On Some Properties of the Topological Entropy of Dynamic Systems on the Cantor Set and the Segment

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Following [2], we give the definition of topological entropy that will be necessary hereafter. Let X be a compact metric space with a metric d and $f : X \to X$ a continuous mapping. Along with the original metric d, we define an additional system of metrics on X:

$$d_n^f(x,y) = \max_{0 \le i < n} d(f^i(x), f^i(y)), \quad x, y \in X, \quad n \in \mathbb{N},$$

where f^i, $i \in \mathbb{N}$, is the i-th iteration of f, $f^0 \equiv \mathrm{id}_X$. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, denote by $N_d(f, \varepsilon, n)$ the maximum number of points in X, pairwise d_n^f-distances between which are greater than ε. Then *the topological entropy* of the mapping f is defined by the formula [2]

$$h_d(f,x) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{1}{n} \ln N_d(f, \varepsilon, n).$$

Let $C(X,X)$ denote the space of continuous functions from X to X with metric

$$\rho(f,g) = \max_{x \in X} d(f(x), g(x)).$$

Theorem 1 ([5,6])**.** *The function $h_{\mathrm{top}} : C(X,X) \to [0;+\infty]$ belongs to the second Baire class on the space $C(X,X)$, and its set of lower semicontinuity points is an everywhere dense set of type G_δ.*

Let $\mathcal{K}$ denote the Cantor set on the interval $[0;1]$ with the metric induced by the natural metric of the real line.

Theorem 2 ([10])**.** *If $X = \mathcal{K}$, then the function $h_{\mathrm{top}} : C(X,X) \to [0;+\infty)$ does not belong to the first Baire class subspace of homeomorphisms.*

Let us denote by $E_h(f)$ the set of *limiting realizable* values of topological entropy, i.e. those that are obtained for arbitrarily small uniform perturbations of the mapping f:

$$E_h(f) = \bigcap_{n \in \mathbb{N}} \big\{ h_{\mathrm{top}}(g) : \ \rho(f,g) < n^{-1} \big\}.$$

Theorem 3 ([9])**.** *For each continuous mapping $f : \mathcal{K} \to \mathcal{K}$, the equality $E_h(f) = [0;+\infty]$ holds.*

From Theorems 1 and 3 we obtain

Theorem 4 ([9])**.** *For the function $h_{\mathrm{top}} : C(\mathcal{K},\mathcal{K}) \to [0;+\infty]$ it is true:*

1. *the set of points of continuity is empty;*

2. *the set of points of lower semicontinuity coincides with the set of points where the equality* $h_{\text{top}}(f) = 0$ *holds and is an everywhere dense set of type* G_δ*;*
3. *the set of upper semicontinuity points coincides with the set of points where the equality* $h_{\text{top}}(f) = +\infty$ *holds and is a set of type* $F_{\sigma\delta}$*.*

From the results of the work [4] for any continuous mapping f we obtain the equalities

$$\min E_h(f) = h_{\text{top}}(f),$$

and from the results of the book [1] we have

$$\max E_h(f) = h_{\text{top}}(f).$$

Theorem 5. *If the mapping* $f \in C([0;1],[0;1])$ *satisfies the condition* $f(0) = 0$*, then the equality* $E_h(f) = [h_{\text{top}}(f);+\infty]$ *holds.*

On the space of the sequence of continuous mappings

$$\widetilde{f} \equiv (f_1, f_2, \dots), \quad f_i : X \to X$$

we introduce the metric

$$\widetilde{\rho}(\widetilde{f},\widetilde{g}) = \sum_{k=1}^{\infty} 2^{-k} \frac{\rho(f_k, g_k)}{1+\rho(f_k, g_k)},$$

we denote the resulting metric space by $\widetilde{C}(X,X)$. The topological entropy of the sequence of mappings $\widetilde{f}$ is defined in [3].

Theorem 6 ([7])**.** *The function* $h_{\text{top}} : \widetilde{C}(X,X) \to [0;+\infty]$ *belongs to the third Baire class.*

Theorem 7 ([7])**.** *If* $X = \mathcal{K}$*, then the function* $h_{\text{top}} : \widetilde{C}(X,X) \to [0;+\infty]$ *does not belong to the second Baire class on the subspace of sequences consisting of homeomorphisms.*

Theorem 8 ([8])**.** *If* $X = [0;1]$*, then the function* $h_{\text{top}} : \widetilde{C}(X,X) \to [0;+\infty]$ *does not belong to the second Baire class.*

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