Approximation of Functional Differential Equations with and without Delay by a System of Equations without Delay

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1 Introduction

An approximation approach for functional differential equations with delays in infinite-dimensional Banach spaces is considered. Particular attention is devoted to the replacement of the original delayed system by an associated evolutionary system of delay-free differential equations. Such an approach provides a foundation for numerical implementations and further qualitative analysis of infinite-dimensional control systems. The proposed approach involves partitioning the delay interval into subintervals and constructing the corresponding system of equations that approximates the behavior of the original system. The main result of the study shows that as the mesh size of the partition tends to zero the distance between the solutions of the delayed equation and those of the delay-free system also tends to zero.

2 Setting of the problem and the main results

Let X be a Banach space with norm $\|\cdot\|_X$. By $C_h = C([-h,0];X)$ we denote the space of X-valued, continuous functions $\varphi : [-h,0] \mapsto X$ with norm

$$\|\varphi\|_C = \sup_{t\in[-h,0]} \|\varphi(t)\|,$$

where $h > 0$ is the delay interval.

Let $A : X \mapsto X$ be an unbounded, closed, linear operator and A is a infinite small generator of strongly continuos semigroup $S(t) = e^{tA}$, $t \geq 0$ in H.

Consider an infinite-dimensional functional differential equation of the form

$$\begin{cases} du(t) = Au(t) + f\Big(t, u(t), \int\limits_{-h}^{0} u(t+\theta)\,d\theta\Big), & t \in [-h,T], \\ u(t) = \varphi(t), & t \in [-h,0]. \end{cases} \tag{2.1}$$

We impose the following conditions on mapping f:

(A1) $f : [0,T] \times X \times X \mapsto X$ is continuous with respect to the set of variables;

(A2) (linear growth condition) $\exists\, L > 0$ - const:

$$\|f(t,u,v)\| \leq L\big(1 + \|u\| + \|v\|\big),$$

for all t, u, v from the domain of definition;

(A3) (Lipshitz condition)

$$\|f(t,u_1,v_1)-f(t,u_2,v_2\|\le L\big(\|u_1-u_2\|+\|v_1-v_2\|\big),$$

for $t\in[0,T]$ and arbitrary t,u_1,u_2,v_1,v_2 from the domain of definition;

(A4) initial function $\varphi:[-h,0]\mapsto X$ is continuous.

The solution to the initial problem (2.1) will be understood in a mild sense.

Definition 2.1. A function $u(t)\in X$ is called a mild solution of the initial value problem (2.1) on $[0,T]$ if:

(1) $u(t)=\varphi(t)$, $t\in[-h,0]$;

(2) $u\in C([0,T],X)$;

(3) $u(t)$ satisfies the integral equation

$$u(t)=S(0)\varphi(0)+\int\limits_0^t S(t-s)f\Bigg(s,u(s),\int\limits_{-h}^0 u(s+\theta)\,d\theta\Bigg)\,ds.$$

It follows from the work of [5] that if conditions (A1)–(A4) are met, the initial problem (2.1) has a unique soft solution on $[0,T]$ that satisfies the inequality

$$\sup_{t\in[0,T]}\|u(t)\|_X<\infty.$$

We construct the following system of evolutionary equations without delays, which we will call approximating, using the equation (2.1).

Let us fix $m\in N$ and divide the interval $[-h,0]$ by the points $-\frac{hj}{m}$, $j=\overline{0,m}$ into m parts.

We define the functions $z_j(t)\in X$ as solutions to the following Cauchy problems:

$$\begin{aligned} dz_0&=Az_0+f\Big(t,z_0(t),\frac{h}{m}\sum_{j=1}^m z_j(t)\Big)\,dt,\\ dz_j(t)&=\frac{m}{h}\,(z_{j-1}(t)-z_j(t)),\ \ t\in[0,T],\\ z_j(0)&=\varphi\Big(-\frac{hj}{m}\Big),\ \ j=\overline{0,m}. \end{aligned}\tag{2.2}$$

Here $z_0(t)$ the solution of the first equation, is understood in the mild sense, while the remaining m equations are considered in the classical sense. The derivative $\frac{dz_j(t)}{dt}$ is considered as a strong derivative with respect to the norm of the space X.

From [4] it follows that the Cauchy problem (2.2) has a unique solution on $[0,T]$, where $z_0(t)$ satisfies (2.2) in the mild sense, while $z_j(t)$ satisfy the remaining m equations in the classical sense.

Definition 2.2. The system (2.2) is called an approximating system for (2.1) if

$$\sup_{t\in[0,T]}\Big\|u\Big(t-\frac{h}{m}j\Big)-z_j(t)\Big\|\longmapsto 0,\ \ m\mapsto\infty,\ \ j=\overline{0,m}.$$

In what follows, we will need a lemma on the modulus of continuity.

Lemma. *Under assumptions* (A1)–(A4)*, the following inequality holds for the solution of the initial value problem*

$$\sup_{|t_1-t_2|\leq l,\ t_1,t_2\in[-h,T]} \|u(t_2)-u(t_1)\| \leq C\big(T,\|\varphi\|_C,h,l\big) \longmapsto 0, \ \ l\mapsto 0.$$

The proof follows directly from the uniform continuity of $u(t)$ on $[-h,T]$.

The main result is the following theorem.

Theorem. *Under assumptions* (A1)–(A4)*, the system* (2.2) *is an approximating system for the initial value problem* (2.1)*, uniformly with respect to $j=\overline{0,m}$, that is*

$$\sup_{j=\overline{0,m}}\ \sup_{t\in[0,T]} \Big\|u\Big(t-\frac{h}{m}j\Big)-z_j(t)\Big\| \longmapsto 0, \ \ m\mapsto\infty.$$

Example. Let Q be a bounded domain in R^d with a bound of ∂Q satisfying the Lyapunov condition. The operator A is a second-order differential operator of the elliptic type

$$Au=\sum_{i,j=1}^{d}(a_{ij}(x)u_{x_i})_{x_j}=\operatorname{div}(a(x)\nabla u)).$$

Here, a_{ij} are Gelder-continuous coefficients with the Gelder exponent $\beta\in(0.1)$, symmetric, bounded, satisfying the condition of uniform ellipticity

$$\sum_{i,j=1}^{d}a_{ij}\eta_i\eta_j\geq C_0|\eta|^2, \ \ \eta\in R^d$$

for some constant $C_0>0$, and $|\cdot|$ is the Euclidean norm in R^d.

Let $X=L^2(Q)$, $D(A)=H^2(Q)\cap H$.

Consider the following equation

$$\begin{cases} du(t,x)=\bigg(Au+f(t,u(t),\displaystyle\int_{-h}^{0}u(t+\theta)\,d\theta\bigg)\,dt, & \\ u(t,x)=\varphi(t,x), & t\in[-h,0],\\ u(0,x)=\varphi_0(x), & x\in Q,\\ u(t,x)=0, & x\in\partial Q, \ \ t\geq 0, \end{cases} \tag{2.3}$$

where $\varphi(t,x)\in C_0=C([0,T];L^2(Q))$.

The real-valued function $f(t,x,y)$ is defined for $t\in[0,T]$, $x\in Q$, $y\in[0,l]$, $l>0$, with a value of R^1. The function $f(t,x,y)$ is continuous over the set of variables and Lipschitz over the variables x and y with constant L.

The domain $D\in[-h,T]\times C$ is the set $\{(t,\varphi):\ t\in[-h,T],\varphi\in G\}$, where G is the set of functions $\varphi\in C$ such that $\int_{-h}^{0}\varphi(\theta,\cdot\,)\,d\theta\in(0,l)$, and ∂G is the set of functions $\varphi\in C$ such that $\int_{-h}^{0}\varphi(\theta,\cdot\,)\,d\theta=l$ or $\varphi(\theta,x)=0$ a.e.

It follows from [4] that the operator A is a generator of C_0 - a semigroup of operators $S(t):X\mapsto X$.

It is not difficult to see that the conditions (A1)–(A4) for the equation (2.3) are fulfilled.

References

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