Approximation of Stochastic Functional-Differential Equations of Neutral Type by a System of Equations without Delay in Infinite-Dimensional Spaces

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We consider the following stochastic functional-differential neutral equation on Hilbert space with delay parameter $h \in (0,1]$:

$$d(u(t)+g(u(t-h),u(t))) = (f(u(t-h),u(t))+Au(t))\,dt + \sigma(u(t-h),u(t))\,dW(t), \quad t \geq 0, \tag{0.1}$$

$$u(t)=\phi(t), \quad t \in [-h,0]. \tag{0.2}$$

Here A is an infinitesimal generator of a strong continious semigroup $\{S(t), t \geq 0\}$ of bounded linear operators in real separable Hilbert space H. The noise $W(t)$ is a Q-Wiener process on separable Hilbert space K. For any $h \in (0,1)$, denote by $C_h := C([-h,0],H)$ the space of continious H-valued functions with the norm

$$\|\phi\|_{C_h} := \sup_{t\in[-h,0]} \|\phi(t)\|_H.$$

Below, we will denote $\|\cdot\|_H$ by $\|\cdot\|$. The functions f and g map $H\times H$ into H and $\sigma{:}H\times H \to L_2^0$, where $L_2^0 = L(Q^{1/2}K,H)$ is the space of Hilbert–Schmidt operators from $Q^{1/2}K$ to H. Finally, $\phi : [-h,0]\times\Omega \to H$ is the initial condition on probability space $(\Omega,\mathcal{F},P)$.

The main goal of this work is to study an approximation systems for neutral type stochastic delay equations in Hilbert space of the form (0.1), (0.2).

The proposed approach consists of splitting the delay interval and constructing a corresponding approximation system. It is important to note that the number of equations in this system grows depending on the number of subintervals. The main result of this work shows that if the number of subintervals tends to infinity, the mean square distance between the solution of equations (0.1)–(0.2) and the solutions of the approximation system tends to zero.

1 Preliminaries

Throughout this paper H and K are separable Hilbert spaces with the norms $\|\cdot\|$ and $\|\cdot\|_K$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space, and Q be a linear bounded covariance operator such that $tr(Q) < \infty$. Introduce

$$W(t) := \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \beta_k(t) e_k, \quad t \geq 0,$$

which is Q-Wiener process on $t \geq 0$. Here $\beta_k(t)$ are standard, one dimensional, independent Wiener processes, $\{e_k, k \geq 1\}$ is an orthonormal system in K, and a sequence of real nonnegative numbers λ_k satisfying

$$Qe_k = \lambda_k e_k, \quad k = 1, 2, \dots,$$

and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

Also, let $\{\mathcal{F}_t, t \geq 0\}$ be a normal filtration satisfying:

1. $W(t)$ is $\mathcal{F}_t$-measurable;
2. $W(t+h) - W(t)$ is independent of $\mathcal{F}_t$ for all $h \geq 0$ and $t \geq 0$.

Let $U_0 = Q^{\frac{1}{2}}(K)$ and $L_2^0 = L_2(U_0, H)$ be the space of all Hilbert–Schmidt operators from U_0 to H with the inner product $(\Phi, \Psi)_{L_2^0} = tr[\Phi Q \Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively.

Definition 1.1 (Mild solution)**.** A continious $\mathcal{F}_t$ adapted stochastic process $u : [-h, T] \times \Omega \to H$ is a mild solution of $(0.1), (0.2)$ for $t \in [0, T]$ if it satisfies the integral equation

$$u(t) = S(t)(\phi(0) + g(\phi(-h), \phi(0))) - g(u(t-h), u(t)) - \int_0^t AS(t-s) g(u(s-h), u(s))\, ds + \int_0^t S(t-s) f(u(s-h), u(s))\, ds + \int_0^t S(t-s)\sigma(u(s-h), u(h)) dW(s),$$

and $u(t) = \phi(t)$ a.s. for $t \in [-h, 0]$.

Introduce the following system

$$\begin{cases} d\big(z_0(t) - g(z_0(t), z_m(t))\big) = \big(Az_0 + f(z_0(t), z_m(t))\big)\, dt + \sigma(z_0(t), z_m(t))\, dW(t), \\ dz_j(t) = \dfrac{m}{h}\,(z_{j-1}(t) - z_j(t)), \quad t \in [0, T], \\ z_j(0) = \phi\Big(-\dfrac{hj}{m}\Big), \quad j = 0, \dots, m. \end{cases} \tag{1.1}$$

Definition 1.2. System (1.1) is called approximating system for $(0.1), (0.2)$ in mean square if

$$\sup_{t\in[0,T]} \mathbf{E}\Big\| u\Big(t - \frac{h}{m} j\Big) - z_j(t)\Big\|^2 \longrightarrow 0, \quad m \to \infty, \quad j = 0, \dots, m.$$

2 Conditions on functions

Condition (H1). *If $\sigma(-A)$ is the spectrum of $(-A)$, we have*

$$Re\, \sigma(-A) > \delta > 0,$$

and A^{-1} is compact in H.

It follows from [4] that for $0 \le \alpha \le 1$ one can define fractional power $(-A)^\alpha$, which is closed linear operator with domain $D(-A)^\alpha$. We denote H_α to be a Banach space $D(-A)^\alpha$ with a norm

$$\|u\|_\alpha := \|(-A)^\alpha u\|,$$

which is equivalent to the graph norm of $(-A)^\alpha$. This way $H_0 = H$. It follows from [2, Section 1.4] that if A^{-1} is compact, then $S(t)$ is compact for $t > 0$. Next, it follows from [4, Theorem 3.2] that under assumption (H1) semigroup $S(t)$ is continious with respect to uniform operator topology for $t > 0$. Thus, using [4, Theorem 3.3] we may conclude that the operator A has a compact resolvent. Consequently, from [2, Theorem 1.4.8] we have the following result.

Proposition 2.1. *Under condition* (H1) *the embedding $H_\alpha \subset H_\beta$ is compact if $0 \le \beta < \alpha \le 1$.*

Proposition 2.2 ([2, Theorem 1.4.3])**.** *Under condition* (H1)*, for every $\alpha \ge 0$ there exists $C_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \le C_\alpha t^{-\alpha} e^{-\delta t}$$

for $t > 0$.

In particular

$$\|S(t)\| \le C_0 e^{-\delta t}$$

for $t > 0$.

Proposition 2.3 ([8])**.** *Let $p > 2$, $T > 0$ and let Φ be an L_2^0 valued, predictable process such that*

$$\mathbf{E} \int_0^T \|\Phi(t)\|_{L_2^0}^p \, dt < \infty.$$

Then there is a constant $M_T > 0$ such that

$$\mathbf{E} \sup_{t \in [0,T]} \bigg\| \int_0^t S(t-s)\Phi(s)\, dW(s) \bigg\| \le M_T \mathbf{E} \int_0^T \|\Phi(s)\|_{L_2^0}^p \, ds.$$

Condition (H2). *The mappings $f : H \times H \to H$ and $\sigma : H \times H \to L_2^0$ are continious and satisfy:*

1. *There exist a positive constant $K > 0$ such that*

$$\|f(u,v)\| + \|\sigma(u,v)\|_{L_2^0} \le K\big(1 + \|u\| + \|v\|\big)$$

for all $u, v \in H$.

2. *There exist positive constant $L > 0$ such that*

$$\|f(u,v) - f(u_1,v_1)\|^2 + \|\sigma(u,v) - \sigma(u_1,v_1)\|_{L_2^0}^2 \le L\big(1 + \|u - u_1\|^2 + \|v - v_1\|^2\big)$$

for all $u, v, u_1, v_1 \in H$.

Condition (H3). *There exist positive constants $\alpha \in (\frac{1}{2}, 1)$ and $M_g \in (0, \frac{1}{2\sqrt{2}})$ such that for all $u, v, u_1, v_1 \in H$ the function $g : H \times H \to H_\alpha$ satisfies*

$$\|g(u,v) - g(u_1,v_1)\|_{H_\alpha}^2 \le M_g\big(\|u - u_1\|^2 + \|v - v_1\|^2\big).$$

Condition (H4). *The initial condition $\phi : [-h, 0] \times \Omega \to H$ is a $\mathcal{F}_0$-measurable random variable, independent of W, which has continious trajectories.*

Remark. It is easy to see from [5] that under these conditions the above equation (0.1), (0.2) have a unique mild solution, and this solution have an invariant measure.

3 Main results

Lemma 3.1 (Continuity modulo lemma)**.** *If conditions* (H1)–(H3) *hold, then for the solution* $u(t)$ *of equation* (0.1)*,* (0.2) *the following holds*

$$\sup_{t_1\in[-h,T]} \mathbf{E} \sup_{t_2\in[t_1,t_1+l]} \|u(t_2)-u(t_1)\|^2 \le C(T,\|\phi\|_{C_h},l) \longrightarrow 0, \quad l\to 0. \tag{3.1}$$

Sketch of Proof. *Step 1.* From Definition 1 we have the following

$$\begin{aligned}\|u(t)\|^2 &\le 8\big\|S(t)(\phi(0)-g(\phi(-h)))\big\|^2 + 2\|g(u(t),u(t-h))\|^2\\ &\quad + 8\bigg\|\int_0^t AS(t-s)g(u(s),u(s-h))\,ds\bigg\|^2 + 8\bigg\|\int_0^t S(t-s)f(u(s),u(s-h))\,ds\bigg\|^2\\ &\quad + 8\bigg\|\int_0^t S(t-s)\sigma(u(s),u(s-h))\,dW(s)\bigg\|^2.\end{aligned}$$

Therefore, from Proposition 2.1 and conditions (H1)–(H4), using Grownwall lemma we get the estimate

$$\mathbf{E}\sup_{s\in[0,t]}\|u(s)\|^2 \le K(T,\|\phi\|^2_{C_h}).$$

And consequently,

$$\mathbf{E}\sup_{s\in[-h,T]}\|u(s)\|^2 \le K(T,\|\phi\|^2_{C_h}).$$

Step 2. To complete the last step, it is crucial to prove that there exist $C_1(T,\|\phi\|_{C_h},l)\to 0$ as $l\to 0$ such that

$$\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(t_1)\|^2 \le C_1\big(T,\|\phi\|_{C_h},l\big) \tag{3.2}$$

for $-l < t_1 < 0$.

Step 3. Let‘s split interval $[0,T]$ by h. This way there exist some $N=[\frac{T}{h}]$ such that sequence $h_n = nh$ is the following $0\le h_1 < h_2 < \cdots < h_{N-1} < h_N = T$.

After that it can be proved that estimation (3.1) on intervals $[h_i,h_{i+1}]$ depends only on previous step and (3.2). The estimation can be extended to the whole interval and (3.1) holds true. □

Then we have the following result.

Theorem (Approximation system)**.** *Assume that conditions* (H1)–(H3) *hold. Then system* (1.1) *is approximating system for* (0.1), (0.2) *in mean square in sense of Definition 1.2.*

References

[1] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.

[2] J. Hale, *Theory of Functional Differential Equations*. Second edition. Applied Mathematical Sciences, Vol. 3. Springer-Verlag, New York–Heidelberg, 1977.

[3] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*. Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin–New York, 1981.

[4] S. Mehri and M. Scheutzow, A stochastic Gronwall lemma and well-posedness of path-dependent SDEs driven by martingale noise. *Preprint* arXiv:1908.10646, 2019; `https://arxiv.org/abs/1908.10646`

[5] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[6] G. O. Petryna, O. M. Stanzhytskiy and O. V. Martynyuk, On the approximation of stochastic delay equations in infinite-dimensional spaces. (Ukrainian) *Bukovyn. Mat. Zh.* **12** (2024), no. 2, 168–181.

[7] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations.* Lecture Notes in Mathematics, 1905. Springer, Berlin, 2007.

[8] M. Röckner, R. Zhu and X. Zhu, Existence and uniqueness of solutions to stochastic functional differential equations in infinite dimensions. *Nonlinear Anal.* **125** (2015), 358–397.

[9] A. Stanzhytsky, O. Misiats and O. Stanzhytskyi, Invariant measure for neutral stochastic functional differential equations with non-Lipschitz coefficients. *Evol. Equ. Control Theory* **11** (2022), no. 6, 1929–1953.

[10] M.-K. von Renesse and M. Scheutzow, Existence and uniqueness of solutions of stochastic functional differential equations. *Random Oper. Stoch. Equ.* **18** (2010), no. 3, 267–284.