Dirichlet Problem for Singular First-Order Differential and Functional Differential Equations

Svatoslav Staněk

Department of Mathematical Analysis, Faculty of Science, Palacký University
Olomouc, Czech Republic
E-mail: svatoslav.stanek@upol.cz

1 Introduction

Let $\mathcal{B}$ be the set of decreasing functions $h \in C(0,1]$, $\lim\limits_{v\to 0+} h(v) = \infty$ and $h(1) = 0$. For example, the functions $(1-v)/v$, $e^{1/v} - e$ and $-\ln v$ belong to $\mathcal{B}$.

Let $\Omega = \{x \in C[0,1] :\ 0 < x(t) < 1 \text{ for } t \in (0,1)\}$ and $\mathcal{S}$ be the set of (generally nonlinear) functionals $\Lambda : \Omega \to (0,1)$ satisfying

(a) $x, y \in \Omega$, $x(t) < y(t)$ for $t \in (0,1) \Longrightarrow \Lambda(x) < \Lambda(y)$,

(b) $\{x_n\} \subset \Omega$, $x \in \Omega$, $\lim\limits_{n\to\infty} x_n(t) = x(t)$ locally uniformly on $(0,1] \Longrightarrow \lim\limits_{n\to\infty} \Lambda(x_n) = \Lambda(x)$.

We study the singular differential equation

$$y'(t) = \frac{r(t)(f(t) - y(t))}{t^\eta (y(t) - g(t))} \tag{1.1}$$

and the singular functional differential equation

$$y'(t) = h(\Lambda(y))\, \frac{r(t)(f(t) - y(t))}{t^\eta (y(t) - g(t))}\,, \ \ h \in \mathcal{B}, \ \ \Lambda \in \mathcal{S}, \tag{1.2}$$

where $\eta \in [0,\infty)$, $r \in C[0,1]$ is positive and the functions f and g satisfy the condition

(H) $f \in C[0,1] \cup C^1(0,1]$, $g \in C^1[0,1]$, f, g are increasing, $f(0) = g(0) = 0$, $f(1) = g(1) = 1$, $f > g$ on $(0,1)$ and $\sup\{tf'(t) :\ t \in (0,1]\} = K < \infty$.

Together with equations (1.1) and (1.2) the Dirichlet boundary condition

$$y(0) = 0, \quad y(1) = 1 \tag{1.3}$$

is considered.

Definition 1.1. We say that a function $y : [0,1] \to \mathbb{R}$ is a solution of problem $(1.1), (1.3)$ if $y \in C[0,1] \cup C^1(0,1)$, y satisfies the boundary condition (1.3) and (1.1) holds for $t \in (0,1)$. Similarly for solutions of problem $(1.2), (1.3)$.

The aim of this paper is to study the existence and uniqueness of solutions to problems $(1.1), (1.3)$ and $(1.2), (1.3)$. The key to obtain solutions of the singular equation (1.1) is to give a convex combination φ (see (2.1) later) of functions f and g and analyze solutions of (1.1) "between" φ and f. By this way the singularity of (1.1) is eliminated. It is interesting that although (1.1) and (1.2) are first-order differential and functional differential equations, their solutions are discussed under the two-point Dirichlet boundary condition (1.3).

2 Preliminaries

Let $J \subset (0,1)$ be an interval. We say that y is a solution of (1.1) on J if $y \in C^1(J)$ and (1.1) holds for $t \in J$.

Lemma 2.1.

(i) *If* $\mathcal{D} = \{(t,y) \in (0,1) \times \mathbb{R} : \; y > g(t)\}$, *then for each* $(t_0, y_0) \in \mathcal{D}$ *the initial value problem* (1.1), $y(t_0) = y_0$ *has a unique solution on a neighborhood of* $t = t_0$.

(ii) *If* y_1 *and* y_2 *are solutions of* (1.1) *on intervals* J_1 *and* J_2 *and* $y_1(t_0) = y_2(t_0)$ *for some* $t_0 \in J_1 \cap J_2$, *then* $y_1 = y_2$ *on* $J_1 \cap J_2$.

(iii) *If* y_1 *and* y_2 *are solutions of* (1.1) *on intervals* J_1 *and* J_2 *and* $y_1(t_0) < y_2(t_0)$ *for some* $t_0 \in J_1 \cap J_2$, *then* $y_1(t) < y_2(t)$, $y_1'(t) > y_2'(t)$ *for* $t \in J_1 \cap J_2$.

Let $m_- = \min\{r(t) : \; t \in [0,1]\}$ and $S = \max\{g'(t) : \; t \in [0,1]\}$. The function

$$w(\lambda) = \lambda(S + \lambda(1 + S + K) + \lambda^2 K)$$

is continuous on $\mathbb{R}$, where K is from the condition (H). Since w is increasing on $[0, \infty)$, $w(0) = 0$ and $\lim\limits_{\lambda \to \infty} w(\lambda) = \infty$, there exist a unique positive value θ such that $w(\theta) = m_-$, that is, $m_- = \theta(S + \theta(1 + S + K) + \theta^2 K)$.

Lemma 2.2. *Let*

$$\varphi(t) = \frac{\theta t f(t) + g(t)}{1 + \theta t}, \quad t \in [0,1]. \tag{2.1}$$

Then

$$\varphi(0) = 0, \;\; \varphi(1) = 1, \;\; g(t) < \varphi(t) < f(t), \;\; t \in (0,1)$$

and

$$\varphi(t) - g(t) = \frac{\theta t (f(t) - g(t))}{1 + \theta t}, \quad t \in [0,1]. \tag{2.2}$$

Lemma 2.3. *Let* φ *be given in* (2.1). *Let* $t_0 \in (0,1)$, y *be a solution of the initial value problem* (1.1)$, y(t_0) = f(t_0)$ *and* J *be the right maximal interval of existence for* y. *Then* $J = [t_0, 1)$ *and*

$$\varphi(t) < y(t) \le f(t) \;\; \textit{for} \;\; t \in [t_0, 1).$$

If we define $y(1) = 1$, *then* $y \in C[t_0, 1]$. *We observe that* y *is the unique solution of problem* (1.1)$, y(t_0) = f(t_0)$ *on* $[t_0, 1)$ *by Lemma 2.1*(ii).

3 Problem (1.1), (1.3)

We are now in the position to give the main result for the solvability of problem (1.1), (1.3).

Theorem 3.1. *There exists a unique solution* y *of problem* (1.1), (1.3). *This solution* y *satisfies the inequality* $\varphi(t) \le y(t) < f(t)$ *for* $t \in (0,1)$, *where* φ *is from Lemma 2.2.*

Sketch of the proof. Since $f \in C[0,1] \cap C^1(0,1]$ is increasing on $[0,1]$, there exists a decreasing sequence $\{t_n\} \subset (0,1)$, $\lim\limits_{n \to \infty} t_n = 0$, such that $f'(t_n) > 0$ for $n \in \mathbb{N}$. Then, by Lemma 2.3, for each $n \in \mathbb{N}$ there exists a unique solution y_n of (1.1) on $[t_n, 1)$ satisfying $y_n(t_n) = f(t_n)$ and $\varphi < y_n \le f$ on $[t_n, 1)$, $y_n(1) = 1$. It can be shown that

$$y_{n+1}(t) < y_n(t) \;\; \text{for} \;\; t \in [t_n, 1) \;\; \text{and} \;\; n \in \mathbb{N}.$$

Chose $T \in (0,1)$ and let $n_T = \min\{n \in \mathbb{N} : \ t_n < T\}$. Then

$$g(t) < \varphi(t) < y_{n+1}(t) < y_n(t) < f(t) \ \text{ for } \ t \in [T,1), \ \ n \geq n_T$$

and (see (2.2))

$$0 < y_n'(t) = \frac{r(t)(f(t)-y_n(t))}{t^\eta(y_n(t)-g(t))} < \frac{r(t)(f(t)-g(t))}{t^\eta(\varphi(t)-g(t))} = \frac{(1+\theta t)r(t)}{t^{\eta+1}\theta}\,, \ \ t \in [t_n,1), \ \ n \in \mathbb{N}.$$

The Arzelà–Ascoli theorem and the Dini theorem now imply $\lim\limits_{n\to\infty} y_n(t) = u(t)$ uniformly on $[T,1]$ for some $u \in C[T,1]$. Then $\varphi \leq u < f$ on $[T,1)$ and letting $n \to \infty$ in the equality

$$y_n(t) = y_n(T) + \int\limits_T^t \frac{r(s)(f(s)-y_n(s))}{s^\eta(y_n(s)-g(s))}\,\mathrm{d}s, \ \ t \in [T,1], \ \ n \in \mathbb{N}$$

we arrive at

$$u(t) = u(T) + \int\limits_T^t \frac{r(s)(f(s)-u(s))}{s^\eta(u(s)-g(s))}\,\mathrm{d}s, \ \ t \in [T,1]$$

by the Lebesgue dominated convergent theorem. Differentiating the last equality, it follows that u is a solution of (1.1) on $[T,1)$. Since $T \in (0,1)$ is arbitrary, repeating the above procedures we show that $\lim\limits_{n\to\infty} y_n(t) = y(t)$ locally uniformly on $(0,1]$, where y is a solution of (1.1) on $(0,1)$ and $g \leq \varphi \leq y < f$ on this interval. Setting $y(0) = 0$, $y(1) = 1$, we conclude that y is a solution of problem (1.1), (1.3).

In order to prove that y is a unique solution of (1.1), (1.3), suppose that w is another solution of this problem and let $w \neq y$. Then Lemma 2.1(ii) gives $w(t) \neq y(t)$ for $t \in (0,1)$, say for instance that $w > y$ on $(0,1)$. Then $w' < y'$ on this interval by Lemma 2.1(iii), contrary to $w(0) = y(0) = 0$, $w(1) = y(1) = 1$. ☐

Example 3.1. Let $0 < a \leq 1 < b$. Then the functions $f(t) = t^a$ and $g(t) = t^b$ satisfy the condition (H). Hence Theorem 3.1 guarantees the existence of a unique solution y of the equation

$$y'(t) = \frac{r(t)(t^a - y(t))}{t^\eta(y(t)-t^b)}\,,$$

satisfying condition (1.3). Moreover, $t^b < y(t) < t^a$ for $t \in (0,1)$.

4 Problem (1.2), (1.3)

We begin by investigating solutions of the differential equation

$$y'(t) = \frac{\kappa r(t)(f(t)-y(t))}{t^\eta(y(t)-g(t))}\,, \tag{4.1}$$

depending on the parameter $\kappa \in (0,\infty)$.

Analyzing the function $w(\lambda) = \lambda(S + \lambda(1+S+K) + \lambda^2 K)$ (see Section 2), we conclude that the equation $w(\lambda) = \kappa m_-$ has a unique positive solution θ_κ, that is,

$$\kappa m_- = \theta_\kappa\big(S + \theta_\kappa(1+S+K) + \theta_\kappa^2 K\big).$$

Let (see (2.1) with θ replaced by θ_κ)

$$\varphi_\kappa(t) = \frac{\theta_\kappa t f(t) + g(t)}{1 + \theta_\kappa t}\,, \quad t \in [0,1], \quad \kappa > 0.$$

Then $\varphi_\kappa(0) = 0$, $\varphi_\kappa(1) = 1$ and $\varphi_{\kappa_1}(t) < \varphi_{\kappa_2}(t)$ for $t \in (0,1)$, $0 < \kappa_1 < \kappa_2$.

By Theorem 3.1 (with r replaced by κr), for $\kappa > 0$ there exists a unique solution y_κ of problem (4.1), (1.3) and

$$g(t) < \varphi_\kappa(t) \le y_\kappa(t) < f(t), \quad t \in (0,1), \quad \kappa > 0. \tag{4.2}$$

Lemma 4.1. *Let* $0 < \kappa_1 < \kappa_2$*. Then* $y_{\kappa_1}(t) < y_{\kappa_2}(t)$ *for* $t \in (0,1)$*.*

Lemma 4.2. *Let* $\kappa \in (0,\infty)$ *and let* $\lim\limits_{n\to\infty} \lambda_n = \kappa$*. Then*

$$\lim_{n\to\infty} y_{\lambda_n}(t) = y_\kappa(t) \ \ \textit{locally uniformly on } (0,1].$$

Theorem 4.1. *Let* $h \in \mathcal{B}$ *and* $\Lambda \in \mathcal{S}$*. Then problem* (1.2), (1.3) *has a unique solution* y*. Moreover,* $g(t) < y(t) < f(t)$ *for* $t \in (0,1)$*.*

Sketch of proof. Let $\phi : (0,\infty) \to (0,\infty)$ be defined by the formula

$$\phi(\kappa) = h(\Lambda(y_\kappa)).$$

Then ϕ is continuous and decreasing and problem (1.2), (1.3) has a solution if and only if the equation $\phi(\kappa) = \kappa$ has a solution in $(0,\infty)$.

Consequently, in order to prove the existence and uniqueness of solutions to problem (1.2), (1.3), we have to show that the equation $\phi(\kappa) = \kappa$ has a unique solution in $(0,\infty)$.

By (4.2), $\Lambda(g) < \Lambda(y_\kappa) < \Lambda(f)$ for $\kappa > 0$. Let $\gamma_* = h(\Lambda(g))$ and $\gamma^* = h(\Lambda(f))$. Then $\gamma_* > \phi(\kappa) > \gamma^*$ for $\kappa > 0$. In particular, $\gamma_* > \phi(\gamma_*)$ and $\gamma^* < \phi(\gamma^*)$. Let $\Psi(\gamma) = \gamma - \phi(\gamma)$. Then Ψ is continuous and increasing on $(0,\infty)$ and since $\Psi(\gamma_*) > 0$, $\Psi(\gamma^*) < 0$, $\gamma_* > \gamma^*$ we conclude that $\Psi(\gamma) = 0$ for a unique $\gamma \in (0,\infty)$, that is, the equation $\phi(\kappa) = \kappa$ has a unique solution in $(0,\infty)$.

If y is a solution of (1.2), (1.3) and $\kappa = h(\Lambda(y))$, then $y = y_\kappa$ and (see (4.2)) $g < y < f$ on $(0,1)$. ☐

Example 4.1. Let $h(v) = (1-v)/v$, $\Lambda(x) = (x(\xi))^\nu/(1 + (x(\xi))^\nu)$, where $\xi \in (0,1)$ and $\nu > 0$. Then $h \in \mathcal{B}$, $\Lambda \in \mathcal{S}$ and $h(\Lambda(x)) = 1/(x(\xi))^\nu$. Let a, b be from Example 3.1. By Theorem 4.1, the functional differential equation

$$y'(t) = \frac{r(t)(t^a - y(t))}{(y(\xi))^\nu t^\eta (y(t) - t^b)}$$

has a unique solution y satisfying (1.3). Moreover, $t^b < y(t) < t^a$ on $(0,1)$.

Example 4.2. Let $f(t) = t(1 - \ln t)$, $g(t) = t^b$, $h(v) = -\ln v$ and $\Lambda(x) = \int_0^1 |p(s)|(x(s))^\nu\,\mathrm{d}s$, where $b \ge 1$, $\nu > 0$ and $p \in C[0,1]$, $|p| \le 1$. Then f, g satisfy the condition (H), $h \in \mathcal{B}$, $\Lambda \in \mathcal{S}$ and $h(\Lambda(x)) = -\ln \int_0^1 |p(s)|(x(s))^\nu\,\mathrm{d}s$. By Theorem 4.1, there exists a unique solution y of the equation

$$y'(t) = \ln \int_0^1 |p(s)|(x(s))^\nu\,\mathrm{d}s\, \frac{r(t)(y(t) - t(1 - \ln t))}{t^\eta (y(t) - t^b)}\,,$$

satisfying condition (1.3). Moreover, $t^b < y(t) < t(1 - \ln t)$ for $t \in (0,1)$.