Mean-Square Exponential Dichotomy for Infinite-Dimensional Stochastic Systems

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1 Introduction and problem formulation

The study of exponential dichotomy (ED) properties for linear evolutionary equations plays a key role in the analysis of stability and asymptotic behavior of solutions. Classical results, beginning with the works of Daleckii and Krein, describe the conditions for the existence of dichotomy for deterministic systems [1].

In stochastic systems, the natural generalization is the mean-square (MS) exponential dichotomy. However, the situation becomes fundamentally more complex in infinite-dimensional Hilbert spaces. Even in the presence of a deterministic ED for an operator A, the addition of small (in the operator sense) stochastic or deterministic perturbations can lead to its destruction. This effect is particularly noticeable for unbounded operators. Thus, the robustness of the classical exponential dichotomy under small perturbations is not guaranteed in the infinite-dimensional case.

The main objective of this paper is to establish and analyze sufficient conditions for the existence of a *weak mean-square dichotomy* for a class of stochastic evolutionary equations. Throughout the paper we consider the stochastic system

$$\mathrm{d}x(t) = A(t)x(t)\,\mathrm{d}t + D(t)x(t)\,\mathrm{d}W(t), \quad t \geq 0, \tag{1.1}$$

on a separable Hilbert space H, where the components of the equation satisfy the following assumptions:

(A) $A : D(A) \subset H \to H$ is a closed (possibly unbounded) linear operator that generates a strongly continuous (i.e. C_0) semigroup $\{S(t)\}_{t\geq 0}$ on H.

(B) $B : [0,\infty) \to L(H)$ is a measurable and locally bounded mapping whose operator norm is uniformly bounded:

$$\sup_{t\geq 0} \|B(t)\| < \infty.$$

(D) $D : [0,\infty) \to L(H, L^2(U_0, H))$ is a measurable and locally bounded mapping such that

$$\sup_{t\geq 0} \|D(t)\|_{L(H,L^2(U_0,H))} < \infty.$$

Here $U_0 = Q^{1/2}U$, and $L^2(U_0, H)$ denotes the space of Hilbert–Schmidt operators from U_0 into H.

(W) $W(t)$ is a Q-Wiener process on a separable Hilbert space U, where the covariance operator $Q \in L(U)$ is self-adjoint, non-negative, and of trace class ($\operatorname{Tr} Q < \infty$). Let $\{e_j\}_{j\ge1}$ be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues $\{\lambda_j\}_{j\ge1}$. If $\{\beta_j(t)\}_{j\ge1}$ is a family of independent real-valued standard Brownian motions adapted to a filtration $\{\mathcal{F}_t\}_{t\ge0}$, then the series

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \beta_j(t)\, e_j$$

defines the associated Q-Wiener process.

To understand the limitation, we first define the classical concept.

Definition 1.1 (Mean-Square Exponential Dichotomy)**.** System (1.1) is called *exponentially dichotomous in the mean square* (for $t \ge t_0$) if there exists a direct sum decomposition $H = H^- \oplus H^+$, and positive constants $K_1, K_2, \gamma_1, \gamma_2 > 0$ such that for any $t \ge \tau \ge t_0$:

1. if $x(t_0) \in H^-$, then
$$\mathbb{E}|x(t)|^2 \le K_1 e^{-\gamma_1(t-\tau)}\, \mathbb{E}|x(\tau)|^2;$$
2. if $x(t_0) \in H^+$, then
$$\mathbb{E}|x(t)|^2 \ge K_2 e^{\gamma_2(t-\tau)}\, \mathbb{E}|x(\tau)|^2.$$

As noted, this classical definition is not robust; small perturbations can destroy the dichotomous structure. To overcome this limitation, we introduce a modified, weaker definition.

Definition 1.2 (Weak Mean-Square Exponential Dichotomy)**.** System (1.1) is called *weakly exponentially dichotomous in the mean square* (for $t \ge t_0$) if there exists a direct sum decomposition $H = H^- \oplus H^+$, and positive constants $K, \gamma_1, \gamma_2 > 0$ such that for any $t \ge \tau \ge t_0$:

1. if $x(t_0) \in H^-$, then
$$\mathbb{E}|x(t)|^2 \le K e^{-\gamma_1(t-\tau)}\, \mathbb{E}|x(\tau)|^2;$$
2. if $x(t_0) \in H^+$ and $x(t_0) \ne 0$, then
$$\lim_{t\to\infty} \mathbb{E}|x(t)|^2 e^{-\gamma_2 t} = \infty.$$

This new definition retains the exponential decay in the stable subspace but critically relaxes the condition on the unstable subspace. It only requires an asymptotic growth that eventually overtakes any exponential with rate γ_2, rather than a strict exponential bound from below. This allows the property to be preserved under the small, time-damping perturbations investigated in this paper, which is demonstrated by our main theorem and the provided counterexamples.

2 Main results

Our main result establishes that this weak dichotomy is preserved under perturbations that vanish over time. We consider system (1.1) in the form

$$\mathrm{d}x(t) = (A + B(t))x(t)\,\mathrm{d}t + D(t)x(t)\,\mathrm{d}W(t), \tag{2.1}$$

where A is the unperturbed operator and $B(t)$, $D(t)$ are perturbations.

Theorem 2.1. *Let the closed operator A in equation* (2.1) *be exponentially dichotomous (in the classical deterministic sense). If the norms of the operators $B(t), D(t)$ tend to zero, in the sense that*

$$\|B(t)\|, \|D(t)\|_{\mathrm{HS}} \to 0 \ \text{ as } \ t \to \infty, \tag{2.2}$$

then there exists a time $t_0 > 0$ such that the perturbed system (2.1) *is weakly exponentially dichotomous in the mean square on the half-axis $t \geq t_0$.*

Sketch of Proof. The full proof is quite technical. The main idea is as follows.

1. Since the unperturbed operator A possesses a classical exponential dichotomy, there exist projections P_- (onto the stable subspace) and $P_+ = I - P_-$ (onto the unstable subspace) and constants $N \geq 1$, $\alpha > 0$ satisfying the standard exponential estimates.

2. The solution $y(t)$ of the perturbed system (2.1) is decomposed using these projections: $y(t) = P_- y(t) + P_+ y(t)$. By applying the variation of constants formula and Itô's isometry, a set of fundamental integral inequalities is derived for $\mathbb{E}|P_- y(t)|^2$ and $\mathbb{E}|P_+ y(t)|^2$.

3. These estimates are combined to yield a central inequality that relates the mean-square norm of the solution at different time points $t_1 \leq t' \leq t_2$:

$$\mathbb{E}(|y(t')|^2) \leq 6N^2 e^{-2\alpha(t_2 - t')}\mathbb{E}(|y(t_2)|^2) + 6N^2 e^{-2\alpha(t' - t_1)}\mathbb{E}(|y(t_1)|^2) + \beta' \cdot \sup_{\tau} \mathbb{E}(|y(\tau)e^{-\alpha'|t' - \tau|}|^2),$$

where $\beta' \to 0$ as $t_0 \to \infty$ due to the damping condition (2.2).

4. The proof proceeds by contradiction. It is assumed that the system does not possess a weak MS dichotomy. This implies the existence of a non-trivial solution $y(t)$ that is "trapped" between two exponentials, i.e., it does not grow or decay exponentially.

5. An auxiliary lemma is proven, which shows that under the conditions of the theorem, any solution must eventually be either exponentially increasing or exponentially decreasing.

6. This lemma is used to show that the "trapped" solution assumed in step 4) leads to a contradiction with the central inequality from step 3) for a sufficiently large time t_0 (where β' becomes small enough).

7. Finally, the stable subspace H^- for the weak dichotomy is explicitly constructed as the set of all initial values $y(0)$ that generate bounded solutions ($\mathbb{E}(|y(t)|^2) \leq C$). This set is shown to be a closed subspace and to satisfy the exponential decay property (Part 1 of Definition 1.2). The unstable subspace H^+ is its complement, and Part 2 of the definition follows from the contradiction argument. □

Remark 2.2. The analysis of the proof shows that the damping condition (2.2) can be replaced by a smallness condition in the uniform norm. If $\sup_{t \geq 0}(\|B(t)\| + \|D(t)\|_{\mathrm{HS}})$ is sufficiently small, the weak MS dichotomy is preserved globally (i.e., from $t_0 = 0$).

Remark 2.3. If A has a dichotomy with constants (N, α), then for sufficiently small perturbations, the new exponential rate α' of the weak dichotomy for system (2.1) can be made arbitrarily close to α.

3 Example: failure of strong dichotomy

The following example demonstrates why the classical (strong) MS dichotomy (Definition 1.1) fails and motivates the need for our weak definition.

Example 3.1. Let ℓ^2 be the Hilbert space of real sequences with orthonormal basis $\{e_k\}_{k=1}^{\infty}$. We define a linear continuous operator $A \in L(\ell^2)$ by its action on the basis vectors

$$Ae_1 = e_1, \ \ Ae_k = -\alpha_k \, e_k \ \text{ for } \ k \geq 2,$$

where $\alpha_k > 0$ and $\inf\limits_{k\geq 2}(\alpha_k) = \alpha_{\text{inf}} > 0$.

The unperturbed system $dx = Ax \, dt$ clearly possesses an exponential dichotomy. The unstable subspace is $H^+ = \operatorname{span}\{e_1\}$ and the stable subspace is $H^- = \overline{\operatorname{span}}\{e_k : \ k \geq 2\}$.

Now, consider a stochastic perturbation $D(t)$ defined by

$$D(t) \, e_{k+1} = \frac{f(t-(k-1))}{k} \, e_1, \ \ D(t) \, e_1 = 0,$$

where $f(\cdot)$ is a bounded "bump" function (e.g., $f(t) = (1 - |2t-1|) I_{[0,1]}(t)$). We analyze the perturbed system

$$dx = Ax \, dt + D(t)x \, d\beta(t), \tag{3.1}$$

where $\beta(t)$ is a 1D Wiener process.

Note that this perturbation is small and vanishes at infinity:

$$\|D(t)\| \leq \frac{1}{t} \to 0 \ \text{ as } \ t \to \infty.$$

We can show that for the perturbed system (3.1):

(a) All non-trivial solutions are unbounded in the mean square.

By analyzing the solution components, it can be shown that $\mathbb{E}|y(t)|^2 \to \infty$ for any $y(0) \neq 0$.

(b) The system is NOT exponentially dichotomous in the strong sense.

From (a), any potential unstable subspace must be the entire space, $H^+ = H$. If the system had a strong MS dichotomy, there would exist constants $N, \delta > 0$ such that for all y_0 with $|y_0| = 1$, we have $\mathbb{E}|y(t, y_0)|^2 \geq N \, e^{\delta t}$.

However, by choosing the initial condition $y_0 = e_k$ at time $t_0 = k-2$, the solution is purely decaying: $\mathbb{E}|y(t)|^2 = e^{-2\alpha_k(k-2)}$. This leads to the inequality $e^{-2\alpha_{inf}(k-2)} \geq N e^{\delta(k-2)}$, which is impossible for all k given any $N, \delta > 0$.

This contradiction proves that system (3.1) does not possess a strong MS dichotomy, even though the perturbation is small and vanishes. This example satisfies the conditions of Theorem 2.1 and thus possesses a *weak* MS dichotomy.

Remark 3.2. It should be noted that properties (a) and (b) of the counterexample can also hold in the case of a compact semigroup $\{S(t)\}_{t>0}$. Indeed, by choosing the sequence $\alpha_k = k$, we obtain an analogous example where the operator A generates a semigroup of Hilbert–Schmidt operators which are compact.

Remark 3.3. Example 3 demonstrates that the decomposition $H = H^- \oplus H^+$ can change radically under arbitrarily small random perturbations. In the unperturbed system, the unstable component is $H^+ = \operatorname{span}\{e_1\}$, while for the perturbed system, it becomes the entire space: $H^+ = H$. This highlights the instability of the subspaces themselves.

Example 3.4 (Application to a Stochastic PDE)**.** Consider the stochastic problem with Dirichlet boundary conditions on $x \in (0, \pi)$:

$$du(t,x) = (u_{xx}(t,x) + \lambda u(t,x))\,dt + f(t)u(t,x)\,d\beta(t),$$
$$u(t,0) = 0, \quad u(t,\pi) = 0,$$

where $\lambda \in \mathbb{R} \setminus \{n^2 : \ n \in \mathbb{N}\}$ and the continuous function $f(t) \to 0$ as $t \to \infty$.

This system is weakly exponentially dichotomous. We set $H = L^2(0,\pi)$ and $A = \frac{d^2}{dx^2} + \lambda I$. The spectrum $\sigma(A) = \{\lambda - n^2 : \ n \in \mathbb{N}\}$ is discrete and does not contain 0, so the unperturbed operator A is deterministically dichotomous. The result follows directly from Theorem 2.1.

4 Conclusion

In this work, we demonstrate that the classical mean-square exponential dichotomy (MS ED) is not robust under small perturbations in infinite-dimensional Hilbert spaces. This lack of robustness presents a significant challenge for the stability analysis of such systems.

To address this limitation, we propose a modified definition: the *weak mean-square exponential dichotomy*. This concept replaces the strict requirement of exponential growth on the unstable subspace with a more general asymptotic growth condition.

The main result of the paper is a theorem proving that this weak dichotomy is preserved (i.e., robust) under small, time-damping perturbations. This key finding establishes a robust framework for analyzing the long-term asymptotic behavior of solutions.

Furthermore, the provided counterexample highlights the necessity of this weaker definition. Our approach thus significantly expands the class of stochastic evolutionary equations for which an exponential decomposition of the phase space can be established.

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