On Nonlocal Boundary Value Problems for First Order Singular Advanced Differential Equations

Nino Partsvania[1,2]

[1]*Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University Tbilisi, Georgia*

[2]*Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University Tbilisi, Georgia*

E-mail: nino.partsvania@tsu.ge

In the present report, the problem on the existence of a solution to the differential equation

$$u'(t) = f(t, u(\tau(t))), \tag{1}$$

satisfying the nonlocal condition

$$\sum_{i=1}^{m} \ell_i u(t_i) = c_0, \tag{2}$$

is considered.

Below everywhere it is assumed that $f : [a, b[\times \mathbb{R} \to \mathbb{R}$ and $\tau : [a, b[\to [a, b[$ are continuous functions,

$$\ell_i > 0 \;\; (i = 1, \dots, m),$$
$$t_1 = a \;\text{ and }\; t_i < t_{i+1} \;\; (i = 1, \dots, m-1) \;\text{ if }\; m > 2,$$

c_0 is a positive constant.

The particular case of (2) is the initial condition

$$u(a) = c_0. \tag{2_1}$$

We are mainly interested in the case where the function f has a nonintegrable singularity in the time variable at the point b, i.e. the case where

$$\int_a^b |f(t, x)|\, dt = +\infty.$$

In this case problems of type $(1), (2)$ (including the initial value problem $(1), (2_1)$) have not been studied previously (see, [1–7] and the references therein). The results below to some extent fill the existing gap.

Theorem 1. *If the condition*

$$f(t, 0) = 0, \;\; f(t, x) \leq 0 \;\text{ for }\; t \in [a, b[\,, \;\; x > 0 \tag{3}$$

holds, then problem $(1), (2)$ *has at least one nonnegative solution.*

Remark 1. In Theorem 1, the absolute value of the function f may have an arbitrary order of growth with respect to the phase variable at infinity. For example, if

$$f(t,x) = p(t)(e_n(0) - e_n(x)),$$

where $p : [a,b[\to]0,+\infty[$ is a continuous function, n is a natural number,

$$e_1(x) = \exp(x), \quad e_{k+1}(x) = \exp(e_k(x)), \quad k = 1,2,\dots,$$

then problem $(1),(2)$ has at least one nonnegative solution.

Remark 2. In Theorem 1, the restriction $t_1 = a$ is unimprovable in some sense. To verify this, we consider the problem

$$u'(t) = p(t)|u(\tau(t))|^\lambda, \tag{4}$$

$$u(a_0) = c_0, \tag{5}$$

where $a < a_0 < b$, $\lambda > 1$, and $p : [a,b] \to]-\infty,0[$ is a continuous function such that

$$(\lambda - 1)\int_a^{a_0} |p(t)|\,dt \ge c_0^{1-\lambda}.$$

This problem can be obtained from $(1),(2)$ in the case, where

$$m = 1, \quad t_1 = a_0, \quad f(t,x) \equiv p(t)|x|^\lambda.$$

It is obvious that all the conditions of Theorem 1 are satisfied except the condition $t_1 = a$. Nevertheless, problem $(4),(5)$ has no nonnegative solution. Indeed, if we assume that such a solution u exists, then we obtain

$$u(t) > 0, \quad -u'(t)u^{-\lambda}(t) = p(t) \text{ for } a < t < b.$$

Hence we get a contradiction

$$c_0^{1-\lambda} = (\lambda - 1)\int_a^{a_0} \big(-u'(t)u^{-\lambda}(t)\big)\,dt + u^{1-\lambda}(a) > (\lambda - 1)\int_a^{a_0} |p(t)|\,dt > c_0^{1-\lambda}.$$

Under the conditions of Theorem 1, every nonnegative solution to problem $(1),(2)$ is nonincreasing and, therefore, has the limit as $t \to b$. Theorem 2 below contains the sufficient condition for that limit to be zero. To formulate this theorem, we need to introduce the following notation:

$$r = c_0/\ell_1,$$

$$f_*(t,x) = \min\big\{|f(t,y)| : \ 0 < x \le y \le r\big\}.$$

Theorem 2. *If along with* (3) *the condition*

$$\int_a^b f_*(t,x)\,dt = +\infty \quad \textit{for } 0 < x < r$$

holds, then problem $(1),(2)$ *has at least one nonnegative solution and the left limit at* b *of every such solution is equal to zero.*

We are able to establish sufficient conditions for the uniqueness of a solution to problem (1), (2) only in the case where the function f is continuous on the set $[a,b]\times\mathbb{R}$. More precisely, the following theorem is valid.

Theorem 3. *Let the function $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ be continuous and let it satisfy condition* (3)*. If, moreover, f is nonincreasing and locally Lipschitz in the phase variable, then problem* (1), (2) *has a unique nonnegative solution.*

An important particular case of (1) is the equation

$$u'(t)=p(t)f_0(u(\tau(t))), \tag{6}$$

where $p:[a,b[\,\to\mathbb{R}$ and $f_0:\mathbb{R}\to\mathbb{R}$ are continuous functions.

Theorems 1–3 yield the following statements.

Corollary 1. *Let*

$$p(t)\geq 0 \ \text{ for } \ a<t<b, \quad f_0(0)=0, \quad f_0(x)<0 \ \text{ for } \ x>0. \tag{7}$$

Then problem (6), (2) *has at least one nonnegative solution. And if along with* (7) *the condition*

$$\int\limits_a^b p(t)\,dt=+\infty$$

holds, then the left limit at b of that solution is equal to zero.

Corollary 2. *Let conditions* (7) *be satisfied. If, moreover, p is continuous on $[a,b]$, and f_0 is nonincreasing, locally Lipschitz function, then problem* (6), (2) *has a unique nonnegative solution.*

Condition (3) in Theorem 1 is essential and it cannot be replaced by the condition

$$f(t,0)=0, \quad f(t,x)\geq 0 \ \text{ for } \ t\in[a,b[\,, \quad x\in\mathbb{R}.$$

More precisely, the following propositions are true.

Proposition 1. *Let*

$$f(t,x)\geq p(t)|x|^\lambda \ \text{ for } \ t\in[a,a_0], \quad x\in\mathbb{R},$$

where $\lambda>1$, $a<a_0<b$, and $p:[a,a_0]\to\,]0,+\infty[$ is a continuous function. If, moreover,

$$c_0>\bigg((\lambda-1)\int\limits_a^{a_0} p(t)\,dt\bigg)^{\frac{1}{1-\lambda}},$$

then problem (1), (2) *has no solution.*

Proposition 2. *Let*

$$f(t,x)\geq p(t)|x| \ \text{ for } \ t\in[a,a_0], \quad x\in\mathbb{R},$$
$$\tau(t)=a_0 \ \text{ for } \ t\in[a,a_0],$$

where $a<a_0<b$, and $p:[a,a_0]\to\,]0,+\infty[$ is a continuous function such that

$$\int\limits_a^{a_0} p(t)\,dt\geq 1.$$

Then problem (1), (2) *has no solution.*

At the end, we give the sketch of the proof of our main result.

Sketch of the Proof of Theorem 1. Let n be an arbitrary natural number. Set

$$b_{1n} = b - (b-a)/2n, \;\; b_{2n} = b - (b-a)/4n, \;\; b_n = b + 1/n, \;\; \tau_n(t) = \tau(t) + 1/n,$$
$$\delta_n(t) = \begin{cases} 1 & \text{for } \; a \le t \le b_{1n}, \\ (b_{2n} - t)/(b_{2n} - b_{1n}) & \text{for } \; b_{1n} < t < b_{2n}, \end{cases}$$
$$f_n(t,x) = \begin{cases} \delta_n(t) f(t,x) & \text{for } \; a \le t \le b_{2n}, \\ 0 & \text{for } \; t > b_{2n}, \end{cases}$$

and consider the strictly advanced differential equation

$$u'(t) = f_n(t, u(\tau_n(t))). \tag{8}$$

We prove:

1. If the conditions of Theorem 1 are satisfied, then for any positive number x Eq. (8) has a unique solution $u_n(\,\cdot\,; x)$, defined on $[a, b_n]$, such that $u_n(b_n; x) = x$;

2. For every n there exists $x_n > 0$ such that

$$\sum_{i=1}^{m} \ell_i u_n(t_i; x_n) = c_0;$$

3. From the sequence $(u_n(\,\cdot\,; x_n))_{n=1}^{+\infty}$ we can get a uniformly converging on every closed interval contained in $[a, b[$ subsequence $(u_{n_k}(\,\cdot\,; x_{n_k}))_{k=1}^{+\infty}$ whose limit is a solution to $(1), (2)$. □

References

[1] N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina, *Introduction to the Theory of Functional-Differential Equations.* (Russian) Nauka, Moscow, 1991.

[2] N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina, *Elements of the Modern Theory of Functional Differential Equations. Methods and Applications.* (Russian) Institute for Computer Studies, Moscow, 2022.

[3] N. V. Azbelev and L. F. Rakhmatullina, Theory of linear abstract functional-differential equations and applications. *Mem. Differential Equations Math. Phys.* **8** (1996), 1–102.

[4] R. Hakl, A. Lomtatidze and J. Šremr, *Some Boundary Value Problems for First Order Scalar Functional Differential Equations.* Masaryk University, Brno, 2002.

[5] P. Hartman, *Ordinary Differential Equations.* John Wiley, New York, 1964.

[6] I. Kiguradze and B. Půža, *Boundary Value Problems for Systems of Linear Functional Differential Equations.* Masaryk University, Brno, 2003.

[7] N. Partsvania, Initial value problem for first order advanced differential equations. *Trans. A. Razmadze Math. Inst.* **179** (2025), no. 3.