Prestack of Spaces of Pseudoanalytic Functions

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Abstract

A way to fit together various spaces of pseudoanalytic functions into a prestack over the category of open subsets of $\mathbb{C}$ is presented, which is done by way of defining suitable morphisms between the spaces of pseudoanalytic functions over the same open subset using pseudodifferentiation and showing that assigning to open sets the appropriate categories of spaces of pseudoanalytic functions is a 2-presheaf satisfying gluing of morphisms. A sufficient condition for a descent datum to be effective is presented.

It is shown how, after appropriately defining morphisms, spaces of pseudoanalytic functions discussed in detail in [1–3], can be put together into a prestack, as presented in [4–6]. Nowhere below does the sign $'$ denote derivative and no function is assumed to be complex-differentiable.

Assigning to each open subset $U \subseteq \mathbb{C}$ the set of generating pairs [2,3] defined on it, $GP(U)$, and to each inclusion of open subsets $U \subseteq V$ the natural restriction map, $GP(V) \to GP(U)$, defines a presheaf on $\mathbb{C}$, which we will denote by GP; so the presheaf $GP : \mathrm{Open}(\mathbb{C})^{\mathrm{op}} \longrightarrow \mathrm{Set}$ is defined as follows:

$$U \longmapsto GP(U) := \big\{(F,G) \in C(U,\mathbb{C})^2 \mid \ (F,G) \text{ is a generating pair}\big\}$$
$$(U \subseteq V) \longmapsto \Big(GP(V) \longrightarrow GP(U), \ \ (F,G) \longmapsto (F,G)\big|_U := \big(F|_U, G|_U\big)\Big).$$

This preasheaf is in fact a sheaf, due to the fact that the property of being a generating pair is local. For generating pairs $(F,G) \in GP(U)$ and $(F',G') \in GP(U)$ we denote the relation of equipotency [2,3]) by $(F,G) \sim_{\mathrm{ep}} (F',G')$ and if (F',G') is a successor of (F,G), we denote this by $(F,G) \preccurlyeq (F,G)$.

For each $U \subset \mathbb{C}$ we denote the map assigning to generating pairs defined on U their characteristic coefficients by

$$\mathrm{ch}_U : GP(U) \longrightarrow C(U,\mathbb{C}) \times C(U,\mathbb{C}) \times C(U,\mathbb{C}) \times C(U,\mathbb{C}),$$
$$(F,G) \longmapsto \bigg(\frac{\frac{\partial F}{\partial \overline{z}}\,\overline{G} - \overline{F}\,\frac{\partial G}{\partial \overline{z}}}{F\overline{G} - \overline{F}G}, \frac{F\,\frac{\partial G}{\partial \overline{z}} - \frac{\partial F}{\partial \overline{z}}\,G}{F\overline{G} - \overline{F}G}, \frac{\frac{\partial F}{\partial z}\,\overline{G} - \overline{F}\,\frac{\partial G}{\partial z}}{F\overline{G} - \overline{F}G}, \frac{F\,\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\,G}{F\overline{G} - \overline{F}G}\bigg),$$

and the associated projections onto components by $\mathrm{ch}_{U,k}$, $k \in \{1,2,3,4\}$.

Proposition 1. *The collection of maps*

$$\big\{\mathrm{ch}_U : GP(U) \longrightarrow C(U,\mathbb{C}) \times C(U,\mathbb{C}) \times C(U,\mathbb{C}) \times C(U,\mathbb{C})\big\}_{U \in \mathrm{Open}(\mathbb{C})}$$

defines a morphism of sheaves:

$$\mathrm{ch} : GP \longrightarrow C(-,\mathbb{C}) \times C(-,\mathbb{C}) \times C(-,\mathbb{C}) \times C(-,\mathbb{C}).$$

Naturally, the same holds for the components $\{\mathrm{ch}_{U,k}\}_{U\in\mathrm{Open}(\mathbb{C})}$, $k\in\{1,2,3,4\}$. The following lemma is important in considering a sufficient condition for a descent datum to be effective, given below.

Lemma 1. *Suppose that on each member of an open covering* $U=\bigcup\limits_{j\in J}U_j$ *a generating pair* $(F_j,G_j)\in GP(U_j)$ *is given such that for each* $j,k\in J$

$$(F_j,G_j)\big|_{U_j\cap U_k}\sim_{\mathrm{ep}}(F_k,G_k)\big|_{U_j\cap U_k}$$

and the functions resulting from gluing together their first two characteristic coefficients

$$\mathrm{ch}_1(F_j,G_j),\ \ \mathrm{ch}_2(F_j,G_j),\ \ j\in J$$

are admissible [3]*; then there exists a generating pair defined on the whole* U*,* $(F,G)\in GP(U)$ *such that for each* $j\in J$

$$(F,G)\big|_{U_j}\sim_{\mathrm{ep}}(F_j,G_j).$$

For each open subset $U\subset\mathbb{C}$ we will define the category of spaces of pseudoanalytic functions on U as follows: its collection of objects will be the set

$$\mathrm{Ob}(\mathcal{PA}(U)):=\big\{\mathcal{PA}_{F,G}(U)\ |\ \ (F,G)\in GP(U)\big\},$$

and for each pair of objects $\mathcal{PA}_{F,G}(U)$ and $\mathcal{PA}_{F',G'}(U)$ the set of morphisms between them $\mathrm{Mor}_{\mathcal{PA}(U)}(\mathcal{PA}_{F,G}(U),\mathcal{PA}_{F',G'}(U)$ will consist of all such maps of sets $\omega:\mathcal{PA}_{F,G}(U)\to\mathcal{PA}_{F',G'}(U)$ that the following condition holds: there exists an open covering $U=\bigcup\limits_{j\in J}U_j$ and a collection

$$\Big\{\Big(U_j,(F_{j,1},G_{j,1})\preccurlyeq(F_{j,2},G_{j,2})\preccurlyeq\cdots\preccurlyeq(F_{j,k_j},G_{j,k_j})\Big)\Big\}_{j\in J},$$

where

$$k:J\to\mathbb{N},\ \ j\mapsto k(j)\equiv k_j\ \ \text{and}\ \ (F_{j,m},G_{j,m})\in GP(U_j),\ \ m\in\{1,\dots,k_j\},$$

and also

$$(F_{j,1},G_{j,1})\sim_{\mathrm{ep}}(F,G)\big|_{U_j}\ \ \text{and}\ \ (F_{j,k_j},G_{j,k_j})\preccurlyeq(F',G')\big|_{U_j},\ \ j\in J,$$

such that for each pair $j,l\in J$ of indices and every pseudoanalytic function $f\in\mathcal{PA}_{F,G}(U)$ the following holds:

$$D_{(F_{j,1},G_{j,1})|_{U_j\cap U_l}\preccurlyeq\cdots\preccurlyeq(F_{j,k_j},G_{j,k_j})|_{U_j\cap U_l}}\big(f|_{U_j\cap U_l}\big)=D_{(F_{l,1},G_{l,1})|_{U_j\cap U_l}\preccurlyeq\cdots\preccurlyeq(F_{l,k_l},G_{l,k_l})|_{U_j\cap U_l}}\big(f|_{U_j\cap U_l}\big),$$

which condition allows the maps

$$D_{(F_{j,1},G_{j,1})\preccurlyeq\cdots\preccurlyeq(F_{j,k_j},G_{j,k_j})}:\mathcal{PA}_{F|_{U_j},G|_{U_j}}(U_j)\longrightarrow\mathcal{PA}_{F'|_{U_j},G'|_{U_j}}(U_j)$$

to be glued together into a map $\mathcal{PA}_{F,G}(U)\to\mathcal{PA}_{F',G'}(U)$ which should be equal to $\omega:\mathcal{PA}_{F,G}(U)\to\mathcal{PA}_{F',G'}(U)$; we say that ω is given locally by pseudodifferentiations, since it is characterized by the property

$$(\omega(f))\big|_{U_j}=D_{(F_{j,1},G_{j,1})\preccurlyeq\cdots\preccurlyeq(F_{j,k_j},G_{j,k_j})}\big(f|_{U_j}\big),$$

for every $f\in\mathcal{PA}_{F,G}(U)$. Besides such maps, for every $\mathcal{PA}_{F,G}(U)\in\mathrm{Ob}(\mathrm{PA}(U)$ we require that

$$\big(\mathrm{id}_{\mathcal{PA}_{F,G}(U)}:\mathcal{PA}_{F,G}(U)\longrightarrow\mathcal{PA}_{F,G}(U)\big)\in\mathrm{Mor}_{\mathcal{PA}(U)}\big(\mathcal{PA}_{F,G}(U),\mathcal{PA}_{F,G}(U)\big).$$

The composition of morphisms is defined as the usual compositions of maps and it is shown that composition of two maps defined as above is again of the same kind and thus the datum $\mathcal{PA}(U)$ is indeed a category. It naturally is a subcategory of the category of $\mathbb{R}$-linear spaces and $\mathbb{R}$-linear maps $\mathrm{Vec}_{\mathbb{R}}$. One has the following lemma, used later on:

Lemma 2. *In the category $\mathcal{PA}(U)$ only the identity morphisms*

$$\mathrm{id}_{\mathcal{PA}_{F,G}(U)} \in \mathrm{Mor}_{\mathcal{PA}(U)}\big(\mathcal{PA}_{F,G}(U), \mathcal{PA}_{F,G}(U)\big), \quad \mathcal{PA}_{F,G}(U) \in \mathrm{Ob}(\mathcal{PA}(U))$$

are isomorphisms.

One may extend the assignment to each open subset $U \subseteq \mathbb{C}$ of its appropriate category $\mathcal{PA}(U)$ to a strict 2-preasheaf

$$\mathcal{PA} : \mathrm{Open}(\mathbb{C})^{\mathrm{op}} \to \mathrm{Cat}, \ \ U \mapsto \mathcal{PA}(U)$$

assigning to the morphism $\iota_{U,V}$ associated to each inclusion $U \subseteq V$ the functor

$$\mathcal{PA}(\iota_{U,V}) \equiv \iota^*_{U,V} : \mathcal{PA}(V) \longrightarrow \mathcal{PA}(U),$$

defined on objects by

$$\mathcal{PA}_{F,G}(V) \longmapsto \iota^*_{U,V}\left(\mathcal{PA}_{F,G}(V)\right) := \mathcal{PA}_{F|_U,G|_U}(U),$$

and on morphisms as follows: for a morphism $\omega \in \mathrm{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F,G}(V), \mathcal{PA}_{F',G'}(V))$ given by the open covering $V = \bigcup\limits_{j\in J} V_j$ and the collection

$$\big\{(F_{j,1}, G_{j,1}) \preccurlyeq \cdots \preccurlyeq (F_{j,k_j}, G_{j,k_j})\big\}_{j\in J}$$

define the morphism $\iota^*_{U,V}(\omega)$ by the open covering $U = \bigcup\limits_{j\in J} U_j$, where $U_j := V_j \cap U$m and the collection

$$\Big\{(F_{j,1}, G_{j,1})\big|_{U_j} \preccurlyeq \cdots \preccurlyeq (F_{j,k_j}, G_{j,k_j})\big|_{U_j}\Big\}_{j\in J};$$

this map is well defined since for each $j \in J$ hold the following implications

$$(F_{j,k_1}, G_{j,k_1}) \sim_{\mathrm{ep}} (F,G)\big|_{V_j} \implies (F_{j,k_1}, G_{j,k_1})\big|_{U_j} \sim_{\mathrm{ep}} (F,G)\big|_{U_j}$$

and

$$(F_{j,k_j}, G_{j,k_j}) \preccurlyeq (F',G')\big|_{V_j} \implies (F_{j,k_j}, G_{j,k_j})\big|_{U_j} \preccurlyeq (F',G')\big|_{U_j},$$

and the equality

$$D_{(F_{j,1},G_{j,1})|_{V_j\cap V_l}\preccurlyeq\cdots\preccurlyeq(F_{j,k_j},G_{j,k_j})|_{V_j\cap V_l}} = D_{(F_{l,1},G_{l,1})|_{V_j\cap V_l}\preccurlyeq\cdots\preccurlyeq(F_{l,k_l},G_{l,k_l})|_{V_j\cap V_l}}$$

implies, due to the locality of pseudodifferentiations,

$$D_{(F_{j,1},G_{j,1})|_{U_j\cap U_l}\preccurlyeq\cdots\preccurlyeq(F_{j,k_j},G_{j,k_j})|_{U_j\cap U_l}} = D_{(F_{l,1},G_{l,1})|_{U_j\cap U_l}\preccurlyeq\cdots\preccurlyeq(F_{l,k_l},G_{l,k_l})|_{U_j\cap U_l}},$$

therefore,

$$\iota^*_{U,V}(\omega) \in \mathrm{Mor}_{\mathcal{PA}(U)}\big(\mathcal{PA}_{F|_U,G|_U}(U), \mathcal{PA}_{F'|_U,G'|_U}(U)\big),$$

and thus we get the map

$$\mathrm{Mor}_{\mathcal{PA}(V)}(\mathcal{PA}_{F,G}(V), \mathcal{PA}_{F',G'}(V)) \longrightarrow \mathrm{Mor}_{\mathcal{PA}(U)}\big(\mathcal{PA}_{F|_U,G|_U}(U), \mathcal{PA}_{F'|_U,G'|_U}(U)\big),$$
$$\omega \longmapsto \iota^*_{U,V}(\omega).$$

Proposition 2. *The assignment*

$$\mathcal{PA} : \mathrm{Open}(\mathbb{C})^{\mathrm{op}} \longrightarrow \mathrm{Cat}$$

is a strict 2-presheaf.

Moreover, one has the following:

Theorem 1. *The* 2*-presheaf* $\mathcal{PA} : \mathrm{Open}(\mathbb{C})^{\mathrm{op}} \to \mathrm{Cat}$ *is a prestack.*

For the total category of the associated fibered category

$$p : \int\limits_{\mathrm{Open}(\mathbb{C})^{\mathrm{op}}} \mathcal{PA} \longrightarrow \mathrm{Open}(\mathbb{C})$$

we will use the notation

$$PSA := \int\limits_{\mathrm{Open}(\mathbb{C})^{\mathrm{op}}} \mathcal{PA}.$$

Thus, an object of PSA is given by an open subset $U \subseteq \mathbb{C}$ and a space of pseudoanalytic functions defined on it $\mathcal{PA}_{F,G}(U)$, and so is a pair

$$(U, \mathcal{PA}_{F,G}(U)) \in \mathrm{Ob}(PSA), \ \ U \subseteq \mathbb{C}, \ \ (F,G) \in GP(U);$$

If $U \nsubseteq V$, the set of morphisms between $(U, \mathcal{PA}_{F,G}(U))$ and $(V, \mathcal{PA}_{F',G'}(V))$ is empty, whereas if $U \subseteq V$, then each morphism between $(U, \mathcal{PA}_{F,G}(U))$ and $(V, \mathcal{PA}_{F',G'}(V))$ is given by a morphism in $\mathcal{PA}(U)$

$$\omega : \mathcal{PA}_{F'|_U, G'|_U}(U) \longrightarrow \mathcal{PA}_{F,G}(U) \in \mathrm{Mor}_{\mathcal{PA}(U)}\big(\mathcal{PA}_{F'|_U, G'|_U}(U), \mathcal{PA}_{F,G}(U)\big),$$

and so,

$$\mathrm{Mor}_{PSA}\big((U, \mathcal{PA}_{F,G}(U)), (V, \mathcal{PA}_{F',G'}(V))\big) \cong \mathrm{Mor}_{\mathcal{PA}(U)}\big(\mathcal{PA}_{F'|_U, G'|_U}(U), \mathcal{PA}_{F,G}(U)\big);$$

for a morphism between $(U, \mathcal{PA}_{F,G}(U))$ and $(V, \mathcal{PA}_{F',G'}(V))$ and a morphism between $(V, \mathcal{PA}_{F',G'}(V))$ and $(W, \mathcal{PA}_{F'',G''}(W))$, with $U \subseteq V \subseteq W$, given by, respectively, the morphisms:

$$\begin{aligned} \omega &\in \mathrm{Mor}_{\mathcal{PA}(U)}\big(\mathcal{PA}_{F'|_U, G'|_U}(U), \mathcal{PA}_{F,G}(U)\big), \\ \eta &\in \mathrm{Mor}_{\mathcal{PA}(V)}\big(\mathcal{PA}_{F''|_V, G''|_V}(V), \mathcal{PA}_{F',G'}(V)\big), \end{aligned}$$

the composition is given by:

$$\omega \circ \iota^*_{U,V}(\eta) : \mathcal{PA}_{F''|_U, G''|_U}(U) \longmapsto \mathcal{PA}_{F,G}(U) \in \mathrm{Mor}_{\mathcal{PA}(U)}\big(\mathcal{PA}_{F''|_U, G''|_U}(U), \mathcal{PA}_{F,G}(U)\big).$$

Observation 1. A globally defined generating pair $(F,G) \in GP(\mathbb{C})$ naturally induces a section of the prestack $p : PSA \to \mathrm{Open}(\mathbb{C})$:

$$s_{F,G} : \mathrm{Open}(\mathbb{C}) \longrightarrow PSA, \quad U \longmapsto (U, \mathcal{PA}_{F|_U, G|_U}(U)),$$

which assigns to the morphism associated with the inclusion $U \subseteq V$ the identity $\mathrm{id}_{\mathcal{PA}_{F|_U, G|_U}(U)}$; since equipotent generating pairs define the same section and also, conversely, if two generating pairs define the same section, then they are equipotent, i.e., for generating pairs $(F,G), (F',G') \in GP(\mathbb{C})$

$$(F,G) \sim_{\mathrm{ep}} (F',G') \iff s_{F,G} = s_{F',G'},$$

this assignment factors through the assignment

$$GP(\mathbb{C})/\sim_{\mathrm{ep}} \longrightarrow \Gamma(\mathrm{Open}(\mathbb{C}), PSA), \ \ [(F,G)] \longmapsto s_{F,G}.$$

Observation 2. Consider a descent datum for the prestack, that is, an open covering $U=\bigcup\limits_{j\in J}U_j$, on each U_j an object from the fiber above it $\mathcal{PA}_{F_j,G_j}(U_j)\in\mathrm{Ob}(\mathcal{PA}(U_j))$ and for each pair $j,k\in J$ an isomorphism in the category $\mathcal{PA}(U_j\cap U_k)$ between the objects resulting from the appropriate restrictions

$$\iota^*_{V_j\cap V_k,V_j}(\mathcal{PA}_{F_j,G_j}(U_j))=\mathcal{PA}_{F_j|_{U_j\cap U_k},G_j|_{U_j\cap U_k}}(U_j\cap U_k),$$
$$\iota^*_{V_j\cap V_k,V_k}(\mathcal{PA}_{F_k,G_k}(U_k))=\mathcal{PA}_{F_k|_{U_j\cap U_k},G_k|_{U_j\cap U_k}}(U_j\cap U_k),$$

i.e., an invertible morphism in this category

$$\theta_{jk}:\mathcal{PA}_{F_j|_{U_j\cap U_k},G_j|_{U_j\cap U_k}}(U_j\cap U_k)\longrightarrow\mathcal{PA}_{F_k|_{U_j\cap U_k},G_k|_{U_j\cap U_k}}(U_j\cap U_k),$$

but, by Lemma 2, each θ_{jk} must be an identity morphism, which means in particular that for each $j,k\in J$

$$\mathcal{PA}_{F_j|_{U_j\cap U_k},G_j|_{U_j\cap U_k}}(U_j\cap U_k)=\mathcal{PA}_{F_k|_{U_j\cap U_k},G_k|_{U_j\cap U_k}}(U_j\cap U_k),$$

and this in turn means that the generating pairs for this spaces are equipotent:

$$\big(F_j|_{U_j\cap U_k},G_j|_{U_j\cap U_k}\big)\sim_{\mathrm{ep}}\big(F_k|_{U_j\cap U_k},G_k|_{U_j\cap U_k}\big).$$

If the functions resulting from gluing together the first two characteristic coefficients of these generating pairs $\mathrm{ch}_1(F_j,G_j)$, $\mathrm{ch}_2(F_j,G_j)$, $j\in J$, defined on the whole U, are admissible, then the conditions of Lemma 1 are satisfied, by which lemma there exists a generating pair $(F,G)\in GP(U)$ such that

$$\big(F|_{U_j},G|_{U_j}\big)\sim_{\mathrm{ep}}(F_j,G_j),$$

for each $j\in J$, therefore, one has the associated object in the fiber over the entire U, $\mathcal{PA}_{F,G}(U)\in\mathrm{Ob}(\mathcal{PA}(U))$, for which

$$\mathcal{PA}_{F|_{U_j},G|_{U_j}}(U_j)=\mathcal{PA}_{F_j,G_j}(U_j),$$

for each $j\in J$, which means that the descent datum

$$\Big\{\Big(\mathcal{PA}_{F_j,G_j}(U_j),\theta_{jk}=\mathrm{id}_{\mathcal{PA}_{F_j|_{U_j\cap U_k},G_j|_{U_j\cap U_k}}(U_j\cap U_k)}\Big)\Big\}_{j\in J}$$

is effective.

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