On the Classical Fundamental Solution of the Cauchy Problem for a Dissipative Ultraparabolic Equation with One Group in Degenerate Variables and Differential Coefficients Independent of the Degenerate Variables

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The fundamental solution of the Cauchy problem for parabolic equations according to Petrovskii and Eidelman, when the coefficients of the equation are increasing coefficients of $|x| \to \infty$, has been well studied. Paper [2] presents results concerning Eidelman parabolic equations with increasing coefficients. For certain non-degenerate and degenerate ultraparabolic equations, the fundamental solution of the Cauchy problem and the estimates of its derivatives are obtained in explicit form [3, 5]. Article [4] considers a degenerate ultraparabolic equation with one degeneracy group. In this case, the equation's coefficients are independent of the degeneracy variables, and their growth depends on an increasing function. In [4], as in [1], L-fundamental solutions are studied. Note that in [2] L-fundamental solutions for ultraparabolic equations with coefficients independent of spatial variables are investigated.

Here we consider the classical fundamental solution for an ultraparabolic equation with two groups of spatial variables.

In a layer $\{(t,x) \mid \ t \in (0,T], \ x \in \mathbb{R}^n\}$ of finite thickness $T > 0$ we consider the ultraparabolic equation

$$\Big(S - \sum_{j,s=1}^{n_1} a_{js} - \sum_{j=1}^{n_1} a_j(t,x_1)\partial_{x_{1j}} - a_0(t,x_1)\Big)u(t,x) = 0, \tag{1}$$

where

$$S := \partial_t - \sum_{j=1}^{n_2} x_{1j}\partial_{x_{2j}}.$$

Here n_1, n_2 are given natural numbers such that $n_2 \le n_1$; $n := n_1 + n_2$; the variable $x \in \mathbb{R}^n$ consists of three groups of variables $x_l := (x_{l1}, \dots, x_{ln_l}) \in \mathbb{R}^{n_l}$, $l \in \{1,2\}$, so that $x := (x_1, x_2)$.

The equation (1) is a degenerate Kolmogorov equation of the second order, its coefficients a_{js}, $\{j,s\} \subset \{1,\dots,n_1\}$, a_j, $j \in \{1,\dots,n_1\}$, and a_0 do not depend on the degeneracy variables x_{2j}, $j \in \{1,\dots,n_2\}$.

The coefficients of the equation (1) are increasing functions of $|x_1| \to +\infty$. Their growth depends on the increasing function D with $|x_1| \to +\infty$.

Definition. The equation (1) will be called a dissipative ultraparabolic equation of Kolmogorov type in $\Pi_{[0,T]}$ if there exists a continuous function $D : \mathbb{R}^{n_1} \to [1,\infty)$, which satisfies the following conditions:

1) $D(x_1) \to \infty$ for $|x_1| \to \infty$;

2) functions
$$\begin{aligned} b_{js}(t,x_1) &\equiv a_{js}(t,x_1), \ \ \{j,s\} \subset \{1,\dots,n_1\}, \\ b_j(t,x_1) &\equiv a_j(t,x_1)D(x_1)^{-1}, \ \ j \in \{1,\dots,n_1\}, \\ b_0(t,x_1) &\equiv a_0(t,x_1)D(x_1)^{-2}, \ \ x_1 \in \mathbb{R}^{n_1}, \ \ 0 \le t \le T \end{aligned}$$
are bounded;

3) for the equation
$$\Big(\partial_t - \sum_{j=1}^{n_2} x_{1j}\partial_{x_{2j}} - \sum_{j,s=1}^{n_1} b_{js}(t,x_1)\partial_{x_{1j}}\partial_{x_{1s}} - \sum_{j=1}^{n_1} b_j(t,x_1)\partial_{x_{1j}}(-i\partial_{x_{n+1}}) - b_0(t,x_1)(-i\partial_{x_{n+1}})^2\Big)v(t,x) = 0,$$
with bounded coefficients and an additional spatial variable x_{n+1} the parabolicity condition is satisfied:
$$\exists\,\delta > 0 \ \ \forall\,\sigma_1 := (\sigma_{11},\dots,\sigma_{1n_1}) \in \mathbb{R}^{n_1} \ \ \forall\,\mu \in \mathbb{R} : \\ \operatorname{Re}\Big(- \sum_{j,s=1}^{n_1} b_{js}(t,x_1)\sigma_{1j}\sigma_{1s} - \sum_{j=1}^{n_1} b_j(t,x_1)\sigma_{1j}\mu - b_0(t,x_1)\mu^2\Big) \ge \delta\Big(|\sigma_1|^2 + \mu^2\Big).$$

The function D is called the dissipation characteristic of the equation (1).

It is assumed that the following conditions are satisfied for the coefficients of the equation (1).

$\mathbf{A}_1$**.** The equation (1) is a dissipative ultraparabolic equation of Kolmogorov type in $\Pi_{[0,T]}$ with dissipation characteristic D.

$\mathbf{A}_2$**.** There are continuous derivatives $\partial_{x_1}^{k_1} a_{js}$, $\{j,s\} \subset \{1,\dots,n_1\}$, $\partial_{x_1}^{k_1} a_j$, $j \in \{1,\dots,n_1\}$, $\partial_x^{k_1} a_0$, $|k_1| \le 2$, for which evaluations are verified
$$\begin{aligned} |\partial_{x_1}^{k_1} a_{js}(t,x_1)| &\le C(D(x_1))^{|k_1|(1-\varepsilon)}, \\ |\partial_{x_1}^{k_1} a_j(t,x_1)| &\le C(D(x_1))^{1+|k_1|(1-\varepsilon)}, \\ |\partial_{x_1}^{k_1} a_0(t,x_1)| &\le C(D(x_1))^{2+|k_1|(1-\varepsilon)}, \ \ t \in [0,T], \ \ x_1 \in \mathbb{R}^{n_1}, \end{aligned}$$
where $C > 0$, $\varepsilon \in (0,1)$; functions $b_{js}(t,x_1)$, $\{j,s\} \subset \{1,\dots,n_1\}$, $b_j(t,x_1)$, $j \in \{1,\dots,n_1\}$, $b_0(t,x_1)$ as functions of t are continuous uniformly with respect to $x_1 \in \mathbb{R}^{n_1}$.

$\mathbf{A}_3$**.** Derivatives $\partial_{x_1}^{k_1} a_{js}$, $\{j,s\} \subset \{1,\dots,n\}$, $\partial_{x_1}^{k_1} a_j$, $j \in \{1,\dots,n\}$, $\partial_{x_1}^{k_1} a_0$, $|k_1| \le 2$, satisfy the local Hölder condition for x with exponent $\lambda \in (0,1)$ uniformly with respect to $t \in [0,T]$, i.e.,
$$\forall\, R > 0 \ \ \exists\, C > 0 \ \ \forall \{x_1, z_1\} \subset B_R^1 \ \ \forall\, t \in [0,T] : \ \ \big|\Delta_{x_1}^{z_1} \partial_{x_1}^{k_1} a(t,x_1)\big| \le C|x_1 - z_1|^{\lambda},$$
where the symbol a denotes the coefficients of the equation; $B_R^1 = \{x_1 \in \mathbb{R}^{n_1} \mid |x_1| \le R\}$.

Let $g : \mathbb{R}^{n_1} \to [1,\infty)$ be a function that is related to the dissipation characteristic D by the condition

$\mathbf{A}_4$**.** $g(x_1) \to \infty$ for $|x_1| \to \infty$; there exist locally Hölder continuous with exponent λ from condition $\mathbf{A}_3$ derivatives $\partial_{x_1}^{k_1} g$, $0 < |k_1| \le 4$, which are related to the dissipation characteristic D by the condition
$$|\partial_{x_1}^{k_1} g(x_1)| \le C\eta(D(x_1))^{|k_1|(1-\varepsilon)}, \ \ x_1 \in \mathbb{R}^{n_1}, \ \ 0 < |k_1| \le 4,$$

where $C > 0$, $\varepsilon \in (0, 1)$, η is a sufficiently small positive number, the choice of which we will dispose of in each specific situation.

Fundamental solution of the Cauchy problem for the equation (1) we will search by the Levy method in the form

$$\begin{gathered}G(t, x; \tau, \xi) = \widehat{G}(t, x; \tau, \xi; x_1)\\ + \int\limits_{\tau}^{t} d\theta \int\limits_{\mathbb{R}^n} \widehat{G}(t, x; \theta, y; x_1) \varphi(\theta, y; \tau, \xi)\, dy, \ \ 0 \leq \tau < t \leq T, \ \ \{x, \xi\} \subset \mathbb{R}^n, \quad (2)\end{gathered}$$

where $\varphi(\,\cdot\,, \,\cdot\,; \tau, \xi) : \Pi_{[0,T]} \to \mathbb{C}$ is an unknown function that we choose so that the function $G(\,\cdot\,, \,\cdot\,; \tau, \xi_1) : \Pi_{[0,T]} \to \mathbb{C}$ was a solution of the equation (1) for any fixed point $(\tau, \xi) \in \Pi_{[0,T)}$. Here $G(\,\cdot\,, \,\cdot\,; \tau, y_1)$ is a fundamental solution of the Cauchy problem for the equation

$$\Big(S - \sum_{j,s=1}^{n_1} a_{js} - \sum_{j=1}^{n_1} a_j(t, y_1) \partial_{x_{1j}} - a_0(t, y_1)\Big) u(t, x) = 0, \ \ y_1 \in \mathbb{R}^{\ltimes\nvdash}.$$

Theorem. *Let the equation* (1) *satisfy the conditions* $\mathbf{A}_1$–$\mathbf{A}_3$. *Then for it there exists a fundamental solution of the Cauchy problem* $G(t, x; \tau, \xi)$, $0 \leq \tau < t \leq T$, $\{x, \xi\} \subset \mathbb{R}^n$, *for which assessments are verified*

$$|\partial_{x_1}^{k_1} G(t, x; \tau, \xi)| \leq C(t - \tau)^{-M - M_{k_1 0}} E_c(t - \tau, x, \xi),$$

and

$$\begin{gathered}|\partial_{x_1}^{k_1} G(t, x; \tau, \xi)| \leq C \sum_{j=0}^{|k_1|} (t - \tau)^{-M - M_{k_1 0} + j/2} (D(x_1))^{j(1-\varepsilon)} E_c(t - \tau, x, \xi) \exp\{g(x) - g(\xi)\},\\ 0 \leq \tau < t \leq T, \ \ \{x, \xi\} \subset \mathbb{R}^n, \ \ |k_1| \leq 2,\end{gathered}$$

where C *and* c *are positive constants, and* g *is any function satisfying condition* $\boldsymbol{A}_4$. *Here*

$$\begin{gathered}E_c(t, x, \xi) = \exp\Big\{ - c\Big(\frac{|X_1(t) - \xi_1|^2}{t} + \frac{|X_2(t) - \xi_2|^2}{t^3}\Big)\Big\}, \ \ t > 0, \ \ \{x, \, \xi\} \subset \mathbb{R}^n,\\ M_{kl} = \frac{|k_1 + |l_1|}{2} + 3\,\frac{|k_2 + |l_2|}{2}\,.\end{gathered}$$

Using (2) the existence of derivatives $\partial_{x_{2j}} G$, $j \in \{1, \dots, n_2\}$, is also proved and their estimates are obtained.

References

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