theorem is bounded in the space setting $\mathbf{K}_c^{m,*} : \widetilde{\mathbb{H}}_{p'}^{-s}(\mathbb{R}^+) \longrightarrow \mathbb{H}_{p'}^{-s}(\mathbb{R}^+)$, $p' := p/(p-1)$, since $-s > 0$. The initial operator $\mathbf{K}_c^m : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+)$ is dual to $\mathbf{K}_c^{m,*}$ and, therefore, is bounded as well. ■

Corollary 3.6 *Let $1 < p < \infty$ and $s \in \mathbb{R}$. A Mellin convolution operator $\mathfrak{M}_a^0$ with an admissible kernel described in Definition* **??** (*also see Example* 3.3 *and Theorem* 3.4 *is bounded in the Bessel potential spaces*

$$\mathfrak{M}_a^0 : \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+).$$

The bounddness property

$$\mathfrak{M}_a^0 : \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+).$$

does not hold in general for even a simplest Mellin convolution operator $\boldsymbol{K}_c$, except the case when the spaces $\widetilde{\mathbb{H}}_p^s(\mathbb{R}^+$ and $\mathbb{H}_p^s(\mathbb{R}^+)$ can be identified, i.e., except the case $1/p - 1 < s < 1/p$. Indeed, to check this consider a smooth function with a compact support $\varphi \in C_0^\infty(\mathbb{R}^+$ which is constant on the unit interval: $\varphi(t) = 1$ for $0 < t < 1$. Obviously, $\varphi \in \mathbb{H}_p^s(\mathbb{R}^+)$ and $\varphi \notin \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+$ for all $s > 1/p$. Then,

$$\boldsymbol{K}_c\varphi(t) = \frac{1}{\pi} \int_0^\infty \frac{\varphi(\tau)\, d\tau}{t - c\,\tau} = \frac{1}{\pi} \int_0^1 \frac{d\tau}{t - c\,\tau} + \frac{1}{\pi} \int_1^\infty \frac{\varphi(\tau)\, d\tau}{t - c\,\tau} = c^{-1} \ln \tau + \varphi_0(t),$$

where $\varphi_0 \in \mathbb{H}_p^s(\mathbb{R}^+) \cap C^\infty(\mathbb{R}^+)$, while the first summand $\ln \tau$ does not belong to $\mathbb{H}^s(\mathbb{R}^+)$ since all functions in this space are continuous and uniformly bounded for $s > 1/p$.

We can prove the following very partial result, which has important practical applications.

Theorem 3.7 *Let* $1 < p < \infty$, $c \in \mathbb{C}$ *and* $\mathbb{X}^s_p(\mathbb{R}^+)$ *denote one of the spaces* $\mathbb{H}^r_p(\mathbb{R}^+)$ *or* $\mathbb{W}^r_p(\mathbb{R}^+)$, *while* $\widetilde{\mathbb{X}}^s_p(\mathbb{R}^+)$ *denote one of the spaces* $\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$ *or* $\widetilde{\mathbb{W}}^r_p(\mathbb{R}^+)$.

If $\dfrac{1}{p} - 1 < r < \dfrac{1}{p} + 1$, *the operator*

$$\begin{aligned} \boldsymbol{A}_c := c\boldsymbol{K}_c - c^{-1}\boldsymbol{K}_{c^{-1}} \quad &: \quad \mathbb{X}^r_p(\mathbb{R}^+) \longrightarrow \mathbb{X}^r_p(\mathbb{R}^+), \\ &: \quad \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+) \end{aligned} \tag{3.18}$$

is bounded while for $\dfrac{1}{p} - 2 < r < \dfrac{1}{p}$ *the operator*

$$\begin{aligned} \boldsymbol{A}^\#_c := \boldsymbol{K}_c - \boldsymbol{K}_{c^{-1}} \quad &: \quad \mathbb{X}^r_p(\mathbb{R}^+) \longrightarrow \mathbb{X}^r_p(\mathbb{R}^+), \\ &: \quad \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{X}}^r_p(\mathbb{R}^+), \end{aligned} \tag{3.19}$$

is bounded.

Proof: If $\dfrac{1}{p} - 1 < r < \dfrac{1}{p}$ the spaces $\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+$ and $\mathbb{H}^r_p(\mathbb{R}^+)$ can be identified and the boundedness (3.18), (3.19) follows from Theorem 3.5.

Now let $\dfrac{1}{p} < r < \dfrac{1}{p} + 1$ Due to (2.12) the following diagrams

$$\begin{array}{ccc} \mathbb{H}^r_p(\mathbb{R}^+) & \xrightarrow{\boldsymbol{A}_c} & \mathbb{H}^r_p(\mathbb{R}^+) \\ \uparrow{\scriptstyle \boldsymbol{\Lambda}^{-1}_{-1}} & & \downarrow{\scriptstyle \boldsymbol{\Lambda}^{1}_{-1}} \\ \mathbb{H}^{r-1}_p(\mathbb{R}^+) & \xrightarrow[\boldsymbol{\Lambda}^1_{-1}\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{-1}]{} & \mathbb{H}^{r-1}_p(\mathbb{R}^+) \end{array}, \qquad \begin{array}{ccc} \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) & \xrightarrow{\boldsymbol{A}_c} & \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \\ \uparrow{\scriptstyle \boldsymbol{\Lambda}^{-1}_{1}} & & \downarrow{\scriptstyle \boldsymbol{\Lambda}^{1}_{1}} \\ \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+) & \xrightarrow[\boldsymbol{\Lambda}^1_{1}\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{1}]{} & \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+) \end{array} \tag{3.20}$$

are commutative. The diagrams (3.20) provide equivalent lifting of the operator $\boldsymbol{A}_c$ from the spaces $\mathbb{H}^r_p(\mathbb{R}^+)$ and $\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$ to the operator $\boldsymbol{A}^+_c := \boldsymbol{\Lambda}^1_1\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_1$ in the space $\widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+)$ and the operator $\boldsymbol{A}^-_c := \boldsymbol{\Lambda}^1_{-1}\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{-1}$ in the space $\mathbb{H}^{r-1}_p(\mathbb{R}^+)$. On the other hand, $\boldsymbol{\Lambda}^1_{\pm 1} = i\partial_t \pm I$ (see (2.11)) and it can be checked easily, using the integration by parts, that $\partial_t\boldsymbol{A}_c = -\boldsymbol{A}^\#_c\partial_t$. Then,

$$\begin{aligned} \boldsymbol{A}^\pm_c = \boldsymbol{\Lambda}^1_{\pm 1}\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{\pm 1} = (i\partial_t \pm I)\boldsymbol{A}_c\boldsymbol{\Lambda}^{-1}_{\pm 1} = (\pm\boldsymbol{A}_c - \boldsymbol{A}^\#_c)\boldsymbol{\Lambda}^{-1}_1 + \boldsymbol{A}^\#_c(i\partial_t \pm I)\boldsymbol{\Lambda}^{-1}_{\pm 1} \\ = (\pm\boldsymbol{A}_c - \boldsymbol{A}^\#_c)\boldsymbol{\Lambda}^{-1}_1 + \boldsymbol{A}^\#_c \end{aligned}$$

Since $\dfrac{1}{p} - 1 < r - 1 < \dfrac{1}{p}$ and the embeddings

$$\begin{aligned} \boldsymbol{\Lambda}^{-1}_{-1}\mathbb{H}^{r-1}_p(\mathbb{R}^+) = \mathbb{H}^r_p(\mathbb{R}^+) \subset \mathbb{H}^{r-1}_p(\mathbb{R}^+), \\ \boldsymbol{\Lambda}^{-1}_{1}\widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+) = \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \subset \widetilde{\mathbb{H}}^{r-1}_p(\mathbb{R}^+) \end{aligned}$$

are continuous, the operators

$$\boldsymbol{A}_c^- = (-\boldsymbol{A}_c - \boldsymbol{A}_c^{\#})\boldsymbol{\Lambda}_1^{-1} + \boldsymbol{A}_c^{\#} \;:\; \mathbb{H}_p^{r-1}(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{r-1}(\mathbb{R}^+),$$
$$\boldsymbol{A}_c^+ = (\boldsymbol{A}_c - \boldsymbol{A}_c^{\#})\boldsymbol{\Lambda}_1^{-1} + \boldsymbol{A}_c^{\#} \;:\; \widetilde{\mathbb{H}}_p^{r-1}(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{H}}_p^{r-1}(\mathbb{R}^+)$$

are bounded. Then, according the commutative diagrams (3.20), the operator $\boldsymbol{A}_c$ in (3.18) is bounded for $\mathbb{X}_p^r = \mathbb{H}_p^r$. For $\mathbb{X}_p^r = \mathbb{W}_p^r$ the boundedness is proved similarly or, alternatively, with the help of the interpolation theorems (see below Corollary **??** for similar arguments).

Now let $\dfrac{1}{p} - 2 < r < \dfrac{1}{p}$. Then

$$\frac{1}{p'} - 1 = -\frac{1}{p} < -r < \frac{1}{p'} + 1 = 2 - 1/p, \qquad p' := \frac{p}{p-1}. \tag{3.21}$$

The pair of the operator $\boldsymbol{K}_c$ and $-\overline{c}^{-1}\boldsymbol{K}_{\overline{c}^{-1}}$ are adjoint to each other. Therefore, the operator

$$\begin{aligned}\boldsymbol{A}_{\overline{c}} := \overline{c}\boldsymbol{K}_{\overline{c}} - \overline{c}^{-1}\boldsymbol{K}_{\overline{c}^{-1}} \;&:\; \mathbb{X}_p^r(\mathbb{R}^+) \longrightarrow \mathbb{X}_p^r(\mathbb{R}^+),\\ &:\; \widetilde{\mathbb{X}}_p^r(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{X}}_p^r(\mathbb{R}^+)\end{aligned} \tag{3.22}$$

is the adjoint to the operator $\boldsymbol{A}_c^{\#}$ in (3.19). Since the parameters $\{-r, p'\}$ satisfy the condition of the first part of the present theorem (see (3.21), the operator $\boldsymbol{A}_{\overline{c}}$ in (3.22) is bounded and jaustifies the boundedness of the adjoint operator $\boldsymbol{A}_c^{\#}$ in (3.19). ■

The next result is crucial in the present investigation. Note that, the case $\arg c = 0$ is essentially different and will be considered in Theorem 5.4 below.

Theorem 3.8 *Let* $0 < \arg c < 2\pi$ *and* $0 < \arg(-c\,\gamma) < \pi$. *Then*

$$\boldsymbol{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi = c^{-s}\mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\gamma}\varphi, \qquad \varphi \in \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+), \tag{3.23}$$

where $c^{-s} = |c|^{-s}e^{-is\arg c}$.

Proof: First of all note, that due to the mapping properties of the Bessel potential operators (see (2.6)) and the mapping properties of a Mellin convolution operator with an admissible kernel both operators

$$\begin{aligned}\boldsymbol{\Lambda}^s_{-\gamma}\mathbf{K}^1_c \;&:\; \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{r-s}_p(\mathbb{R}^+),\\ \mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\gamma} \;&:\; \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^{r-s}_p(\mathbb{R}^+)\end{aligned} \tag{3.24}$$

are correctly defined and bounded for all $s \in \mathbb{R}$, $1 < p < \infty$, since $-\pi < \arg(-\gamma) < 0$ and $0 < \arg(-c\,\gamma) < \pi$.

Second, let us consider the positive integer values $s = n = 1, 2, \ldots$. Then, with the help of formulae (2.11) and (3.15) it follows that:

$$\boldsymbol{\Lambda}^n_{-\gamma}\mathbf{K}^1_c\varphi = \Big(i\,\frac{d}{dt} - \gamma\Big)^n\mathbf{K}^1_c\varphi = \sum_{k=0}^{n}\binom{n}{k} i^k(-\gamma)^{n-k}\,\frac{d^k}{dt^k}\,\mathbf{K}^1_c\varphi$$

$$= \sum_{k=0}^{n} \binom{n}{k} i^k (-\gamma)^{n-k} c^{-k} \Big(\mathbf{K}^1_c \frac{d^k}{dt^k} \varphi \Big)(t) =$$

$$= c^{-n} \mathbf{K}^1_c \Big(\sum_{k=0}^{n} \binom{n}{k} i^k \, (-c\,\gamma)^{n-k} \, \frac{d^k}{dt^k} \varphi \Big)(t) =$$

$$= c^{-n} \mathbf{K}^1_c \mathbf{\Lambda}^n_{-c\gamma} \varphi, \qquad \varphi \in \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$$

and we have proven formula (3.23) for positive integers $s = n = 1, 2, \ldots$.

For negative $s = -1, -2, \ldots$ formulae (3.23) follows if we apply the inverse operator $\mathbf{\Lambda}^{-n}_{-\gamma}$ and $\mathbf{\Lambda}^{-n}_{-c\gamma}$ to the proved ope–rator equality

$$\mathbf{\Lambda}^n_{-\gamma} \mathbf{K}^1_c = c^{-n} \mathbf{K}^1_c \mathbf{\Lambda}^n_{-c\gamma}$$

for positive $n = 1, 2, \ldots$ from the left and from the right, respectively. We obtain

$$\mathbf{K}^1_c \mathbf{\Lambda}^{-n}_{-c\gamma} = c^{-n} \mathbf{\Lambda}^{-n}_{-\gamma} \mathbf{K}^1_c \quad \text{or} \quad \mathbf{\Lambda}^{-n}_{-\gamma} \mathbf{K}^1_c = c^n \mathbf{K}^1_c \mathbf{\Lambda}^{-n}_{-c\gamma}$$

and (3.23) is proved also for a negative $s = -1, -2, \ldots$.

In order to derive formula (3.23) for non-integer values of s, we can confine ourselves to the case $-2 < s < -1$. Indeed, any non-integer value $s \in \mathbb{R}$ can be represented in the form $s = s_0 + m$, where $-2 < s_0 < -1$ and m is an integer. Therefore, if for $s = s_0 + m$ the operators in (3.24) are correctly defined and bounded, and if the relations in question are valid for $-2 < s_0 < -1$, then we can write

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma} \mathbf{K}^1_c = \mathbf{\Lambda}^{s_0+m}_{-\gamma} \mathbf{K}^1_c = c^{-m} \mathbf{\Lambda}^{s_0}_{-\gamma} \mathbf{K}^1_c \mathbf{\Lambda}^m_{-c\gamma} &= c^{-s_0-m} \mathbf{K}^1_c \mathbf{\Lambda}^{s_0}_{-c\gamma} \mathbf{\Lambda}^m_{-c\gamma} \\ &= c^{-s_0-m} \mathbf{K}^1_c \mathbf{\Lambda}^{s_0+m}_{-c\gamma} = c^{-s} \mathbf{K}^1_c \mathbf{\Lambda}^s_{-c\gamma}. \end{aligned}$$

Thus let us assume that $-2 < s < -1$ and consider the expression

$$\mathbf{\Lambda}^s_{-\gamma} \mathbf{K}^1_c \varphi(t) \;= \frac{1}{2\pi^2}\, r_+ \int_{-\infty}^{\infty} e^{-i\xi t} (\xi - \gamma)^s \int_0^{\infty} e^{i\xi y} \int_0^{\infty} \frac{\varphi(\tau)}{y - c\tau} \, d\tau \, dy \, d\xi, \tag{3.25}$$

where r_+ is the restriction to $\mathbb{R}^+$. It is clear that the integral in the right-hand-side of (3.25) exists. Indeed, if $\varphi \in \mathbb{L}_2$, then $\mathbf{K}^1_c \varphi \in \mathbb{L}_2 \cap C^\infty$ and $\mathbf{\Lambda}^s_{-\gamma} \mathbf{K}^1_c \varphi \in \mathbb{H}^{-s} \cap C^\infty \subset \mathbb{L}_2 \cap C^\infty$.

Now consider the function $e^{-izt}(z - \gamma)^s e^{izy}$, $z \in \mathbb{C}$. Since $\mathrm{Im}\gamma \neq 0$, $s < -1$, then for sufficiently small $\varepsilon > 0$ this function is analytic in the strip between the lines $\mathbb{R}$ and $\mathbb{R} + i\varepsilon$ and vanishes at the infinity for all finite $t \in \mathbb{R}$ and for all $y > 0$. Therefore, the integration over the real line $\mathbb{R}$ in the first integral of (3.25) can be replaced by the integration over the line $\mathbb{R} + i\varepsilon$, i.e.

$$\mathbf{\Lambda}^s_{-\gamma} \mathbf{K}^1_c \varphi(t) = \frac{1}{2\pi^2}\, r_+ \!\int_{-\infty}^{\infty} e^{-i\xi t + \varepsilon t} (\xi + i\varepsilon - \gamma)^s \int_0^{\infty} e^{i\xi y - \varepsilon y} \int_0^{\infty} \frac{\varphi(\tau)}{y - c\tau} \, d\tau \, dy \, dx. \tag{3.26}$$

Let us use the density of the set $C^\infty_0(\mathbb{R}^+)$ in $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$. Thus for all finite $t \in \mathbb{R}$ and for all functions $\varphi \in C^\infty_0(\mathbb{R})$ with compact supports the integrand in the corresponding triple integral for (3.26) is absolutely integrable. Therefore, for such functions one can use Fubini-Tonelli theorem and change the order of integration in (3.26). Thereafter, one returns to the integration over the real line $\mathbb{R}$ and obtains

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = \frac{1}{2\pi^2}\, r_+ \int_0^\infty \varphi(\tau) \int_0^\infty \frac{1}{y-c\tau} \int_{-\infty}^\infty e^{i\xi(y-t)}(\xi-\gamma)^s d\xi\, dy\, d\tau, \tag{3.27}$$

In order to study the expression in the right-hand side of (3.27), one can use a well known formula

$$\int_{-\infty}^\infty (\beta+ix)^{-\nu}e^{-ipx}\,dx = \begin{cases} 0 & \text{for } \; p>0, \\ -\dfrac{2\pi(-p)^{\nu-1}e^{\beta p}}{\Gamma(\nu)} & \text{for } \; p<0, \end{cases} \qquad \operatorname{Re}\nu>0, \qquad \operatorname{Re}\beta>0,$$

[GR94, Formula 3.382.6]. It can be rewritten in a more convenient form–viz.,

$$\int_{-\infty}^\infty e^{i\mu\,\xi}(\xi-\gamma)^s\,d\xi = \begin{cases} 0 & \text{if } \; \mu<0,\; \operatorname{Im}\gamma>0, \\ \dfrac{2\pi\,\mu^{-s-1}e^{-\frac{\pi}{2}si+\mu\,\gamma i}}{\Gamma(-s)} & \text{if } \; \mu>0,\; \operatorname{Im}\gamma>0. \end{cases} \tag{3.28}$$

Applying (3.28) to the last integral in (3.27), one obtains

$$\begin{aligned} \mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) &= \frac{e^{-\frac{\pi}{2}si}}{\pi\Gamma(-s)} r_+ \int_0^\infty \varphi(\tau)\,d\tau \int_t^\infty \frac{e^{i(y-t)\gamma}dy}{(y-t)^{1+s}(y-c\tau)} \\ &= \frac{e^{-\frac{\pi}{2}si}}{\pi\Gamma(-s)} r_+ \int_0^\infty \varphi(\tau)\,d\tau \int_0^\infty \frac{y^{-s-1}e^{i\gamma\,y}dy}{y+t-c\tau}, \end{aligned} \tag{3.29}$$

where the integrals exist since $-s-1>-1$ and $0<\arg\gamma<\pi$ (i.e., $\operatorname{Im}\gamma>0$).

Let us recall the formula

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu\,x}\,dx}{x+\beta} = \beta^{\nu-1}e^{\beta\,\mu}\Gamma(\nu)\Gamma(1-\nu,\beta\mu), \qquad \operatorname{Re}\nu>0, \quad \operatorname{Re}\mu>0, \quad |\arg\,\beta|<\pi \tag{3.30}$$

(cf. [GR94, formula 3.383.10]). Due to the conditions $0<\arg\,c<2\pi$, $t>0$, $\tau>0$ we have $|\arg(t-c\tau)|<\pi$ and, therefore, we can apply (3.30) to the equality (3.29). Then (3.29) acquires the following final form:

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t) = \frac{e^{-\frac{\pi}{2}si}}{\pi} r_+ \int_0^\infty \frac{e^{-i\gamma(t-c\tau)}\Gamma(1+s,-i\gamma(t-c\tau))\varphi(\tau)\,d\tau}{(t-c\tau)^{1+s}}. \tag{3.31}$$

Consider now the reverse composition $\mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\gamma}\varphi(t)$. Changing the order of integration in the corresponding expression (see (3.27) for a similar motivation), one obtains

$$\begin{aligned}\mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\gamma}\varphi(t) &:= \frac{1}{2\pi^2}\,r_+\int_0^\infty \frac{1}{t-c\,y}\int_{-\infty}^\infty e^{-i\xi\,y}(\xi-c\,\gamma)^s\int_0^\infty e^{i\xi\,\tau}\varphi(\tau)d\tau\,d\xi\,dy\\ &= \frac{1}{2\pi^2}r_+\int_0^\infty \varphi(\tau)\int_0^\infty \frac{1}{t-c\,y}\int_{-\infty}^\infty e^{i\xi(\tau-y)}(\xi-c\,\gamma)^s d\xi\,dy\,d\tau. \qquad (3.32)\end{aligned}$$

In order to compute the expression in the right-hand side of (3.32), let us recall formula 3.382.7 from [GR94]:

$$\int_{-\infty}^\infty (\beta-ix)^{-\nu}e^{-ipx}\,dx = \begin{cases}0 & \text{for}\quad p<0,\\ \dfrac{2\pi\,p^{\nu-1}e^{-\beta\,p}}{\Gamma(\nu)} & \text{for}\quad p>0,\end{cases}\quad \mathrm{Re}\,\nu>0,\quad \mathrm{Re}\beta>0$$

and rewrite it in a form more suitable for our consideration–viz.,

$$\int_{-\infty}^\infty e^{i\mu\,\xi}(\xi+\omega)^s\,d\xi = \begin{cases}0 & \mu>0,\ \mathrm{Im}\,\omega>0,\\ \dfrac{2\pi\,(-\mu)^{-s-1}e^{\frac{\pi}{2}si-\mu\,\omega i}}{\Gamma(-s)} & \mu<0,\ \mathrm{Im}\,\omega>0,\end{cases}$$
$$\mathrm{Re}\,s<0,\quad \mu\in\mathbb{R},\quad \omega,\,s\in\mathbb{C}. \qquad (3.33)$$

Using (3.33), we represent (3.32) in the form

$$\begin{aligned}\mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\gamma}\varphi(t) &= \frac{e^{\frac{\pi}{2}si}}{\pi\Gamma(-s)}\,r_+\int_0^\infty\varphi(\tau)\,d\tau\int_\tau^\infty\frac{e^{-ic\,\gamma(y-\tau)}\,dy}{(y-\tau)^{s+1}(t-c\,y)}\\ &= -\frac{e^{\frac{\pi}{2}si}}{\pi c\Gamma(-s)}r_+\int_0^\infty\varphi(\tau)\,d\tau\int_0^\infty\frac{y^{-s-1}e^{-ic\gamma\,y}\,dy}{y+\tau-c^{-1}t},\end{aligned} \qquad (3.34)$$

where the integrals exist since $-s-1>-1$ and $-\pi<\arg(c\,\gamma)<0$ (i.e., $\mathrm{Im}\,c\,\gamma<0$).

Due to the conditions $0<\arg\,c<2\pi$, $t>0$, $\tau>0$ we have $|\arg(\tau-c^{-1}t)|<\pi$, Therefore, we can apply formula (3.30) to (3.34) and get the following representation:

$$\begin{aligned}\mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\gamma}\varphi(t) &= -\frac{c^{-1}e^{\frac{\pi}{2}si}}{\pi}r_+\int_0^\infty\frac{e^{-ic\gamma(c^{-1}t-\tau)}\Gamma(1+s,-ic\gamma(c^{-1}t-\tau))\varphi(\tau)\,d\tau}{(\tau-c^{-1}t)^{1+s}}\\ &= \frac{c^se^{-\frac{\pi}{2}si}}{\pi}r_+\int_0^\infty\frac{e^{-i\gamma(t-c\,\tau)}\Gamma(1+s,-i\gamma(t-c\,\tau))\varphi(\tau)\,d\tau}{(t-c\,\tau)^{1+s}}. \qquad (3.35)\end{aligned}$$

If we multiply (3.35) by c^{-s} we get precisely the expression in (3.31) and, therefore, $\boldsymbol{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\varphi(t)=c^{-s}\mathbf{K}^1_c\boldsymbol{\Lambda}^s_{-c\gamma}\varphi(t)$, which proves the claimed equality (3.23) for $-2<s<-1$ and accomplishes the proof. □

Corollary 3.9 *Let $0 < \arg c < 2\pi$ and $0 < \arg\gamma < \pi$. Then for arbitrary $\gamma_0 \in \mathbb{C}$ such that $0 < \arg\gamma_0 < \pi$ and $-\pi < \arg(c\,\gamma_0) < 0$, one has*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}W_{g_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma_0}, \tag{3.36}$$

where

$$g^s_{-\gamma,-\gamma_0}(\xi) := \left(\frac{\xi-\gamma}{\xi-\gamma_0}\right)^s.$$

If, in addition, $1 < p < \infty$ and $1/p - 1 < r < 1/p$ then equality (3.36) *can be supplemented as follows:*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\left[\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} + \mathbf{T}\right]\mathbf{\Lambda}^s_{-c\,\gamma_0}, \tag{3.37}$$

where $\mathbf{T} : \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ is a compact operator, and if c is a real negative number, then $c^{-s} := |c|^{-s}e^{-\pi s i}$.

PROOF. It follows from equalities e4.2.13 and (3.23) that

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = \mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{-\gamma_0}\mathbf{\Lambda}^s_{-\gamma_0}\mathbf{K}^1_c = c^{-s}W_{g_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma_0}$$

and (3.36) is proved. If $1 < p < \infty$ and $1/p - 1 < r < 1/p$, then the commutator

$$\mathbf{T} := W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c - \mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} \ : \ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$$

of Mellin and Fourier convolution operators is correctly defined and bounded. It is compact for $r = 0$ and all $1 < p < \infty$ (see [Du74, Du87]). Due to Krasnoselsky's interpolation theorem (see [Kr60] and also [**?**, Sections 1.10.1 and 1.17.4]), the operator $\mathbf{T}$ is compact in all $\mathbb{L}_r$-spaces for $1/p - 1 < r < 1/p$. Therefore, the equality (3.36) can be rewritten as

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c = c^{-s}\left[\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} + \mathbf{T}\right]\mathbf{\Lambda}^s_{-c\,\gamma_0},$$

and we are done. ■

Remark 3.10 *The assumption $1/p - 1 < r < 1/p$ in* (3.37) *cannot be relaxed. Indeed, the operator $W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c = \mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{-\gamma_0}\mathbf{K}^1_c \ : \ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ is bounded for all $r \in \mathbb{R}$ (see* (3.24)*). But the operator $\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}} \ : \ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+)$ is bounded only for $1/p - 1 < r < 1/p$ because the function $g^s_{-\gamma,-\gamma_0}(\xi)$ has an analytic extension into the lower half-plane but not into the upper one.*

4 A LOCAL PRINCIPLE

In the present section we expose well known, but slightly modified local principle from [Si65], which we apply intensively.

Let $\mathfrak{B}_1(\Omega)$and $\mathfrak{B}_2(\Omega)$ be Banach spaces of functions on a domain $\Omega\subset\mathbb{R}^n$ and multiplication by uniformly bounded $C^\infty(\overline{\Omega})$-functions are bounded operators in both spaces. If $\Omega=\mathbb{R}^n$, we consider one point compactification $\overline{\Omega}:=\dot{\mathbb{R}}^n$ of $\Omega=\mathbb{R}^n$.

Let $x\in\overline{\Omega}$ and consider the class of multiplication operators by functions

$$\begin{aligned}\Delta_x:=\Big\{vI\ :\ v\in C^\infty(\Omega),\ \ v(t)=1\ \text{ for }\ |t-x|<\varepsilon_1,\quad v(x)\geqslant 0\\ \text{and }\ v(t)=0\ \text{ for }\ |t-x|>\varepsilon_2\Big\}\end{aligned}\tag{4.1}$$

where $\varepsilon_2>\varepsilon_1>0$ are not fixed and vary from function to function. Δ_x is, obviously, a localizing class in the algebra of bounded linear operators $\mathscr{L}(\mathfrak{B}_1(\Omega),\mathfrak{B}_2(\Omega))$ and $\{\Delta_x\}_{x\in\overline{\Omega}}$ is a covering class. Indeed, for a system $\{v_xI\}_{x\in\overline{\Omega}}$ we consider the related covering

$$\overline{\Omega}=\bigcup_{x\in\overline{\Omega}}U_x,\quad U_x:=\{y\in\overline{\Omega}\ :\ v_x(y)=1\}\,.$$

The set $\overline{\Omega}$ is compact and there exists a finite covering system $\overline{\Omega}=\bigcup_{j=1}^N U_{x_j}$. The corresponding sum is strictly positive

$$\inf_{y\in\mathbb{R}^n}g(y)\geqslant 1\quad\text{for}\quad g(y):=\sum_{j=1}^N v_{x_j}(y)\tag{4.2}$$

and the multiplication operator $\sum_{j=1}^N v_{x_j}I=gI$ has the inverse $g^{-1}I$. Thus, the system of localizing classes $\{\Delta_x\}_{x\in\overline{\Omega}}$ is covering.

Most probably we would have to deal with the quotient space $\mathscr{L}_0'(\mathfrak{B}_1(\Omega),\mathfrak{B}_2(\Omega)):=\mathscr{L}(\mathfrak{B}_1(\Omega),\mathfrak{B}_2(\Omega))/\mathscr{C}(\mathfrak{B}_1(\Omega),\mathfrak{B}_2(\Omega))$ of linear bounded operators with respect to the compact operators.

Definition 4.1 *A quotient class $[\boldsymbol{A}]\in\mathscr{L}'(\mathfrak{B}_1(\Omega),\mathfrak{B}_2(\Omega))$ is called Δ_x-invertible if there exists a quotient class $[\mathbf{R}_x]\in\mathscr{L}'(\mathfrak{B}_2(\Omega),\mathfrak{B}_1(\Omega))$ and $v_x\in\Delta_x$ such that the operator equalities $[\mathbf{R}_x\mathbf{A}v_xI_1]=[v_xI_1]$ and $[v_x\mathbf{A}\mathbf{R}_x]=[v_xI_2]$ holds, where I_1 and I_2 are the identity operators in the spaces $\mathfrak{B}_1(\Omega)$ and $\mathfrak{B}_2(\Omega)$.*

Consider a pair of operators

$$\mathbf{A}_j\ :\ \mathfrak{B}_1(\Omega_j)\to\mathfrak{B}_2(\Omega_j),\qquad j=1,2,\tag{4.3}$$

in the same pairs of function spaces $\mathfrak{B}_1(\Omega_1)$, $\mathfrak{B}_2(\Omega_1)$ and $\mathfrak{B}_1(\Omega_2)$, $\mathfrak{B}_2(\Omega_2)$ defined on different domains $\Omega_1,\Omega_2\subset\mathbb{R}^n$. For this we assume that for any pair of points $x_1\in\overline{\Omega}_1$ and $x_2\in\overline{\Omega}_2$ there exist there exists a local diffeomorphism of neighbourhoods

$$\beta\ :\ \omega(x_1)\to\omega(x_2),\qquad x_j\in\omega(x_j)\subset\Omega_j,\quad j=1,2.\tag{4.4}$$

The operators

$$\beta_*\varphi(x) := \varphi(\beta(x)), \qquad \beta_*^{-1}\psi(y) := \psi(\beta_(^{-1}y))$$

are inverses to each-other and map the spaces

$$\beta_* \ : \ \mathfrak{B}_j(\omega_2) \to \mathfrak{B}_j(\omega_1), \qquad \beta_*^{-1} \ : \ \mathfrak{B}_j(\omega_1) \to \mathfrak{B}_j(\omega_2).$$

Definition 4.2 (Quasi localization) *Let multiplication by uniformly bounded C^∞ functions on corresponding closed domains $\overline{\Omega_1}$ and $\overline{\Omega_1}$ are bounded operators in all respective spaces $\mathfrak{B}_2(\Omega_1)$ and $\mathfrak{B}_1(\Omega_2)$, $\mathfrak{B}_2(\Omega_2)$.*

Two classes from the quotient spaces $[\boldsymbol{A}_1], [\boldsymbol{A}_2] \in \mathscr{L}'(\mathfrak{B}_1, \mathfrak{B}_2)$ (see (4.3)) are called locally quasi equivalent at $x_1 \in \overline{\Omega}_1$ and $x_2 \in \overline{\Omega}_2$, if

$$\inf_{v^1_{x_1}, v^2_{x_1} \in \Delta_{x_1}} \|\|\|[v^1_{x_1}][\mathbf{A}_1 - \beta_*\mathbf{A}_2\beta_*^{-1}]\|\|\| = \inf_{v^1_{x_1}, v^2_{x_1} \in \Delta_{x_1}} \|\|\|[\mathbf{A}_1 - \beta_*\mathbf{A}_2\beta_*^{-1}][v^2_{x_1}I]\|\|\| = 0, \tag{4.5}$$

where the norm in the quotient space $\mathscr{L}'(\mathfrak{B}_1, \mathfrak{B}_2) = \mathscr{L}(\mathfrak{B}_1, \mathfrak{B}_2)/\mathscr{C}(\mathfrak{B}_1, \mathfrak{B}_2)$ coincides with the essential norm

$$\|[\boldsymbol{A}]\| := \|\|\boldsymbol{A}\|\| := \inf_{T \in \mathscr{C}(\mathfrak{B}_1, \mathfrak{B}_2)} \|A + T\|.$$

Such an equivalence we denote as follows $[\mathbf{A}_1] \overset{\Delta_{x_1}}{\sim} \beta \overset{\Delta_{x_2}}{\sim} [\mathbf{A}_2]$ or also $[\mathbf{A}_1] \overset{x_1}{\sim} \beta \overset{x_2}{\sim} [\mathbf{A}_2]$.

If $\Omega_1 = \Omega_2 = \Omega$ and $\beta(x) = x$ is the identity map, the equivalence at the point $x \in \Omega$ is denoted as follows $[\mathbf{A}_1] \overset{\Delta_x}{\sim} [\mathbf{A}_2]$ or also $[\mathbf{A}_1] \overset{x}{\sim} [\mathbf{A}_2]$.

Definition 4.3 *Let $\mathbf{A}$, $\mathfrak{B}_1(\Omega)$ and $\mathfrak{B}_2(\Omega)$ be the same as in Definition 4.1. An operator $\mathbf{A} \ : \ \mathfrak{B}_1(\Omega) \to \mathfrak{B}_2(\Omega)$ is called of local type if $v_1\boldsymbol{A}v_2I \ : \ \mathfrak{B}_1(\Omega) \to \mathfrak{B}_2(\Omega)$ is compact for all $v_1, v_2 \in C^\infty(\Omega)$, provided $\operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \emptyset$ (see [Se66]); Or, equivalently, if $v\boldsymbol{A} - \boldsymbol{A}vI \ : \ \mathfrak{B}_1(\Omega) \to \mathfrak{B}_2(\Omega)$ is compact for all $v \in C^\infty(\Omega)$ (see [Se66]).*

Proposition 4.4 (Localization principle) *Let $\mathbf{A}$, $\mathfrak{B}_j(\Omega_k)$, $j,k = 1,2$, be a group of four function spaces as in Definition 4.2 and*

$$\mathbf{A} \ : \ \mathfrak{B}_1(\Omega_1) \to \mathfrak{B}_2(\Omega_1), \qquad \mathbf{B}_x \ : \ \mathfrak{B}_1(\Omega_2) \to \mathfrak{B}_2(\Omega_2), \qquad x \in \Omega_1,$$

be operators of local type.

If the quasi equivalence $[\mathbf{A}] \overset{x}{\sim} \beta_x \overset{y}{\sim} [\mathbf{B}_x]$ holds for some diffeomorphism $\beta_x \ : \ \omega(x) \to \omega(y(x))$, $y(x) \in \Omega_2$, then $[\mathbf{A}]$ is locally invertible at $x \in \Omega_1$ if and only if $[\mathbf{B}_x]$ is locally invertible at $y(x)$.

If the quasi equivalence $[\mathbf{A}] \overset{x}{\sim} \beta_x \overset{y(x)}{\sim} [\mathbf{B}_x]$ holds for all $x \in \overline{\Omega_1}$ and $[\mathbf{B}_x] \in \mathscr{L}'(\mathfrak{B}_1(\Omega_2), \mathfrak{B}_2(\Omega_2))$ are locally invertible at $y(x) \in \Omega_2$ for all $y(x) \in \overline{\Omega}$, than the quotient class $[\boldsymbol{A}]$ is globally invertible (i.e., $\mathbf{A} \ : \ \mathfrak{B}_1(\Omega_1) \to \mathfrak{B}_2(\Omega_1)$ is a Fredholm operator).

Remark 4.5 *If in the foregoing Proposition 4.4 we drop the condition that $\mathbf{A}$ and $\mathbf{B}_x$ are of local type, then from the left (from the right) quasi equivalence and the left invertibility of $[\mathbf{B}_x] \in \mathscr{L}'(\mathfrak{B}_1(\Omega_2), \mathfrak{B}_2(\Omega_2))$ at $y(x) \in \Omega_2$ for all $y(x) \in \overline{\Omega}$, follows the global invertibility of the quotient class $[\boldsymbol{A}]$ from the left (from the right), i.e., the existence of the left (of the right) regularizer for the operator $\mathbf{A} \ : \ \mathfrak{B}_1(\Omega_1) \to \mathfrak{B}_2(\Omega_1)$).*

5 Algebra Generated by Mellin and Fourier Convolution Operators

Let $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ denote one point compactification of the real axes $\mathbb{R}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$-the two point compactification of $\mathbb{R}$. By $C(\dot{\mathbb{R}})$ (by $C(\overline{\mathbb{R}})$, respectively) we denote the space of continuous functions $g(x)$ on $\mathbb{R}$ which have the equal limits at the infinity $g(-\infty) = g(+\infty)$ (limits at the infinity can be different $g(-\infty) \neq g(+\infty)$. By $PC(\dot{\mathbb{R}})$is denoted the space of piecewise-continuous functions on $\dot{\mathbb{R}}$, having the limits $a(t\pm 0)$ at all points $t \in \dot{\mathbb{R}}$, including the infinity.

Unlike the operators W^0_a and $\mathfrak{M}^0_a$ (see Section 1), possessing the property

$$W^0_a W^0_b = W^0_{ab}, \quad \mathfrak{M}^0_a \mathfrak{M}^0_b = \mathfrak{M}^0_{ab} \ \text{ for all } \ a, b \in \mathfrak{M}_p(\mathbb{R}), \tag{5.1}$$

the composition of the convolution operators on the semi-axes W_a and W_b cannot be computed by the rules similar to (5.1). Nevertheless, the following propositions hold.

Proposition 5.1 ([Du79],§ 2) *Let $1 < p < \infty$ and $a, b \in \mathfrak{M}_p(\overline{\mathbb{R}}^+) \cap PC(\dot{\mathbb{R}})$ be scalar $\mathbb{L}_p$-multipliers, piecewise-continuous on $\mathbb{R}$ including infinity. Then the commutant $[W_a, W_b] := W_a W_b - W_b W_a$ of the operators W_a and W_b is a compact operator in the Lebesgue space $[W_a, W_b] : \mathbb{L}_p(\mathbb{R}^+) \longmapsto \mathbb{L}_p(\mathbb{R}^+)$.*

Moreover, if, in addition, the symbols $a(\xi)$ and $b(\xi)$ of the operators W_a and W_b have no common discontinuity points, i.e., if

$$\big[a(\xi+0) - a(\xi+0)\big]\big[b(\xi+0) - b(\xi+0)\big] = 0 \ \textit{ for all } \ \xi \in \dot{\mathbb{R}},$$

then $\mathbf{T} = W_a W_b - W_{ab}$ is a compact operator in $\mathbb{L}_p(\mathbb{R}^+)$.

Note that the algebra of $N \times N$ matrix multipliers $\mathfrak{M}_2(\mathbb{R})$ coincides with the algebra of $N \times N$ matrix functions essentially bounded on $\mathbb{R}$. For $p \neq 2$, the algebra $\mathfrak{M}_p(\mathbb{R})$ is rather complicated. There are multipliers $g \in \mathfrak{M}_p(\mathbb{R})$ which are elliptic, i.e. $\operatorname{ess\,inf} |g(x)| > 0$, but $1/g \notin \mathfrak{M}_p(\mathbb{R})$. In connection with this, let us consider the subalgebra $PC\mathfrak{M}_p(\mathbb{R})$ which is the closure of the algebra of piecewise-constant functions on $\mathbb{R}$ in the norm of multipliers $\mathfrak{M}_p(\mathbb{R})$

$$\big\|a \mid \mathfrak{M}_p(\mathbb{R})\big\| := \big\|W^0_a \mid \mathbb{L}_p(\mathbb{R})\big\|.$$

Note that any function $g \in PC\mathfrak{M}_p(\mathbb{R}) \subset PC(\mathbb{R})$ has limits $g(x\pm 0)$ for all $x \in \overline{\mathbb{R}}$, including the infinity. Let

$$C\mathfrak{M}_p(\overline{\mathbb{R}}) := C(\overline{\mathbb{R}}) \cap PC\mathfrak{M}^0_p(\mathbb{R}), \quad C\mathfrak{M}^0_p(\dot{\mathbb{R}}) := C(\dot{\mathbb{R}}) \cap PC\mathfrak{M}_p(\mathbb{R}),$$

where functions $g \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ (functions $h \in C(\dot{\mathbb{R}})$) might have jump only at the infinity $g(-\infty) \neq g(+\infty)$ (are continuous at the infinity $h(-\infty) = h(+\infty)$).

$PC\mathfrak{M}_p(\mathbb{R})$ is a Banach algebra and contains all functions of bounded variation as a subset for all $1 < p < \infty$ (Stechkin's theorem, see [Du79, Section 2]). Therefore, $\coth \pi(i\beta + \xi) \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ for all $p \in (1, \infty)$.

Proposition 5.2 ([Du79], § 2) *If $g \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$ is an $N \times N$ matrix multiplier, then its inverse $g^{-1} \in PC\mathfrak{M}_p(\overline{\mathbb{R}})$ if and only if it is elliptic, i.e.* $\det g(x \pm 0) \neq 0$ *for all $x \in \overline{\mathbb{R}}$. If this is the case, the corresponding Mellin convolution operator $\mathfrak{M}^0_g : \mathbb{L}_p(\mathbb{R}^+) \longmapsto \mathbb{L}_p(\mathbb{R}^+)$ is invertible and $(\mathfrak{M}^0_g)^{-1} = \mathfrak{M}^0_{g^{-1}}$.*

Moreover, any $N \times N$ matrix multiplier $b \in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ can be approximated by polynomials

$$r_n(\xi) := \sum_{j=-m}^{m} c_m \left(\frac{\xi - i}{\xi + i}\right)^m, \quad r_m \in C\mathfrak{M}^0_p(\overline{\mathbb{R}}),$$

with constant $N \times N$ matrix coefficients, whereas any $N \times N$ matrix multiplier $g \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$ having a jump discontinuity at infinity can be approximated by $N \times N$ matrix functions $d \coth \pi(i\beta + \xi) + r_m(\xi)$, $0 < \beta < 1$.

Due to the connection between the Fourier and Mellin convolution operators (see Introduction, (**??**)), the following is a direct consequence of Proposition 3.2.

Corollary 5.3 *The Mellin convolution operator*

$$\boldsymbol{A} = \mathfrak{M}^0_{\mathscr{A}_\beta} : \mathbb{L}_p(\mathbb{R}, t^\gamma),$$

in (2.2) *with the symbol $\mathscr{A}_\beta(\xi)$ in* (**??**) *is invertible if and only if the symbol is elliptic,*

$$\inf_{\xi \in \mathbb{R}} \left|\det \mathscr{A}_\beta(\xi)\right| > 0 \tag{5.2}$$

and the inverse is then written as $\mathbf{A}^{-1} = \mathfrak{M}^0_{\mathscr{A}^{-1}_{1/p}}$.

The Hilbert transform on the semi-axis

$$S_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i} \int_0^\infty \frac{\varphi(y)\, dy}{y - x} \tag{5.3}$$

is the Fourier convolution $S_{\mathbb{R}^+} = W_{-\text{sign}}$ on the semi-axis $\mathbb{R}^+$ with the discontinuous symbol $-\text{sign}\xi$ (see [Du79, Lemma 1.35]), and it is also the Mellin convolution

$$S_{\mathbb{R}^+} = \mathfrak{M}^0_{s_\beta} = \mathbf{Z}_\beta W^0_{s_\beta} \mathbf{Z}^{-1}_\beta, \tag{5.4}$$

$$s_\beta(\xi) := \coth \pi(i\beta + \xi) = \frac{e^{\pi(i\beta+\xi)} + \boldsymbol{e}^{-\pi(i\beta+\xi)}}{e^{\pi(i\beta+\xi)} \boldsymbol{e}^{-\pi(i\beta+\xi)}} = -i \cot \pi(\beta\imath\xi), \ \xi \in \mathbb{R}$$

(cf. (**??**) and (1.7)). Indeed, to verify (5.4) rewrite $S_{\mathbb{R}^+}$ in the following form

$$S_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i} \int_0^\infty \frac{\varphi(y)}{1 - \frac{x}{y}} \frac{dy}{y} = \int_0^\infty K\left(\frac{x}{y}\right) \varphi(y) \frac{dy}{y},$$

where $K(t) := (1/\pi i)(1-t)^{-1}$. Further, using the formula

$$\int_0^\infty \frac{t^{z-1}}{1-t}\,dt = \pi \cot \pi z, \quad \operatorname{Re} z < 1,$$

cf. [GR94, formula 3.241.3], one shows that the Mellin transform $\mathscr{M}_\beta K(\xi)$ coincides with the function $s_\beta(\xi)$ from (5.4).

Next Theorem 5.4 is an enhancement of Theorem 2.4.

Theorem 5.4 *Let* $1 < p < \infty$ *and* $s \in \mathbb{R}$*. For arbitrary* $\gamma_j \in \mathbb{C}$, $\operatorname{Im} \gamma_j > 0$ *(j=1,2) the Hilbert transform*

$$\boldsymbol{K}^1_1 = -iS^+_{\mathbb{R}} = -iW_{-\mathrm{sign}} = W_{i\mathrm{sign}} \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+) \tag{5.5}$$

(see (3.14)*,* (3.15) *and* (5.3)*; the case* $c = 1$*,* $\arg c = 0$*, Theorem 3.8).* $\boldsymbol{K}^1_1$ *is a Fourier convolution operator and*

$$\mathbf{\Lambda}^s_{-\gamma_1}\mathbf{K}^1_1\mathbf{\Lambda}^{-s}_{\gamma_2} = W_{i\mathrm{sign}g^s_{-\gamma_1,\gamma_2}} \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+), \tag{5.6}$$

where $g^s_{-\gamma_1,\gamma_2}(\xi)$ *is defined in* (2.10).

Proof: Formula (5.6) follows from (2.8) and (5.5). ∎

We need certain results concerning the compactness of Mellin and Fourier convolutions in $\mathbb{L}_p$-spaces. These results are scattered in literature. For the convenience of the reader, we reformulate them here as Propositions 5.5–5.9. For more details, the reader can consult [Co69, Du79, Du87].

Proposition 5.5 ([Du87], Proposition 1.6) *Let* $1 < p < \infty$, $a \in C(\dot{\mathbb{R}}^+)$, $b \in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ *and* $a(0) = a(\infty) = b(\infty) = 0$*. Then the operators* $a\mathfrak{M}^0_b, \mathfrak{M}^0_b\, aI \,:\, \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proposition 5.6 ([Du79], Lemma 7.1 and [Du87], Proposition 1.2) *Let* $1 < p < \infty$, $a \in C(\dot{\mathbb{R}}^+)$, $b \in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ *and* $a(\infty) = b(\infty) = 0$*. Then the operators* $aW_b, W_b\, aI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proposition 5.7 ([Du87], Lemma 2.5, Lemma 2.6 and [Co69]) *Assume that* $1 < p < \infty$*. Then*

1. *If* $g \in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ *and* $g(\infty) = 0$*, the Hankel operator* $H_g : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *is compact;*

2. *If functions* $a \in C(\dot{\mathbb{R}})$, $b \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, $c \in C(\overline{\mathbb{R}}^+)$ *fulfill one of the conditions*

   (i) $c(0) = b(+\infty) = 0$ *and* $a(\xi) = 0$ *for all* $\xi > 0$,

   (ii) $c(0) = b(-\infty) = 0$ *and* $a(\xi) = 0$ *for all* $\xi < 0$,

(iii) $c(0) = b(\pm\infty) = a(0) = 0$,

the operators $cW_a\mathfrak{M}^0_b$, $c\mathfrak{M}^0_b W_a$, $W_a\mathfrak{M}^0_b\, cI$, $\mathfrak{M}^0_b W_a\, cI\ :\ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ *are all compact.*

Proof: Let us comment on item 1 which is actually well known. The kernel $k(x+y)$ of the operator H_a is approximated by the Laguerre polynomials $k_m(x+y) = e^{-x-y}p_m(x+y)$, $m = 1, 2, \dots$, where $p_m(x+y)$ are polynomials of order m so that the corresponding Hankel operators converge in norm $\|H_a - H_{a_m}|\,|\mathscr{L}(\mathbb{L}_p(\mathbb{R}^+))\|$
$longrightarrow 0$, where $a_m = \mathscr{F}k_m$ are the Fourier transforms of the Laguerre polynomials (see, e.g. [GF74]). Since

$$|k_m(x+y)| = \left|e^{-x-y}p_m(x+y)\right| \leqslant C_m e^{-x}e^{-y}x^m y^m, \ \ m = 1, 2, \dots ,$$

for some constant C_m, the condition on the kernel

$$\int_0^\infty \Bigg[\int_0^\infty |k_m(x+y)|^{p'}\, dy\Bigg]^{p/p'} dx < \infty, \ \ p' := \frac{p}{p-1}\,,$$

holds and ensures the compactness of the operator $H_{a_m} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$. Then the limit operator $H_a = \lim\limits_{m\longrightarrow\infty} H_{a_m}$ is compact as well.

Items (i) and (ii) are proved in [Du87].

The item (iii) follows from (i) and (ii) and the representation $cW_a\mathfrak{M}^0_b = cW_{\chi_- a}\mathfrak{M}^0_b + cW_{\chi_+ a}\mathfrak{M}^0_b$, where $\chi_\pm$ are the characteristic functions of the semi-axes $\mathbb{R}^\pm$. ■

Proposition 5.8 ([Du79], Lemma 7.1 and [Du87], Proposition 1.2) *Let* $1 < p < \infty$, $a \in C(\dot{\mathbb{R}}^+)$, $b \in C\mathfrak{M}^0_p(\dot{\mathbb{R}})$ *and* $a(\infty) = b(\infty) = 0$. *Then the operators* $aW_b, W_b\, aI : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

Proposition 5.9 ([Du79], Lemma 7.4 and [Du87], Lemma 1.2) *Let* $1 < p < \infty$ *and let* a *and* b *satisfy at least one of the conditions*

(i) $a \in C(\overline{\mathbb{R}}^+)$, $b \in \mathfrak{M}^0_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$,

(ii) $a \in PC(\overline{\mathbb{R}}^+)$, $b \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$.

Then the commutants $[aI, W_b]$ *and* $[aI, \mathfrak{M}^0_b]$ *are compact operators in the space* $\mathbb{L}_p(\mathbb{R}^+)$.

Remark 5.10 *Note that, if both, a symbol* b *and a function* a, *have jumps at finite points, the commutants* $[aI, W_b]$ *and* $[aI, \mathfrak{M}^0_b]$ *are not compact. Only jumps of a symbol at the infinity does not matter.*

Proposition 5.11 ([Du87]) *The Banach algebra, generated by the Cauchy singular integral operator* $S_{\mathbb{R}^+}$ *and the identity operator* I *on the semi-axis* $\mathbb{R}^+$, *contains Fourier convolution operators with symbols having discontinuity of the jump type only at zero and at the infinity and Mellin convolution operators with continuous symbols on* $\dot{\mathbb{R}}$ *(including the uinfinity).*

Moreover, the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ generated by the Cauchy singular integral operators with "shifts"

$$S^c_{\mathbb{R}^+}\varphi(x) := \frac{1}{\pi i}\int_0^\infty \frac{e^{-ic(x-y)}\varphi(y)\,dy}{y-x} = W_{-\mathrm{sign}(\xi-c)}\varphi(x) \ \textit{for all}\ \ c\in\mathbb{R}$$

and by the identity operator I on the semi-axis $\mathbb{R}^+$ over the field of $N\times N$ complex valued matrices coincides with the Banach algebra generated by Fourier convolution operators with piecewise-constant $N\times N$ matrix symbols contains all Fourier convolution W_a and hankel H_b operators with $N\times N$ matrix symbols (multipliers) $a, b\in PC\mathfrak{M}_p(\overline{\mathbb{R}})$.

Let us consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ generated by Mellin convolution and Fourier convolution operators, i.e. by the operators

$$\mathbf{A} := \sum_{j=1}^m \mathfrak{M}^0_{a_j} W_{b_j}, \tag{5.7}$$

and there compositions, in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$. Here $\mathfrak{M}^0_{a_j}$ are Mellin convolution operators with continuous $N\times N$ matrix symbols $a_j\in C\mathfrak{M}_p(\dot{\mathbb{R}})$, W_{b_j} are Fourier convolution operators with $N\times N$ matrix symbols $b_j\in C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^-\cup\overline{\mathbb{R}}^+)$. The algebra of $N\times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\})$ consists of those piecewise-continuous $N\times N$ matrix multipliers $b\in\mathfrak{M}_p(\mathbb{R})\cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axis $\mathbb{R}^-$ and $\mathbb{R}^+$ but might have finite jump discontinuities at 0 and at the infinity.

This and more general algebras were studied in [Du87] and also in earlier works [Du74, Du86, Th85].

Remark 5.12 *If in* (5.7) *we admit more general symbols $a_j\in C\mathfrak{M}_p(\overline{\mathbb{R}})$ which have different limits at the infinity $a_j(-\infty)\neq a_j(+\infty)$, this will not be a generalization.*

Indeed, if $a_j\in C\mathfrak{M}_p(\overline{\mathbb{R}})$ has different limits at the infinity $a_j(-\infty)\neq a_j(+\infty)$ we can represent

$$a_j(\xi) = a^0_j(\xi) + a_j(-\infty)\frac{1-\coth\pi\left(\frac{i}{p}+\xi\right)}{2} + a_j(+\infty)\frac{1+\coth\pi\left(\frac{i}{p}+\xi\right)}{2},$$
$$a^0_j(\pm\infty)=0$$

and the corresponding Mellin operator is written as follows

$$\begin{aligned}\mathfrak{M}^0_{a_j} &= \mathfrak{M}^0_{a^0_j} + \frac{a_j(-\infty)}{2}\left[I - S_{\mathbb{R}^+}\right] + \frac{a_j(-\infty)}{2}\left[I + S_{\mathbb{R}^+}\right]\\ &= \mathfrak{M}^0_{a^0_j} + \frac{a_j(-\infty)}{2}\left[I - W_{-\mathrm{sign}}\right] + \frac{a_j(-\infty)}{2}\left[I + W_{-\mathrm{sign}}\right]\end{aligned}$$

*(see (**??**) and (3.15)). Therefore, the discontinuity at the infinity of the symbols of Mellin convolution operators is taken over in Fourier convolution operators and we can even assume in* (5.7) *that $a^0_j(\pm\infty)=0$ for all $j=1,\dots,m$.*

In order to keep the exposition self-contained and to improve formulations from [Du87], the results concerning the Banach algebra generated by the operators (5.7) are presented here with some modification and the proofs.

Note that the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ is actually a subalgebra of the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ generated by the Fourier convolution operators W_a with piecewise-constant symbols $a(\xi)$ in the space $\mathbb{L}_p(\mathbb{R}^+)$ (cf. Proposition 5.9). Let $\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denote the ideal of all compact operators in $\mathbb{L}_p(\mathbb{R}^+)$. Since the quotient algebra $\mathfrak{F}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative in the scalar case $N=1$, the following is true.

Corollary 5.13 *The quotient algebra* $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ *is commutative in the scalar case* $N=1$.

To expose the symbol of the operator (5.7), consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$, where (cf. Figure 1)

$$\Gamma_1 := \{\infty\} \times \overline{\mathbb{R}}, \qquad \Gamma_2^{\pm} := \overline{\mathbb{R}}^+ \times \{\pm\infty\}, \qquad \Gamma_3 := \{0\} \times \overline{\mathbb{R}}.$$

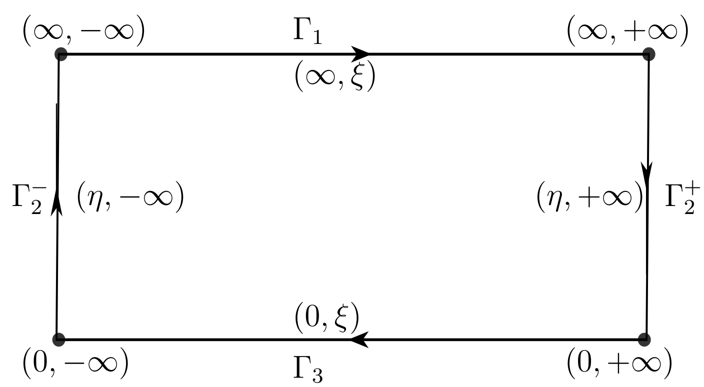

Fig. 1. The domain $\mathfrak{R}$ of definition of the symbol $\mathscr{A}_p^s(\omega)$.

The symbol $\mathscr{A}_p(\omega)$ of the operator $\boldsymbol{A}$ in (5.7) is a function on the set $\mathfrak{R}$, viz.

$$\mathscr{A}_p(\omega) := \begin{cases} \displaystyle\sum_{j=1}^{m} a_j(\xi)(b_j)_p(\infty,\xi), & \omega=(\infty,\xi)\in\overline{\Gamma_1}, \\ \displaystyle\sum_{j=1}^{m} a_j(\infty)b_j(\eta), & \omega=(\eta,+\infty)\in\Gamma_2^+, \\ \displaystyle\sum_{j=1}^{m} a_j(\infty)b_j(-\eta), & \omega=(\eta,-\infty)\in\Gamma_2^-, \\ \displaystyle\sum_{j=1}^{m} a_j(\xi)(b_j)_p(0,\xi), & \omega=(0,\xi)\in\overline{\Gamma_3}. \end{cases} \tag{5.8}$$

The symbol $\mathscr{A}_p(\omega)$, when $\omega = (\infty, \xi)$ ranges through the infinite interval Γ_1 (see Fig. 1) fills the gap between the values

$$\sum_{j=1}^{m} a_j(\infty) b_j(-\infty) \qquad \text{and} \qquad \sum_{j=1}^{m} a_j(\infty) b_j(+\infty)$$

and when $\omega = (0, \xi)$ ranges through the infinite interval Γ_3 (see Fig. 1) it fills the gap between the values

$$\sum_{j=1}^{m} a_j(\xi) b_j(0-0) \quad \text{and} \quad \sum_{j=1}^{m} a_j(\xi) b_j(0+0).$$

The connecting function $g_p(\infty, \xi)$ in (5.8) for a piecewise continuous function $g \in PC(\overline{\mathbb{R}})$ is defined as follows

$$\begin{aligned} g_p(x,\xi) &:= \frac{g(x+0)+g(x-0)}{2} + \frac{g(x+0)-g(x-0)}{2i} \cot \pi\Big(\frac{1}{p} - i\xi\Big) \\ &= e^{i\pi \frac{g_x^+ + g_x^-}{2}} \frac{\cos \pi \Big(\frac{1}{p} - \frac{g_x^+ + g_x^-}{2} - i\xi\Big)}{\sin \pi \Big(\frac{1}{p} - i\xi\Big)}, \qquad \xi \in \mathbb{R}, \qquad (5.9) \\ g_x^{\pm} &:= \frac{1}{\pi i} \ln\, g(x \pm 0), \quad \operatorname{Re} g_x^{\pm} = \frac{1}{\pi} \arg\, g(x \pm 0), \quad x \in \dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}. \end{aligned}$$

The function $g_p(\infty, \xi)$ fills up the discontinuity (the jump) of $g(\xi)$ at ∞ between $g(-\infty)$ and $g(+\infty)$ with an oriented arc of the circle such that from every point of the arc the oriented interval $[g(-\infty), g(+\infty)]$ is seen under the angle π/p. Moreover, the oriented arc lies above the oriented interval if $1/2 < 1/p < 1$ (i.e., if $1 < p < 2$) and the oriented arc is under the oriented interval if $0 < 1/p < 1/2$ (i.e., if $2 < p < \infty$). For $p = 2$ the oriented arc coincides with the oriented interval (see Figure 2).

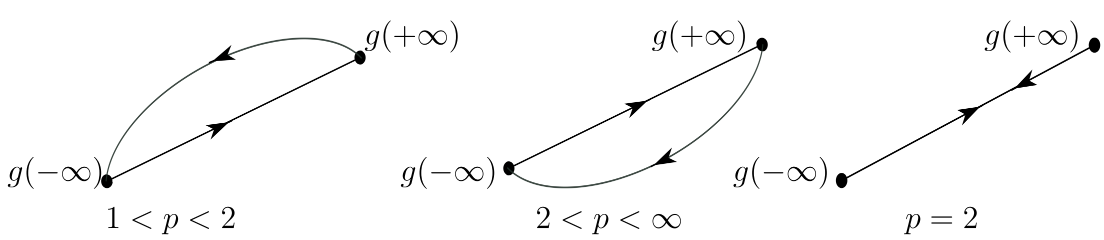

Figure 2: Arc condition

A similar geometric interpretation is valid for the function $g_p(t, \xi)$, which connects the point $g(t-0)$ with $g(t+0)$ at the point t where $g(\xi)$ has a jump discontinuity.

To make the symbol $\mathscr{A}_p(\omega)$ continuous, we endow the rectangle $\mathfrak{R}$ with a special topology. Thus let us define the distance on the curves Γ_1, $\Gamma_2^{\pm}$, Γ_3 and on $\overline{\mathbb{R}}$ by

$$\rho(x,y) := \Big| \arg \frac{x-i}{x+i} - \arg \frac{y-i}{y+i} \Big| \ \text{ for arbitrary } \ x, y \in \overline{\mathbb{R}}.$$

In this topology, the length $|\mathfrak{R}|$ of $\mathfrak{R}$ is 6π, and the symbol $\mathscr{A}_p(\omega)$ is continuous everywhere on $\mathfrak{R}$. The image of the function $\det\mathscr{A}_p(\omega)$, $\omega \in \mathfrak{R}$ ($\det\mathscr{B}_p(\omega)$) is a closed curve in the complex plane. It follows from the continuity of the symbol at the angular points of the rectangle $\mathfrak{R}$ where the one-sided limits coincide. Thus

$$\begin{aligned} \mathscr{A}_p(\pm\infty,\infty) &= \textstyle\sum_{j=1}^{m} a_j(\infty)b_j(\mp\infty), \\ \mathscr{A}_p(\pm\infty,0) &= \textstyle\sum_{j=1}^{m}[a_j(\infty)b_j(0\mp 0). \end{aligned}$$

Hence, if the symbol of the corresponding operator is elliptic, i.e. if

$$\inf_{\omega\in\mathfrak{R}} \left|\det\mathscr{A}_p(\omega)\right| > 0, \tag{5.10}$$

the increment of the argument $(1/2\pi)\arg\mathscr{A}_p(\omega)$ when ω ranges through $\mathfrak{R}$ in the positive direction is an integer, is called the winding number or the index and it is denoted by $\operatorname{inddet}\mathscr{A}_p$.

Theorem 5.14 *Let $1 < p < \infty$ and let $\mathbf{A}$ be defined by* (5.7)*. The operator $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if its symbol $\mathscr{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, the index of the operator has the value*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{inddet}\mathscr{A}_p. \tag{5.11}$$

The operator $\mathbf{A}$ is locally invertible at $0 \in \mathbb{R}^+$ if and only if its symbol $\mathscr{A}_p^s(\omega)$, defined in (5.8)*, is elliptic on Γ_1, i.e.*

$$\inf_{\omega\in\Gamma_1} \left|\det\mathscr{A}_p^s(\omega)\right| = \inf_{\xi\in\mathbb{R}} \left|\det\mathscr{A}_p^s(\xi,\infty)\right| > 0.$$

Proof: Note that our study is based on a localization technique. For more details concerning this approach we refer the reader to [Du79, Du84a, GK79, Si65].

Let us apply the Gohberg–Krupnik local principle to the operator $\mathbf{A}$ in (5.9), "freezing" the symbol of $\mathbf{A}$ at a point $x \in \overline{\mathbb{R}} := \mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$. For $x \in \mathbb{R}$ and $\ell \in \mathbb{N}$, $\ell \geq 1$, let $C_x^\ell(\overline{\mathbb{R}})$ denote the set of all ℓ-times differentiable non-negative functions which are supported in a neighborhood of $x \in \mathbb{R}$ and are identically one everywhere in a smaller neighborhood of x. For $x \in \{-\infty\}\cup\{+\infty\}\cup\{\infty\}$, the functions from the corresponding classes $C_{+\infty}^\ell(\overline{\mathbb{R}})$ and $C_{-\infty}^\ell(\overline{\mathbb{R}})$ vanish on semi-infinite intervals $[-\infty, c)$ and $(-c,\infty]$, respectively, for certain $c > 0$ and are identically one in smaller neighborhoods. It is easily seen that the system of localizing classes $\{C_x^\ell(\overline{\mathbb{R}})\}_{x\in\overline{\mathbb{R}}}$ is covering in the algebras $C(\overline{\mathbb{R}})$, $\mathfrak{M}_p(\overline{\mathbb{R}})$, respectively (cf. [Du79, Du84a, DS08, GK79]).

Let us now consider a system of localizing classes $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. These localizing classes depend on two variables, viz. on

$\omega \in \mathfrak{R}$ and $x \in \overline{\mathbb{R}^+}$. In particular, the class $\mathfrak{L}_{\omega,x}$ contains the operator $\Lambda_{\omega,x}$,

$$\Lambda_{\omega,x} := \begin{cases} \left[h_0\mathfrak{M}^0_{v_\xi}W_{g_\infty}\right] = \left[h_0\mathfrak{M}^0_{v_\xi}\right] \\ \qquad\qquad \text{if } \ \omega = (\xi,\infty) \in \Gamma_1, \ \ x = 0; \\ \left[h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_\infty}\right] = \left[h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_{\mp\infty}}\right] \\ \qquad\qquad \text{if } \ \omega = (\pm\infty,\infty) \in \Gamma_2^{\pm} \cap \Gamma_1, \ \ x \in \mathbb{R}^+; \\ \left[h_\infty\mathfrak{M}^0_{v_{\pm\infty}}W_{g_\eta}\right] = \left[h_\infty\mathfrak{M}^0_{v_{\pm\infty}}W_{g_{\mp\eta}}\right] \\ \qquad\qquad \text{if } \ \omega = (\pm\infty,\eta) \in \Gamma_2^{\pm}, \ \ x = \infty; \\ \left[h_\infty\mathfrak{M}^0_{v_\xi}W_{g_0}\right] = \left[\mathfrak{M}^0_{v_\xi}W_{g_0}\right] \\ \qquad\qquad \text{if } \ \omega = (\xi,0) \in \overline{\Gamma}_3, \ \ x = \infty, \end{cases} \tag{5.12}$$

where $h_x \in C^1_x(\overline{\mathbb{R}}^+)$, $v_\xi \in C^1_\xi(\overline{\mathbb{R}}^+)$, $g_\eta \in C^1_\eta(\overline{\mathbb{R}}^+)$, and $[\mathbf{A}] \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denotes the coset containing the operator $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)$.

To verify the equalities in (5.12), one has to show that the difference between the operators in the square brackets is compact.

Consider the first equality in (5.12): The operator

$$h_0W_{g_\infty} - h_0I = h_0W_{(g_\infty-1)} = h_0W_{g_0}$$

is compact, since both functions h_0 and $1 - g_\infty = g_0$ have compact supports, so Proposition 5.5 applies.

To check the second equality in (5.12), let us note that $h_x(0) = 0$, $v_{\pm\infty}(\mp\infty) = 0$ and $g_{\pm\infty}(\xi) = 0$ for all $\mp\xi > 0$. From the fourth part of Proposition 5.7 we conclude that for any $x \in \mathbb{R}^+$ the operator $h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_{\pm\infty}}$ is compact. This leads to the claimed equality since

$$\left[h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_\infty}\right] = \left[h_x\mathfrak{M}^0_{v_{\pm\infty}}\{W_{g_{-\infty}} + W_{g_{+\infty}}\}\right] = \left[h_x\mathfrak{M}^0_{v_{\pm\infty}}W_{g_{\mp\infty}}\right].$$

The third identity in (5.12) can be verified analogously. Concerning the fourth identity in (5.12): one can replace h_∞ by 1 because the difference $h_\infty W_{g_0} - W_{g_0} = (1 - h_\infty)W_{g_0} = h_0W_{g_0}$ is compact due to Proposition 5.5.

Now consider other properties of the system $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$. Propositions 5.5–5.8 imply that

$$\left[h_x\mathfrak{M}^0_{v_\xi}W_{g_\infty}\right] = 0 \ \text{ for all } \ (\xi,\eta,x) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \overline{\mathbb{R}}^+ \setminus \mathfrak{R} \times \overline{\mathbb{R}}^+.$$

Therefore, the system of localizing classes $\{\mathfrak{L}_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ is covering: for a given system $\{\Lambda_{\omega,x}\}_{(\omega,x)\in\mathfrak{R}\times\overline{\mathbb{R}}^+}$ of localizing operators one can select a finite number of points $(\omega_1,x_1) = (\xi_1,\eta_1,x_1),\dots,(\omega_s,x_s) = (\xi_s,\eta_s,x_s) \in \mathfrak{R}$ and add appropriately chosen terms $[h_{x_{s+j}}\mathfrak{M}^0_{v_{\xi_{s+j}}}W_{g_{s+j}}] = 0$ with $(\xi_{s+j},\eta_{s+j},x_{s+j})) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \overline{\mathbb{R}}^+ \setminus (\mathfrak{R} \times \overline{\mathbb{R}}^+)$, $j = 1,2,\dots,r$ so, that the equality

$$\sum_{j=1}^{r}\sum_{k=1}^{s}\left[c_{x_j}\mathfrak{M}^0_{a_{\xi_j}}W_{b_{\eta_k}}\right] = \left[c\mathfrak{M}^0_aW_b\right] \tag{5.13}$$

holds and the functions $c \in C(\overline{\mathbb{R}}^+)$, $a \in C\mathfrak{M}_p(\overline{\mathbb{R}})$, $b \in C\mathfrak{M}_p(\overline{\mathbb{R}})$ are all elliptic. This implies the invertibility of the coset $[c\mathfrak{M}^0_a W_b]$ in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ and the inverse coset is $[c\mathfrak{M}^0_a W_b]^{-1} = [c^{-1}\mathfrak{M}^0_{a^{-1}} W_{b^{-1}}]$.

Note that the choice of a finite number of terms in (5.13) is possible due to the Borel–Lebesgue lemma and the compactness of the sets $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^+$ (two point and one point compactification of $\mathbb{R}$ and of $\mathbb{R}^+$, respectively).

Moreover, localization in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ leads to the following local representatives of the cosets containing Mellin and Fourier convolution operators with symbols $a, b \in C\mathfrak{M}_p(\overline{\mathbb{R}})$:

$$[\mathfrak{M}^0_a] \overset{\mathfrak{M}^0_{v_{\xi_0}}}{\sim} [\mathfrak{M}^0_{a(\xi_0)}] = [a(\xi_0)I] \ \text{ if } \ \xi_0 \in \overline{\mathbb{R}}, \tag{5.14a}$$
$$[\mathfrak{M}^0_a] \overset{v_{x_0}I}{\sim} [\mathfrak{M}^0_{a_\infty}] \ \text{ if } \ x_0 \in \overline{\mathbb{R}^+}, \ \ x_0 \neq 0, \tag{5.14b}$$
$$[\mathfrak{M}^0_a] \overset{v_\infty I}{\sim} [\mathfrak{M}^0_a] \ \text{ if } \ x_0 = \infty, \tag{5.14c}$$
$$[\mathfrak{M}^0_a] \overset{v_0 I}{\sim} [\mathfrak{M}^0_a] \ \text{ if } \ x_0 = 0, \tag{5.14d}$$
$$[W_b] \overset{W_{v_{\eta_0}}}{\sim} [W_{b(\eta_0)}] = [b(\eta_0)I] \ \text{ if } \ \eta_0 \in \mathbb{R} \setminus \{0\}, \tag{5.14e}$$
$$[W_b] \overset{W_{v_0}}{\sim} [W_{b^0}] = [\mathfrak{M}^0_{b_p(0,\cdot)}] \ \text{ if } \ \eta = 0, \tag{5.14f}$$
$$[W_b] \overset{W_{v_\infty}}{\sim} [W_{b^\infty(\infty,\cdot)}] = [\mathfrak{M}^0_{b_p(\infty,\cdot)}] \ \text{ if } \ \eta_0 = \infty, \tag{5.14g}$$
$$[W_b] \overset{v_{x_0}I}{\sim} [W_{b^\infty}] = [\mathfrak{M}^0_{b_p(\infty,\cdot)}] \ \text{ if } \ x_0 \in \mathbb{R}^+, \tag{5.14h}$$
$$[W_b] \overset{v_\infty I}{\approx} [W_b] \ \text{ if } \ x_0 = \infty, \tag{5.14i}$$

where

$$\begin{aligned} g^\infty(\xi) &:= \tfrac{g(+\infty)+g(-\infty)}{2} + \tfrac{g(+\infty)-g(-\infty)}{2}\text{sign}\xi \\ &= g(-\infty)\chi_-(\xi) + g(+\infty)\chi_+(\xi), \\ g^0(\xi) &:= \tfrac{g(0+0)+g(0-0)}{2} + \tfrac{g(0+0)+g(0-0)}{2}\text{sign}\xi \\ &= g(0-0)\chi_-(\xi) + g(0+0)\chi_+(\xi), \end{aligned} \tag{5.15}$$

and $\chi_\pm(\xi) := (1/2)(1 \pm \text{sign}\xi)$. Note that in the equivalency relations (5.14e)–(5.14g) we used the identities, cf. (5.3) and (5.9),

$$\begin{aligned} W_{g^\infty} &= \tfrac{g(-\infty)-g(+\infty)}{2} - \tfrac{g(-\infty)-g(+\infty)}{2} S_{\mathbb{R}^+} = \mathfrak{M}_{g_p(\infty,\cdot)}, \\ W_{g^0} &= \tfrac{g(0+0)+g(0-0)}{2} - \tfrac{g(0+0)-g(0-0)}{S}{}_{\mathbb{R}^+} = \mathfrak{M}_{g_p(0,\cdot)}, \end{aligned}$$

which means that the Fourier convolution operators with homogeneous of order 0 symbols $g^\infty(\xi)$ and $g^0(\xi)$ are, simultaneously, Mellin convolutions with the symbols $g_p(\infty, \xi)$, $g_p(0, \xi)$.

Using the equivalence relations (5.14a)–(5.14h) and the compactness of the corresponding operators, cf. Propositions 5.5–5.7, one finds easily the following local representatives of the operator (coset) $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}\mathbb{L}_p(\mathbb{R}^+)$ (see (5.9) for the operator $\mathbf{A}$):

$$[\mathbf{A}] \overset{\Lambda_{(\xi_0,\infty),0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)} W_{(b_j)^\infty}\Big] =$$

$$= \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)(b_j)_p(\infty,\cdot)}\Big] \overset{\Lambda_{(\xi_0,\infty),0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\xi_0)(b_j)_p(\infty,\xi_0)}\Big] =$$

$$= \big[\mathscr{A}_p(\xi_0,\infty)I\big] \ \text{ if } \ \omega=(\xi_0,\infty)\in\Gamma_1, \ \ x_0=0, \tag{5.16a}$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\infty),x_0}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{(b_j)^\infty}\Big] = \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)(b_j)_p(\infty,\cdot)}\Big] =$$

$$= \big[\mathfrak{M}^0_{\mathscr{A}_p(\pm\infty,\cdot)}\big] \overset{\Lambda_{(\pm\infty,\infty),x_0}}{\sim} \big[\mathscr{A}_p(\pm\infty,\infty)I\big] \tag{5.16b}$$

$$\text{if } \ \omega=(\pm\infty,\infty)\in\overline{\Gamma_2^{\pm}}\cap\overline{\Gamma_1}, \ \ 0<x_0<\infty;$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\mp\eta_0),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{b_j(\mp\eta_0)}\Big] = \Big[\sum_{j=1}^{m} a_j(\pm\infty) b_j(\mp\eta_0) I\Big] =$$

$$= \big[\mathscr{A}_p(\pm\infty,\mp\eta_0)I\big] \ \text{ if } \ \eta_0>0, \ \omega=(\pm\infty,\mp\eta_0)\in\Gamma_2^{\pm}, \ x_0=\infty; \tag{5.16c}$$

$$[\mathbf{A}] \overset{\Lambda_{(\xi_0,0),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j} W_{b_j^0}\Big] =$$

$$= \Big[\sum_{j=1}^{m} a_j(\xi_0)\mathfrak{M}_{(b_j)_p(0,\cdot)}\Big] \overset{\Lambda_{(\xi_0,0),\infty}}{\sim} \Big[\sum_{j=1}^{m} a_j(\xi_0)(b_j)_p(0,\xi_0)\Big] =$$

$$= \big[\mathscr{A}_p(\xi_0,0)I\big] \ \text{ if } \ \omega=(\xi_0,0)\in\overline{\Gamma}_3, \ \ x_0=\infty; \tag{5.16d}$$

$$[\mathbf{A}] \overset{\Lambda_{(\pm\infty,\eta),\infty}}{\sim} \Big[\sum_{j=1}^{m} \mathfrak{M}^0_{a_j(\pm\infty)} W_{b_j(0)}\Big] = \Big[\sum_{j=1}^{m} a_j(\pm\infty) b_j(0) I\Big] =$$

$$= \big[\mathscr{A}_p(\pm\infty,0)I\big] \ \text{ if } \ \omega=(\pm\infty,0)\in\overline{\Gamma}_3, \ \ x_0=\infty. \tag{5.16e}$$

It is remarkable that the local representatives (5.16a)–(5.16e) are just the quotient classes of multiplication operators by constant $N\times N$ matrices $[\mathscr{A}_p(\xi_0,\eta_0)I]$. If $\det\mathscr{A}_p(\xi_0,\eta_0)=0$, these representatives are not invertible, both locally and globally. On the other hand, they are globally invertible if $\det\mathscr{A}_p(\xi_0,\eta_0)\neq 0$. Thus, the conditions of the local invertibility for all points $\omega_0=(\xi_0,\eta_0)\in\mathfrak{R}$ and the global invertibility of the operators under consideration coincide with the ellipticity condition for the symbol $\inf\limits_{(\xi_0,\eta_0)\in\mathfrak{R}} \det\mathscr{A}_p(\xi_0,\eta_0)\neq 0$.

The index $\mathrm{Ind}\mathbf{A}$ is a continuous integer-valued multiplicative function $\mathrm{Ind}\mathbf{AB}=\mathrm{Ind}\mathbf{A}+\mathrm{Ind}\mathbf{B}$ defined on the group of Fredholm operators of $\mathfrak{A}_p(\mathbb{R}^+)$. On the other hand, the index function $\mathrm{inddet}\mathscr{A}_p$ defined on L_p-symbols $\mathscr{A}_p$ possesses the same property $\mathrm{inddet}\mathscr{A}_p\mathscr{B}_p=\mathrm{inddet}\mathscr{A}_p+\mathrm{inddet}\mathscr{B}_p$, see explanations after (5.10). Moreover, the set of operators (5.9) is dense in the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ and the corresponding set of their symbols is dense in the algebra $C(\mathfrak{R})$ of all continuous functions on $\mathfrak{R}$. For $p=2$ these algebras even coincide. Therefore, there is an algebraic homeomorphism between the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ and the algebra of their symbols which is a dense subalgebra of $C(\mathfrak{R})$. Hence, two various index functions can be only connected by the relation $\mathrm{Ind}\mathbf{A}=M_0\mathrm{inddet}\mathscr{A}_p$ with an integer constant M_0 independent of $\mathbf{A}\in\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Since for any Fourier convolution operator $\mathbf{A}=W_a$ the index formula is $\mathrm{Ind}\mathbf{A}=-\mathrm{inddet}\mathscr{A}_p$ [Du74, Du75a, Du79], the constant $M_0=-1$, and the index formula (5.11) is proved.

Concerning the concluding assertion of the theorem: $\mathbf{A}$ is, after lifting to $\mathbb{L}_p$-space, locally equivalent at 0 to the Mellin convolution operator $\mathfrak{M}^0_{\mathscr{A}^s_p(\infty,\xi)}$, which commutes with the dilation

$$\mathfrak{M}^0_a V_\lambda = V_\lambda \mathfrak{M}^0_a, \qquad V_\lambda\varphi(t) := \varphi(\lambda t) \qquad \text{for all} \quad \lambda > 0$$

and, therefore, is locally invertible at 0 if and only if is globally invertible (see [Du77, Du79, Si65]) and this is the case iff $\inf_{\xi\in\mathbb{R}} |\mathscr{A}^s_p(\infty,\xi)| > 0$. ∎

Remark 5.15 *Let us emphasize that the formula* (5.11) *does not contradict the invertibility of "pure Mellin convolution" operators* $\mathfrak{M}^0_a : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *with an elliptic matrix symbol* $a \in C\mathfrak{M}^0_p(\mathbb{R})$*,* $\inf_{\xi\in\mathbb{R}} |a(\xi)| > 0$*, stated in Proposition 1.1, even if* $\operatorname{ind} a \neq 0$*.*

In fact, computing the symbol of $\mathfrak{M}^0_a$ *by formula* (5.8)*, one obtains*

$$(\mathfrak{M}^0_a)_p(\omega) := \begin{cases} a(\xi), & \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ a(+\infty), & \omega = (+\infty,\eta) \in \Gamma^+_2, \\ a(-\infty), & \omega = (-\infty,\eta) \in \Gamma^-_2, \\ a(\xi), & \omega = (\xi,0) \in \overline{\Gamma_3}. \end{cases}$$

Noting that on the sets Γ_1 *and* Γ_3 *the variable* ω *runs in opposite direction, the increment of the argument* $[\arg\det(\mathfrak{M}^0_a)_p(\omega)]_{\mathfrak{R}} = 0$ *is zero, implying* $\operatorname{Ind}\mathfrak{M}^0_a = 0$*.*

In contrast to the above, the pure Fourier convolution operators $W_b : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *with elliptic matrix symbol* $b \in C\mathfrak{M}^0_p(\mathbb{R})$*,* $\inf_{\xi\in\mathbb{R}} |b_p(\xi,\eta)| > 0$ *can possess non-zero indices. Since*

$$b_p(\omega) := \begin{cases} b_p(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ b(-\eta), & \omega = (+\infty,\eta) \in \Gamma^+_2, \\ b(\eta), & \omega = (-\infty,\eta) \in \Gamma^-_2, \\ b(0), & \omega = (\xi,0) \in \overline{\Gamma_3}, \end{cases}$$

one arrives at the well-known formula

$$\operatorname{Ind} W_b = -\operatorname{ind} b_p.$$

Moreover, in the case where the symbol $b(-\infty) = b(+\infty)$ *is continuous, one has* $b_p(\xi,\eta) = b(\xi)$*. Thus the ellipticity of the corresponding operator leads to the formula*

$$\operatorname{ind} b_p = \operatorname{ind}\det b.$$

If $\mathscr{A}_p(\omega)$ is the symbol of an operator $\mathbf{A}$ of (5.7), the set $\mathscr{R}(\mathscr{A}_p) := \{\mathscr{A}_p(\omega) \in \mathbb{C} : \omega \in \mathfrak{R}\}$ coincides with the essential spectrum of $\mathbf{A}$. Recall that the essential spectrum $\sigma_{ess}(\mathbf{A})$ of a bounded operator $\mathbf{A}$ is the set of all $\lambda \in \mathbb{C}$ such that the operator $\mathbf{A} - \lambda I$ is not Fredholm in $\mathbb{L}_p(\mathbb{R}^+)$ or, equivalently, the coset $[\mathbf{A} - \lambda I]$ is not invertible in the quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$. Then, due to Banach theorem, the essential norm $|||\mathbf{A}|||$ of the operator $\mathbf{A}$ can be estimated as follows

$$\sup_{\omega\in\omega} |\mathscr{A}_p(\omega)| \leqslant |||\mathbf{A}||| := \inf_{\mathbf{T}\in\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))} \big\|(\mathbf{A}+\mathbf{T}) \mid \mathscr{L}(\mathbb{L}_p(\mathbb{R}^+))\big\|. \tag{5.17}$$

The inequality (5.17) enables one to extend continuously the symbol map (5.8)

$$[\mathbf{A}] \longrightarrow \mathscr{A}_p(\omega), \quad [\mathbf{A}] \in \mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+)) \tag{5.18}$$

on the whole Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$. Now, applying Theorem 5.14 and atandard methods, cf. [Du87, Theorem 3.2], one can derive the following result.

Corollary 5.16 *Let $1 < p < \infty$ and $\mathbf{A} \in \mathfrak{A}_p(\mathbb{R}^+)$. The operator $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ is Fredholm if and only if it's symbol $\mathscr{A}_p(\omega)$ is elliptic. If $\mathbf{A}$ is Fredholm, then*

$$\mathrm{Ind}\mathbf{A} = -\mathrm{ind}\mathscr{A}_p.$$

Corollary 5.17 *The set of maximal ideals of the commutative Banach quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ generated by scalar $N = 1$ operators in* (5.7)*, is homeomorphic to $\mathfrak{R}$, and the symbol map in* (5.8)*,* (5.18) *is a Gelfand homeomorphism of the corresponding Banach algebras.*

Proof: The proof is based on Theorem 5.14 and Corollary 5.16 and if similar to [Du87, Theorem 3.1]. The details of the proof is left to the reader. ■

Remark 5.18 *All the above results are valid in a more general setting viz., for the Banach algebra $\mathfrak{PA}^{N\times N}_{p,\alpha}(\mathbb{R}^+)$ generated in the weighted Lebesgue space of N-vector-functions $\mathbb{L}^N_p(\mathbb{R}^+, x^\alpha)$ by the operators*

$$\mathbf{A} := \sum_{j=1}^{m} \left[d^1_j \mathfrak{M}^0_{a^1_j} W_{b^1_j} + d^2_j \mathfrak{M}^0_{a^2_j} H_{c^1_j} + d^3_j W^0_{b^2_j} H_{c^2_j} \right] \tag{5.19}$$

when coefficients $d^1_j, d^2_j, d^3_j \in PC^{N\times N}(\overline{\mathbb{R}})$ are piecewise-continuous $N\times N$ matrix functions, symbols of Mellin convolution operators $\mathfrak{M}^0_{a^1_j}$, $\mathfrak{M}^0_{a^2_j}$, Winer–Hopf (Fourier convolution) operators $W_{b^1_j}$, $W_{b^2_j}$ and Hankel operators $H_{c^1_j}$, $H_{c^2_j}$ are $N \times N$ piecewise-continuous matrix $\mathbb{L}_p$-multipliers $a^k_j, b^k_j, c^k_j \in PC^{N\times N}\mathfrak{M}_p(\mathbb{R})$.

The spectral set $\Sigma(\mathfrak{PA}^{N\times N}_{p,\alpha}(\mathbb{R}^+))$ of such Banach algebra (viz., the set where the symbols are defined, e.g. $\mathfrak{R}$ for the Banach algebra $\mathfrak{A}^{N\times N}_p(\mathbb{R}^+)$ investigated above) is more sophisticated and described in the papers [Du77, Du78, Du87, Th85]. Let $\mathfrak{CA}_{p,\alpha}(\mathbb{R}^+)\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ be the sub-algebra of $\mathfrak{PA}_{p,\alpha}(\mathbb{R}^+) = \mathfrak{PA}^{1\times 1}_{p,\alpha}(\mathbb{R}^+)$ generated by scalar operators (5.19) *with continuous coefficients $c_j, h_j \in C(\overline{\mathbb{R}})$ and scalar piecewise-continuous $\mathbb{L}_p$-multipliers) $a_j, b_j, d_j, g_j \in PC\mathfrak{M}_p(\mathbb{R})$. The quotient-algebra $\mathfrak{CA}_{p,\alpha}(\mathbb{R}^+)\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ with respect to the ideal of all compact operators is a commutative algebra and the spectral set $\Sigma(\mathfrak{PA}_{p,\alpha}(\mathbb{R}^+))$ is homeomorphic to the set of maximal ideals.*

We will not elaborate more on further details concerning the Banach algebra $\mathfrak{PA}^{N\times N}_{p,\alpha}(\mathbb{R}^+)$, since the result exposed above are sufficient for the purpose of this and subsequent papers dealing with the BVPs in domains with corners at the boundary.

6 MELLIN CONVOLUTION OPERATORS IN THE BESSEL POTENTIAL SPACES. THE BOUNDEDNESS AND LIFTING

As it was already mentioned, the primary aim of the present paper is to study Mellin convolution operators $\mathfrak{M}^0_a$ acting in the Bessel potential spaces,

$$\mathfrak{M}^0_a : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+). \tag{6.1}$$

The symbols of these operators are $N \times N$ matrix functions $a \in C\mathfrak{M}^0_p(\overline{\mathbb{R}})$, continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

Theorem 6.1 *Let $s \in \mathbb{R}$ and $1 < p < \infty$.*

If conditions of Theorem 3.8 hold, the Mellin convolution operator between the Bessel potential spaces

$$\mathbf{K}^1_c \ : \ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+) \tag{6.2}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c W_{g^s_{-c\gamma,\gamma}} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{6.3}$$

where $c^{-s} = |c|^{-s}e^{-is\arg c}$ and the function $g^s_{-c\gamma,\gamma}$ is defined in (2.10).

If conditions of Corollary 3.9 hold, the Mellin convolution operator between the Bessel potential spaces (6.2) *is lifted to the equivalent operator*

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{g^s_{-c\gamma_0,\gamma}} = c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\gamma_0,\gamma}} + \mathbf{T}\\ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+),\end{aligned} \tag{6.4}$$

where $\mathbf{T} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator.

Proof: The equivalent operator after lifting is

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$$

(see Theorem 2.2). To proceed we need two formulae

$$\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = W_{g^s_{-c\gamma,\gamma}}, \qquad W_{g^s_{-\gamma,-\gamma_0}}W_{g^s_{-c\gamma_0,\gamma}} = W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\gamma_0,\gamma}}. \tag{6.5}$$

The first one holds because $0 < \arg\gamma < \pi$ (see (2.8)) and the second one holds because $g^s_{-\gamma,-\gamma_0}(\xi)$ has a smooth, uniformly bounded analytic extension in the complex lower half plane (see (2.13)).

If conditions of Theorem 3.8 hold, we apply formula (3.24), the first formula in (6.5) and derive the equality in (6.3):

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\mathbf{K}^1_c W_{g^s_{-c\gamma,\gamma}}.$$

If conditions of Corollary 3.9 hold, we apply formulae (3.36), (3.37), both formula in (6.5) and derive the equality in (6.4):

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{g^s_{-c\gamma_0,\gamma}}\\ &= c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}}W_{g^s_{-c\gamma_0,\gamma}} + \mathbf{T}.\end{aligned}$$

■

Remark 6.2 *The case of operator* $\mathbf{K}^1_1$ *is not covered by the foregoing Theorem 6.1, where* $\arg\, c \neq 0$*. This case is essentially different as underlined in Theorem 5.4 because* $\mathbf{K}^1_1$ *is a Hilbert transform* $\mathbf{K}^1_1 = -\pi i S_{\mathbb{R}^+} = \pi i W_{\rm sign}$ *and* $\mathbf{K}^1_1$ *between the Bessel potential spaces* (6.2) *is lifted to the equivalent Fourier convolution operator*

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_1\mathbf{\Lambda}^{-s}_{\gamma} = W_{\pi i g^s_{-\gamma,\gamma}{\rm sign}} \ :\ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{6.6}$$

as it follows from Theorem 5.4.

Theorem 6.3 *Let* $c_j, d_j \in \mathbb{C}$, $0 < \arg\, c_j < 2\pi$, $0 < \arg\gamma < \pi$, $-\pi < \arg(c_j\gamma) < 0$ *for* $j = 1, \ldots, m$ *and* $0 < \arg(c_j\gamma) < \pi$ *for* $j = m+1, \ldots, n$.

The Mellin convolution operator between the Bessel potential spaces

$$\boldsymbol{A} = \sum_{j=1}^{n} d_j\mathbf{K}^1_{c_j} \ :\ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+) \tag{6.7}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\boldsymbol{A}\mathbf{\Lambda}^{-s}_{\gamma} = \sum_{j=0}^{m} d_j c_j^{-s}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma,-\gamma}} + \sum_{j=m+1}^{n} d_j c_j^{-s} W_{g^s_{-\gamma,-\gamma_j}}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma_j,\gamma}} \tag{6.8a}$$

$$= \sum_{j=0}^{m} d_j c_j^{-s}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma,\gamma}} + \sum_{j=m+1}^{n} d_j c_j^{-s}\mathbf{K}^1_{c_j}W_{g^s_{-\gamma,-\gamma_j}g^s_{-c_j\gamma_j,\gamma}} + \mathbf{T}, \tag{6.8b}$$

in the $\mathbb{L}_p(\mathbb{R}^+)$ *space, where* $c^{-s} = |c|^{-s}e^{-is\arg c}$ *and* γ_j *are such that* $0 < \arg\gamma_j < \pi$, $-\pi < \arg(c_j\,\gamma_j) < 0$ *for* $j = m+1, \ldots, n$. $\mathbf{T}\ :\ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ *is a compact operator.*

Proof: The proof is a direct consequence of Theorem 6.1. ■

Theorem 6.4 *Let* $s \in \mathbb{R}$ *and* $1 < p < \infty$.

If conditions of Theorem 3.8 hold, the Mellin convolution operator between the Bessel potential spaces

$$\mathbf{K}^2_c \ :\ \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+) \to \mathbb{H}^r_p(\mathbb{R}^+) \tag{6.9}$$

is lifted to the equivalent operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_{\gamma} = c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma,\gamma}}\,, \tag{6.10}$$

in $\mathbb{L}_p(\mathbb{R}^+)$ *space, where* $c^{-s} = |c|^{-s}e^{-is\arg c}$ *and the function* $g^s_{-c\gamma,\gamma}$ *is defined in* (2.10) *and the last summand in* (6.10)

$$\mathbf{T} := s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{g^{s-1}_{-c\gamma,\gamma}}\mathbf{\Lambda}^{-1}_{\gamma} \ :\ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{6.11}$$

is a compact operator.

If conditions of Corollary 3.9 hold, the Mellin convolution operator between the Bessel potential spaces (6.9) *is lifted to the equivalent operator in* $\mathbb{L}_p(\mathbb{R}^+)$ *space*

$$\begin{aligned}&\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_{\gamma} = \qquad (6.12)\\ &= c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-c\gamma_0,\gamma}} + s\,\gamma\,c^{-s}W_{g^s_{-\gamma,-\gamma_0}}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma_0,\gamma}}\\ &= c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\,\gamma_0,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{(\xi-c\gamma_0)^{-1}g^s_{-c\gamma_0,-\gamma_0}g^s_{-\gamma,\gamma}} + \mathbf{T},\end{aligned}$$

where $\mathbf{T}\,:\,\mathbb{L}_p(\mathbb{R}^+)\to\mathbb{L}_p(\mathbb{R}^+)$ *is a compact operator.*

Proof: Let conditions of Theorem 3.8 hold (that means $\operatorname{Im}\gamma > 0$ and $\operatorname{Im} c\,\gamma < 0$. Then

$$\frac{1}{(t-c)^2} = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\left[\frac{1}{t-c-\varepsilon i} - \frac{1}{t-c+\varepsilon i}\right]$$

and we have

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^2_c\mathbf{\Lambda}^{-s}_{\gamma} &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\mathbf{\Lambda}^s_{-\gamma}\left[\mathbf{K}^1_{c+\varepsilon i} - \mathbf{K}^1_{c-\varepsilon i}\right]\mathbf{\Lambda}^{-s}_{\gamma}\\ &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon i}\left[(c+\varepsilon i)^{-s}\mathbf{K}^1_{c+\varepsilon i}\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} - (c-\varepsilon i)^{-s}\mathbf{K}^1_{c-\varepsilon i}\mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\right]\mathbf{\Lambda}^{-s}_{\gamma}\\ &= \lim_{\varepsilon\to 0}\left\{\frac{(c+\varepsilon i)^{-s} - (c-\varepsilon i)^{-s}}{2\varepsilon i}\mathbf{K}^1_{c+\varepsilon i}\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma}\right.\\ &\qquad -(c-\varepsilon i)^{-s}\frac{1}{2\varepsilon i}\left[\mathbf{K}^1_{c+\varepsilon i} - \mathbf{K}^1_{c-\varepsilon i}\right]\mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\\ &\qquad \left.-(c-\varepsilon i)^{-s}\mathbf{K}^1_{c-\varepsilon i}\frac{1}{2\varepsilon i}\left[\mathbf{\Lambda}^s_{-(c+\varepsilon i)\gamma} - \mathbf{\Lambda}^s_{-(c-\varepsilon i)\gamma}\right]\right\}\mathbf{\Lambda}^{-s}_{\gamma}\\ &= -s\,c^{-s-1}\mathbf{K}^1_c\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{\Lambda}^{-s}_{\gamma} + c^{-s}\mathbf{K}^2_c\mathbf{\Lambda}^s_{-c\,\gamma}\mathbf{\Lambda}^{-s}_{\gamma}\\ &\qquad +c^{-s}\mathbf{K}^1_c\lim_{\varepsilon\to 0}\mathscr{F}^{-1}\frac{(\xi - c\,\gamma - \varepsilon\gamma i)^s - (\xi - c\,\gamma + \varepsilon\gamma i)^s}{2\varepsilon i}\mathscr{F}\mathbf{\Lambda}^{-s}_{\gamma}\\ &= c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c\mathbf{\Lambda}^{s-1}_{-c\gamma}\mathbf{\Lambda}^{-s}_{\gamma}\\ &= c^{-s}\left[\mathbf{K}^2_c - sc^{-1}\mathbf{K}^1_c\right]W_{g^s_{-c\gamma,\gamma}} + s\,\gamma\,c^{-s}\mathbf{K}^1_c W_{(\xi+\gamma)^{-1}g^{s-1}_{-c\gamma,\gamma}}\end{aligned}$$

Formula (6.10) is proved.

Formula (6.12) is derived from (6.10) as in Theorem 6.1. ∎

Remark 6.5 *The case of operators* $\mathbf{K}^n_c$, $n = 3, 4, \ldots$, *can be treated similarly as in Corollary 6.4: with the help of perturbation the operator* $\mathbf{K}^n_c$ *can be represented in the form*

$$\begin{aligned}&\mathbf{K}^n_c\varphi = \lim_{\varepsilon\to 0}\mathbf{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\varphi, \qquad \forall\,\varphi\in\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)\\ &\mathbf{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\varphi(t) := \int_0^\infty \mathscr{K}_{c_{1,\varepsilon},\ldots,c_{n,\varepsilon}}\left(\frac{t}{\tau}\right)\varphi(\tau)\frac{d\tau}{\tau} = \sum_{j=1}^n d_j(\varepsilon)\mathbf{K}^1_{c_{j,\varepsilon}}\varphi(t),\end{aligned}$$

$$\mathscr{K}_{c_{1,\varepsilon},\ldots,c_{m,\varepsilon}}(t) := \frac{1}{(t-c_{1,\varepsilon})\cdots(t-c_{n,\varepsilon})} = \sum_{j=1}^{n}\frac{d_j(\varepsilon)}{t-c_{j,\varepsilon}}, \tag{6.13}$$
$$c_{j,\varepsilon} = c(1+\varepsilon e^{i\omega_j}),\ \ \omega_j\in(-\pi,\pi),\ \ \arg c_{j,\varepsilon}, \arg c_{j,\varepsilon}\,\gamma_j \neq 0,\ \ j=1,\ldots,m.$$

The points $\omega_1,\ldots,\omega_n \in (-\pi,\pi]$ are pairwise different, i.e., $\omega_j \neq \omega_k$ for $j \neq k$ (we remind that $\arg c \neq 0$ because $n = 3,4,\ldots$). By equating the numerators in the formula (6.13) we find the coefficients $d_1(\varepsilon),\ldots,d_{n-1}(\varepsilon)$.

Since the operators $\mathbf{K}^3_c, \mathbf{K}^4_c, \ldots$ encounter in applications rather rarely, we have confined ourselves with the exact formulae only for the operators $\mathbf{K}^1_c$ and $\mathbf{K}^2_c$.

7 Mellin Convolution Operators in the Bessel Potential Spaces. Fredholm Properties

Let us write the symbol of a model operator

$$\mathbf{A} := d_0 I + W_{a_0} + \sum_{j=1}^{n} W_{a_j}\mathbf{K}^1_{c_j}W_{b_j}, \tag{7.1}$$

acting in the Bessel potential spaces $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$, compiled of the identity I, of Fourier $W_{a_0},\ldots,W_{a_n}, W_{b_1},\ldots,W_{b_n}$ and Mellin $\mathbf{K}^1_{c_1},\ldots,\mathbf{K}^1_{c_n}$ convolution operators.

We assume that $a_0,\ldots,a_n,b_1,\ldots,b_n \in C\mathfrak{M}_p(\overline{\mathbb{R}}\setminus\{0\})$, $c_1,\ldots,c_n\in\mathbb{C}$ and, if $s \leqslant \dfrac{1}{p}-1$ or $s \geqslant \dfrac{1}{p}$, the functions $a_1(\xi),\ldots,a_n(\xi)$ have bounded analytic extensions in the lower half plane $\operatorname{Im}\xi < 0$, while the functions $b_1(\xi),\ldots,b_n(\xi)$ have bounded analytic extensions in the upper half plane $\operatorname{Im}\xi > 0$ to ensure the proper mapping properties of the operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$. For $\dfrac{1}{p}-1 < s < \dfrac{1}{p}$ such constraints are not necessary.

Now we describe the symbol $\mathscr{A}^s_p(\omega)$ of the operator $\mathbf{A}$. For this we lift the operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$ to the $\mathbb{L}_p$-setting and apply equality (2.13):

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{7.2}$$
$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} = d_0\mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{\gamma} + \mathbf{\Lambda}^s_{-\gamma}W_{a_0}\mathbf{\Lambda}^{-s}_{\gamma} + \sum_{j=1}^{n} W_{a_j}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_{c_j}\mathbf{\Lambda}^{-s}_{\gamma}W_{b_j}$$
$$= d_0 W_{\left(\frac{\xi-\gamma}{\xi+\gamma}\right)^s} + W_{a_0(\xi)\left(\frac{\xi-\gamma}{\xi+\gamma}\right)^s} + \sum_{j=1}^{n} W_{a_j}\mathbf{K}^1_{c_j}W_{\left(\frac{\xi-c\gamma}{\xi+\gamma}\right)^s}W_{b_j} \tag{7.3}$$

(see Theorem **??**, diagram (2.7)) if conditions of Theorem 3.8 hold (see (6.4)) and to the operator

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} = d_0 W_{\left(\frac{\xi-\gamma}{\xi+\gamma}\right)^s} + W_{a_0(\xi)\left(\frac{\xi-\gamma}{\xi+\gamma}\right)^s} + \sum_{j=1}^{n} W_{a_j}\mathbf{K}^1_{c_j}W_{\left(\frac{\xi-\gamma}{\xi-\gamma_0}\right)^s\left(\frac{\xi-c\gamma_0}{\xi+\gamma}\right)^s}W_{b_j} + \mathbf{T}, \tag{7.4}$$

where $\mathbf{T} : \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ is a compact operator, if conditions of Corollary **??** hold (see (6.5)).

The symbol of the lifted operator (7.2)– (7.4) in $\mathbb{L}_p(\mathbb{R}^+)$-space we declare the symbol of the operator $\mathbf{A}$ in the Bessel potential space. This symbol, written according formulae (5.8) and (5.9), has the form:

$$\mathscr{A}^s_p(\omega) := d_0\mathscr{I}^s_p(\omega) + \mathscr{W}^s_{a_0,p}(\omega) + \sum_{j=1}^{n} \mathscr{W}^0_{a_j,p}(\omega)\mathscr{K}^{1,s}_{c_j,p}(\omega)\mathscr{W}^0_{b_j,p}(\omega), \tag{7.5}$$

where $\mathscr{I}^s_p(\omega)$, $\mathscr{W}^s_{a_0,p}(\omega)$, $\mathscr{W}^0_{a_j,p}(\omega)$, $\mathscr{K}^{1,s}_{c_j,p}(\omega)$ and $\mathscr{W}^0_{b_j,p}(\omega)$ are the symbols of the operators $W_{\left(\frac{\xi-\gamma}{\xi-\gamma}\right)^s}$ in $\mathbb{L}_p$ (of I in $\mathbb{H}^s_p$), of $W_{a_0(\xi)\left(\frac{\xi-\gamma}{\xi-\gamma}\right)^s}$ in $\mathbb{L}_p$ (of W_{a_0} in $\mathbb{H}^s_p$), of W_{a_j} in $\mathbb{L}_p$ (and in $\mathbb{H}^s_p$) of $\mathbf{K}^1_{c_j}W_{\left(\frac{\xi-c\gamma}{\xi-\gamma}\right)^s}$ in $\mathbb{L}_p$ (of $\mathbf{K}^1_{c_j}$ in $\mathbb{H}^s_p$), of W_{b_j} in $\mathbb{L}_p$ (and in $\mathbb{H}^s_p$).

Now it suffices to expose the symbols $\mathscr{I}^s_p(\omega)$, $\mathscr{W}^s_{a_0,p}(\omega)$, $\mathscr{W}^0_{a_j,p}(\omega)$ and $\mathscr{K}^{1,s}_{c_j,p}(\omega)$ of the operators I, W_{a_0}, $W_{a_j}(j = 1, 2, \dots, n)$ and $\mathbf{K}^1_c$ separately (the symbol $\mathscr{W}^0_{b_j,p}(\omega)$ of $W_{b_j}(j = 1, 2, \dots, n)$ is written analogously):

$$\mathscr{I}^s_p(\omega) := \begin{cases} g^s_{-\gamma,\gamma,p}(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma}_1, \\ \left(\dfrac{\eta-\gamma}{\eta+\gamma}\right)^{\mp s}, & \omega = (+\infty,\eta) \in \Gamma^{\pm}_2, \\ e^{\pi s i}, & \omega = (\xi,0) \in \overline{\Gamma}_3, \end{cases} \tag{7.6a}$$

$$\mathscr{W}^s_{a,p}(\omega) := \begin{cases} a^s_p(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma}_1, \\ a(\mp\eta)\left(\dfrac{\eta-\gamma}{\eta+\gamma}\right)^{\mp s}, & \omega = (+\infty,\eta) \in \Gamma^{\pm}_2, \\ e^{\pi s i}a_p(0,\xi), & \omega = (\xi,0) \in \overline{\Gamma}_3, \end{cases} \tag{7.6b}$$

$$\mathscr{W}^0_{a,p}(\omega) := \begin{cases} a_p(\infty,\xi), & \omega = (\xi,\infty) \in \overline{\Gamma}_1, \\ a(\mp\eta), & \omega = (+\infty,\eta) \in \Gamma^{\pm}_2, \\ a_p(0,\xi), & \omega = (\xi,0) \in \overline{\Gamma}_3, \end{cases} \tag{7.6c}$$

$$\mathscr{K}^{1,s}_{c,p}(\omega) := \begin{cases} \dfrac{c^{-s}e^{-i\pi(\frac{1}{p}-i\xi-1)}c^{\frac{1}{p}-i\xi-1}}{\sin\pi(\frac{1}{p}-i\xi)}, & \omega = (\xi,\infty) \in \overline{\Gamma}_1,, \ \ \omega = (\xi,0) \in \overline{\Gamma}_3 \\ 0, & \omega = (\pm\infty,\eta) \in \Gamma^{\pm}_2 \quad \text{for} \quad \arg c \neq 0, \end{cases} \tag{7.6d}$$

$$\mathscr{K}^{1,s}_{1,p}(\omega) := \begin{cases} -i\cot\pi(\frac{1}{p}-i\xi), & \omega = (\xi,\infty) \in \overline{\Gamma}_1,, \\ \pm 1, & \omega = (\pm\infty,\eta) \in \Gamma^{\pm}_2, \\ i\cot\pi(\frac{1}{p}-i\xi), & \omega = (\xi,0) \in \overline{\Gamma}_3, \end{cases} \tag{7.6e}$$

$$a^s_p(\infty,\xi) := \frac{e^{2\pi si}a(\infty)+a(-\infty)}{2} - \frac{e^{2\pi si}a(\infty)-a(-\infty)}{2i}\cot\pi\left(\frac{1}{p}-i\xi\right),$$

$$a_p(x,\xi) := \tfrac{a(x+0)+a(x-0)}{2} - \tfrac{a(x+0)-a(x-0)}{2i}\cot\pi\Big(\tfrac{1}{p} - i\xi\Big), \quad x = 0, \infty,$$

$$g^s_{-\gamma,\gamma,p}(\infty,\xi) := \tfrac{e^{2\pi si}+1}{2} - \tfrac{e^{2\pi si}-1}{2i}\cot\pi\Big(\tfrac{1}{p} - i\xi\Big) = e^{\pi si}\frac{\sin\pi\Big(\frac{1}{p}-s-i\xi\Big)}{\sin\pi\Big(\frac{1}{p}-i\xi\Big)},$$
$$\xi\in\mathbb{R}, \quad \eta\in\mathbb{R}^+,$$

where
$$0 < \arg\, c < 2\pi, \quad -\pi < \arg(c\,\gamma) < 0, \quad 0 < \arg\gamma < \pi$$
and $c^s = |c|^s e^{is\arg\, c}$, $(-c)^\delta = |c|^\delta e^{i\delta(\arg\, c\mp\pi)}$ for $c,\delta\in\mathbb{C}$; the sign ”-” is chosen for $\pi < \arg\, c < 2\pi$ and the sign ”+” is chosen for $0 < \arg\, c < \pi$.

Note that, we got the equal symbol $\mathscr{K}^{1,s}_{c,p}(\omega)$ of the operator $\mathbf{K}^1_{c_j}$ in the cases (7.3) and (7.4) because the functions

$$g^s_{-\gamma,-\gamma_0}(\xi)g^s_{-c\gamma_0,\gamma}(\xi) := \left(\frac{\xi-\gamma}{\xi-\gamma_0}\right)^s\left(\frac{\xi-c\gamma_0}{\xi+\gamma}\right)^s \quad\text{and}\quad g^s_{-c\gamma,\gamma}(\xi) := \left(\frac{\xi-c\gamma}{\xi+\gamma}\right)^s$$

have equal limits at infinity $g^s_{-c\gamma,\gamma}(\pm\infty) = g^s_{-\gamma,-\gamma_0}(\pm\infty)g^s_{-c\gamma_0,\gamma}(\pm\infty) = 1$ and $g^s_{-c\gamma,\gamma}(0) = g^s_{-\gamma,-\gamma_0}(0)g^s_{-c\gamma_0,\gamma}(0) = (-c)^s$.

If $a(-\infty) = 1$ and $a(+\infty) = e^{2\pi\alpha i}$, then $a^-_\infty = 0$, $a^+_\infty = 2\alpha$ and the symbol $a^s_p(\infty,\xi)$ acquires the form:

$$a^s_p(\infty,\xi) = e^{\pi(s+\alpha)i}\frac{\sin\pi\left(\frac{1}{p}-s+\alpha-i\xi\right)}{\cos\pi\Big(\frac{1}{p}-i\xi\Big)}. \tag{7.6f}$$

Note that, the Mellin convolution operator

$$\mathbf{K}^1_{-1}\varphi(t) := \int_0^\infty\frac{\varphi(\tau)\,d\tau}{t+\tau} = \mathfrak{M}^0_{\mathscr{M}_{\frac{1}{p}}\mathscr{K}^1_{-1}}, \qquad \mathscr{M}_{\frac{1}{p}}\mathscr{K}^1_{-1}(\xi) = \frac{\pi d^{\,\beta-i\xi-1}}{\sin\pi(\beta-i\xi)}$$

(see (3.10b)), which we encounter in applications, has a rather simple symbol in the Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^+)$ (see (7.6b), where $c = -1 = e^{i\pi}$):

$$\mathscr{K}^{1,s}_{-1,p}(\omega) := \begin{cases}\dfrac{e^{-\pi si}}{\sin\pi(\beta-i\xi)}, & \omega = (\xi,\infty)\in\overline{\Gamma_1}\cup\overline{\Gamma_3},\\ 0, & \omega = (\pm\infty,\eta)\in\Gamma^\pm_2, .\end{cases}$$

Theorem 7.1 *Let $1 < p < \infty$, $s\in\mathbb{R}$. The operator*

$$\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+), \tag{7.7}$$

defined in (7.1)*, is Fredholm if and only if its symbol $\mathscr{A}^s_p(\omega)$ defined in* (7.5) *and* (7.6a)–(7.6f)*, is elliptic.*

If $\mathbf{A}$ is Fredholm, the index of the operator has the value

$$\mathrm{Ind}\mathbf{A} = -\mathrm{inddet}\mathscr{A}^s_p.$$

Proof: Let $c_j, d_j \in \mathbb{C}$, $0 < \arg\, c_j < 2\pi$. Lifting the operator $\mathbf{A}$ to the $\mathbb{L}_p(\mathbb{R}^+)$ space we get

$$\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} = d_0\mathbf{\Lambda}^s_{-\gamma}\mathbf{\Lambda}^{-s}_{\gamma} + \mathbf{\Lambda}^s_{-\gamma}W_{a_0}\mathbf{\Lambda}^{-s}_{\gamma} + \sum_{j=1}^{n} W_{a_j}\mathbf{\Lambda}^s_{-\gamma}\mathbf{K}^1_{c_j}\mathbf{\Lambda}^{-s}_{\gamma}W_{b_j}, \tag{7.8}$$

where $c^{-s} = |c|^{-s}e^{-is\arg\, c}$ and γ_j are such that $0 < \arg\gamma_j < \pi$, $-\pi < \arg(c_j\,\gamma_j) < 0$ for $j = m+1, \ldots, n$.

To derive (7.8) we have applied the following property of convolution operators $\mathbf{\Lambda}^s_{-\gamma}W_{a_j} = W_{a_j}\mathbf{\Lambda}^s_{-\gamma}$ and $W_{b_j}\mathbf{\Lambda}^s_{\gamma} = \mathbf{\Lambda}^s_{\gamma}W_{b_j}$, $\mathbf{\Lambda}^{\mp s}_{\pm\gamma} = W_{\lambda^{\mp s}_{\pm\gamma}}$, which are based on the analytic extension properties of the symbols $\lambda^s_{-\gamma}, a_1(\xi), \ldots, a_n(\xi)$ in the lower half plane $\operatorname{Im}\xi < 0$ and of symbols $\lambda^{-s}_{\gamma}, b_1(\xi), \ldots, b_n(\xi)$ in the upper half plane $\operatorname{Im}\xi > 0$ (see (2.6)).

The model operators I, $\mathbf{K}^1_c$ and W_a Lifted to the space $\mathbb{L}_p(\mathbb{R}^+)$ acquire the form

$$\begin{gathered}\mathbf{\Lambda}^s_{\gamma}I\mathbf{\Lambda}^{-s}_{\gamma} = W_{g^s_{-\gamma,\gamma}}, \qquad \mathbf{\Lambda}^s_{\gamma}W_{a_k}\mathbf{\Lambda}^{-s}_{\gamma} = W_{a_k g^s_{-\gamma,\gamma}},\\ \mathbf{\Lambda}^s_{\gamma}\mathbf{K}^1_c\mathbf{\Lambda}^{-s}_{\gamma} = \begin{cases} c^{-s}\mathbf{K}^1_c W_{g^s_{-c\,\gamma,\gamma}} & \text{for } -\pi < \arg(c\,\gamma) < 0,\\ c^{-s}\mathbf{K}^1_c W_{g^s_{-\gamma,-\gamma_0}g^s_{-c\,\gamma_0,\gamma}} + \mathbf{T} & \text{for } \ 0 < \arg(c\,\gamma) < \pi,\\ & \quad -\pi < \arg(c\,\gamma_0)| < 0,\end{cases}\end{gathered} \tag{7.9}$$

where $\mathbf{T}$ is a compact operator. Here, as above, $0 < \arg\, c < 2\pi$, $0 < \arg\,\gamma < \pi$, $0 < \arg\,\gamma_0 < \pi$ and either $-\pi < \arg(c\,\gamma) < 0$ or, if $-\pi < \arg(c\,\gamma) < 0$, then $-\pi < \arg(c\,\gamma_0)| < 0$. Here $c^{-s} = |c|^{-s}e^{-is\arg\, c}$.

Therefore the operator $\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma}$ in (7.8) is rewritten as follows:

$$\begin{aligned}\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma} &= d_0 W_{g^s_{-\gamma,\gamma}} + W_{a_0 g^s_{\gamma,\gamma}} + \sum_{j=1}^{m} c_j^{-s}W_{a_j}\mathbf{K}^1_{c_j}W_{g^s_{-c_j\gamma,\gamma}}W_{b_j}\\ &+ \sum_{j=m+1}^{n} c_j^{-s}W_{a_j}\mathbf{K}^1_{c_j}W_{g^s_{-\gamma,-\gamma_j}g^s_{-c_j\gamma_j,\gamma}}W_{b_j} + \mathbf{T}\ :\ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+),\end{aligned} \tag{7.10}$$

where $\mathbf{T}$ is a compact operator and we ignore it when writing the symbol of $\mathbf{A}$.

We declare the symbol of the lifted operator $\mathbf{\Lambda}^s_{-\gamma}\mathbf{A}\mathbf{\Lambda}^{-s}_{\gamma}$ (see (7.10)) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ as the symbol of the initial operator $\mathbf{A}\ :\ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$ in (7.1).

The function $g^s_{-\gamma,\gamma} \in C(\mathbb{R})$ is continuous on $\mathbb{R}$, but has different limits at the infinity

$$g^s_{-\gamma,\gamma}(-\infty) = 1, \quad g^s_{-\gamma,\gamma}(+\infty) = e^{2\pi si}, \qquad g^s_{-\gamma,\gamma}(0) = e^{\pi si}, \tag{7.11a}$$

while the functions $g^s_{-\gamma,-\gamma_0}$, $g^s_{-c\gamma,\gamma}, g^s_{-c\gamma_0,\gamma} \in C(\mathbb{R})$ are continuous on $\mathbb{R}$ including the infinity

$$\begin{gathered} g^s_{-c\,\gamma,\gamma}(\pm\infty) = g^s_{-\gamma,-\gamma_0}(\pm\infty) = g^s_{-c\,\gamma_0,\gamma}(\pm\infty) = 1,\\ g^s_{-\gamma,-\gamma_0}(0)g^s_{-c\,\gamma_0,\gamma}(0) = \left(\frac{-\gamma}{-\gamma_0}\right)^s\left(\frac{-c\gamma_0}{\gamma}\right)^s = (-c)^s\,,\\ g^s_{-c\,\gamma,\gamma}(0) = (-c)^s \quad \text{if} \quad 0 < \arg\, c < 2\pi.\end{gathered} \tag{7.11b}$$

In the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$, the symbols of the first two operators in (7.10), are written according the formulae (5.8)–(5.9) by taking into account the equalities (7.11a) and (7.11a). The symbols of these operators have, respectively, the form (7.6a) and (7.6b).

The symbols of operators $W_{a_1}, \dots, W_{a_n}$ and $W_{b_1}, \dots, W_{b_n}$ are written with the help of the formulae (5.8)–(5.9) and have the form (7.6c).

The lifted Mellin convolution operators

$$\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_{c_j} \mathbf{\Lambda}^{-s}_\gamma \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow, \mathbb{L}_p(\mathbb{R}^+)$$

are of mixed type and comprise both the Mellin convolution operators $\mathbf{K}^1_{c_j} = \mathfrak{M}^0_{\mathscr{K}^1_{c_j,p}(\xi)}$, where the symbol $\mathscr{K}^1_{c_j,p}(\xi) := \mathscr{M}_{1/p}\mathscr{K}^1_{c_j}(\xi)$ is defined in (3.10b) and (3.10c), and the Fourier convolution operators $W_{g^s_{-c_j\,\gamma_0,\gamma}}$ and $W_{g^s_{-\gamma,-\gamma_0}g^s_{-c_j\,\gamma_0,\gamma}}$. The symbol of the operators $\mathbf{\Lambda}^s_\gamma \mathbf{K}^1_{c_j} \mathbf{\Lambda}^{-s}_\gamma$ from (7.9) in the Lebesgue space $\mathbb{L}_p(\mathbb{R}^+)$ is found according formulae (5.8)–(5.9), has the form (7.6d) and is declared the symbol of $\mathbf{K}^1_{c_j} \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$. The symbols of Fourier convolution factors $W_{g^s_{-c_j\,\gamma_0,\gamma}}$ and $W_{g^s_{-\gamma,-\gamma_0}g^s_{-c_j\,\gamma_0,\gamma}}$, which contribute the symbol of $\mathbf{K}^1_{c_j} = \mathfrak{M}^0_{\mathscr{K}^1_{c_j,p}}$ are written again according formulae (5.8)–(5.9) by taking into account the equalities (7.11a) and (7.11b).

To the lifted operator applies Theorem 6.3 and gives the result formulated in Theorem 7.1. ■

Corollary 7.2 *Let $1 < p < \infty$, $s \in \mathbb{R}$. The operator*

$$\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+),$$

defined in (5.17)*, is locally invertible at $0 \in \mathbb{R}^+$ if and only if its symbol $\mathscr{A}^s_p(\omega)$, defined in* (7.5) *and* (7.6a)–(7.6f)*, is elliptic on Γ_1, i.e.*

$$\inf_{\omega\in\Gamma_1} \left|\det \mathscr{A}^s_p(\omega)\right| = \inf_{\xi\in\mathbb{R}} \left|\det \mathscr{A}^s_p(\xi,\infty)\right| > 0.$$

Proof:

■

For the definition of the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p(\Omega) = \mathbb{B}^s_{p,p}(\Omega)$, $\widetilde{\mathbb{W}}^s_p(\Omega) = \widetilde{\mathbb{B}}^s_{p,p}(\Omega)$ for an arbitrary domain $\Omega \subset \mathbb{R}^n$, including the half axes $\mathbb{R}^+$, we refer to the monograph [**?**].

Corollary 7.3 *Let $1 < p < \infty$, $s \in \mathbb{R}$. If the operator $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$, defined in* (5.17)*, is Fredholm* (*is invertible*) *for all $a \in (s_0, s_1)$ and $p \in (p_0, p_1)$, where $-\infty < s_0 < s_1 < \infty$, $1 < p_o < p_1 < \infty$, then*

$$\mathbf{A} : \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{W}^s_p(\mathbb{R}^+), \quad s \in (s_0, s_1), \ \ p \in (p_0, p_1) \tag{7.12}$$

is Fredholm and has the equal index

$$\mathrm{Ind}\mathbf{A} = -\mathrm{ind}\det\mathscr{A}^s_p. \tag{7.13}$$

(*is invertible, respectively*) *in the Sobolev–Slobodeckij* (*Besov*) *spaces $\mathbb{W}^s_p = \mathbb{B}^s_{p,p}$.*

Proof: First of all recall that the Sobolev–Slobodeckij (Besov) spaces $\mathbb{W}^s_p = \mathbb{B}^s_{p,p}$ emerge as the result of interpolation with the real interpolation method between the Bessel potential spaces

$$\begin{aligned} \left(\mathbb{H}^{s_0}_{p_0}(\Omega), \mathbb{H}^{s_1}_{p_1}(\Omega)\right)_{\theta,p} &= \mathbb{W}^s_p(\Omega), \quad s := s_0(1-\theta) + s_1\theta, \\ \left(\widetilde{\mathbb{H}}^{s_0}_{p_0}(\Omega), \widetilde{\mathbb{H}}^{s_1}_{p_1}(\Omega)\right)_{\theta,p} &= \widetilde{\mathbb{W}}^s_p(\Omega), \quad p := \tfrac{1}{p_0}(1-\theta) + \tfrac{1}{p_1}\theta, \quad 0 < \theta < 1. \end{aligned} \tag{7.14}$$

If $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$ is Fredholm (or is invertible) for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$, it has a regularizer $\mathbf{R}$ (has the inverse $\mathbf{A}^{-1} = \mathbf{R}$, respectively), which is bounded in the setting

$$\mathbf{R} : \mathbb{W}^s_p(\mathbb{R}^+) \longrightarrow \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$$

due to the interpolation (7.14) and

$$\mathbf{RA} = I + \mathbf{T}_1, \quad \mathbf{AR} = I + \mathbf{T}_2,$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and in $\mathbb{H}^s_p(\mathbb{R}^+)$, respectively ($\mathbf{T}_1 = \mathbf{T}_2 = 0$ if $\mathbf{A}$ is invertible).

Due to the Krasnoselskij interpolation theorem (see [**?**]), $\mathbf{T}_1$ and $\mathbf{T}_2$ are compact in $\widetilde{\mathbb{W}}^s_p(\mathbb{R}^+)$ and in $\mathbb{W}^s_p(\mathbb{R}^+)$, respectively for all $s \in (s_0, s_1)$ and $p \in (p_0, p_1)$ and, therefore, $\mathbf{A}$ in (7.12) is Fredholm (is invertible, respectively).

The index formulae (7.13) follows from the embedding properties of the Sobolev–Slobodeckij and the Bessel potential spaces by standard well-known arguments. ■

Chapter 5

BVPs FOR THE LAPLACE-BELTRAMI EQUATIONS ON SURFACES WITH LIPSCHITZ BOUNDARY

In the present chapter we present results on boundary value problems for the Laplace-Beltrami equation on a surface with the Lipschitz boundary $\mathscr{C}$ in a non-classical setting, when solutions are sought in the Bessel potential spaces $\mathbb{H}^s_p(\mathscr{C})$, $1/p < s < 1 + 1/p$, $1 < p < \infty$. The Fredholm and unique solvability criteria is found. By the localization the problem is reduced to the investigation of Model Dirichlet, Neumann and mixed boundary value problems for the Laplace equation in a planar angular domain $\Omega_\alpha \subset \mathbb{R}^2$ of magnitude α. Results are presented on the model Dirichlet, Neumann, Mixed Dirichlet-Neumann and impedance boundary value problems in a non-classical setting. The problems are investigated by the potential method and reduced to locally equivalent 2×2 systems of Mellin convolution equations with meromorphic kernels on the semi-infinite axes $\mathbb{R}^+$ in the Bessel potential spaces. Such equations were studied recently by R. Duduchava in [Du15] and V. Didenko & R. Duduchava in [DD16].

1 INTRODUCTION AND FORMULATION OF THE PROBLEMS

Many problems in mathematical physics e.g., cracks in elastic media, electromagnetic scattering by surfaces etc., are reformulated in the form of a boundary value problem for an elliptic partial differential equation in domains and surfaces with angular points at the boundary. In the recent paper [BDKT13] investigation of such BVPs are reduced with the help of localization to the investigation of a family of model problems in plane with finite number of angular points on the boundary of magnitude $\alpha_j \in (0, 2\pi)$, $j = 1, \dots, m$ are reduced to the investugation of the associated model BVPs in angles with vertex at 0 and the same magnitude.

Consider a hypersurface $\mathscr{C} \subset \mathbb{R}^3$ with the Lipschitz boundary Γ and by $\mathscr{M}_\Gamma$ denote the the angular points (the knots) of Γ. Let $\boldsymbol{\nu} := (\nu_1, \nu_2, \nu_3)^\top$ be the normal vector field on the surface $\mathscr{C}$,

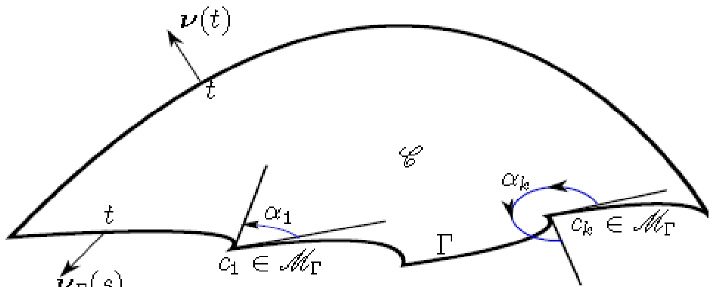

Fig. 1

On $\mathscr{C}$ we consider the mixed BVP

$$\begin{cases} \Delta_{\mathscr{C}} u(t) = f(t), & t \in \mathscr{C}, \\ u^+(s) = g(s), & \text{on} \quad \Gamma_D, \\ (\partial_{\boldsymbol{\nu}_\Gamma} u)^+(s) = h(s), & \text{on} \quad \Gamma_N, \end{cases} \tag{1.1}$$

where $\Delta_{\mathscr{C}}$ is the Laplace-Beltrami operator

$$\Delta_{\mathscr{C}} := \mathscr{D}_1^2 + \mathscr{D}_2^2 + \mathscr{D}_3^2$$

and $\mathscr{D}_j := \partial_j - \nu_j \partial_{\boldsymbol{\nu}}$, $j = 1, 2, 3$, are the Günter's tangential derivatives on the surface (cf. Chapter 1, § 3). $\boldsymbol{\nu}_\Gamma := (\nu_{\Gamma,1}, \nu_{\Gamma,2}, \nu_{\Gamma,3})^\top$ is the normal vector field to the boundary Γ, tangential to $\mathscr{S}$ and $\partial_{\boldsymbol{\nu}_\Gamma} = \nu_{\Gamma,1}\mathscr{D}_1 + \nu_{\Gamma,2}\mathscr{D}_2 + \nu_{\Gamma,3}\mathscr{D}_3$ is the normal derivative.

Problem (1.1) is considered in the non-classical setting

$$u \in \mathbb{H}^s_p(\mathscr{C}), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\mathscr{C}), \quad g \in \mathbb{H}^{s-1/p}_p(\Gamma_D), \quad h \in \mathbb{H}^{s-1-1/p}_p(\Gamma_N), \\ \Gamma = \Gamma_D \cup \Gamma_N, \quad 1 < p < \infty, \quad s > \frac{1}{p}. \tag{1.2}$$

Note that, the upper constraint in $\frac{1}{p} < s < 1 + \frac{1}{p}$ is necessary to ensure an invariant definition of the Bessel potential and Besov spaces on non-smooth boundary Γ, while the lower constraint ensures the existence of the Dirichlet trace u^+ and, together with the Green formulae, also the existence of the Neumann trace $(\partial_{\boldsymbol{\nu}} u)^+$ of a solution on the boundary. These constraints can not be relaxed.

For the definitions of the Bessel potential $\mathbb{H}^s_p(\mathscr{C})$, $\widetilde{\mathbb{H}}^s_p(\mathscr{S})$, $\mathbb{H}^r_p(\mathscr{C})$, $\widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$ and Sobolev-Slobodečkii $\widetilde{\mathbb{W}}^r_p(\mathbb{R}^+)$ etc. spaces for $r \in \mathbb{R}$, $1 < p < \infty$ we refer to the classical source [Tr92] and also [Du01, DS93, Hr83, Ta96].

Here we define only the space $\widetilde{\mathbb{H}}^{-1}_{p,0}(\mathscr{C})$ mentioned above. Let $\widetilde{\mathbb{H}}^{-1}_0(\mathscr{C})$ be a subspace of $\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$, orthogonal to

$$\widetilde{\mathbb{H}}^{-1}_{p,0}(\mathscr{C}) := \left\{ f \in \mathbb{H}^{-1}_0(\mathscr{C}) \ : \ \langle f, \varphi \rangle = 0 \text{ for all } \varphi \in C^1_0(\mathscr{C}) \right\}.$$

$\widetilde{\mathbb{H}}^{-1}_\Gamma(\mathscr{C})$ consists of those distributions on $\mathscr{C}$, belonging to $\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ which have their supports just on Γ and $\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ is decomposed into the direct sum of the subspaces:

$$\widetilde{\mathbb{H}}^{-1}(\mathscr{C}) = \widetilde{\mathbb{H}}^{-1}_\Gamma(\mathscr{C}) \oplus \widetilde{\mathbb{H}}^{-1}_0(\mathscr{C}).$$

The space $\widetilde{\mathbb{H}}^{-1}_{\Gamma}(\mathscr{C})$ is non-empty (see [HW08, § 5.1] and excluding it from $\widetilde{\mathbb{H}}^{-1}(\mathscr{C})$ is necessary to make BVPs uniquely solvable (cf. [HW08] and the next Theorem 1.1).

Let
$$\widetilde{\mathbb{H}}^{-r}_{p,0}(\mathscr{C}) = \widetilde{\mathbb{H}}^{-1}_{0}(\mathscr{C}) \bigcap \widetilde{\mathbb{H}}^{-1}_{0}(\mathscr{C}), \quad r > \frac{1}{p}.$$

Theorem 1.1 (Theorem 2.1, Remark 2.2 and Remark 2.3, [DTT14]) . *The BVP* (1.1) *has a unique solution in the classical weak setting:*

$$u \in \mathbb{H}^1(\mathscr{C}), \qquad f \in \widetilde{\mathbb{H}}^{-1}_{0}(\mathscr{C}), \qquad g \in \mathbb{H}^{1/2}(\Gamma), \qquad h \in \mathbb{H}^{-1/2}(\Gamma). \tag{1.3}$$

A natural question is raied: why we investigate BVP (1.1) in the non-classical setting, when in the classical setting the solvability result is easily obtainable. Besides that this is an interesting mathematical problem, in many cases, for example in approximation methods, it is important to know a maximal smoothness of a solution. From the solvability results in non-classical setting it is possible to conclude smoothness property of a solution.

To formulate the appropriate main theorems of the present work we need the following definition.

Definition 1.2 *The BVP* (1.1) *in the setting* (1.2) *(the BVP* (1.10)*, the BVP* (1.11)*) is Fredholm if the homogeneous problem $f = g = 0$ ($f = h = 0$, respectively) has a finite number of solutions and the BVP has a solution if and only if the data f, g, h satisfy a finite number of orthogonality conditions.*

Next is the main theorem of the present chapter.

Theorem 1.3 *The BVP on a surface* (1.1) *in the non-classical setting* (1.2) *is Fredholm if and only if the following holds:*

i. If at $c_j \in \mathscr{M}_D$ collide the Dirichlet conditions, then

$$e^{i2\pi(s-1/p)} \sin^2 \pi(s - i\xi) + e^{-i2\pi s} \sin^2(\alpha_j - \pi)(1/p - s - 1 - i\xi) \neq 0 \quad \text{for all} \quad \xi \in \mathbb{R}. \tag{1.4}$$

ii. If at $c_j \in \mathscr{M}_N$ collide the Neumann conditions, then

$$e^{i2\pi(s-1/p)} \sin^2 \pi(s - i\xi) + e^{-i2\pi s} \sin^2(\alpha_j - \pi)(1/p - s - i\xi) \neq 0 \quad \text{for all} \quad \xi \in \mathbb{R}. \tag{1.5}$$

iii. If at $c_j \in \mathscr{M}_N$ collide the Dirichlet and Neumann conditions, then

$$e^{4\pi i/p} \sin^2 \pi(2/p - i\xi - s) + \cos^2[(\pi - \alpha_j)(2/p - i\xi - s)] \neq 0 \quad \text{for all} \quad \xi \in \mathbb{R}. \tag{1.6}$$

If conditions (1.4), (1.5) *and* (1.6) *hold (i.e. the BVP* (1.1), (1.2) *is Fredholm), the subset $(1/p, \infty) \times (1, \infty)$ of the Euclidean plane $\mathbb{R}^2$, where the pairs (s, p) range, decomposes into an infinite union $\mathscr{R}_o \cup \mathscr{R}_1 \cup \cdots$ of non-intersecting connected subsets of regular pairs, for which the BVP* (1.1) *is Fredholm in the setting* (1.2).

If the connected subset $\mathscr{R}_0$ contains the point $(1, 2)$ (i.e. $s = 1, p = 2$) then BVP (1.1) *is uniquely solvable in the setting* (1.2) *for all pairs $(s, p) \in \mathscr{R}_0$.*

The formulated theorem is proved at the end of the last § 4. Theorem is proved based on a local principle, which reduces the proof to the investigation of model problems: Dirichlet, Neumann and Mixed BVPs on a model domain, which is an angle of magnitude α. We will investigate model Dirichlet, Neumann and Mixed BVPs in § 3-5. Next we formulate what is a model domain and model BVPs.

To the set of knots $\mathscr{M}_\Gamma$ we add all those smoothness points on Γ where the Dirichlet and Neumann boundary conditions collide.

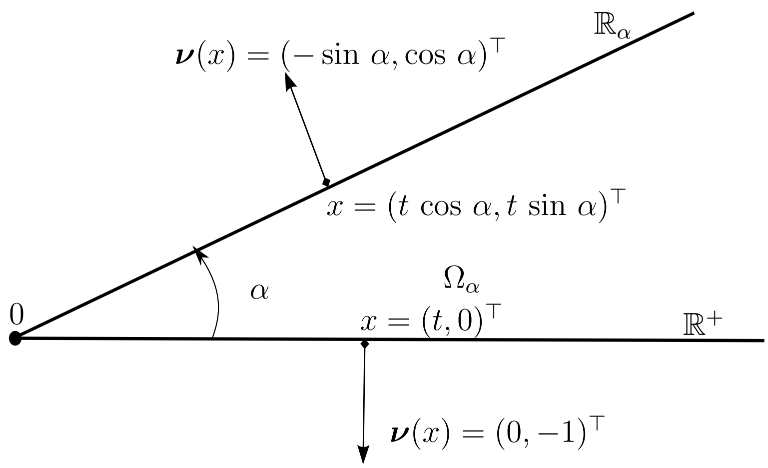

Fig. 2

With the BVP (1.1) we associate at all knots $c_j \in \mathscr{M}_\Gamma$ the model domain Ω_{α_j} (see Fig. 2), which represents the angle of magnitude α_j and the corresponding boundary is a model curve:

$$\begin{aligned} &\Gamma_{\alpha_j} := \partial\Omega_{\alpha_j} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_j}, \qquad \mathbb{R}^+ = [0, \infty), \\ &\mathbb{R}_{\alpha_j} := \{e^{i\alpha_j t} = (t\cos\,\alpha_j, t\sin\,\alpha_j)\ :\ t \in \mathbb{R}^+\}. \end{aligned} \tag{1.7}$$

$\boldsymbol{\nu}$ denotes the unit normal vector field on the boundary $\Gamma_{\alpha_j} := \partial\Omega_{\alpha_j} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_j}$

$$\boldsymbol{\nu}(t) = (\nu_1(t), \nu_2(t))^\top = \begin{cases} (0, -1)^\top & \text{for} \quad t \in \mathbb{R}^+ \\ (-\sin\,\alpha_j, \cos\,\alpha_j) & \text{for} \quad t \in \mathbb{R}_{\alpha_j} \end{cases} \tag{1.8}$$

and $\partial_{\boldsymbol{\nu}} := \nu_1\partial_1 + \nu_2\partial_2$-the corresponding normal derivative.

The set of knots $\mathscr{M}_\Gamma$ we divide in three subsets: $\mathscr{M}_\Gamma = \mathscr{M}_D \cup \mathscr{M}_N \cup \mathscr{M}_{DN}$, where $\mathscr{M}_D$ consists of all knots c_j where Dirichlet conditions collide and $\alpha_j \neq \pi$; $\mathscr{M}_N$ consists of all knots c_j where Neumenn conditions collide and $\alpha_j \neq \pi$; $\mathscr{M}_{DN}$ consists of all knots c_j where Dirichlet and Neumenn conditions collide and here α_j can be arbitrary $0 < \alpha_j < 2\pi$.

Consider the following model BVPs, associated with the BVP (1.1).

The Model Dirichlet BVP

$$\begin{cases} \Delta u(t) = f(t), & t \in \Omega_{\alpha_j}, \\ u^+(s) = g(s), & \text{on} \quad \Gamma_{\alpha_j} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_j}, \end{cases} \tag{1.9}$$

$$u \in \mathbb{H}^s_p(\Omega_{\alpha_j}), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_j}), \quad g \in \mathbb{H}^{s-1/p}_p(\Gamma_{\alpha_j}), \quad 1 < p < \infty, \quad s > \frac{1}{p}$$

at a knot $c_j \in \mathscr{M}_D$ (where Dirichlet conditions collide).

The Model Neumann BVP

$$\begin{cases} \Delta u(t) = f(t), & t \in \Omega_{\alpha_j}, \\ (\partial_{\boldsymbol{\nu}} u)^+(s) = h(s), & \text{on} \quad \Gamma_{\alpha_j} = \mathbb{R}^+ \cup \mathbb{R}_{\alpha_j}, \end{cases} \tag{1.10}$$

$$u \in \mathbb{H}^s_p(\Omega_{\alpha_j}), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_j}), \quad h \in \mathbb{H}^{s-1-1/p}_p(\Gamma_{\alpha_j}), \quad 1 < p < \infty, \quad s > \frac{1}{p}$$

at a knot $c_j \in \mathscr{M}_N$ (where the Neumann conditions collide).

The model Mixed BVP

$$\begin{cases} \Delta u(t) = f(t), & t \in \Omega_{\alpha_j}, \\ u^+(s) = g(s), & \text{on} \quad \mathbb{R}^+, \\ (\partial_{\boldsymbol{\nu}} u)^+(s) = h(s), & \text{on} \quad \mathbb{R}_{\alpha_j}, \end{cases} \tag{1.11}$$

$$u \in \mathbb{H}^s_p(\Omega_{\alpha_j}), \quad f \in \widetilde{\mathbb{H}}^{s-2}_{p,0}(\Omega_{\alpha_j}), \quad g \in \mathbb{H}^{s-1/p}_p(\mathbb{R}^+), \quad h \in \mathbb{H}^{s-1-1/p}_p(\mathbb{R}_{\alpha_j}),$$
$$1 < p < \infty, \quad s > \frac{1}{p}$$

at a knot $c_j \in \mathscr{M}_{DN}$ (where the Dirichlet and Neumann conditions collide).

Theorem 1.4 (Local Principle, [BDKT13]) *The initial mixed boundary value problem* (1.1) *in the non-classical setting is Fredholm if and only if the boundary value problems* (1.9)*,* (1.10) *and* (1.11) *are Fredholm in the non-classical setting for all knots $c_j \in \mathscr{M}_\Gamma$.*

As a particular case of Theorem 1.1 we get the following.

Corollary 1.5 *The boundary value problems* (1.9)*,* (1.10) *and* (1.11) *have unique solutions in the classical weak setting $p = 2$, $s = 1$.*

Results for the model mixed BVP (1.11) was obtained in [DT18, DT19] (Fredholm criteria, the unique solability). Similar results for the BVPs with mixed impedance conditions are proved in [**?**].

The purpose of the present paper is to write the criteria of the solvability of the BVP on a surface (1.1) in the non-classical setting (1.2). For this we need to study first model Dirichlet (1.9) and the model Neumann (1.10) boundary value problems.

Investigations of the boundary integral equations run into difficulties due to the absence of results on Mellin convolution equations in the Bessel potential space setting $\varphi \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $f \in \mathbb{H}^s_p(\mathbb{R}^+)$. In the recent papers [**?**] L. Castro & D. Kapanadze reduce BVPs (1.9) and (1.10) in the $\mathbb{H}^{1+\varepsilon}(\Omega_\alpha)$ space settings to equivalent Wiener-Hopf $\pm$ Hankel operators, by manipulating with the even and odd extensions and the reflection operators. The obtained equations were investigated in $\mathbb{L}_2(\mathbb{R}^+)$ space and, in the last paper [**?**], in the special potential space, defined by Mellin transforms.

In a series of papers [?] P.A. Krutitskii investigated Boundary value problems for the Helmholtz equation in a planar 2D domain Ω outer to a finite number of domains and cuts, with Dirichlet, Neumann, mixed and impedance conditions on the boundary and faces of cuts. Unique solvability was proved in classical strong setting $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ by reducing the problems to boundary Fredholm integral equations. Singularities at the tips of cuts were described as well.

In the present paper we apply the potential method and reduce investigation of BVPs (1.9) and (1.10) to the investigation of simpler equivalent systems.

The paper is organized as follows.First we recall auxiliary materials on potential operators and representation of solutions to BVPs in model domains (see In §,2) and then on Mellin convolution operators in the Bessel potential spaces (In §,3). Then we prove criteria of Fredholmi proprty and unique solvability of model Dirichlet problem (see §, 4) nad of model Neumann problem (see §, 5). In conclusion, at the end of the last § 4, we prove the main Theorem 1.1.

2 POTENTIAL OPERATORS

It is well known that the Laplace operator Δ has the Fundamental solution $\mathscr{K}_\Delta$

$$\mathscr{K}_\Delta(x) := \frac{1}{2\pi}\ln|x|, \qquad \Delta\mathscr{K}_\Delta(x) = \delta(x), \qquad x \in \mathbb{R}^2,$$

which is used to define the standard double layer $\boldsymbol{W}_\Delta$, the single layer $\boldsymbol{V}_\Delta$ and the Newton $\boldsymbol{N}_\Delta$ potentials on the angle Ω_α:

$$\begin{aligned}
\boldsymbol{V}_\Delta\varphi(x) &:= \frac{1}{2\pi}\int_{\Gamma_\alpha} \ln|x-\tau|\varphi(\tau)d\sigma,\\
\boldsymbol{W}_\Delta\varphi(x) &:= \frac{1}{2\pi}\int_{\Gamma_\alpha} \partial_{\boldsymbol{\nu}(\tau)}\ln|x-\tau|\varphi(\tau)d\sigma,\\
\boldsymbol{N}_\Delta\varphi(x) &:= \frac{1}{2\pi}\int_{\Omega_\alpha} \ln|x-y|\varphi(y)\,dy, \qquad x \in \Omega_\alpha.
\end{aligned} \tag{2.1}$$

For the standard properties of these potentials we refer to [Du01].

Any solution $u \in \mathbb{H}^s_p(\Omega_\alpha)$ to the BVP (1.9) (and also of the BVP (1.10)) is represented as follows

$$u(x) = \boldsymbol{N}_\Delta f(x) + \boldsymbol{W}_\Delta u^+(x) - \boldsymbol{V}_\Delta[\partial_{\boldsymbol{\nu}} u]^+(x) \qquad x \in \Omega_\alpha \tag{2.2}$$

(see [?, Du01]), where u^+ and $[\partial_{\boldsymbol{\nu}} u]^+$ are the Dirichlet and the Neumann traces of the solution u on the boundary Γ_α (cf. Fig. 2, Formulae (1.7) and (1.8)).

Let us recall the Plemelji formulae

$$\begin{aligned}
&(\boldsymbol{W}_\Delta\varphi)^\pm(t) = \pm\frac{1}{2}\varphi(t) + \boldsymbol{W}_{\Delta,0}\varphi(t), \quad (\partial_{\boldsymbol{\nu}_\Delta(t)}\boldsymbol{V}_\Delta\psi)^\pm)(t) = \mp\frac{1}{2}\psi(t) + \boldsymbol{W}^*_{\Delta,0}\psi(t),\\
&(\partial_{\boldsymbol{\nu}_\Delta(t)}\boldsymbol{W}_\Delta\psi)^\pm(t) = \boldsymbol{V}_{\Delta,+1}\psi(t), \quad (\boldsymbol{V}_\Delta\varphi)^\pm(t) = \boldsymbol{V}_{\Delta,-1}\varphi(t) \qquad t \in \Gamma_\alpha := \partial\Omega_\alpha,
\end{aligned} \tag{2.3}$$

where the pseudodifferential operators (ΨDO)

$$\begin{aligned} \boldsymbol{V}_{\Delta,-1}\varphi(t) &:= \frac{1}{2\pi}\int_{\Gamma_\alpha} \ln|t-\tau|\varphi(\tau)d\sigma, \\ \boldsymbol{W}_{\Delta,0}\varphi(t) &:= \frac{1}{2\pi}\int_{\Gamma_\alpha} \partial_{\boldsymbol{\nu}(\tau)} \ln|t-\tau|\varphi(\tau)d\sigma, \\ \boldsymbol{W}^*_{\Delta,0}\varphi(t) &:= \frac{1}{2\pi}\int_{\Gamma_\alpha} \partial_{\boldsymbol{\nu}(t)} \ln|t-\tau|\varphi(\tau)d\sigma, \\ \boldsymbol{V}_{\Delta,+1}\varphi(t) &:= \frac{1}{2\pi}\int_{\Gamma_\alpha} \partial_{\boldsymbol{\nu}(t)}\partial_{\boldsymbol{\nu}(\tau)} \ln|t-\tau|\varphi(\tau)d\sigma, \qquad t\in\Gamma_\alpha \end{aligned} \tag{2.4}$$

of orders -1, 0, 0 and $+1$, are associated with the layer potentials of the Helmholtz equation. The operator $\boldsymbol{V}_{\Delta,-1}$ has weakly singular kernel and the integral exists in the Lebesgue sense, while the operators $\boldsymbol{W}_{\Delta,0}$ and $\boldsymbol{W}^*_{\Delta,0}$ have singular kernel of order -1 and the integrals exists in the Cauchy Mean Value sense. $\boldsymbol{V}_{\Delta,+1}$nis a hypersingular integral operator and it is interpreted in [DT18, § 1]. The standard mapping property is listed below (see [**?**, Du01, HW08] for details):

$$\begin{aligned} \boldsymbol{V}_{\Delta,-1} &: \mathbb{H}^s_p(\Gamma_\alpha) \longrightarrow \mathbb{H}^{s+1}_p(\Gamma_\alpha), \\ \boldsymbol{W}_{\Delta,0} &: \mathbb{H}^s_p(\Gamma_\alpha) \longrightarrow \mathbb{H}^{s}_p(\Gamma_\alpha), \\ \boldsymbol{W}^*_{\Delta,0} &: \mathbb{H}^s_p(\Gamma_\alpha) \longrightarrow \mathbb{H}^{s}_p(\Gamma_\alpha), \\ \boldsymbol{V}_{\Delta,+1} &: \mathbb{H}^s_p(\Gamma_\alpha) \longrightarrow \mathbb{H}^{s-1}_p(\Gamma_\alpha), \qquad s\in\mathbb{R}, \quad 1<p<\infty. \end{aligned} \tag{2.5}$$

Next we will write some pseudodifferential operators (PsDOs) in explicit form for the later use in §§ 3-5. For this consider the pull back operator $\boldsymbol{J}_\alpha\ :\ \mathbb{H}^s_p(\mathbb{R}_\alpha) \to \mathbb{H}^s_p(\mathbb{R}^+)$ and its inverse $\boldsymbol{J}^{-1}\ :\ \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}_\alpha)$ are defined as follows

$$\begin{aligned} \boldsymbol{J}_\alpha\varphi(t) &= \varphi(t\cos\alpha, t\sin\alpha), \qquad t\in\mathbb{R}^+, \\ \boldsymbol{J}^{-1}_\alpha\psi(x_1,x_2) &= \psi\left(\sqrt{x_1^2+x_2^2}\right), \qquad (x_1,x_2)^\top\in\mathbb{R}_\alpha. \end{aligned} \tag{2.6}$$

First let us consider the PsDOs $r_{\mathbb{R}^+}\boldsymbol{V}_{\Delta,+1}r_{\mathbb{R}_\alpha}$. By applying the equality

$$\partial_{\boldsymbol{\nu}(x)}\partial_{\boldsymbol{\nu}(y)}\ln|x-y| = -\partial^2_{\boldsymbol{\nu}(y)}\ln|x-y| = -\delta(x-y) + \partial^2_{\ell(y)}\ln|x-y|,$$

proved similarly to (5.7), we get:

$$\begin{aligned} \boldsymbol{V}_{\Delta,+1}\varphi(t) &:= \int_{\Gamma_\alpha} \partial_{\boldsymbol{\nu}(t)}\partial_{\boldsymbol{\nu}(\tau)} \ln|t-\tau|\varphi(\tau)d\sigma = -\varphi(t) + \int_{\Gamma_\alpha}\partial^2_{\ell(\tau)}\ln|t-\tau|\varphi(\tau)d\sigma \\ &= -\varphi(t) - \int_{\Gamma_\alpha}\partial_{\ell(\tau)}\ln|t-\tau|\partial_{\ell(\tau)}\varphi(\tau)d\sigma, \quad t\in\Gamma_\alpha. \end{aligned}$$

By using the parametrization $x=(x_1,x_2)^\top=(t,0)^\top$ of $\mathbb{R}^+$, the parametrization $y=(y_1,y_2)^\top=(\tau\cos\alpha, \tau\sin\alpha)^\top$ of $\mathbb{R}_\alpha$, recalling that $\mathbb{R}_\alpha$ is oriented from $-\infty$ to 0, using the equality

$$\partial_{\boldsymbol{\nu}(x)}=\begin{cases}-\lim\limits_{(x_1,x_2)\to t,0}\partial_{x_2} & \text{for } x \text{ on } \mathbb{R}^+,\\ \lim\limits_{(x_1,x_2)\to(t\cos\,\alpha,t\sin\,\alpha)}[-\sin\,\alpha\,\partial_{x_1}+\cos\,\alpha\,\partial_{x_2}] & \text{for } x \text{ on } \mathbb{R}_\alpha.\end{cases}\tag{2.7}$$

snd the equalities

$$\partial_{\ell(y)}=-\cos\,\alpha\,\partial_{y_1}-\sin\,\alpha\,\partial_{y_2},\quad \ln|x-y|=\frac{1}{2}\ln\left[(x_1-y_1)^2+(x_2-y_2)^2\right]\tag{2.8}$$

for $t\in\mathbb{R}^+$, $y\in\Gamma_\alpha$, we proceed as follows:

$$\begin{aligned}&\boldsymbol{W}_{\Delta,0}\varphi(x)=\frac{1}{2\pi}\int_{\Gamma_\alpha}\partial_{\boldsymbol{\nu}(y)}\ln\,|(x_1,x_2)-(y_1,y_2)|\varphi(y)d\sigma\\&=-\frac{1}{2\pi}\int_0^\infty\partial_{y_2}\ln\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\Big|_{y=(\tau,0)}\varphi(\tau)d\tau\\&+\frac{1}{2\pi}\int_0^\infty[-\sin\,\alpha\,\partial_{y_1}+\cos\,\alpha\,\partial_{y_2}]\ln\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\Big|_{y=(\tau\,\cos\,\alpha,\tau\sin\,\alpha)}\varphi_1(\tau)d\tau\\&=\frac{1}{2\pi}\int_0^\infty\frac{x_2\varphi(\tau)d\tau}{(x_1-\tau)^2+x_2^2}+\frac{1}{2\pi}\int_0^\infty\frac{(x_1\sin\,\alpha-x_2\cos\,\alpha)\varphi_1(\tau)d\tau}{(x_1-\tau\cos\,\alpha)^2+(x_2-\tau\sin\,\alpha)^2}.\end{aligned}\tag{2.9}$$

Here $\varphi_1(\tau):=\varphi(\tau\,\cos\,\alpha,\tau\,\sin\,\alpha)$ because in the second summand we have changed first the orientation ($\mathbb{R}_\alpha$ is oriented from ∞ to 0), than variables and have reduced integration on $\mathbb{R}_\alpha$ to integration on $\mathbb{R}^+$. Note, that the first integral summand in (2.9) vanishes if we restrict variable $x=(x_1,x_2)^\top=(t,0)^\top$ to the axes $\mathbb{R}^+$ (because $x_2=0$) and, vice versa, the second integral summand in (2.9) vanishes if we restrict variable $x=(x_1,x_2)^\top=(t\cos\,\alpha,t\sin\,\alpha)^\top$ to the axes $\mathbb{R}_\alpha$ (because then $x_1\sin\,\alpha-x_2\cos\,\alpha=t\,\sin\,\alpha\,\cos\,\alpha-t\,\sin\,\alpha\,\cos\,\alpha=0$) and the integrals differ only in sign:

$$\begin{aligned}r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}\varphi(t)=-\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}\varphi(t)&=\frac{\sin\,\alpha}{2\pi}\int_0^\infty\frac{t\,\varphi_1(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\,\alpha}\\&=\frac{1}{4\pi i}\int_0^\infty\left[\frac{e^{i\alpha}}{t-e^{i\alpha}\tau}-\frac{e^{-i\alpha}}{t-e^{-i\alpha}\tau}\right]\varphi_1(\tau)\,d\tau\\&=\frac{1}{4i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}}-e^{-i\alpha}\boldsymbol{K}_{e^{-i\alpha}}\right]\varphi_1(t),\qquad\varphi_1(t):=(\boldsymbol{J}_\alpha\varphi)(t),\qquad t\in\mathbb{R}^+,\end{aligned}\tag{2.10}$$

where $\boldsymbol{J}_\alpha$ is the pull back operator (see (2.6)) and $r_{\mathbb{R}^+}$ and $r_{\mathbb{R}_\alpha}$ are the restriction operators to the spaces on the corresponding subsets $\mathbb{R}^+$ and $\mathbb{R}_\alpha$.

By a similar calculation for the dual operator $\boldsymbol{W}^*_{\Delta,0}$ we get the following:

$$\begin{aligned}r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0}r_{\mathbb{R}^+}\varphi(t)=-\boldsymbol{J}_\alpha r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0}r_{\mathbb{R}^+}\varphi(t)&=\frac{\sin\,\alpha}{2\pi}\int_0^\infty\frac{\tau\,\varphi(\tau)\,d\tau}{t^2+\tau^2-2t\tau\cos\,\alpha}\\&=\frac{1}{4\pi i}\int_0^\infty\left[\frac{1}{t-e^{i\alpha}\tau}-\frac{1}{t-e^{-i\alpha}\tau}\right]\varphi(\tau)\,d\tau\end{aligned}$$

$$= \frac{1}{4i}\left[\boldsymbol{K}_{e^{i\alpha}} - \boldsymbol{K}_{e^{-i\alpha}}\right]\varphi(t), \qquad t \in \mathbb{R}^+. \tag{2.11}$$

For the singular integral operators $\boldsymbol{W}_{\Delta,0}$ and its dual $\boldsymbol{W}^*_{\Delta,0}$ we have proved the following (see (2.10), (2.11)):

$$r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}\varphi(t) = -\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}\varphi(t) = \frac{1}{4i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}} - e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\varphi(t), \tag{2.12a}$$

$$r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0}r_{\mathbb{R}^+}\varphi(t) = -\boldsymbol{J}_\alpha r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0}r_{\mathbb{R}^+}\varphi(t) = \frac{1}{4i}\left[\boldsymbol{K}_{e^{i\alpha}} - \boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\varphi(t), \quad t \in \mathbb{R}^+, \tag{2.12b}$$

$$r_{\mathbb{R}^+}\boldsymbol{W}_\Delta r_{\mathbb{R}^+} = r_{\mathbb{R}_\alpha}\boldsymbol{W}_\Delta r_{\mathbb{R}_\alpha} = r_{\mathbb{R}^+}\boldsymbol{W}^*_\Delta r_{\mathbb{R}^+} = r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_\Delta r_{\mathbb{R}_\alpha} = 0, \tag{2.12c}$$

where $\boldsymbol{J}_\alpha$ is the pull back operator (see (2.6)) and $r_{\mathbb{R}^+}$ and $r_{\mathbb{R}_\alpha}$ are the restriction operators to the spaces on the corresponding subsets $\mathbb{R}^+$ and $\mathbb{R}_\alpha$.

By using the equality (2.7) we proceed as follows

$$\begin{aligned}
\partial_{\boldsymbol{\nu}(t)}\,\partial_{\boldsymbol{\nu}(y)}\mathscr{K}_\Delta(t-y) &= \partial_{\boldsymbol{\nu}(x)}\partial_{\boldsymbol{\nu}(y)}\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= -\partial_{x_2}(-\sin\alpha\,\partial_{y_1} + \cos\alpha\,\partial_{y_2})\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \left\{-\sin\alpha\,\partial_{y_1}\partial_{y_2} + \cos\alpha\,\partial^2_{y_2}\right\}\mathscr{K}_\Delta(x-y)\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \left[\cos\alpha\Delta\mathscr{K}_\Delta(x-y) - \partial_{y_1}\left\{\cos\alpha\,\partial_{y_1} + \sin\alpha\,\partial_{y_2}\right\}\mathscr{K}_\Delta(x-y)\right]\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \left[\cos\alpha\delta(x-y) + \partial_{y_1}\partial_{\ell(y)}\mathscr{K}_\Delta(x-y)\right]\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \left[\cos\alpha\,\delta(0) + \frac{1}{4\pi}\partial_{\ell(y)}\partial_{y_1}\ln\left[(x_1-y_1)^2 + (x_2-y_2)^2\right]\right]\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= \left[\cos\alpha\,\delta(0) - \frac{1}{2\pi}\partial_{\ell(y)}\frac{x_1-y_1}{2\pi[(x_1-y_1)^2 + (x_2-y_2)^2]}\right]\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}
\end{aligned}$$

Now integrating by parts (see (**??**)) we continue as follows:

$$\begin{aligned}
r_{\mathbb{R}^+}\boldsymbol{V}_{\Delta,+1}r_{\mathbb{R}_\alpha}v(t) &= \frac{1}{2\pi}r_{\mathbb{R}^+}\int_{\mathbb{R}_\alpha}\frac{(x_1-y_1)\partial_{\ell(y)}v(y)d\sigma}{(x_1-y_1)^2 + (x_2-y_2)^2}\Big|_{\substack{x=(t,0)\\ y=(\tau\cos\alpha,\tau\sin\alpha)}}\\
&= -\frac{1}{2\pi}\int_0^\infty\frac{t-\tau\cos\alpha}{t^2+\tau^2-2t\tau\,\cos\alpha}(\boldsymbol{J}_\alpha\partial_\ell v)(\tau)d\tau\\
&= -\frac{1}{4\pi}\int_0^\infty\left[\frac{1}{t-e^{i\alpha}\tau} + \frac{1}{t-e^{-i\alpha}\tau}\right](\boldsymbol{J}_\alpha\partial_\ell v)(\tau)d\tau,
\end{aligned}$$

$$= \frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right]\partial_\tau v_1(t), \qquad t \in \mathbb{R}^+, \tag{2.13}$$

since $(\boldsymbol{J}_\alpha \partial_\ell v)(\tau) = -(\partial_\tau v_1)(\tau)$, where $v_1 := \boldsymbol{J}_\alpha v$. $\boldsymbol{K}_c$ is the Mellin convolution operator (see [Du79, Du84b, Du86, Du82] and Chapterv 4, § 2):

$$\mathbf{K}^1_c\phi(t) := \frac{1}{\pi}\int\limits_0^\infty \frac{\phi(\tau)\,d\tau}{t - c\,\tau}, \qquad 0 < \arg\, c < 2\pi, \quad \phi \in \mathbb{L}_p(\mathbb{R}^+). \tag{2.14}$$

The formula

$$\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha} \boldsymbol{V}_{\Delta,+1} r_{\mathbb{R}^+} w(t) = -\frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right](\partial_\tau w)(t), \quad t \in \mathbb{R}^+ \tag{2.15}$$

is proved similarly.

Now we look to the singular integral operators $r_{\mathbb{R}_\alpha}\partial_\ell \boldsymbol{V}_{\Delta,-1} r_{\mathbb{R}^+}$ and $r_{\mathbb{R}^+}\partial_\ell \boldsymbol{V}_{\Delta,-1} r_{\mathbb{R}_\alpha}$. We proceed as in (2.13):

$$\begin{aligned}
\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\partial_\ell \boldsymbol{V}_{\Delta,-1} r_{\mathbb{R}^+} w(t) &= \frac{1}{2\pi}\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\int_{\mathbb{R}^+} \partial_{\ell(x)} \ln|x-y|\, w(y) d\sigma\Big|_{\substack{x=(t\cos\,\alpha, t\sin\,\alpha))\\ y=(\tau,0)}} \\
&= -\frac{1}{2\pi}\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\int_{\mathbb{R}^+} \frac{\cos\,\alpha(x_1-\tau) + x_2 \sin\,\alpha}{(x_1-\tau)^2 + x_2^2} w(\tau)\,d\tau\Big|_{x=(t\cos\,\alpha, t\sin\,\alpha)} \\
&= -\frac{1}{2\pi}\int_0^\infty \frac{\cos\,\alpha(t\cos\,\alpha - \tau) + t\,\sin^2\alpha}{(t\,\cos\alpha - \tau)^2 + t\sin^2\alpha} w(\tau) d\tau \\
&= -\frac{1}{2\pi}\int_0^\infty \frac{t - \tau\,\cos\alpha}{(t\,\cos\alpha - \tau)^2 + t^2\sin^2\alpha} w(\tau) d\tau \\
&= -\frac{1}{4\pi}\int_0^\infty \left[\frac{1}{t - e^{i\alpha}\tau} + \frac{1}{t - e^{-i\alpha}\tau}\right] w(\tau) d\tau, \\
&= -\frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right] w(t), \quad t \in \mathbb{R}^+.
\end{aligned} \tag{2.16}$$

The formulae

$$r_{\mathbb{R}^+}\partial_\ell \boldsymbol{V}_{\Delta,-1} r_{\mathbb{R}_\alpha} w(t) = \frac{1}{4}\left[\boldsymbol{K}_{e^{i\alpha}} + \boldsymbol{K}_{e^{-i\alpha}}\right]\boldsymbol{J}_\alpha w(\tau), \quad t \in \mathbb{R}^+ \tag{2.17}$$

is proved similarly.

3 MELLIN CONVOLUTION EQUATION IN BESSEL POTENTIAL SPACES

99 Let us recall from [DD16] results on the Fredholm properties of operators

$$\mathbf{A} := d_0 I + \sum_{j=1}^n d_j \mathbf{K}^1_{c_j} \;:\; \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+), \tag{3.1}$$

where $\mathbf{K}^1_{c_1}, \dots, \mathbf{K}^1_{c_n}$ are admissible Mellin convolution operators and $d_0, \dots, d_n$ are $m \times m$ constant matrix coefficients. $\widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$ and $\mathbb{H}^s_p(\mathbb{R}^+)$ are the spaces of m-vector functions.

To this end consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$, where (cf. Figure 3)

$$\Gamma_1 := \{+\infty\} \times \overline{\mathbb{R}}, \quad \Gamma_2^\pm := \overline{\mathbb{R}}^+ \times \{\pm\infty\}, \quad \Gamma_3 := \{0\} \times \overline{\mathbb{R}}.$$

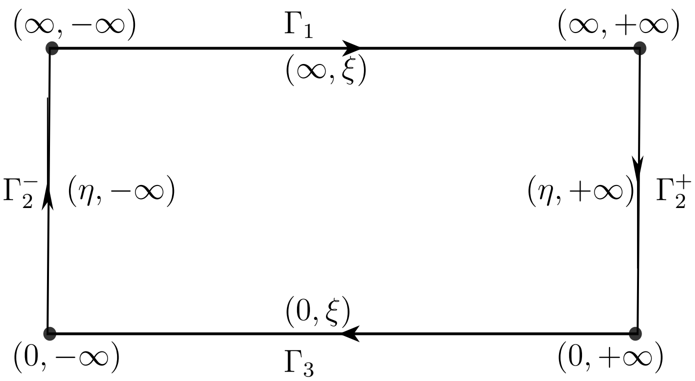

Fig. 3. The domain $\mathfrak{R}$ of definition of the symbol $\mathscr{A}^s_p(\omega)$.

According to [DD16, formulae (52)-(53d)] the symbol $\mathscr{A}^s_p(\omega)$ of the operator $\mathbf{A}$ is

$$\mathscr{A}^s_p(\omega) := d_0 \mathscr{I}^s_p(\omega) + \sum_{j=1}^{n} d_j \mathscr{K}^{1,s}_{c_j,p}(\omega), \tag{3.2}$$

where

$$\mathscr{I}^s_p(\omega) := \begin{cases} g^s_{-\gamma,\gamma,p}(\infty,\xi), & \omega = (\infty,\xi) \in \overline{\Gamma}_1, \\ \left(\dfrac{\pm\eta - \gamma}{\pm\eta + \gamma}\right)^s, & \omega = (\eta, \pm\infty) \in \Gamma_2^\pm, \\ e^{\pi s i}, & \omega = (0,\xi) \in \overline{\Gamma}_3, \qquad \xi, \eta \in \mathbb{R}, \end{cases} \tag{3.3a}$$

$$g^s_{-\gamma,\gamma,p}(\infty,\xi) := \frac{e^{2\pi s i} + 1}{2} + \frac{e^{2\pi s i} - 1}{2i} \cot \pi\left(\frac{1}{p} - i\xi\right) = e^{\pi s i} \frac{\sin \pi\left(\frac{1}{p} + s - i\xi\right)}{\sin \pi\left(\frac{1}{p} - i\xi\right)}, \quad \xi \in \mathbb{R},$$

$$\mathscr{K}^{1,s}_{c,p}(\omega) := \begin{cases} -\dfrac{e^{-i\pi\left(\frac{1}{p} - i\xi\right)} c^{\frac{1}{p} - i\xi - s - 1}}{\sin \pi(\frac{1}{p} - i\xi)}, & \omega = (\infty,\xi) \in \overline{\Gamma}_1, \\ 0, & \omega = \eta, \pm\infty) \in \Gamma_2^\pm, \\ -\dfrac{e^{-i\pi\left(\frac{1}{p} + s - i\xi\right)} c^{\frac{1}{p} - i\xi - s - 1}}{\sin \pi(\frac{1}{p} - i\xi)}, & \omega = (0,\xi) \in \overline{\Gamma}_3, \end{cases} \tag{3.3b}$$

where

$$0 < \arg c < 2\pi, \qquad c^{-s} = |c|^{-s} e^{i(2\pi - \arg c)s}, \quad c^{\gamma} = |c|^{\gamma} e^{i\gamma \arg c}. \tag{3.3c}$$

The function $\det \mathscr{A}_p^s(\omega)$ is continuous on the rectangle $\mathfrak{R}$. The statement is easy to verify analyzing the symbols in (3.2), (3.3a)-(3.3b) and taking into account that

$$\mathscr{I}_p^s(-\infty,-\infty)=1, \quad \mathscr{I}_p^s(0,-\infty)=\mathscr{I}_p^s(0,+\infty)=e^{\pi s i}, \quad \mathscr{I}_p^s(+\infty,+\infty)=e^{2\pi s i},$$
$$\mathscr{K}^{1,s}_{-1,p}(-\infty,-\infty)=\mathscr{K}^{1,s}_{-1,p}(0,-\infty)=\mathscr{K}^{1,s}_{-1,p}(0,+\infty)=\mathscr{K}^{1,s}_{-1,p}(+\infty,+\infty)=0,$$
$$g^s_{-\gamma,\gamma,p}(\infty,-\infty)=1, \qquad g^s_{-\gamma,\gamma,p}(\infty,+\infty)=e^{2\pi s i}.$$

Therefore, the image of the function $\det \mathscr{A}_p^s(\omega)$ is a closed curve in the complex plane and, if the symbol is elliptic

$$\inf_{\omega\in\mathfrak{R}} \left|\det \mathscr{A}_p^s(\omega)\right| > 0,$$

the increment of the argument $(1/2\pi)\arg \mathscr{A}_p^s(\omega)$ when ω ranges through $\mathfrak{R}$ in the direction of orientation, is an integer. It is called the winding number or the index of the curve $\Gamma := \{z\in\mathbb{C} : z = \det \mathscr{A}_p(\omega),\ \omega\in\mathfrak{R}\}$ and is denoted by $\operatorname{ind}\det \mathscr{A}_p^s$.

Propositions 3.1-3.3, exposed below, are well known and will be applied in the next section in the proof of the main theorems.

Proposition 3.1 ([Du15] and Theorem 5.4, [DD16]) *Let* $1 < p < \infty$, $s \in \mathbb{R}$. *The operator*

$$\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+) \tag{3.4}$$

defined in (3.1) *is Fredholm if and only if its symbol* $\mathscr{A}_p^s(\omega)$ *defined in* (3.2), (3.3a)– (3.3b), *is elliptic. If* $\mathbf{A}$ *is Fredholm, then*

$$\operatorname{Ind}\mathbf{A} = -\operatorname{inddet}\mathscr{A}_p^s.$$

The operator $\mathbf{A}$ *in* (3.4) *is locally invertible at* 0 *if and only if it is globally invertible.*

The operator $\mathbf{A}$ *in* (3.4) *is locally invertible at* 0 *if and only if its symbol* $\mathscr{A}_p^s(\omega)$ *is elliptic on the set* Γ_1 *only,* $\inf_{\omega\in\Gamma_1}\left|\det \mathscr{A}_p^s(\omega)\right| > 0$.

Proposition 3.2 ([Du15], Corollary 6.3) *Let* $1 < p < \infty$, $s \in \mathbb{R}$ *and let* $\mathbf{A}$ *be defined by* (3.1). *If the operator* $\mathbf{A} : \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$ *is Fredholm (is invertible) for all* $s \in (s_0, s_1)$ *and* $p \in (p_0, p_1)$, *where* $-\infty < s_0 < s_1 < \infty$, $1 < p_o < p_1 < \infty$, *then* $\boldsymbol{A}$ *is Fredholm (is invertible, respectively) in the Sobolev-Slobodečkii space setting*

$$\mathbf{A} \ : \ \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{W}^s_p(\mathbb{R}^+), \qquad \textit{for all} \quad s\in(s_0,s_1) \quad \textit{and} \quad p\in(p_0,p_1)$$

and has the same index

$$\operatorname{Ind}\mathbf{A} = -\operatorname{ind}\det \mathscr{A}_p^s.$$

Proposition 3.3 ([?, ?]) *Let two pairs of parameter-dependent Banach spaces* $\mathfrak{B}^s_1$ *and* $\mathfrak{B}^s_2$*,* $s_1 < s < s_2$*, have intersections* $\mathfrak{B}^{s'}_j \cap \mathfrak{B}^{s''}_j$ *dense in* $\mathfrak{B}^{s'}_j$ *and in* $\mathfrak{B}^{s''}_j$ *for all* $j = 1, 2$*,* $s', s'' \in (s_1, s_2)$*.*

If a linear bounded operator $A : \mathfrak{B}^s_1 \to \mathfrak{B}^s_2$ *is Fredholm for all* $s \in (s_1, s_2)$*, it has the same kernel and co-kernel for all values of this parameter* $s \in (s_1, s_2)$*.*

In particular, if $A : \mathfrak{B}^s_1 \to \mathfrak{B}^s_2$ *is Fredholm for all* $s \in (s_1, s_2)$ *and is invertible for only one value* $s_0 \in (s_1, s_2)$*, it is invertible for all values of this parameter* $s \in (s_1, s_2)$*.*

4 MODEL DIRICHLET BVP

In the present section we derive an equivalent boundary integral equation for the model Dirichlet problem (1.9) and investigate it. For this we need the following auxiliary result.

Let $C^s_0(\Gamma_\alpha)$ denote the set of Hölder continuous functions with with exponent s and compact supports. It is well known that $C^s_0(\Gamma_\alpha)$ is a dense subset of $\mathbb{H}^s_p(\Gamma_\alpha)$ for $0 < s < 1 + 1/p$.

Lemma 4.1 *Let* $1 < p < \infty$*,* $-1 - \dfrac{1}{p} < s < 1 + \dfrac{1}{p}$*,* $g_0 \in C^s_0(\Gamma_\alpha)$*,* $g_0(0) = 1$*, is a fixed function. Let us consider the linear functional*

$$F_0(\varphi) := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{\Gamma_{\alpha,\varepsilon}} \psi(\tau)d\sigma, \qquad \psi \in \mathbb{H}^s_p(\Gamma_\alpha),$$

where $\Gamma_{\alpha,\varepsilon}$ *is the intersection of* Γ_α *with the circle of radius* ε *centered at the vertex* $0 \in \Gamma_\alpha$*.*

Then for arbitrary $\varphi \in \mathbb{H}^s_p(\Gamma_\alpha)$ *and* $\psi \in \mathbb{W}^s_p(\Gamma_\alpha)$ *the following representations hold:*

$$\begin{aligned} &\varphi = F_0(\varphi)g_0 + \varphi_+ + \varphi_\alpha, \qquad \varphi_+ \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+), \quad \varphi_\alpha \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}_\alpha), \\ &\psi = F_0(\psi)g_0 + \psi_+ + \psi_\alpha, \qquad \psi_+ \in \widetilde{\mathbb{W}}^s_p(\mathbb{R}^+), \quad \psi_\alpha \in \widetilde{\mathbb{W}}^s_p(\mathbb{R}_\alpha), \qquad \text{(4.1)span} \\ &F_0(\varphi_+) = F_0(\varphi_\alpha) = F_0(\psi_+) = F_0(\psi_\alpha) = 0. \end{aligned}$$

Proof: Easy to check that for $\varphi \in C^s_0(\Gamma_\alpha)$ holds $F_0(\varphi) = \varphi(0)$ and, since $g_0(0) = 1$, we get $\varphi_+(0) = 0$, $\varphi_\alpha(0) = 0$. The inclusions $\varphi_+ \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+)$, $\varphi_\alpha \in \widetilde{\mathbb{H}}^s_p(\mathbb{R}_\alpha)$ follow automatically. Since the subset $C^s_0(\Gamma_\alpha)$ is dense in $\mathbb{H}^s_p(\Gamma_\alpha)$ (also in $\mathbb{W}^s_p(\Gamma_\alpha)$) and F_0 is a linear bounded functional in $\mathbb{H}^s_p(\Gamma_\alpha)$ (also in $\mathbb{W}^s_p(\Gamma_\alpha)$), the both representations in (4.1) remain valid for arbitrary function $\varphi \in \mathbb{H}^s_p(\Gamma_\alpha)$ (for arbitrary function $\psi \in \mathbb{W}^s_p(\Gamma_\alpha)$). ■

We remind that the Dirichlet trace $u^+ = g \in \mathbb{W}^{s-1/p}_p(\Gamma_\alpha)$ is a known function and let $(\partial_\nu u)^+ = \psi \in \mathbb{W}^{s-1-1/p}_p(\Gamma_\alpha)$ denote the unknown Neuman's trace. Then the representation formula (2.2) for a solution to the Dirichlet BVP (1.9) has the following form

$$u = \boldsymbol{N}_\Delta f + \boldsymbol{W}_\Delta g - \boldsymbol{V}_\Delta \psi. \tag{4.2}$$

By applying the Plemelji Formulae (2.3) to (4.2) we get

$$(\partial_\nu u)^+ = \psi = (\partial_\nu \boldsymbol{N}_\Delta f)^+ + \boldsymbol{V}_{\Delta,+1} g + \frac{1}{2}\psi - \boldsymbol{W}^*_{\Delta,0}\psi, \qquad \psi \in \Gamma_\alpha$$

and rewrite the obtained equality as follows:

$$\begin{gathered}\frac{1}{2}\psi + \boldsymbol{W}^*_{\Delta,0}\psi = G,\\ G := (\partial_\nu \boldsymbol{N}_\Delta f)^+ + \boldsymbol{V}_{\Delta,+1} g, \qquad \psi,\ G \in \mathbb{W}^{s-1-1/p}_p(\Gamma_\alpha).\end{gathered} \tag{4.3}$$

Since $I = r_{\mathbb{R}^+} + r_{\mathbb{R}_\alpha}$, applying the equalities (2.12c) we rewrite the equation (4.3) as follows:

$$\begin{gathered}\frac{1}{2}\psi + r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}_\alpha}\psi + r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}^+}\psi = G,\\ G, \psi \in \mathbb{W}^{s-1-1/p}_p(\Gamma_\alpha).\end{gathered} \tag{4.4}$$

Now we recall the representation (2.12b), restrict equation (4.4) to $\mathbb{R}^+$ by applying $r_{\mathbb{R}^+}$, which gives us the first equation in (4.5) below. Then restrict equation (4.4) to $\mathbb{R}_\alpha$ and apply the pull back operator $\boldsymbol{J}_\alpha$ and its inverse (see (2.6)) and get the second equation in (4.5). Thus, we get the system of two equations on the half axes with two unknown functions:

$$\left\{\begin{aligned} &\frac{1}{2}\psi_1 + (r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}_\alpha}\boldsymbol{J}^{-1}_\alpha)\psi_2 + F_0(\psi)g_2 = G_1,\\ &\frac{1}{2}\psi_2 + (\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}^+})\psi_1 + F_0(\psi)g_1 = G_2,\end{aligned}\right. \tag{4.5}$$

$$\begin{gathered} g_1 := r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}_\alpha} g_0, \quad g_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}^+} g_0,\\ \psi_1 := r_{\mathbb{R}^+}\psi, \quad \psi_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\psi, \quad G_1 := r_{\mathbb{R}^+}G, \quad G_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha} G,\\ \psi_1, \psi_2, \in \widetilde{\mathbb{W}}^{s-1-1/p}_p(\mathbb{R}^+), \quad g_1, g_2, G_1, G_2 \in \mathbb{W}^{s-1-1/p}_p(\mathbb{R}^+).\end{gathered} \tag{4.6}$$

Since one dimentional operator $F_0(\cdot)$ does not influence Fredholm property of the system (4.5), the system

$$\left\{\begin{aligned} &\frac{1}{2}\psi_1 + (r_{\mathbb{R}^+}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}_\alpha}\boldsymbol{J}^{-1}_\alpha)\psi_2 = G_1,\\ &\frac{1}{2}\psi_2 + (\boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}^*_{\Delta,0} r_{\mathbb{R}^+})\psi_1 = G_2,\end{aligned}\right. \tag{4.7}$$

$$\psi_1, \psi_2, \in \widetilde{\mathbb{W}}^{s-1-1/p}_p(\mathbb{R}^+), \quad G_1, G_2 \in \mathbb{W}^{s-1-1/p}_p(\mathbb{R}^+)$$

is Fredholm-equivalent to the system (4.6).

Due to the formula (2.12b) the system (4.7) of boundary integral equation coincides with the following system of integral equations of Mellin type:

$$\left\{\begin{aligned} &\psi_1 - \frac{1}{2i}\left[\boldsymbol{K}^1_{e^{i\alpha}} - \boldsymbol{K}^1_{e^{i(2\pi-\alpha)}}\right]\psi_2 = G_1,\\ &\psi_2 + \frac{1}{2i}\left[\boldsymbol{K}^1_{e^{i\alpha}} - \boldsymbol{K}^1_{e^{i(2\pi-\alpha)}}\right]\psi_1 = G_2,\end{aligned}\right. \tag{4.8}$$

$$\psi_1, \psi_2, \in \widetilde{\mathbb{W}}^{s-1-1/p}_p(\mathbb{R}^+), \quad G_1, G_2 \in \mathbb{W}^{s-1-1/p}_p(\mathbb{R}^+).$$

Theorem 4.2 *Let* $1 < p < \infty$, $\dfrac{1}{p} < s < 1 + \dfrac{1}{p}$.

The model Dirichlet boundary value problem in the non-classical setting (1.9) *is Freholm if and only if the system of boundary integral equation* (4.8) *is Fredholm.*

Now we can prove the main teorem of the present section.

Theorem 4.3 *The Model Dirichlet BVP in the non-classical setting* (1.9) *is Fredholm (and the system of boundary integral equation* (4.8) *is Fredholm) if and only if the following holds:*

$$e^{i2\pi(s-1/p)}\sin^2\pi(s-i\xi) + e^{-i2\pi s}\sin^2(\alpha-\pi)(1/p-s-1-i\xi) \neq 0 \quad \forall\, \xi \in \mathbb{R}. \tag{4.9}$$

If the condition (4.9) *holds, the semi-strip* $(1/p, \infty) \times (0, 1)$ *of the Euclidean plane* $\mathbb{R}^2$, *where the pairs* $(s, 1/p)$ *range, decomposes into an infinite union* $\mathscr{R}_0 \cup \mathscr{R}_1 \cup \cdots$ *of non-intersecting connected subsets of regular pairs, for which the BVP* (1.9) *is Fredholm.*

If the point $(1, 1/2)$ *(i.e.* $s = 1, p = 2$*) belongs to the connected subset* $\mathscr{R}_0$*, then BVP* (1.9) *is uniquely solvable for all pairs* $(s, 1/p) \in \mathscr{R}_0$.

The same unique solvability holds for the system of integral equations (4.5).

Proof: Let us investigate the Fredholm properties of the system (4.8). An equivalent task is to study the Fredholm property of the corresponding operator ■

$$\boldsymbol{D}_\alpha = I - \frac{1}{2i}d[\boldsymbol{K}^1_{e^{i\alpha}} - \boldsymbol{K}^1_{e^{i(2\pi-\alpha)}}] \;:\; \widetilde{\mathbb{W}}^{s-1-1/p}_p(\mathbb{R}^+) \to \mathbb{W}^{s-1-1/p}_p(\mathbb{R}^+). \tag{4.10a}$$

For this it suffices, due to Proposition 3.2, to prove the same theorem for the operator

$$\boldsymbol{D}_\alpha = I - \frac{1}{2i}d[\boldsymbol{K}^1_{e^{i\alpha}} - \boldsymbol{K}^1_{e^{i(2\pi-\alpha)}}] \;:\; \widetilde{\mathbb{H}}^{s-1-1/p}_p(\mathbb{R}^+) \to \mathbb{H}^{s-1-1/p}_p(\mathbb{R}^+). \tag{4.10b}$$

Here d is the 2×2 constant matrix

$$d := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.11}$$

The symbol of the operator $\boldsymbol{D}_\alpha$ in (4.10b) on the set $\overline{\Gamma}_1$, according to the formulae (3.3a)-(3.3c), reads:

$$\mathscr{D}^{s-1-1/p}_{\alpha,p}(\infty,\xi) = \tag{4.12}$$
$$= \begin{bmatrix} e^{i\pi(s-1/p)}\dfrac{\sin\pi(s-i\xi)}{\sin\pi(1/p-i\xi)} & -e^{-i\pi s}\dfrac{\sin(\alpha-\pi)(1/p-s-1-i\xi)}{\sin\pi(1/p-i\xi)} \\ e^{-i\pi s}\dfrac{\sin(\alpha-\pi)(1/p-s-1-i\xi)}{\sin\pi(1/p-i\xi)} & e^{i\pi(s-1/p)}\dfrac{\sin\pi(s-i\xi)}{\sin\pi(1/p-i\xi)} \end{bmatrix},$$

since

$$
\begin{aligned}
\mathscr{I}_p^{s-1-1/p}(\infty,\xi) &= e^{2\pi(s-1-1/p)i}\frac{\sin\pi\Big(1/p+s-1-1/p-i\xi\Big)}{\sin\pi\Big(1/p-i\xi\Big)}\\
&= e^{i\pi(s-1/p)}\frac{\sin\pi\Big(s-i\xi\Big)}{\sin\pi\Big(1/p-i\xi\Big)};
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
&\frac{1}{2i}\left[\mathscr{K}^{1,s-1-1/p}_{e^{i\alpha},p}(\infty,\xi)-\mathscr{K}^{1,s-1-1/p}_{e^{i(2\pi-\alpha)},p}(\infty,\xi)\right]\\
&\qquad= -e^{-i\pi(1/p-i\xi)}\frac{e^{i\alpha(1/p-s-1-i\xi)}-e^{i(2\pi-\alpha)(1/p-s-1-i\xi)}}{2i\sin\pi(1/p-i\xi)}\\
&\qquad= e^{-i\pi s}\frac{e^{i(\alpha-\pi)(1/p-s-1-i\xi)}-e^{-i(\alpha-\pi)(1/p-s-1-i\xi)}}{2i\sin\pi(1/p-i\xi)}\\
&\qquad= e^{-i\pi s}\frac{\sin(\alpha-\pi)(1/p-s-1-i\xi)}{\sin\pi(1/p-i\xi)}.
\end{aligned}
$$

Since $\det\mathscr{D}_p^{s-1-1/p}(\infty,\xi)$ coincides with the function in (4.9), due to Proposition 3.1 the operator in (4.10b) is locally Fredholm and, therefore, globally Fredholm if the condition (4.9) holds.

The determinant of the symbol

$$
\det\mathscr{D}_p^{s-1-1/p}(\infty,\xi) = e^{i2\pi(s-1/p)}\sin^2\pi(s-i\xi)+e^{-i2\pi s}\sin^2(\alpha-\pi)(1/p-s-1-i\xi)
$$

is a periodic function with respect to the parameters s and $1/p$ and vanishes on curves which divide the strip $(1,\infty)\times(0,1)\subset\mathbb{R}^2$ into connected subsets $\mathscr{R}_0,\mathscr{R}_1,\ldots$. Due to Corollary 1.5 the BVP (1.9) is uniquely solvable for $s=1$ and $p=2$. Then, due to Proposition 3.3, BVP (1.9) is uniquely solvable for all pairs $(s,1/p)\in\mathscr{R}_0$, provided $(1,1/2)\in\mathscr{R}_0$. ■

5 Model Neumann BVP

In the present section we will derive equivalent boundary integral equations for the model Neumann problem (1.10) and investigate it.

If the Neuman's trace $(\partial_\nu u)^+ = h\in\mathbb{W}_p^{s-1-\frac{1}{p}}(\Gamma_\alpha)$ is known and $u^+=\varphi\in\mathbb{W}_p^{s-\frac{1}{p}}(\Gamma_\alpha)$ denotes the unknown Dirichlet trace, the representation formula for a solution to BVP (1.10) has the following form

$$
u = \boldsymbol{N}_\Delta f+\boldsymbol{W}_\Delta\varphi-\boldsymbol{V}_\Delta h. \tag{5.1}
$$

By applying the Plemelji Formulae (2.5) to (5.1) we get

$$
u^+=\varphi=(\boldsymbol{N}_\Delta f)^+ + \frac{1}{2}\varphi+\boldsymbol{W}_{\Delta,0}\varphi-\boldsymbol{V}_{\Delta,-1}h,\qquad \varphi\in\Gamma_\alpha.
$$

Since $I = r_{\mathbb{R}^+} + r_{\mathbb{R}_\alpha}$, rewrite the obtained equation as follows:

$$\begin{gathered}\frac{1}{2}\varphi - r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}\varphi - r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}\varphi = H,\\ H := (\partial_\nu \boldsymbol{N}_\Delta f)^+ - \boldsymbol{V}_{\Delta,-1}h, \qquad \varphi,\ H \in \mathbb{W}^{s-1/p}_p(\Gamma_\alpha).\end{gathered} \tag{5.2}$$

By using the representation (4.1), similarly to (4.4)–(4.6) the equation (5.2) is rewritten as an equivalent system of boundary integral equations on the semi-axes $\mathbb{R}^+$:

$$\begin{cases} \dfrac{1}{2}\varphi_1 - r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}\varphi_2 - F_0(\varphi)h_2 = H_1,\\ \dfrac{1}{2}\varphi_2 - \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}\varphi_1 - F_0(\varphi)h_1 = H_2,\end{cases} \tag{5.3}$$

$$h_1 := r_{\mathbb{R}^+}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}_\alpha}g_0, \quad h_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\boldsymbol{W}_{\Delta,0}r_{\mathbb{R}^+}g_0,$$

$$\varphi_1 := r_{\mathbb{R}^+}\varphi, \quad \varphi_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}\varphi, \quad H_1 := r_{\mathbb{R}^+}H, \quad H_2 := \boldsymbol{J}_\alpha r_{\mathbb{R}_\alpha}H,$$
$$\varphi_1,\varphi_2 \in \widetilde{\mathbb{W}}^{s-1/p}_p(\mathbb{R}^+) \qquad h_1,h_2,H_1,H_2 \in \mathbb{W}^{s-1/p}_p(\mathbb{R}^+).$$

Due to the formula (2.12a) the system (5.3) of boundary integral equation coincides with the following system of integral equations of Mellin type:

$$\begin{cases} \psi_1 - \dfrac{1}{2i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}} - e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\psi_2 = G_1,\\ \psi_2 + \dfrac{1}{2i}\left[e^{i\alpha}\boldsymbol{K}_{e^{i\alpha}} - e^{-i\alpha}\boldsymbol{K}_{e^{i(2\pi-\alpha)}}\right]\psi_1 = G_2,\end{cases} \tag{5.4}$$

$$\psi_1,\psi_2,\in \widetilde{\mathbb{W}}^{s-1-1/p}_p(\mathbb{R}^+), \quad G_1,G_2 \in \mathbb{W}^{s-1-1/p}_p(\mathbb{R}^+).$$

Theorem 5.1 *Let* $1 < p < \infty$, $\dfrac{1}{p} < s < 1 + \dfrac{1}{p}$.

The model Neumann boundary value problem in the non-classical setting (1.10) *is Freholm if and only if the system of boundary integral equation* (5.4) *is Fredholm.*

Now we can prove the main theorem of the present section.

Theorem 5.2 *The Model Neumann BVP in the non-classical setting* (1.10) *is Fredholm (and the system of boundary integral equation* (5.4) *is Fredholm) if and only if the following holds:*

$$e^{i2\pi(s-1/p)}\sin^2\pi(s-i\xi) + e^{-i2\pi s}\sin^2(\alpha-\pi)(1/p-s-i\xi) \neq 0 \quad \forall\,\xi\in\mathbb{R}. \tag{5.5}$$

If the condition (5.5) *holds, the subset* $(1/p,\infty)\times(1,\infty)$ *of the Euclidean plane* $\mathbb{R}^2$*, where the pairs* (s,p) *range, decomposes into an infinite union* $\mathscr{R}_o \cup \mathscr{R}_1 \cup \cdots$ *of non-intersecting connected subsets of regular pairs, for which the BVP* (1.10) *is Fredholm.*

If the point $(1,2)$ *(i.e.* $s=1, p=2$*) belongs to the connected subset* $\mathscr{R}_0$*, then BVP* (1.10) *is uniquely solvable for all pairs* $(s,p)\in\mathscr{R}_0$.

The same unique solvability holds for the system of integral equations (5.3).

Proof: Let us investigate the Fredholm properties of the system (5.4). An equivalent task is to study the Fredholm property of the corresponding operator ■

$$\boldsymbol{N}_\alpha = I - \frac{1}{2i} d\left[e^{i\alpha}\boldsymbol{K}_{e5.^{i\alpha}} - e^{-i\alpha}\boldsymbol{K}_{e5.^{i(2\pi-\alpha)}}\right] \;:\; \widetilde{\mathbb{W}}^{s-1-1/p}_p(\mathbb{R}^+) \to \mathbb{W}^{s-1-1/p}_p(\mathbb{R}^+), \quad (5.6a)$$

For this it suffices, due to Proposition 3.2, to prove the same theorem for the operator

$$\boldsymbol{N}_\alpha = I - \frac{1}{2i} d\left[e^{i\alpha}\boldsymbol{K}_{e5.^{i\alpha}} - e^{-i\alpha}\boldsymbol{K}_{e5.^{i(2\pi-\alpha)}}\right] \;:\; \widetilde{\mathbb{H}}^{s-1-1/p}_p(\mathbb{R}^+) \to \mathbb{H}^{s-1-1/p}_p(\mathbb{R}^+). \quad (5.6b)$$

Here the 2×2 matrix d is defined in (4.11).

The symbol of the operator $\boldsymbol{D}_\alpha$ in (5.6b) on the set $\overline{\Gamma}_1$, according to the formulae (3.3a)-(3.3c), reads:

$$\mathscr{D}^{s-1-1/p}_{\alpha,p}(\infty,\xi) = \quad (5.7)$$
$$= \begin{bmatrix} e^{i\pi(s-1/p)}\dfrac{\sin\pi(s-i\xi)}{\sin\pi(1/p-i\xi)} & -e^{-i\pi s}\dfrac{\sin(\alpha-\pi)(1/p-s-i\xi)}{\sin\pi(1/p-i\xi)} \\ e^{-i\pi s}\dfrac{\sin(\alpha-\pi)(1/p-s-i\xi)}{\sin\pi(1/p-i\xi)} & e^{i\pi(s-1/p)}\dfrac{\sin\pi(s-i\xi)}{\sin\pi(1/p-i\xi)} \end{bmatrix},$$

since

$$\frac{1}{2i}\left[e^{i\alpha}\mathscr{K}^{1,s-1-1/p}_{e5.^{i\alpha},p}(\infty,\xi) - e^{-i\alpha}\mathscr{K}^{1,s-1-1/p}_{e5.^{i(2\pi-\alpha)},p}(\infty,\xi)\right]$$
$$= -e^{-i\pi(1/p-i\xi)}\frac{e5.^{i\alpha(1/p-s-i\xi)} - e^{i(2\pi-\alpha)(1/p-s-i\xi)}}{2i\sin\pi(1/p-i\xi)}$$
$$= e^{-i\pi s}\frac{e5.^{i(\alpha-\pi)(1/p-s-i\xi)} - e^{-i(\alpha-\pi)(1/p-s-i\xi)}}{2i\sin\pi(1/p-i\xi)}$$
$$= e^{-i\pi s}\frac{\sin(\alpha-\pi)(1/p-s-i\xi)}{\sin\pi(1/p-i\xi)}$$

and for $\mathscr{I}^{s-1-1/p}_p(\infty,\xi)$ see (4.13).

Since $\det\mathscr{N}^{s-1-1/p}_p(\infty,\xi)$ coincides with the function in (5.5), due to Proposition 3.1 the operator in (5.6a) is locally Fredholm and, therefore, globally Fredholm if the condition (5.5) holds.

The determinant of the symbol

$$\det\mathscr{N}^{s-1-1/p}_p(\infty,\xi) = e^{i2\pi(s-1/p)}\sin^2\pi(s-i\xi) + e^{-i2\pi s}\sin^2(\alpha-\pi)(1/p-s-i\xi)$$

is a periodic function with respect to the parameters s and $1/p$ and vanishes on curves which divide the strip $(1,\infty)\times(0,1)\subset\mathbb{R}^2$ into connected subsets $\mathscr{R}_0, \mathscr{R}_1, \ldots$. Due to Corollary

1.5 the BVP (1.10) is uniquely solvable for $s = 1$ and $p = 2$. Then, due to Proposition 3.3, BVP (1.10) is uniquely solvable for all pairs $(s, 1/p) \in \mathscr{R}_0$, provided $(1, 1/2) \in \mathscr{R}_0$. ■

Proof of Theorem 1.3: Due to the local principle, Theorem 1.4, the BVP (1.1) is Fredholm if all local representatives (the corresponding BVPs (1.9)–(1.11)) at the knots $c_j \in \mathscr{M} = \mathscr{M}_D \cup \mathscr{M}_N \cup \mathscr{M}_{DN}$ are Fredholm. Due to Theorems 4.3, Theorem 5.2 proved above and Theorem 0.3 proved in [DT19] Conditions (1.4), (1.5) and (1.6) are necessary and sufficient for the corresponding Dirichlet, Neumann and mixed BVPs are Fredholm in appropreate settings.

The determinants of the symbols in (1.4), (1.5) and (1.6) are periodic function with respect to the parameters s and $1/p$ and vanishes on curves which divide the strip $(1, \infty) \times (0, 1) \subset \mathbb{R}^2$ into connected subsets $\mathscr{R}_0, \mathscr{R}_1, \ldots$. Due to Theorem 1.1 the BVP (1.1) is uniquely solvable for $s = 1$ and $p = 2$. Then, due to Proposition 3.3, BVP (1.1) is uniquely solvable for all pairs $(s, 1/p) \in \mathscr{R}_0$, provided $(1, 1/2) \in \mathscr{R}_0$. ■

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