Stability and Bifurcation Analysis of Delay Differential Models in Computing Systems

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Delay differential equations (DDEs) are widely used to model systems where the current state depends on past behavior. Introducing a delay can significantly affect system dynamics, often causing oscillations, bifurcations, or instability. These effects are particularly important in modeling computational processes and communication delays in computer systems. The mathematical foundations of delay equations were established in classical works [1], and further developed in the comprehensive monograph [3], which laid the groundwork for the modern theory of functional-differential equations. A detailed analysis of population models involving delays was presented in [5], demonstrating how time lags can generate periodic or complex behavior. Retarded dynamical systems and delay effects on stability, particularly their role in control theory, were investigated in [8,10]. Further advances were achieved in [4], where semi-discretization techniques for time-delay systems were proposed, and in [9], which presented an extensive overview of recent developments and open problems in delay system analysis.

Recent studies have focused on analytical and numerical aspects of the controlled perturbed functional-differential equations with delay. In particular, [2,6,7] developed representation formulas and investigated the continuity of solutions with respect to initial data for various classes of quasi-linear and neutral equations, including cases with discontinuous initial conditions.

Building upon these results, the present paper is devoted to the analysis of a nonlinear *retarded-type* delay differential equation of the form

$$\dot{x}(t) = -ax(t) + b\,\frac{x(t-\tau)}{1+(x(t-\tau))^p}\,, \quad a,b,p>0, \quad \tau>0, \tag{1}$$

which models delayed feedback effects arising in dynamic systems. Here $x(t)$ denotes the state variable (for example, CPU utilization, queue length, or network congestion level). The term $-ax(t)$ reflects the natural decay or dissipation of the process, while the delayed feedback term $\frac{bx(t-\tau)}{1+(x(t-\tau))^p}$ introduces a nonlinear response representing the influence of past states. The constant delay τ models the time required for the feedback to affect the present state.

The study aims to determine the equilibrium state, analyze its local stability with respect to the delay parameter, and identify conditions under which a Hopf bifurcation occurs, leading to oscillatory behavior. Analytical results will be accompanied by numerical simulations that illustrate the transition from stable equilibrium to periodic oscillations as the delay increases. This approach provides a unified nonlinear framework for exploring the influence of time delays on system stability and for bridging the theoretical analysis of delay equations with their practical applications in control and computational systems. Let $y(t) = x(t) - x^*$ denote a small perturbation from the steady state. Linearization around the equilibrium x^* then yields the standard form (see, e.g., [1, 3]):

$$\dot{y}(t) = -\alpha y(t) + \sum_{i=1}^{n} \beta_i y(t - \tau_i), \tag{2}$$

where

$$\alpha = a + \frac{p(x^*)^p}{(1 + (x^*)^p)^2} \sum_{i=1}^{n} b_i, \quad \beta_i = \frac{b_i}{1 + (x^*)^p}.$$

Remark 1. Here, α represents the effective damping combining natural decay and linearized nonlinear feedback, while β_i quantifies the influence of each delayed term.

Definition 1 (Delay-independent stability)**.** The equilibrium x^* is delay-independently stable if it remains asymptotically stable for all delays $\tau_i > 0$.

Theorem 1 (Delay-independent stability criterion)**.** *The equilibrium x^* of system* (2) *is asymptotically stable for all $\tau_i > 0$ if*

$$\alpha > \sum_{i=1}^{n} |\beta_i|. \tag{3}$$

Sketch of proof. The characteristic equation corresponding to (2) is

$$\lambda + \alpha - \sum_{i=1}^{n} \beta_i e^{-\lambda \tau_i} = 0.$$

If inequality (3) holds, then the real parts of all roots are negative for any $\tau_i > 0$, ensuring asymptotic stability independently of delay magnitudes. □

Remark 2. Condition (3) provides a conservative but easily verifiable stability bound.

When condition (3) is violated, the system may lose stability as delays increase.

Definition 2 (Hopf bifurcation)**.** A Hopf bifurcation occurs when a pair of complex-conjugate eigenvalues of the characteristic equation cross the imaginary axis as a delay parameter varies.

The characteristic equation corresponding to (2) is

$$\lambda + \alpha - \sum_{i=1}^{n} \beta_i e^{-\lambda \tau_i} = 0. \tag{4}$$

To analyze possible Hopf bifurcations, we set $\lambda = i\omega$ ($\omega > 0$) and separate the real and imaginary parts, obtaining

$$\alpha = \sum_{i=1}^{n} \beta_i \cos(\omega \tau_i), \tag{5}$$

$$\omega = -\sum_{i=1}^{n} \beta_i \sin(\omega \tau_i). \tag{6}$$

These equations represent the real and imaginary components of the characteristic equation, respectively, and are used to determine the critical frequency ω at which a Hopf bifurcation occurs.

Theorem 2 (Hopf bifurcation condition)**.** *A Hopf bifurcation occurs if there exists $\omega > 0$ satisfying* (5), (6) *and the transversality condition*

$$\frac{d}{d\tau_i}\,\Re(\lambda)\Big|_{\lambda=i\omega} \neq 0$$

is fulfilled at the corresponding critical delay $\tau_{i,c}$.

Proof. Let $\lambda = \lambda(\tau_i)$ be a root of the characteristic equation (4) with $\lambda(\tau_{i,c}) = i\omega$. Differentiating the characteristic equation implicitly with respect to τ_i gives

$$\frac{d\lambda}{d\tau_i} = -\frac{\partial F/\partial\tau_i}{\partial F/\partial\lambda}\,, \quad F(\lambda,\tau_1,\dots,\tau_n) := \lambda + \alpha - \sum_{k=1}^{n} \beta_k e^{-\lambda\tau_k}.$$

Computing the partial derivatives yields

$$\frac{\partial F}{\partial\lambda} = 1 + \sum_{k=1}^{n} \beta_k\tau_k e^{-\lambda\tau_k}, \quad \frac{\partial F}{\partial\tau_i} = \beta_i\lambda e^{-\lambda\tau_i},$$

and hence

$$\frac{d\lambda}{d\tau_i} = \frac{\beta_i\lambda e^{-\lambda\tau_i}}{1 + \sum\limits_{k=1}^{n} \beta_k\tau_k e^{-\lambda\tau_k}}\,.$$

Taking the real part at $\lambda = i\omega$ gives the rate at which the eigenvalue crosses the imaginary axis, which constitutes the transversality condition required for a Hopf bifurcation. □

For $n = 1$, system (2) reduces to

$$\dot{y}(t) = -\alpha y(t) + \beta y(t-\tau),$$

with characteristic equation $\lambda + \alpha - \beta e^{-\lambda\tau} = 0$. Setting $\lambda = i\omega$ gives

$$\begin{cases} \alpha = \beta\cos(\omega\tau), \\ \omega = -\beta\sin(\omega\tau). \end{cases}$$

The corresponding critical delays are

$$\tau_k = \frac{1}{\omega}\Big[\arctan\Big(-\frac{\omega}{\alpha}\Big) + k\pi\Big], \quad k \in \mathbb{Z}. \tag{7}$$

Remark 3. Equation (7) provides a criterion for the onset of oscillations in the single-delay case. Multiple critical delays may exist, producing successive transitions between stable and oscillatory behavior.

For $n > 1$, the interplay between multiple delayed feedback channels can generate rich and complex dynamics, including quasi-periodic oscillations, multistability, or even chaotic behavior. In this scenario, the coupled system (5), (6) is generally solved numerically for the frequency ω and corresponding delays τ_i.

Continuation and bifurcation analysis tools, such as DDE-Biftool, allow tracing Hopf branches in the (τ_i, b_i)-parameter plane and can reveal coexistence of stable equilibria and periodic orbits.

Remark 4. Increasing the number of feedback delays typically enriches system dynamics. Interacting delay channels can induce amplitude modulation, multiple stable solutions, or complex oscillatory patterns. These effects are particularly relevant for distributed computing networks, where communication, scheduling, and resource latencies coexist.

The Hopf bifurcation analysis delineates the boundary between stable and oscillatory regimes. When effective damping dominates ($\alpha > \sum_{i=1}^{n} |\beta_i|$), stability is guaranteed regardless of delay magnitudes. However, as feedback strength or delays increase, oscillatory modes can appear, leading to self-sustained periodic dynamics.

This mechanism explains why excessive communication latency, memory congestion, or delayed process scheduling can produce cyclic overloading in practical computing systems. Understanding these patterns is crucial for designing delay-resilient architectures and predicting performance in complex networks.

To visually confirm the analytical predictions and illustrate the dynamical consequences of delayed feedback, numerical simulations of the nonlinear DDE model (1) are performed. The parameters are chosen to highlight the transition from stability to oscillation:

$$a = 1.0, \quad b_1 = 2.5, \quad b_2 = 1.5, \quad p = 3,$$

with delays set as $\tau_1 = \tau_2 = \tau$ and τ used as the primary bifurcation parameter.

For these parameters, the equilibrium is

$$x^* \approx 1.44.$$

While the delay-independent stability condition ($\alpha > \sum |\beta_i|$) holds, the system is expected to lose stability via a supercritical Hopf bifurcation as τ increases. The critical delay is found numerically to be near

$$\tau_c \approx 1.5.$$

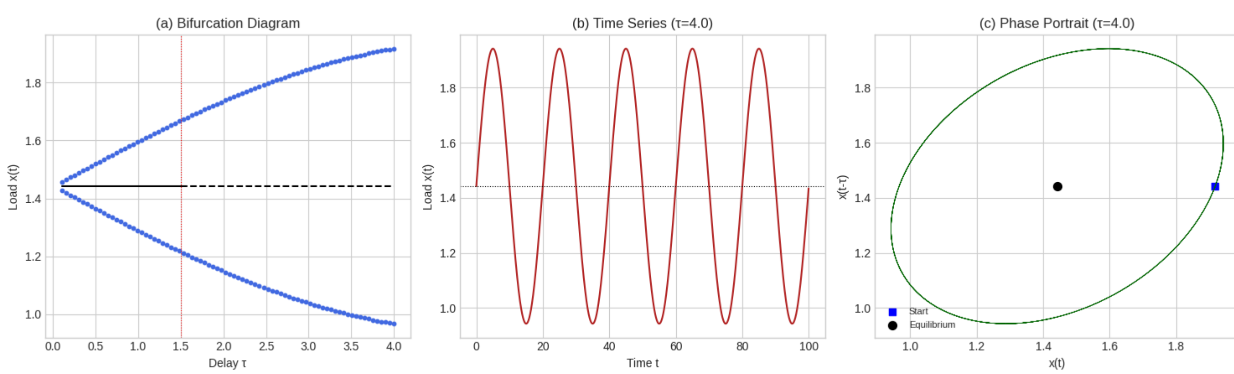

Figure 1. Numerical simulation results showing the effect of increasing delay τ ($a = 1.0$, $b_1 = 2.5$, $b_2 = 1.5$, $p = 3$). **(a) Bifurcation Diagram:** The equilibrium branch loses stability at $\tau_c \approx 1.5$, where stable periodic solutions (limit cycles) emerge. **(b) Time Series** ($\tau = 2.5$)**:** The load variable $x(t)$ exhibits self-sustained periodic oscillations, representing cyclic overloading. **(c) Phase Portrait** ($\tau = 2.5$)**:** The closed trajectory in the $(x(t), x(t-\tau))$ plane confirms the existence of a stable limit cycle, a hallmark of the Hopf bifurcation.

Figure 1 illustrates the resulting dynamics:

- **Panel (a):** The bifurcation diagram displays the steady-state load x^* becoming unstable past τ_c. The emerging branches indicate the amplitude of the stable limit cycle, which grows as the delay increases further. This is the supercritical Hopf bifurcation phenomenon.

- **Panel (b):** The time series for a large delay ($\tau = 2.5$) shows persistent, bounded oscillations. This models the real-world scenario where high network or scheduling latency (τ) causes self-induced cyclic fluctuations in resource utilization ($x(t)$).

- **Panel (c):** The phase portrait visually confirms that the long-term system state is a closed loop, representing a stable limit cycle. The system no longer settles to a constant load, but instead oscillates periodically.

In summary, the numerical experiments strongly validate the analytical framework, confirming that delay is the primary mechanism driving the transition from stable equilibrium to oscillatory dynamics in this class of computing system models.

Remark 5. The compact presentation in Figure 1 emphasizes how multiple visualizations support the same qualitative conclusion: delay enlarges the feedback loop, reducing stability margins and generating self-sustained oscillations. This type of multi-panel figure is useful for conveying the entire dynamical scenario without occupying excessive space.

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