On the Correspondence Between Boundary Value Problems for Integro-Differential Equations and Their Analogues on Time Scales

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1 Introduction

We consider the boundary value problem for a system of integro-differential equations

$$\dot{x}(t) = X\bigg(t, x(t), \int\limits_0^t \varphi(t, s, x(s))\, ds\bigg), \tag{1.1}$$

subject to the boundary condition

$$F(x(0), x(T)) = 0. \tag{1.2}$$

We also study the corresponding boundary value problem for a family of integro-dynamic equations defined on a family of time scales $\mathbb{T}_\lambda$ with $\lambda \in \Lambda \subset \mathbb{R}$, where 0 is assumed to be a limit point of Λ. Denoting $[0, T]_\lambda := [0, T] \cap \mathbb{T}_\lambda$, we assume that the points 0 and T belong to all time scales $\mathbb{T}_\lambda$. The corresponding boundary value problem on $\mathbb{T}_\lambda$ takes the form

$$x_\lambda^\Delta(t) = X\bigg(t, x_\lambda(t), \int\limits_0^t \varphi(t, s, x_\lambda(s))\, \Delta s\bigg), \tag{1.3}$$

$$F(x_\lambda(0), x_\lambda(T)) = 0. \tag{1.4}$$

Here, the function $X(t, x, y)$ is defined and continuous for $t \in [0, T]$, $x \in D \subset \mathbb{R}^d$, and $y \in D_1 \subset \mathbb{R}^m$. The function $\varphi(t, s, z)$ is defined and continuous on $Q_1 = [0, T] \times [0, T] \times D$, with $\varphi : Q_1 \to D_1$. A more precise formulation of the problem will be given below.

Let $\mu_\lambda(t) : \mathbb{T}_\lambda \to [0, \infty)$ denote the graininess function, which characterizes the density of the time scale, and let $\mu_\lambda := \sup\limits_{t \in \mathbb{T}_\lambda} \mu_\lambda(t)$. If $\mu_\lambda \to 0$ as $\lambda \to 0$, then $[0, T]_\lambda$ approaches the continuous interval $[0, T]$, and the boundary value problem (1.3), (1.4) on the family of time scales tends to the integro-differential boundary value problem (1.1), (1.2). This naturally raises the question of the relationship between the solutions of these two types of problems.

The main contribution of this work is to demonstrate that, under suitable assumptions, the solvability of the integro-dynamic boundary value problem (1.3), (1.4) for sufficiently small λ guarantees the solvability of the integro-differential boundary value problem (1.1), (1.2). Conversely, we show that if problem (1.1), (1.2) admits a solution, then the corresponding problems (1.3), (1.4) are also solvable for small λ. Furthermore, we prove that the solutions $x_\lambda(t)$ converge to the solution $x(t)$ of problem (1.1), (1.2) as $\lambda \to 0$.

2 Time scales

We recall some standard notations from the theory of time scales (see [1]). A time scale is defined as a nonempty closed subset of $\mathbb{R}$. For a subset $A \subset \mathbb{R}$ we write $A_{\mathbb{T}} = A \cap \mathbb{T}$.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is given by $\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$. Similarly, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}$. Here we adopt the convention $\inf \varnothing := \sup \mathbb{T}$, $\sup \varnothing := \inf \mathbb{T}$.

A point $t \in \mathbb{T}$ is said to be *left-dense (LD)* if $\rho(t) = t$, *left-scattered (LS)* if $\rho(t) < t$, *right-dense (RD)* if $\sigma(t) = t$, and *right-scattered (RS)* if $\sigma(t) > t$.

When $\mathbb{T}$ possesses a left-scattered maximum M, we put $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{M\}$; if no such point exists, we simply take $\mathbb{T}^{\kappa} = \mathbb{T}$.

The *graininess function* $\mu(t) := \sigma(t) - t$ characterizes the density of the time scale.

We say that a function $f : \mathbb{T} \to \mathbb{R}^d$ is Δ-differentiable at a point $t \in \mathbb{T}^{\kappa}$ whenever the limit

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$. In this case, the value $f^{\Delta}(t)$ is referred to as the Δ-derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}^d$ is said to be *rd-continuous* provided it is continuous at every right-dense point of $\mathbb{T}$ and admits finite left limits at all left-dense points.

We say that $f : \mathbb{T} \to \mathbb{R}^d$ is *regulated* when it admits finite right limits at every right-dense point and finite left limits at every left-dense point of $\mathbb{T}$.

Suppose $f : \mathbb{T} \to \mathbb{R}^d$ is regulated. We call a function F an *antiderivative* of f if

$$F^{\Delta}(t) = f(t), \ \ t \in \mathbb{T}.$$

In this case, the *Cauchy Δ-integral* of f is given by

$$\int_{\tau}^{s} f(t)\, \Delta t = F(s) - F(\tau), \ \ s, \tau \in \mathbb{T}.$$

3 Auxiliary results

The proofs of the main results are based on estimates of the proximity between the Cauchy problems associated with (1.1) and (1.3).

We assume that $X(t, x, y)$ is continuously differentiable in its domain, and that there exists a constant $C > 0$ such that

$$|X(t, x, y)| + \left|\frac{\partial X(t, x, y)}{\partial t}\right| + \left|\frac{\partial X(t, x, y)}{\partial x}\right| + \left|\frac{\partial X(t, x, y)}{\partial y}\right| \le C \tag{3.1}$$

for all $t \in [0, T]$, $x \in D$, and $y \in D_1$.

Moreover, we assume that

$$\left\|\frac{\partial \varphi(t, s, z)}{\partial z}\right\| \le C \tag{3.2}$$

for all $t, s \in [0, T]$ and $z \in D$.

Here, $\frac{\partial X}{\partial x}$, $\frac{\partial X}{\partial y}$, and $\frac{\partial \varphi}{\partial z}$ denote the corresponding Jacobian matrices, while $|\cdot|$ and $\|\cdot\|$ denote the norms of a vector and a matrix, respectively.

Let $t_0, t_1 \in [0, T]_{\lambda}$, and let $x(t)$ and $x_{\lambda}(t)$ denote the solutions of the corresponding Cauchy problems on $[t_0, t_1]$ and $[t_0, t_1]_{\lambda}$, with the initial conditions $x(t_0) = x_0$ and $x_{\lambda}(t_0) = x_0$, respectively.

The following result establishes the closeness of these solutions, with an estimate that is uniform over all time scales.

Lemma 3.1. *Suppose that conditions* (3.1) *and* (3.2) *hold. Then, for all* $t \in [t_0, t_1]_\lambda$*, the following inequality holds:*

$$|x(t) - x_\lambda(t)| \le \mu_\lambda K, \tag{3.3}$$

where

$$K = e^{C(t_1 - t_0 + 1)}\Big(c + \frac{C^2(t_1 - t_0)}{4}\Big) + 3C.$$

The proof of this statement follows from the approach of works [2,3], where it was established for the case of ordinary differential equations. The main difficulty of the proof lies in establishing estimate (3.3) at limit points that are limits of left-scattered points. In doing so, the boundedness of the partial derivatives in (3.1) and (3.2) plays a crucial role.

We will also need results on the continuous dependence and continuous differentiability of the Cauchy problem for (1.1) with respect to the initial data.

Consider, for equation (1.1), the Cauchy problem with the initial condition $x(0) = x_0$, and denote its solution by $x(t, x_0)$ as a function of x_0. The following result holds concerning the continuous dependence of this solution on x_0.

Lemma 3.2 ([5])**.** *Assume that conditions* (3.1) *and* (3.2) *are satisfied. Then the solution* $x(t, x_0)$ *of the Cauchy problem exists in a certain positive neighborhood of the origin and depends continuously on the initial value* $x_0 \in D$.

An important question is also the continuous differentiability of the solution of the Cauchy problem with respect to x_0.

Lemma 3.3 ([5])**.** *Assume that the conditions of Lemma 3.2 are satisfied. Then the function* $x(t, x_0)$ *is continuously differentiable with respect to the parameter* x_0*, and the function* $z(t) = \frac{\partial x}{\partial x_0}(t, x_0)$ *satisfies the linear integro-differential equation*

$$\dot z = \frac{\partial X\big(t, x(t, x_0), \int_0^t \varphi(t, s, x(s, x_0))\, ds\big)}{\partial x}\, z(t) + \frac{\partial X\big(t, x(t, x_0), \int_0^t \varphi(t, s, x(s, x_0))\, ds\big)}{\partial y} \int_0^t \frac{\partial \varphi(t, s, x(s, x_0))}{\partial z}\, z(s)\, ds. \tag{3.4}$$

Remark 3.1. By analogy with ordinary differential equations, equation (3.4) will be called the variational equation.

Regarding the continuous and differentiable dependence of the solution of the Cauchy problem for (1.3) on the initial data, the following result holds.

Lemma 3.4. *Assume that the conditions of Lemma 3.2 are satisfied. Then the function* $x_\lambda(t, x_0)$ *is continuously differentiable with respect to the parameter* x_0*, and the function* $z_\lambda(t) = x_\lambda(t, x_0)$ *satisfies the linear integro-dynamic equation*

$$z_\lambda^\Delta = \frac{\partial X\big(t, x_\lambda(t, x_0), \int_0^t \varphi(t, s, x_\lambda(s, x_0))\, \Delta s\big)}{\partial x}\, z_\lambda(t) + \frac{\partial X\big(t, x_\lambda(t, x_0), \int_0^t \varphi(t, s, x_\lambda(s, x_0))\, \Delta s\big)}{\partial y} \int_0^t \frac{\partial \varphi(t, s, x_\lambda(s, x_0))}{\partial z}\, z_\lambda(s)\, \Delta s. \tag{3.5}$$

Again, by analogy with ordinary differential equations, we will call equation (3.5) the variational equation on time scales.

The proof of this fact is carried out similarly to that in [4], where a corresponding result was obtained for dynamic equations on time scales.

4 Main results

Our main results concern the relationship between the solutions of boundary value problems (1.1), (1.2) and (1.3), (1.4).

Theorem 4.1. *Under assumptions* (3.1) *and* (3.2)*, suppose there exists* $\lambda_0 \in \Lambda$ *such that, for all* $\lambda \in (0, \lambda_0]$*, the boundary value problem* (1.3), (1.4) *admits a solution* $x_\lambda(t)$ *lying in* D *together with a* ρ*-neighborhood* (*where* $\rho > 0$ *is independent of* λ).

Then the boundary value problem (1.1), (1.2) *admits a solution* $x = x(t)$*, and there exists a sequence* $\lambda_n \in (0, \lambda_0)$ *with* $\lambda_n \to 0$ *as* $n \to \infty$*, such that*

$$\sup_{t\in[0,T]_{\lambda_n}} |x(t) - x_{\lambda_n}(t)| \to 0 \ \text{ as } \ \lambda_n \to 0. \tag{4.1}$$

Moreover, if problem (1.1), (1.2) *has a unique solution, then the convergence in* (4.1) *holds for all* $\lambda \in (0, \lambda_0)$ *as* $\lambda \to 0$.

To formulate the converse result, we assume that (1.1) has a solution $x(t) = x(t, x_0)$ with the initial condition $x(0, x_0) = x_0$. Define the function

$$F_0(x_0) := F(x_0, x(T, x_0)),$$

and compute the total derivative of F_0 with respect to x as follows:

$$\frac{\partial F_0(x_0)}{\partial x} = \frac{\partial F(x_0, x(T, x_0))}{\partial x} + \frac{\partial F(x_0, x(T, x_0))}{\partial y} \cdot \frac{\partial x(T, z)}{\partial z}\bigg|_{z=x_0},$$

where $x(t, z)$ denotes the solution of (1.1) with the initial condition $x(0, z) = z$.

Theorem 4.2. *Let Assumption A hold, and suppose that problem* (1.1), (1.2) *admits a solution* $x = x(t)$ *lying in* D *together with a* ρ*-neighborhood. Furthermore, assume that*

$$\det \frac{\partial F_0(x_0)}{\partial x} \neq 0.$$

Then there exist constants $\lambda_0 > 0$ *and* $\sigma \le \rho$ *such that, for all* $\lambda \in (0, \lambda_0)$*, problem* (1.3), (1.4) *has a unique solution* $x_\lambda(t)$ *that lies within the* σ*-neighborhood of* $x(t)$*, i.e.,*

$$|x(t) - x_\lambda(t)| < \sigma, \ \ t \in [0, T]_\lambda,$$

and the following convergence estimate holds:

$$\sup_{t\in[0,T]_\lambda} |x(t) - x_\lambda(t)| \to 0 \ \text{ as } \ \lambda \to 0.$$

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