# On the Coefficients of Sensitivity for a SIR Controlled Differential Model Considering Perturbations of Delays and Control Function

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In the paper, a system of functional-differential equations is established, whose solutions represent the sensitivity coefficients of the SIR differential model. Let $I=[t_0,t_1]$ be a given interval and let $\mathbb{R}^3$ be the 3-dimensional vector space of points $x=(x^1,x^2,x^3)^T$, where $T$ means transpose; suppose that $O\subset\mathbb{R}^3, V\subset\mathbb{R}^1$ are open, convex, bounded sets. Let the 3-dimensional vector function

$$f(t,x,x_1,x_2,v)=\big(f^1(t,x,x_1,x_2,v),f^2(t,x,x_1,x_2,v),f^3(t,x,x_1,x_2,v)\big)^T$$

be continuous on $I\times O^3\times V$ and continuously differentiable with respect to $x$, $x_1$, $x_2$, and $v$. Furthermore, let $\tau_{2i}>\tau_{1i}>0$, $i=1,2$, and $\theta>0$ be given numbers with $t_0+\tau<t_1$, where $\tau=\max\{\tau_{21},\tau_{22},\theta\}$. Let $\varphi(t)=(\varphi^1(t),\varphi^2(t),\varphi^3(t))^T\in O$, $t\in I_1=[\widehat{\tau},t_0]$ be continuously differentiable function, where $\widehat{\tau}=t_0-\tau$ and let $\Omega$ be a set of the scalar piecewise-continuous control function $u(t)\in V$, $t\in I_2=[\widehat{\tau},t_1]$. To each element $w=(\tau_1,\tau_2,u(t))\in W=(\tau_{11},\tau_{21})\times(\tau_{12},\tau_{22})\times\Omega$ we assign the controlled functional-differential equation

$$\dot{x}(t)=f\big(t,x(t),x(t-\tau_1),x(t-\tau_2),u(t-\theta)\big),\ \ t\in I, \qquad (1)$$
$$x(t)=(x^1(t),x^2(t),x^3(t))^T,$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in I_1. \qquad (2)$$

**Definition.** Let $w\in W$. A function $x(t;w)\in O$ for $t\in I_2$, is called a solution of equation (1) with the initial condition (2), or equivalently, a solution corresponding to the element $w$ and defined on the interval $I_2$, if $x(t;w)$ satisfies condition (2), is absolutely continuous on the interval $I$, and satisfies equation (1) almost everywhere on $I$.

Let us introduce the notations:

- $|w|=|\tau_1|+|\tau_2|+\|u\|$, $\|u\|=\sup\{|u(t)|:\ t\in I_2\}$;
- $W_\varepsilon(w_0)=\{w\in W:|w-w_0|\le\varepsilon\}$;

- $\varepsilon > 0$ is a fixed number and $w_0 = (\tau_{10}, \tau_{20}, u_0(t)) \in W$ is a fixed element;

furthermore,

- $\delta\tau_1 = \tau_1 - \tau_{10}$, $\delta\tau_2 = \tau_2 - \tau_{20}$, $\delta u(t) = u(t) - u_0(t)$ and

$$\delta w = w - w_0 = (\delta\tau_1, \delta\tau_2, \delta u(t)), \quad |\delta w| = |\delta\tau_1| + |\delta\tau_2| + \|\delta u\|,$$
$$W - w_0 = \big\{\delta w = w - w_0 : \ w \in W\big\}.$$

**Theorem 1.** *Let $x_0(t) = (x_0^1(t), x_0^2(t), x_0^3(t))^T$ be the solution corresponding to the element $w_0 \in W$, i.e. $x_0(t)$ is the solution of the initial problem*

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_{10}), x(t-\tau_{20}), u_0(t-\theta)\big), \ \ t \in I,$$
$$x(t) = \varphi(t), \ \ t \in I_1,$$

*defined on the interval $I_2$. Then, there exists a number $\varepsilon_1 > 0$ such, that for arbitrary $\delta w \in \delta W_{\varepsilon_1} = \{\delta w \in W - w_0 : \ |\delta w| \le \varepsilon_1\}$ the perturbed problem* (1), (2) *admits a solution $x(t; w) = x(t; w_0 + \delta w)$ defined on the interval $I_2$ and the following representation holds:*

$$x(t; w) = x_0(t) + \delta x(t; \delta w) + o(t; \delta w), \ \ \delta x(t; \delta w) = \big(\delta x^1(t; \delta w), \delta x^2(t; \delta w), \delta x^3(t; \delta w)\big)^T, \ \ t \in I, \quad (3)$$

*where*

$$\lim_{|\delta w| \to 0} \frac{|o(t; \delta w)|}{|\delta w|} = 0 \ \ \textit{uniformly for} \ \ t \in I.$$

*Moreover, the function*

$$\delta x(t) = \big(\delta x^1(t), \delta x^2(t), \delta x^3(t)\big)^T = \begin{cases} 0, & t \in I_1, \\ \delta x(t; \delta w), & t \in (t_0, t_1] \end{cases}$$

*is a solution of the equation*

$$\begin{aligned} \dot{\delta x}(t) = f_x[t]\delta x(t) + f_{x_1}[t]\delta x(t-\tau_{10}) + f_{x_2}[t]\delta x(t-\tau_{20}) \\ - f_{x_1}[t]\dot{x}_0(t-\tau_{10})\delta\tau_1 - f_{x_2}[t]\dot{x}_0(t-\tau_{20})\delta\tau_2 + f_v[t]\delta u(t-\theta), \ \ t \in (t_0, t_1), \end{aligned} \quad (4)$$

*with the initial condition*

$$\delta x(t) = 0, \ \ t \in I_1. \quad (5)$$

*Here*

$$f_x[t] = f_x\big(t, x_0(t), x_0(t-\tau_{10}), x_0(t-\tau_{20}), u_0(t-\theta)\big). \quad (6)$$

*Remark.* The Theorem 1 is a simple corollary of the Theorem 2.1 given in [1]. The function $\delta x(t) = (\delta x^1(t), \delta x^2(t), \delta x^3(t))^T := \delta x(t; w)$ in the formulas (4) and (3) is called the vector coefficient of sensitivity. The functions $\delta x^i(t)$, $i = 1, 2, 3$ are called the scalar coefficients of sensitivity. Finding of the sensitivity coefficient is an important tool for establishing properties of the mathematical models. Equations for the sensitivity coefficients are obtained in [2, 4–8] for the various classes of functional-differential equations.

The SIR model [3, 9] describes the spread of infectious disease within a population. It divides the total population into three main compartments: Susceptible (S), Infected (I), and Recovered

(R). Let us consider the SIR model with delays in state and control variables as the following system of functional-differential equations:

$$\begin{cases} \dot{x}^1(t) = -\beta x^1(t-\tau_1)x^2(t-\tau_1) - u(t-\theta)x^1(t), \\ \dot{x}^2(t) = \beta x^1(t-\tau_1)x^2(t-\tau_1) - \gamma x^2(t-\tau_2), \\ \dot{x}^3(t) = \gamma x^2(t-\tau_2) + u(t-\theta)x^1(t), \end{cases} \qquad t \in I = [t_0, t_1], \tag{7}$$

with the initial condition

$$x^i(t) = \varphi^i(t), \ \ t \in I_1, \ \ i = 1,2,3. \tag{8}$$

Here, $x^1(t)$ denotes the number of susceptible individuals at time $t$; $x^2(t)$ denotes the number of infected individuals at time $t$; and $x^3(t)$ denotes the number of recovered individuals at time $t$; $u(t)$ is the control function; $\beta$ denotes the infection rate; $\gamma$ is the recovery rate; $\tau_1$ represents the delay (represents the incubation period or latent period before an infection has an effect); $\tau_2$ denotes the delay (represents the time from infection to recovery); $\theta$ represents the control delay (represents the time between applying the control and its effect on the system); $\varphi^i(t)$, $t \in I_1$, $i = 1,2,3$, are given initial functions (represents the number of susceptible, infected and recovered individuals during the delay interval); $u(t-\theta))$ is the control function with a time delay $\theta$ (vaccination, quarantine, etc.); $x^1(t-\tau_1)x^2(t-\tau_1)$ represents the interactions between susceptible and infective individuals $\tau_1$ time units ago, that produces new infections at the current time; $u(t-\theta)x^1(t)$ represents the number of susceptible individuals affected today by a control action that was applied $\theta$ time units ago. In addition, we note that the model implies the validity of the following equality $x^1(t)+x^2(t)+x^3(t) = N(t)$. This relation means that the population size is fixed (i.e., no births, deaths due to disease, or deaths by natural causes). $N(t)$ is the total population at time $t$. Incubation period of the infectious agent is instantaneous, and duration of infectivity is same as length of the disease. It also assumes a completely homogeneous population with no age or social structure. For the considered model we suppose that: $\beta > 0$, $\gamma > 0$; $O = (0,a) \times (0,a) \times (0,a)$, $V = (0,b)$; $a > 0$, $b > 0$. Let us introduce notations: $x = (x^1, x^2, x^3)^T$, $x_1 = (x_1^1, x_1^2, x_1^3)^T$, $x_2 = (x_2^1, x_2^2, x_2^3)^T$. It is not difficult to see that using the previous notations we have:

$$f^1(t, x, x_1, x_2, v) = -\beta x_1^1 x_1^2 - v x^1, \quad f^2(\cdot) = \beta x_1^1 x_1^2 - \gamma x_2^2, \quad f^3(\cdot) = \gamma x_2^2 + v x^1.$$

Now, let us find the matrices $f_x[t]$, $f_{x_1}[t]$, $f_{x_2}[t]$, and $f_u[t]$ (see (6))

$$f_x[t] = \begin{pmatrix} f^1_{x^1}[t] & f^1_{x^2}[t] & f^1_{x^3}[t] \\ f^2_{x^1}[t] & f^2_{x^2}[t] & f^2_{x^3}[t] \\ f^3_{x^1}[t] & f^3_{x^2}[t] & f^3_{x^3}[t] \end{pmatrix},$$

where

$$f^1_{x^1}[t] = -u_0(t-\theta), \ \ f^1_{x^2}[t] = 0, \ \ f^1_{x^3}[t] = 0, \ \ f^2_{x^1}[t] = 0, \ \ f^2_{x^2}[t] = 0, \ \ f^2_{x^3}[t] = 0,$$
$$f^3_{x^1}[t] = u_0(t-\theta), \ \ f^3_{x^2}[t] = 0, \ \ f^3_{x^3}[t] = 0,$$
$$f_{x_1}[t] = \begin{pmatrix} f^1_{x_1^1}[t] & f^1_{x_1^2}[t] & f^1_{x_1^3}[t] \\ f^2_{x_1^1}[t] & f^2_{x_1^2}[t] & f^2_{x_1^3}[t] \\ f^3_{x_1^1}[t] & f^3_{x_1^2}[t] & f^3_{x_1^3}[t] \end{pmatrix},$$

here,

$$f^1_{x_1^1}[t] = -\beta x_0^2(t-\tau_{10}), \ \ f^1_{x_1^2}[t] = -\beta x_0^1(t-\tau_{10}), \ \ f^1_{x_1^3}[t] = 0, \ \ f^2_{x_1^1}[t] = \beta x_0^2(t-\tau_{10}),$$

$$f^2_{x^2_1}[t]=\beta x^1_0(t-\tau_{10}),\ \ f^2_{x^3_1}[t]=0, f^3_{x^1_1}[t]=0,\ \ f^3_{x^2_1}[t]=0,\ \ f^3_{x^3_1}[t]=0,$$
$$f_{x_2}[t]=\begin{pmatrix} f^1_{x^1_2}[t] & f^1_{x^2_2}[t] & f^1_{x^3_2}[t]\\ f^2_{x^1_2}[t] & f^2_{x^2_2}[t] & f^2_{x^3_2}[t]\\ f^3_{x^1_2}[t] & f^3_{x^2_2}[t] & f^3_{x^3_2}[t]\end{pmatrix},$$

where

$$f^1_{x^1_2}[t]=0,\ \ f^1_{x^2_2}[t]=0,\ \ f^1_{x^3_2}[t]=0,\ \ f^2_{x^1_2}[t]=0,\ \ f^2_{x^2_2}[t]=-\gamma,\ \ f^2_{x^3_2}[t]=0,$$
$$f^3_{x^1_2}[t]=0,\ \ f^3_{x^2_2}[t]=\gamma,\ \ f^3_{x^3_2}[t]=0,$$
$$f_v[t]=\big(f^1_v[t],f^2_v[t],f^3_v[t]\big)^T,$$
$$f^1_v[t]=-x^1_0(t),\ \ f^2_v[t]=0,\ \ f^3_v[t]=x^1_0(t).$$

After elementary calculations we get (see (4))

$$f_x[t]\delta x(t)=\begin{pmatrix}-u_0(t-\theta)\delta x^1(t)\\ 0\\ u_0(t-\theta)\delta x^3(t)\end{pmatrix},$$
$$f_{x_1}[t]\delta x(t-\tau_{10})=\begin{pmatrix}-\beta x^2_0(t-\tau_{10})\delta x^1(t-\tau_{10})-\beta x^1_0(t-\tau_{10})\delta x^2(t-\tau_{10})\\ \beta x^2_0(t-\tau_{10})\delta x^1(t-\tau_{10})+\beta x^1_0(t-\tau_{10})\delta x^2(t-\tau_{10})\\ 0\end{pmatrix},$$
$$f_{x_2}[t]\delta x(t-\tau_{20})=\begin{pmatrix}0\\ -\gamma\delta x^2(t-\tau_{20})\\ \gamma\delta x^2(t-\tau_{2o})\end{pmatrix},\quad f_v[t]\delta u(t-\theta)=\begin{pmatrix}-x^1_0(t)\delta u(t-\theta)\\ 0\\ x^1_0(t)\delta u(t-\theta)\end{pmatrix},$$
$$f_{x_1}[t]\dot x_0(t-\tau_{10})=\begin{pmatrix}-\beta\big[x^2_0(t-\tau_{10})\dot x^1_0(t-\tau_{10})+x^1_0(t-\tau_{10})\dot x^2_0(t-\tau_{10})\big]\\ \beta\big[x^2_0(t-\tau_{10})\dot x^1_0(t-\tau_{10})+x^1_0(t-\tau_{10})\dot x^2_0(t-\tau_{10})\big]\\ 0\end{pmatrix},$$
$$f_{x_2}[t]\dot x_0(t-\tau_{20})=\begin{pmatrix}0\\ -\gamma\dot x^2_0(t-\tau_{20})\\ \gamma\dot x^3_0(t-\tau_{20})\end{pmatrix}.$$

Using expressions in given above the Theorem 1, it follows

**Theorem 2.** *Let $x^i_0(t)$, $i=1,2,3$ be the solution of the system of functional- differential equations*

$$\begin{cases}\dot x^1(t)=-\beta x^1(t-\tau_{10})x^2(t-\tau_{10})-u_0(t-\theta)x^1(t),\\ \dot x^2(t)=\beta x^1(t-\tau_{10})x^2(t-\tau_{10})-\gamma x^2(t-\tau_{20}), & t\in I\\ \dot x^3(t)=\gamma x^2(t-\tau_{20})+u_0(t-\theta)x^1(t),\end{cases}$$

*with the initial condition $x^i(t)=\varphi^i(t)$, $t\in I_1$, $i=1,2,3$, i.e. $x^i_0(t)$, $i=1,2,3$ is the solution corresponding to the element $w_0\in W$ and defined on the interval $I_2$. Then, there exists $\varepsilon_1>0$ such that, for arbitrary $\delta w\in\delta W_{\varepsilon_1}=\{\delta w=w-w_0:\ |\delta w|\le\varepsilon_1\}$, the perturbed problem (7), (8) admits a solution $x^i(t;w)=x^i(t;w_0+\delta w)$, $i=1,2,3$, defined on the interval $I_2$, and the following representation holds:*

$$x^i(t;w)=x^i_0(t)+\delta x^i(t;\delta w)+o^i(t;\delta w),\ \ t\in I,\ \ i=1,2,3$$

*(see* (3)*), where*

$$\lim_{|\delta w|\to 0}\frac{|o^i(t;\delta w)|}{|\delta u|}=0,\ \ i=1,2,3 \ \textit{uniformly for} \ t\in I.$$

*Moreover, the coefficients of sensitivity $\delta x^i(t):=\delta x^i(t;\delta w)$, $i=1,2,3$, $t\in I$, satisfy the following system of functional-differential equations:*

$$\begin{aligned}
\dot{\delta x}^1(t)&=-\beta x_0^2(t-\tau_{10})\delta x^1(t-\tau_{10})-\beta x_0^1(t-\tau_{10})\delta x^2(t-\tau_{10})-u_0(t-\theta)\delta x^1(t)\\
&\qquad+\beta\big[x_0^2(t-\tau_{10})\dot x_0^1(t-\tau_{10})+x_0^1(t-\tau_{10})\dot x_0^2(t-\tau_{10})\big]\delta\tau_1-x_0^1(t)\delta u(t-\theta),\\
\dot{\delta x}^2(t)&=\beta x_0^2(t-\tau_{10})\delta x^1(t-\tau_{10})+\beta x_0^1(t-\tau_{10})\delta x^2(t-\tau_{10})-\gamma\delta x^2(t-\tau_{20})\\
&\qquad-\beta\big[x_0^2(t-\tau_{10})\dot x_0^1(t-\tau_{10})+x_0^1(t-\tau_{10})\dot x_0^2(t-\tau_{10})\big]\delta\tau_1+\gamma\dot x_0^2(t-\tau_{20})\delta\tau_2,\\
\dot{\delta x}^3(t)&=x_0^1(t)\delta u(t-\theta)+u_0(t-\theta)\delta x^2(t)-\gamma\dot x_0^3(t-\tau_{20})\delta\tau_2,
\end{aligned}$$

*with the initial condition $\delta x^i(t)=0$, $t\in I_1$, $i=1,2,3$ (see* (5)*).*

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