The System of Singularly Perturbed Differential Equations with Differential Second-Order Turning Point

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It is known that a uniform asymptotic solution has not yet been constructed for the Orr–Sommerfeld equation with a differential turning point of the second order, although the problem was posed quite a long time ago [2]. In this direction, the authors have already constructed a uniform asymptotic solution for the system of third-order differential equation with a differential turning point of the first order [3–5], and now the central problem of researches are construct a uniform asymptotic solution for the system of fourth-order differential equation with a differential turning point which is equivalent to the equation of the Orr–Sommerfeld type.

Let us consider a system of SPDE with a stable turning point (SSPDE):

$$\varepsilon^3 y^{(4)}(x,\varepsilon) + a(x)y^2(x,\varepsilon) + b(x)y(x,\varepsilon) = h(x). \tag{1}$$

It should be noted that the problem of type (1) is the subject of research by many authors, i.g., the theories of R. Langer, W. Wasow, C. Lin, M. Nakano, and T. Nishimoto.

Similar to Langer's works, our idea for constructing a uniform asymptotic solution of (1) is accompanied by the general scheme of the theory developed for lower-order systems. The main difference and distinguishing feature of this work from previous ones are:

- the degenerate equation is a second-order differential equation;
- the turning point is located near the second derivative;
- a uniform asymptotic solution is constructed on the entire segment, including the turning point, for a system of singularly perturbed differential equations (SPDE)

  $$\varepsilon Y'(x,\varepsilon) - A(x,\varepsilon)Y(x,\varepsilon) = H(x), \tag{2}$$

  where

  $$A(x,\varepsilon) = A_0(x) + \varepsilon A_1 = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ -c(x) & -b(x) & -a(x) & 0 \end{pmatrix}$$

  when $\varepsilon \to 0$, $x \in [0,l]$, $Y(x,\varepsilon) = colomn(y_1(x,\varepsilon), y_2(x,\varepsilon), y_3(x,\varepsilon), y_4(x,\varepsilon))$ is an unknown vector function, $H(x) = colomn(0,0,0,h(x))$ is a given vector function.

The system (2) to be studied here will be investigated under the following conditions:

1. $\widetilde{a}(x), b(x), c(x), h(x) \in C^\infty[0;l]$;
2. $a(x) \equiv x\widetilde{a}(x)$, $\widetilde{a}(x) > 0$, $b(x) > 0$, $c(x) > 0$.

The scalar reduced equation for this matrix will be

$$x\widetilde{a}(x)\omega''(x)+b(x)\omega'(x)+c(x)\omega(x)=h(x).$$

Using the external feature of the degenerate equation, namely, that the turning point is located near the second-order derivative, we call such a turning point a **second-order differential turning point**.

The analysis of such kind of problems and construction of uniform asymptotic solution on a given segment with a turning point brings certain difficulties and problems in the construction of asymptotic forms [3].

The characteristic equation that corresponds to the SP system (2) is as follows:

$$|A(x,0)-\lambda E|=\begin{vmatrix}-\lambda & 0 & 0 & 0\\ 0 & -\lambda & 0 & 0\\ 0 & 0 & -\lambda & 1\\ -c(x) & -b(x) & -a(x) & -\lambda\end{vmatrix}=\lambda^2(-\lambda^2+x\widetilde{a}(x))=0.$$

The roots of this equation are: $\lambda_{1,2}=0$, $\lambda_{3,4}=\pm i\sqrt{x\widetilde{a}(x)}$.

In order to save all essential singular functions, that appear in the solution of system (2) due to the special point

$$t=\mu^{-2}\cdot\varphi(x),$$

$\mu=\varepsilon^{\frac{1}{3}}$, where exponent p and regularizing function $\varphi(x)$ are to be determined. Instead of function $Y(x,\varepsilon)$ the transformation function $\widetilde{Y}(x,t,\varepsilon)$ will be studied, also the transformation will be performed in such a way that the following identity is true

$$\widetilde{Y}_k(x,t,\varepsilon)\big|_{t=\varepsilon^{-p}\varphi(x)}\equiv Y_k(x,\varepsilon),$$

which is the necessary condition for suggested method. The vector equation (2) can be written as

$$\widetilde{L}_\varepsilon\widetilde{Y}_k(x,t,\varepsilon)\equiv\mu\varphi'\,\frac{\partial\widetilde{y}(x,t,\varepsilon)}{\partial t}+\mu^3\,\frac{\partial\widetilde{y}(x,t,\varepsilon)}{\partial x}-A(x,\varepsilon)\widetilde{Y}_k(x,t,\varepsilon)=H(x). \tag{3}$$

In order to construct the asymptotic expansion of the extended system (3) in the form of power series in a small parameter, it is necessary that $p=\frac{2}{3}$ and $\gamma=\frac{1}{3}$. In this case, to determine the coefficients of the corresponding series, we obtain regularly perturbed systems of equations with respect to the small parameter $\varepsilon>0$.

We describe the space of functions in which it will be possible to construct a uniform asymptotic solution of the transformed system (2).

The set $D_k(x)$ consists of the sum of elements $(\alpha_k(x,\varepsilon)U_i(t))$ and $(\varepsilon^\gamma\beta_k(x,\varepsilon)U_i'(t))$, where $i=1;2$, $k=\overline{1;4}$, which are solutions of the homogeneous vector equation

$$\varepsilon Y'(x,\varepsilon)-A(x,\varepsilon)Y(x,\varepsilon)=0.$$

The analytical functions $\alpha(x,\varepsilon)$ and $\beta(x,\varepsilon)$ depend on a small parameter $\varepsilon>0$ and are infinitely differentiable with respect to the variable $x\in[0;l]$. The functions U_i are Airy–Dorodnitsyn functions. The function $\psi(t)$, its derivative, and the function $\omega(x,\varepsilon)$ are responsible for the particular solution of the non-homogeneous system (2).

The element of this space has the form

$$\widetilde{Y}_k(x,t,\varepsilon)=\sum_{i=1}^{2}\big[\alpha_{ik}(x)U_i(t)+\beta_{ik}(x)U_i'(t)\big]+f_k(x)\nu(t)+\varepsilon^\gamma g_k(x)\nu'(t)+\omega_k(x).$$

Regularizing function is:

$$\varphi(x) = \left(\frac{3}{2}\int\limits_0^x \sqrt{x\widetilde{a}(x)}\,dx\right)^{\frac{2}{3}}.$$

The series

$$\widetilde{y}(x,t,\varepsilon) = \sum_{r=0}^{\infty}\varepsilon^r\left[\sum_{k=1}^{2}\left[\alpha_{kr}(x)U_k(\varepsilon^{-\frac{2}{3}}\cdot\varphi(x)) + \varepsilon^{\frac{1}{3}}\beta_{kr}(x)\,\frac{dU_k(\varepsilon^{\frac{2}{3}}\cdot\varphi(x))}{d(\varepsilon^{-\frac{2}{3}}\cdot\varphi(x))}\right]\right] + \sum_{r=0}^{\infty}\varepsilon^r\left[f_r(x)\nu(\varepsilon^{\frac{2}{3}}\cdot\varphi(x)) + \varepsilon^{\frac{1}{3}}g_r(x)\,\frac{d\nu(\varepsilon^{-\frac{2}{3}}\cdot\varphi(x))}{d(\varepsilon^{-\frac{2}{3}}\cdot\varphi(x))}\right] + \sum_{r=0}^{\infty}\varepsilon^r\omega_r(x)$$

is a formal solution of the SPDE system (2).

References

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