

Nonlinear Problems with Non-Fixed Right Boundary with Application in Economic Modelling

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We consider the boundary value problem for the differential equation with argument deviations

$$x'(t) = f(t, x(t), (Rx)(t)), \quad t \in [a, b]; \quad (1)$$

$$U(x(a), x(b)) = d, \quad \phi(x) = d_0, \quad (2)$$

where $b - a < 2$, $d \in \mathbb{R}^n$, $\phi : C([a, B], \mathbb{R}^n) \rightarrow \mathbb{R}$ is a functional, $d_0 \in \mathbb{R}$, and

$$(Rx)(t) = \text{col}(x_1(\tau_1(t)), \dots, x_n(\tau_n(t)))$$

with continuous $\tau_i : [a, b] \rightarrow [a, b]$, $i = 1, 2, \dots, n$. The value b in (1), (2) is regarded as an unknown parameter in the range $a \leq b \leq B$, where $B \in (a, a + 2)$ is given.

The bounded sets D_0 , D_1 , $D(\varrho)$, where $f : [a, b] \times (D(\varrho))^2 \rightarrow \mathbb{R}^n$ and $U : D_0 \times D_1 \rightarrow \mathbb{R}^n$ are defined and continuous, will be specified below (see (8)). We assume that f , B and ϕ satisfy the Lipschitz conditions:

$$|f(t, x, w) - f(t, \tilde{x}, \tilde{w})| \leq K_1|x - \tilde{x}| + K_2|w - \tilde{w}| \quad (3)$$

for all $t \in [a, b]$, $\{x, \tilde{x}, w, \tilde{w}\} \subset D$;

$$|U(u, v) - U(\tilde{u}, \tilde{v})| \leq K_3|u - \tilde{u}| + K_4|v - \tilde{v}| \quad (4)$$

for all $\{u, \tilde{u}\} \subset D_0$, $\{v, \tilde{v}\} \subset D_1$;

$$|\phi(u) - \phi(v)| \leq L|u - v| \quad (5)$$

for all $(u, v) \subset D(\varrho)$, where K_i , $1 \leq i \leq 4$, and L are non-negative constant matrices of dimensions $n \times n$, $1 \times n$, respectively.

Definition 1. A solution of problem (1), (2) is a pair (x, b) where $b \in [a, B]$ and $x : [a, b] \rightarrow \mathbb{R}^n$ is a continuously differentiable function satisfying (1) and (2).

Let us set $Q = K_1 + K_2$, $Q_0 = K_3 + K_4$ and assume in addition that

$$r(Q) < 1. \tag{6}$$

Fix certain bounded sets D_0, D_1 in \mathbb{R}^n and focus on the solution x of problem (1), (2) with $x(a) \in D_0$ and $x(b) \in D_1$. Consider the set

$$D_{0,1} = \{(1 - \theta)z + \theta\eta : z \in D_0, \eta \in D_1, \theta \in [0, 1]\} \tag{7}$$

and its componentwise vector ϱ -neighbourhood

$$D(\varrho) = O_\varrho(D_{0,1}), \tag{8}$$

where $O_\varrho(M) = \{x : |x - \xi| \leq \varrho \text{ for some } \xi \in M\}$ (the inequalities between vectors are componentwise).

By formally putting $z = x(a)$, $\eta = x(b)$, we introduce the vectors of parameters

$$z = \text{col}(z_1, z_2, \dots, z_n), \quad \eta = \text{col}(\eta_1, \eta_2, \dots, \eta_m).$$

Considering the relations

$$x(a) = z, \quad x(b) = \eta \tag{9}$$

as an independent boundary condition, similarly to [2], we replace problem (1), (2) by the family of “model” problems (1), (9) containing parameters z, η . The idea is to look for solutions of (1), (2) among solutions of (1), (9) with suitable z, η . To study problems (1), (9), introduce the iteration sequence

$$x_{m+1}(t, z, \eta, b) = x_0(t, z, \eta, b) + \int_a^t f(s, x_m(s, z, \eta, b), (Rx_m(\cdot, z, \eta, b))(s)) ds - \frac{t}{b-a} \int_a^b f(s, x_m(s, z, \eta, b), (Rx_m(\cdot, z, \eta, b))(s)) ds, \tag{10}$$

where $m = 0, 1, \dots$ and

$$x_0(t, z, \eta, b) = z + \frac{t-a}{b-a}(\eta - z), \quad t \in [a, b]. \tag{11}$$

Each of the functions (10), (11) satisfies (9) for arbitrary z, η .

Theorem 1. *Assume that conditions (3)–(6) hold and the non-negative vector ϱ in (8) can be chosen so that*

$$\varrho \geq \frac{1}{2} (B - a)\delta_{[a,B] \times (D(\varrho))^2}(f), \tag{12}$$

where

$$\delta_{[a,B] \times (D(\varrho))^2}(f) = \frac{1}{2} \left(\max_{(t,x,y) \in [a,B] \times (D(\varrho))^2} f(t, x, y) - \min_{(t,x,y) \in [a,B] \times (D(\varrho))^2} f(t, x, y) \right).$$

Then, for all fixed $z \in D_0, \eta \in D_1, b \in [a, B]$:

- (a) each function (10) has range in $D(\varrho)$, is continuously differentiable on $[a, b]$ and satisfies the two-point boundary condition (9);

(b) the limit

$$x_\infty(t, z, \eta, b) = \lim_{m \rightarrow \infty} x_m(t, z, \eta, b)$$

exists uniformly in $(t, z, \eta) \in [a, b] \times D_0 \times D_1$;

(c) the function $x_\infty(\cdot, z, \eta, b)$ satisfies the two-point boundary conditions (9);

(d) the function $x_\infty(\cdot, z, \eta, b)$ is a unique continuously differentiable solution of the integral equation

$$x(t) = x_0(t, z, \eta, b) + \int_a^t f(s, x(s), (Rx)(s)) \, ds - \frac{t-a}{b-a} \int_a^b f(s, x(s), (Rx)(s)) \, ds;$$

(e) the following estimate holds for $t \in [a, b]$, $m \geq 0$:

$$|x_\infty(t, z, \eta, b) - x_m(t, z, \eta, b)| \leq \frac{1}{2} (b-a) Q^m (I-Q)^{-1} \delta_{[a, B] \times (D(\varrho))^2}(f),$$

where I is the identity matrix.

Theorem 2. Assume that conditions of Theorem 1 hold. Then $(x_\infty(\cdot, z, \eta, b), b)$ is a solution of problem (1), (2) if and only if the vector parameters z, η and the scalar b satisfy the system of $2n+1$ determining equations

$$\Delta(z, \eta, b) = 0, \quad U(x_\infty(a, z, \eta, b), x_\infty(b, z, \eta, b)) = d, \quad \Lambda(z, \eta, b) = 0,$$

where

$$\begin{aligned} \Lambda(z, \eta, b) &= \phi(x_\infty(\cdot, z, \eta, b)) - d_0, \\ \Gamma(z, \eta, b) &= U(x_\infty(a, z, \eta, b), x_\infty(b, z, \eta, b)) - d, \\ \Delta(z, \eta, b) &= \frac{1}{b-a} \int_a^b f(s, x_\infty(s, z, \eta, b), (Rx_\infty(\cdot, z, \eta, b)(s))) \, ds. \end{aligned}$$

Fix an m and set $\Lambda_m(z, \eta, b) = \phi(x_m(\cdot, z, \eta, b)) - d_0$,

$$\begin{aligned} \Gamma_m(z, \eta, b) &= U(x_m(a, z, \eta, b), x_m(b, z, \eta, b)) - d, \\ \Delta_m(z, \eta, b) &= \frac{1}{b-a} \int_a^b f(s, x_m(s, z, \eta, b), (Rx_m(\cdot, z, \eta, b)(s))) \, ds, \end{aligned}$$

and consider the equations

$$\Delta_m(z, \eta, b) = 0, \quad U(x_m(a, z, \eta, b), x_m(b, z, \eta, b)) = d, \quad \Lambda_m(z, \eta, b) = 0, \quad (13)$$

which can be called approximate determining equations.

Introduce a notation. Let u and v be vector functions with n components defined on a set M . Following [1], we write $u \triangleright_M v$ if one can specify a function $i : M \rightarrow \{1, 2, \dots, n\}$ such that $u_{i(x)}(x) > v_{i(x)}(x)$ for all $x \in M$.

Theorem 3. *Let the conditions of Theorem 1 hold. Assume that one can specify an $m \geq 0$ and open bounded sets $W_0 \subset D_0, W_1 \subset D_1$ such that, on the boundary ∂W of the set $W = W_0 \times W_1 \times (a, B) \subset \mathbb{R}^{2n+1}$, the conditions*

$$\begin{aligned} |\Delta_m| \triangleright_{\partial W} & \frac{1}{2} (b - a)^2 Q^{m+1} (I - Q)^{-1} \delta_{[a,B] \times (D(\varrho))^2} (f), \\ |\Gamma_m| \triangleright_{\partial W} & \frac{1}{2} (b - a) Q_0 Q^m (I - Q)^{-1} \delta_{[a,B] \times (D(\varrho))^2} (f), \\ |\Lambda_m| \triangleright_{\partial W} & \frac{1}{2} (b - a) L Q^m (I - Q)^{-1} \delta_{[a,B] \times (D(\varrho))^2} (f) \end{aligned}$$

hold. If, moreover, the Brouwer degree of the map $\begin{pmatrix} \Delta_m \\ \Gamma_m \\ \Lambda_m \end{pmatrix}$ over W is different from 0, then there exist certain values $z^ \in W_0, \eta^* \in W_1, b^* \in [a, B]$ such that (x^*, b^*) with $x^* = x_\infty(\cdot, z^*, \eta^*, b^*)$ is a solution of problem (1), (2).*

The proof of these statements can be carried out similarly to the corresponding theorems in [1,2].

Let us apply the techniques described above to an example. The dynamic model of a new product introduction studied in [3] has the form of an initial value problem assigned to a system of three non-linear retarded differential equations. Here we examine a version of this model in the boundary value problem setting, where one of the endpoints of the time interval is not fixed beforehand. The unknown right-hand endpoint can be interpreted as the waiting time for the required conditions to be satisfied.

Denote the number of potential investors at time t by $I(t)$, the number of shareholders by $H(t)$, and the intensity of investors by $S(t)$. Consider the mathematical model formulated as a boundary value problem for functional differential equations with an unfixed right endpoint:

$$\begin{aligned} I'(t) &= \lambda - \gamma_1 I(t) - \tau I(t) S(t), \\ H'(t) &= \beta I(\tau_1(t)) S(\tau_2(t)) - \gamma_2 H(t), \end{aligned} \tag{14}$$

$$\begin{aligned} S'(t) &= -\gamma_3 S(t) + k I(t), \quad t \in [0, T]; \\ I(0) &= 70, \quad S(0) = 2500, \quad I(T) = 85, \quad H(T) = 15, \end{aligned} \tag{15}$$

where prime stands for d/dt . The value T , which marks the end of the observation interval, is regarded as an unknown parameter that is to be determined together with the functions I, H , and S defined on $[0, T]$.

Let us take in model (14), (15) the values $\lambda = 15, \gamma_1 = 0.01, \gamma_2 = 0.3, \gamma_3 = 2.4, \beta = 0.0002, k = 0.08, \tau_1(t) = \frac{3}{5}t, \tau_2(t) = \frac{4}{5}t$. Introducing the parameters $I(0) = z_1, H(0) = z_2, S(0) = z_3, I(T) = \eta_1, H(T) = \eta_2, S(T) = \eta_3$, constructing the iterations (10), and numerically solving the approximate determining system (13) for several values of m , we obtain the parameters values shown in the table below.

The residual obtained by substituting the eighth approximation into the differential equations (14) are of order $10^{-6}, 10^{-6},$ and 10^{-4} , respectively, which is evidence of the good quality of approximation. The absolute error does not exceed the value on the right in estimate (e) of Theorem 1.

m	z_1	z_2	z_3	η_1	η_2	η_3	T
0	70	13.27191985	2500	85	15	969.0868551	0.3682997542
1	70	6.216351499	2500	85	15	1005.994120	0.3803373380
3	70	6.217553942	2500	85	15	1005.774925	0.3800173445
5	70	6.217719157	2500	85	15	1005.806224	0.3800309868
6	70	6.217714561	2500	85	15	1005.805487	0.3800306806
7	70	6.217712246	2500	85	15	1005.805307	0.3800306282
8	70	6.217712406	2500	85	15	1005.805327	0.380030635

References

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