On the Descriptive Type of Some Oscillatory Properties of Solutions to Linear Differential Equations and Systems

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Oscillatory properties of solutions are a classical subject of study in the qualitative theory of differential equations. A vast number of works is devoted to the investigation of these properties (see, e.g., the bibliographies in the survey [4] and the monograph [1]). Recently, interest in this topic has increased significantly, driven on the one hand by the needs of modern technologies that deliberately utilize oscillatory regimes, and on the other hand by the problems of vibration protection for machinery and equipment.

Since we will consider solutions of linear differential equations and systems defined on the half-axis $\mathbb{R}_+ \equiv [0,+\infty)$, we introduce the following

Definition 1. A function $y:\mathbb{R}_+\to\mathbb{R}$ is called *oscillatory* (respectively, *strictly oscillatory*) if for any $T>0$ there exist points $t_2>t_1>T$ such that the function y has at least one zero (respectively, changes sign) on the interval $[t_1,t_2]$. We say that a function changes sign on $[t_1,t_2]$ if $y(t_1)y(t_2)<0$.

To quantitatively describe oscillatory and strictly oscillatory behavior, I. N. Sergeev introduced numerical characteristics, now called *Sergeev frequencies*. Let us recall their definition.

Let $\varkappa\in\{-,0,+,*\}$. For a given $t>0$, denote by $\nu^{\varkappa}(y,t)$ the following quantity depending on the value of $\varkappa$:

- the number of sign changes of y on the half-interval $[0,t)$ if $\varkappa=-$;
- the number of zeros of y on $[0,t)$ if $\varkappa=0$;
- the number of roots (i.e., zeros counting multiplicity) of y on $[0,t)$ if $\varkappa=+$;
- the number of hypermultiple roots of y on $[0,t)$ if $\varkappa=*$. In this case, a simple zero is counted once, and a multiple zero is counted infinitely many times, regardless of its actual multiplicity.

Definition 2. The *upper* (respectively, *lower*) *characteristic frequency of signs, zeros, or roots* of a scalar function $y:\mathbb{R}_+\to\mathbb{R}$ is the quantity

$$\widehat{\omega}^{\varkappa}[y]=\varlimsup_{t\to+\infty}\frac{\pi}{t}\,\nu^{\varkappa}(y,t)\quad\Big(\text{respectively, }\ \check{\omega}^{\varkappa}[y]=\varliminf_{t\to+\infty}\frac{\pi}{t}\,\nu^{\varkappa}(y,t)\Big),$$

for $\varkappa=-,0,+$, respectively.

There is an obvious connection between oscillatory properties and Sergeev frequencies: if the Sergeev frequency of zeros (respectively, signs) of a function is positive, then the function is oscillatory (respectively, strictly oscillatory). The converse is not true. For example, the function y defined by $y(t)=\sin(\sqrt{t+1})$, $t\in\mathbb{R}_+$, is strictly oscillatory (and all its zeros are simple), yet $\widehat{\omega}^{\alpha}(y)=\check{\omega}^{\alpha}(y)=0$ for $\alpha\in\{-,0,+\}$.

In what follows, we call a linear homogeneous nth-order equation

$$y^{(n)}+a_1(t)y^{(n-1)}+\cdots+a_{n-1}(t)\dot{y}+a_n(t)y=0,\ \ t\in\mathbb{R}_+, \tag{1}$$

oscillatory (respectively, *strictly oscillatory*) if all its nonzero solutions possess the corresponding property.

A generalization of the concept of characteristic frequency to the case of vector functions, i.e., to solutions of differential systems, was given in [9,10], where the *oscillation exponents* (called there the *total* and *vector frequencies*) were introduced and their basic properties were established. Let us recall their definition.

Definition 3. The *weak upper* (respectively, *lower*) *oscillation exponent of signs, zeros, roots, or hyperroots* of a vector function $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the quantity

$$\widehat{\nu}_\circ^\varkappa(x) = \overline{\lim_{t\to+\infty}} \inf_{a\in\mathbb{R}^n_*} \frac{\pi}{t}\, \nu^\varkappa\bigl(\langle x,a\rangle,t\bigr) \quad \Bigl(\text{respectively, } \ \check{\nu}_\circ^\varkappa(x) = \underline{\lim_{t\to+\infty}} \inf_{a\in\mathbb{R}^n_*} \frac{\pi}{t}\, \nu^\varkappa\bigl(\langle x,a\rangle,t\bigr)\Bigr),$$

for $\varkappa = -,0,+,*$, respectively, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the scalar product and the asterisk in the subscript indicates the exclusion of the zero vector.

Using the approach from Definition 3, we define the oscillatory behavior of a vector function.

Definition 4. A vector function $x : \mathbb{R}_+ \to \mathbb{R}^n$ is called *oscillatory* (respectively, *strictly oscillatory*) if for any $T > 0$ there exist $t_2 > t_1 > T$ such that for every vector $a \in \mathbb{R}^n_*$, the function $\langle x,a\rangle$ has at least one zero (respectively, changes sign) on the interval $[t_1,t_2]$. The system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{2}$$

is called *oscillatory* (respectively, *strictly oscillatory*) if all its nonzero solutions possess the corresponding property.

A natural question arises: what are the descriptive types of the sets of oscillatory equations, systems, and oscillatory solutions in the corresponding function spaces?

For a given $n \in \mathbb{N}$, denote by $\widetilde{\mathcal{M}}^n$ the set of systems (2) with continuous matrix functions $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ (which we identify with the systems they define), endowed with the *compact-open topology* induced by the metric

$$\rho_0(A,B) = \sup_{t\in\mathbb{R}_+} \min\bigl\{|A(t)-B(t)|,(t+1)^{-1}\bigr\}, \quad A,B \in \widetilde{\mathcal{M}}^n.$$

The subset of $\widetilde{\mathcal{M}}^n$ consisting of systems corresponding to linear homogeneous nth-order equations (1) will be denoted by $\widetilde{\mathcal{E}}^n$.

Let $\imath$ denote the mapping that associates to each equation $a \in \widetilde{\mathcal{E}}^n$ and each nonzero vector $\xi \in \mathbb{R}^n_*$ the solution of equation a with the initial vector ξ at $t = 0$. Similarly, let $\jmath$ denote the mapping that associates to each pair $(A,\xi) \in \widetilde{\mathcal{M}}^n \times \mathbb{R}^n_*$ the solution of the corresponding Cauchy problem.

Let M be a metric space and $\mathfrak{G}(M)$ and $\mathfrak{F}(M)$ denote the classes of all open and closed subsets of M, respectively. Then $\mathfrak{G}_\delta(M)$ denotes the class of all countable intersections of open sets, and $\mathfrak{F}_\sigma(M)$ denotes the class of all countable unions of closed sets. The classes $\mathfrak{G}_{\delta\sigma}(M)$, $\mathfrak{F}_{\sigma\delta}(M)$ etc. are defined similarly. When the space M is clear from the context, we omit it.

The mentioned link between oscillation and Sergeev frequencies naturally leads to a comparison of the descriptive type of the sets of oscillatory solutions and those with positive Sergeev frequency.

From a result by I. N. Sergeev [7] it follows that the set of equations satisfying the condition

$$\omega_1(a) = \inf_{y\in\mathcal{S}_*(a)} \widehat{\nu}^{\,-}[y] > 0,$$

where $\mathcal{S}_*(a)$ is the set of nonzero solutions to equation a, is a $\mathfrak{G}_{\delta\sigma}$-set.

It was established in [2, 3] that the set

$$\big\{(a,\xi)\in\widetilde{\mathcal{E}}^n\times\mathbb{R}^n_*:\ \widehat{\omega}^-\circ\imath(a,\xi)>0\big\}$$

is a $\mathfrak{G}_{\delta\sigma}$-set, while the sets

$$\big\{(a,\xi)\in\widetilde{\mathcal{E}}^n\times\mathbb{R}^n_*:\ \widehat{\omega}^0\circ\imath(a,\xi)>0\big\},\quad \big\{(a,\xi)\in\widetilde{\mathcal{E}}^n\times\mathbb{R}^n_*:\ \check{\omega}^0\circ\imath(a,\xi)>0\big\},$$
$$\big\{(a,\xi)\in\widetilde{\mathcal{E}}^n\times\mathbb{R}^n_*:\ \check{\omega}^-\circ\imath(a,\xi)>0\big\}$$

are $\mathfrak{F}_{\sigma\delta\sigma}$-sets.

We have established the following result.

Theorem 1. *For any $n\ge 2$, the following statements hold:*

1) *the set of pairs $(a,\xi)\in\widetilde{\mathcal{E}}^n\times\mathbb{R}^n_*$ corresponding to oscillatory solutions is an $\mathfrak{F}_{\sigma\delta}$-set, and the set corresponding to strictly oscillatory solutions is a $\mathfrak{G}_\delta$-set;*

2) *the set of strictly oscillatory equations* (1) *is a $\mathfrak{G}_\delta$-subset of the space $\widetilde{\mathcal{E}}^n$.*

From the result of [11], one can conclude that the set

$$\Big\{A\in\widetilde{\mathcal{M}}^n:\ \sup_{x\in\mathcal{S}_*(A)}\check{\nu}^*[x]>0\Big\}$$

is an $\mathfrak{F}_\sigma$-set. Regarding the descriptive types of the positivity sets of the minimal oscillation exponents, nothing is known to the authors.

It was stated in [5] that the set

$$\big\{(A,\xi)\in\widetilde{\mathcal{M}}^n\times\mathbb{R}^n_*:\ \check{\nu}^*\circ\jmath(A,\xi)>0\big\}$$

is an $\mathfrak{F}_\sigma$-set, while the set

$$\big\{(A,\xi)\in\widetilde{\mathcal{M}}^n\times\mathbb{R}^n_*:\ \widehat{\nu}^*\circ\jmath(A,\xi)>0\big\}$$

is an $\mathfrak{F}_{\sigma\delta\sigma}$-set.

Regarding the oscillatory properties defined above, we have the following

Theorem 2. *For any $n\ge 2$, the following statements hold:*

1) *the set of pairs $(A,\xi)\in\widetilde{\mathcal{M}}^n\times\mathbb{R}^n_*$ corresponding to oscillatory solutions is an $\mathfrak{F}_{\sigma\delta}$-set, and the set corresponding to strictly oscillatory solutions is a $\mathfrak{G}_\delta$-set;*

2) *the set of strictly oscillatory systems* (2) *is a $\mathfrak{G}_\delta$-subset of the space $\widetilde{\mathcal{M}}^n$.*

The question of whether the assertions of Theorems 1 and 2 are sharp, as well as the question of the descriptive type of the sets of oscillatory linear equations and systems, remain open.

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