

Pairs of Positive Solutions of a Quasilinear Elliptic Neumann Problem Driven by the Mean Curvature Operator

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This contribution is based on our recent paper [26], for which we refer for further details and proofs. Following [26], we discuss here the existence of pairs of positive weak solutions of the model problem

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda w(x)|u|^{p-2}u & \text{in } \Omega, \\ -\frac{\nabla uv}{\sqrt{1+|\nabla u|^2}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

(h_1) Ω is a bounded domain in \mathbb{R}^N , $N \geq 2$, with a $C^{0,1}$ boundary $\partial\Omega$ and a unit outer normal ν to $\partial\Omega$;

(h_2) $p \in (1, 1^*)$, with $1^* = \frac{N}{N-1}$, is a fixed exponent, $w \in L^{\frac{1^*}{1^*-p}}(\Omega)$ is a non-zero function, and $\lambda \in \mathbb{R}$ is a parameter.

In a quite similar fashion, but at the expense of an additional, somewhat cumbersome, technical effort, we could also deal with more general differential equations involving quasilinear operators $-\operatorname{div}A(x, \nabla u)$ patterned on the mean curvature operator $-\frac{1}{N} \operatorname{div}(\nabla u / \sqrt{1+|\nabla u|^2})$, like in [4, 7, 21], and functions $f(x, u)$ modelled on the power function $|u|^{p-2}u$, like in [10, 19, 20, 22–24].

As it is natural for problem (1), two notions of weak solutions shall be used here: bounded variation solutions and classical weak solutions, whose precise definitions are given below.

Notation

Throughout, for any $v \in BV(\Omega)$, $Dv = D^a v + D^s v$ is the Lebesgue decomposition of the measure Dv in an absolutely continuous part $D^a v = \nabla v \mathcal{L}^n$, with ∇v the approximate differential of v and \mathcal{L}^n the N -dimensional Lebesgue measure in \mathbb{R}^N , and a singular part $D^s v$ with respect to \mathcal{L}^n . Moreover, $|Dv|$ stands for the total variation of the measure Dv , with $|Dv| = |Dv|^a + |Dv|^s = |D^a v| + |D^s v|$ the Lebesgue decomposition of $|Dv|$, and $\frac{Dv}{|Dv|}$ is the density of Dv with respect to $|Dv|$. We also denote by \mathcal{H}^{N-1} the $N - 1$ -dimensional Hausdorff measure (cf. [1] for further details on these topics). Finally, a function $v \in L^1(\Omega)$ is said positive if $\operatorname{ess\,inf} v \geq 0$ and $\operatorname{ess\,sup} v > 0$, while it is said sign-changing if $\operatorname{ess\,inf} v \cdot \operatorname{ess\,sup} v < 0$.

Definition 1 (Bounded variation solutions). Assume (h_1) and (h_2). A function $u \in BV(\Omega)$ is a bounded variation solution of (1) if, for all $\phi \in BV(\Omega)$ such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$,

$$\int_{\Omega} \frac{\nabla u \nabla \phi}{\sqrt{1+|\nabla u|^2}} dx + \int_{\Omega} \frac{Du}{|Du|} \frac{D\phi}{|D\phi|} d|D\phi|^s = \lambda \int_{\Omega} w|u|^{p-2}u\phi dx.$$

The next result whose proof is given in [26] establishes some relevant properties enjoyed by the bounded variation solutions of (1). Following [2], we set $X(\Omega)_N = \{z \in L^\infty(\Omega, \mathbb{R}^N) \mid \operatorname{div} z \in L^N(\Omega)\}$. Then, by [2, Theorem 1.9] the integration by parts formula

$$\int_{\Omega} v \operatorname{div} z \, dx = \int_{\partial\Omega} [z \cdot \nu] v \, d\mathcal{H}^{N-1} - \int_{\Omega} d(Dv, z)$$

holds for all $v \in BV(\Omega)$ and $z \in X(\Omega)_N$, where $[z \cdot \nu] \in L^\infty(\partial\Omega)$ is the weak trace on $\partial\Omega$ of the component of z along the outer normal ν to $\partial\Omega$, which exists by [2, Theorem 1.2], and (Dv, z) is the Radon measure defined in [2, Definition 1.4].

Proposition 1. *Let u be a bounded variation solution of (1). Then, the following holds:*

- (1) *the distributional divergence of the vector field $\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \in L^\infty(\Omega)$ satisfies*

$$-\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \lambda w |u|^{p-2} u \text{ almost everywhere in } \Omega,$$

accordingly, $\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \in X(\Omega)_N$;

- (2) *the identity, between measures in Ω ,*

$$\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, Du \right) = \sqrt{1 + |Du|^2} - \sqrt{1 - \frac{|\nabla u|^2}{1 + |\nabla u|^2}} \mathcal{L}^N$$

holds;

- (3) *the weak trace $\left[\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nu_{\partial\Omega} \right] \in L^\infty(\partial\Omega)$ of the component on $\partial\Omega$ of the vector field $\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \in X(\Omega)_N$ along to the unit outer normal ν exists and satisfies*

$$\left[\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nu_{\partial\Omega} \right] = 0 \quad \mathcal{H}^{N-1}\text{-almost everywhere on } \partial\Omega.$$

Remark 1. The third conclusion of Proposition 1 implies that all bounded variation solutions of (1) satisfy the homogeneous Neumann boundary condition in the weak sense explained above.

Remark 2. Proposition 1 shows that the notion of solution expressed by Definition 1 implies the validity of the Neumann counterpart of the definition of solution recently used, first, in [28] and, later, in [6] for the Dirichlet problem for the mean curvature equation.

The introduction of bounded variation functions when investigating prescribed mean curvature problems has become rather customary since the early seventies [8] and reveals absolutely imperative in the present context. Indeed, we shall show in a moment that, already in the one-dimensional case and even for Hölder continuous weight functions w , if (1) admits a pair of positive solutions, then the larger one necessarily displays jump discontinuities for sufficiently small values of the parameter.

The following definition of classical weak solution of (1) is quite standard (cf. [29]), nevertheless we remind it for completeness.

Definition 2 (Classical weak solutions). Assume (h_1) and (h_2) . A function $u \in C^1(\overline{\Omega})$ is a classical weak solution of (1) if, for all $\phi \in C^1(\overline{\Omega})$,

$$\int_{\Omega} \frac{\nabla u \nabla \phi}{\sqrt{1 + |\nabla u|^2}} \, dx = \lambda \int_{\Omega} w |u|^{p-2} u \phi \, dx. \tag{2}$$

Remark 3. Combining the third conclusion of Proposition 1 with [3, Remark 1.8] entails that, if Ω has a C^1 boundary $\partial\Omega$, then any classical weak solution u of (1) satisfies the homogeneous Neumann boundary condition everywhere on $\partial\Omega$. Moreover, from [11, Theorem 2.1] it follows that, if $w \in L^\infty(\Omega)$, then $u \in W_{\text{loc}}^{2,q}(\Omega)$ for all finite $q \geq 1$, and if, in addition $\partial\Omega$ has a $C^{1,1}$ boundary, then $u \in W^{2,q}(\Omega)$ for all finite $q \geq 1$.

This work is mainly motivated by the results we obtained in some recent papers, in collaboration with J. López-Gómez [13–18] and with J. López-Gómez and S. Rivetti [19,20], concerning the one-dimensional counterpart of (1),

$$\begin{cases} -\left(\frac{u'}{\sqrt{1 + |u'|^2}}\right)' = \lambda w(x) |u|^{p-2} u & \text{in } (0, 1), \\ u'(0) = 0, \quad u'(1) = 0 \end{cases} \tag{3}$$

and variations or extensions thereof. A particular example of our previous findings is provided by the following statement.

Theorem 1. *Assume*

(k_1) $p \in (1, 2)$;

(k_2) $w \in L^\infty(0, 1)$ and there is $z \in (0, 1)$ such that $w(x) > 0$ almost everywhere in $(0, z)$ and $w(x) < 0$ almost everywhere in $(z, 1)$;

(k_3) $\int_0^1 w(x) \, dx < 0$.

Then, there exists $\lambda^* \in (0, +\infty)$ such that, for all $\lambda > \lambda^*$, problem (3) has no positive bounded variation solutions and, for all $\lambda \in (0, \lambda^*)$, has at least two positive bounded variation solutions $u_\lambda^{(1)}, u_\lambda^{(2)}$, with $u_\lambda^{(1)} \in W^{2,\infty}(0, 1)$ and $u_\lambda^{(2)} \in W_{\text{loc}}^{2,\infty}[0, z) \cap W_{\text{loc}}^{2,\infty}(z, 1]$, satisfying

$$\|u_\lambda^{(1)}\|_{C^1} \rightarrow 0, \quad \text{ess inf } u_\lambda^{(2)} \rightarrow +\infty \text{ as } \lambda \rightarrow 0.$$

Moreover, the following regularity criterion is valid: if

$$\text{either } \int_0^z \left(\int_z^x w(t) \, dt\right)^{-\frac{1}{2}} \, dx = +\infty \text{ or } \int_z^1 \left(\int_z^x w(t) \, dt\right)^{-\frac{1}{2}} \, dx = +\infty \tag{4}$$

holds, then, for all $\lambda > 0$, all positive solutions of (3) belong to $W^{2,\infty}(0, 1)$, while if (4) fails, i.e.,

$$\text{both } \int_0^z \left(\int_z^x w(t) \, dt\right)^{-\frac{1}{2}} \, dx \in \mathbb{R} \text{ and } \int_z^1 \left(\int_z^x w(t) \, dt\right)^{-\frac{1}{2}} \, dx \in \mathbb{R} \tag{5}$$

hold, then, for all small $\lambda > 0$, $u_\lambda^{(2)}$ exhibits a jump discontinuity at z and

$$\lim_{\lambda \rightarrow 0} (u_\lambda^{(2)}(z^-) - u_\lambda^{(2)}(z^+)) = +\infty.$$

A few comments about Theorem 1 are in order.

- (1) The non-existence of positive solutions follows by adapting the arguments given in [19, Theorem 6.1, Remark 6.1], or [15, Theorem 2.1], whereas, their existence and multiplicity come from [20, Theorem 1.6], [19, Theorem 9.1]. The regularity criterion has been established in [13, Proposition 3.6], [14, Theorem 6.1], [17, Theorems 3.1–3.3]. The two plots of Figure 1 illustrate admissible configurations for the set of the positive solutions of (3), displaying branches which bifurcate from $(\lambda, u) = (0, 0)$ and from $(\lambda, u) = (0, +\infty)$, as guaranteed by Theorem 1.

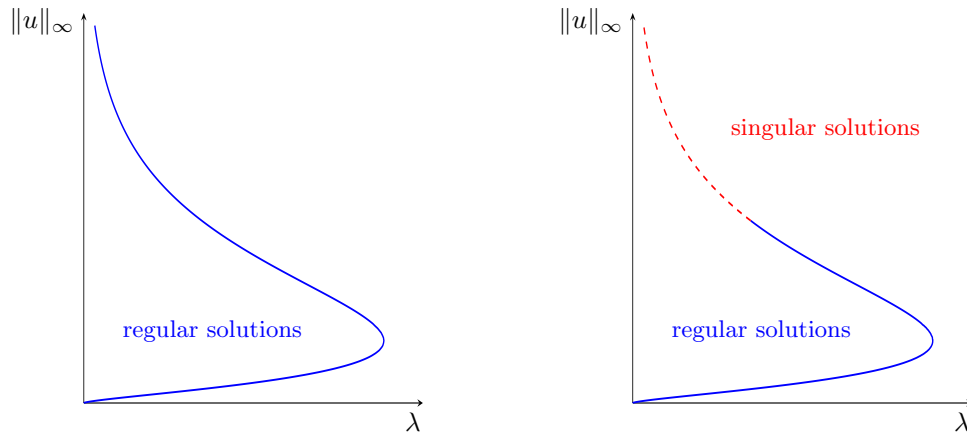


Figure 1. Admissible branches of positive solutions of (3) within the context of Theorem 1: on the left when (4) holds, on the right when (4) fails.

- (2) It has been proven in [20, Proposition 1.1] that the existence of a bounded variation solution u of (1), with $\text{ess inf } u > 0$, entails that the essential support of w^+ must have positive measure and (k_3) must hold. Accordingly, under (k_2) , Assumption (k_3) is necessary for the existence of solutions $u_\lambda^{(2)}$ satisfying $\text{ess inf } u_\lambda^{(2)} \rightarrow +\infty$ as $\lambda \rightarrow 0$.
- (3) Assumption (4) has been introduced in [14] and provides a mild one sided condition guaranteeing the regularity of all positive solutions of (3). We stress that (4) does not imply the continuity of w at z . For instance, it holds if $w(z^-) = 0$ and $w'(z^-) \in \mathbb{R}$, or $w(z^+) = 0$ and $w'(z^+) \in \mathbb{R}$, even though the function w is discontinuous at z . Whereas, it is manifest that (5) is fulfilled by a large class of Hölder continuous functions w . Condition (4) is optimal, as witnessed by the fact that its failure, that is, the validity of (5), yields the existence of singular, discontinuous, solutions of (3). Consequently, conditions (4) and (5) produce a sharp local criterion characterising the regularity properties of the large positive solutions of (3) for all small λ . This way, we also clarify and refine a result obtained in [9, Section 3C] (see also [5, Section 3.3]) where it is shown, through examples, that the membership of w to the Sobolev spaces $W^{1,q}(0,1)$ for any finite $q \geq 1$ is not enough to guarantee the $W^{1,1}$ -regularity of the solutions. Figure 2 displays a picture of the profiles of the positive solutions of (3) in the frame of Theorem 1.

In [26] we provided a partial extension of Theorem 1 to a genuine PDE setting, as formulated by Theorem 2. Yet, we stress that, while we are able to establish for (1) existence and multiplicity conclusions quite similar to those of Theorem 1, extending to (1) the optimal regularity/singularity assertions, based on conditions like (4) and (5), remains so far an extremely challenging unsolved problem.

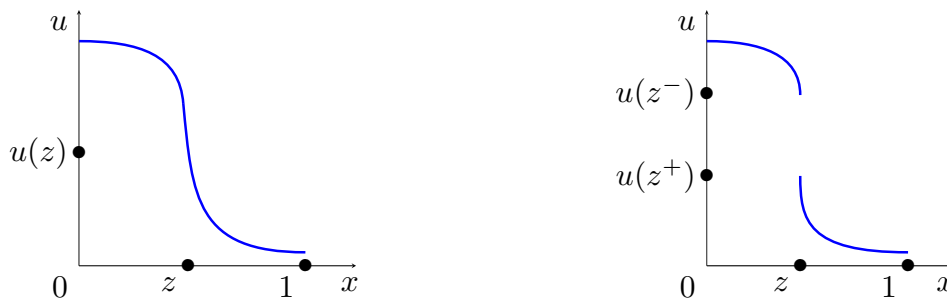


Figure 2. Profiles of the positive solutions of (3) within the context of Theorem 1: regular on the left, singular on the right.

Theorem 2. Assume (h_1) , (h_2) ,

(h_3) $\int_{\Omega} w \, dx < 0$ and the essential support of w^+ has non-empty interior.

Then, there exists $\lambda^* \in (0, +\infty)$ such that, for all $\lambda \in (0, \lambda^*)$, problem (1) has at least two positive bounded variation solutions $u_{\lambda}^{(1)}, u_{\lambda}^{(2)}$ satisfying

$$\|u_{\lambda}^{(1)}\|_{BV} \rightarrow 0, \quad \|u_{\lambda}^{(2)}\|_{L^{1^*}} \rightarrow +\infty \text{ as } \lambda \rightarrow 0.$$

Suppose, in addition, that

(h_4) $\partial\Omega$ is of class $C^{1,\sigma}$ for some $\sigma \in (0, 1]$,

(h_5) $w \in L^{\infty}(\Omega)$.

Then, there exist $\lambda_* \in (0, \lambda^*)$ and $\tau \in (0, 1]$ such that, for all $\lambda \in (0, \lambda_*]$, $u_{\lambda}^{(1)}$ can be chosen to be a classical weak solution satisfying

$$u_{\lambda}^{(1)} \in C^{1,\tau}(\overline{\Omega}) \cap W_{loc}^{2,q}(\Omega), \text{ for all finite } q \geq 1, \text{ and } \|u_{\lambda}^{(1)}\|_{C^{1,\tau}} \rightarrow 0 \text{ as } \lambda \rightarrow 0.$$

Remark 4. Due to Assumptions (h_1) – (h_3) , problem (1) belongs to the class, widely investigated in the literature, of sub-superlinear indefinite elliptic problems, specifically, according to the nature of the mean curvature operator, the right hand side of the differential equation in (1) is “2-sublinear” at zero and “1-superlinear” at infinity in the essential support of w^+ : this feature determines the existence of multiple solutions.

Remark 5. Under (h_1) and (h_2) , it is very easy to verify that Assumption (h_4) is almost necessary for the existence of strictly positive classical weak solutions of (1). Indeed, if, for some $\lambda > 0$, $u \in C^1(\overline{\Omega})$ is a classical weak solution of (1) with $\min u > 0$, then testing Equation (2) against $\phi = 1$ and $\phi = u^{1-p}$, respectively, yields

$$\int_{\Omega} w u^{p-1} \, dx = 0 \text{ and } (1-p) \int_{\Omega} \frac{u^{-p} |\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} \, dx = \lambda \int_{\Omega} w \, dx.$$

From the former identity it is apparent that

$$\operatorname{ess\,inf}_{\Omega} w \cdot \operatorname{ess\,sup}_{\Omega} w < 0,$$

so that, in particular, the essential support of w^+ must have positive measure. Obviously, the support of w must have non-empty interior whenever w is continuous in Ω . On the other hand, the latter identity implies, as $p > 1$, that either

$$\int_{\Omega} w \, dx < 0,$$

or u is a positive constant. This last eventuality is ruled out at once, because otherwise (2) would yield $\int_{\Omega} w \phi \, dx = 0$, for all $\phi \in C^1(\overline{\Omega})$, and thus $w = 0$, while (h_2) requires $w \neq 0$.

As already observed above, it follows from [20, Proposition 1.1] that the same conclusion holds for the positive bounded variation solutions of (3).

Remark 6. The proof of Theorem 2 is variational and combines minimisation and non-smooth critical point theory. Thanks to this features, further information are provided on the nature of $u_{\lambda}^{(1)}, u_{\lambda}^{(2)}$ as (sub-)critical points of the action functional $\mathcal{I}_{\lambda} : BV(\Omega) \rightarrow \mathbb{R}$ associated with problem (1) and defined, for $\lambda > 0$, by

$$\mathcal{I}_{\lambda}(v) = \mathcal{J}(v) - \lambda \int_{\Omega} \frac{1}{p} w |v|^p \, dx,$$

with $\mathcal{J} : BV(\Omega) \rightarrow \mathbb{R}$ the area functional in $BV(\Omega)$ given by

$$\mathcal{J}(v) = \int_{\Omega} \sqrt{1 + |\nabla v|^2} \, dx + \int_{\Omega} d|Dv|^s.$$

Namely, we see that the smaller solution, $u_{\lambda}^{(1)}$, is a local minimiser of \mathcal{I}_{λ} , while the larger one, $u_{\lambda}^{(2)}$, is a (sub-)critical point of mountain pass type. For other variational properties of $u_{\lambda}^{(1)}, u_{\lambda}^{(2)}$ we refer to [26].

Remark 7. The odd symmetry of problem (1) obviously implies that $-u_{\lambda}^{(1)}, -u_{\lambda}^{(2)}$ constitute a pair of negative classical weak solutions of problem (1). The fact of $u_{\lambda}^{(1)}, -u_{\lambda}^{(1)}$ being local minimisers predicts the existence of a third, sign-changing, classical weak solution. Actually, we can say more: from [25] we see that an infinite sequence of classical weak solutions tending to 0 exists for any given $\lambda > 0$. A second sequence of arbitrarily large bounded variation solutions, not necessarily classical, is expected to exist as well. These and other related topics will be investigated in a future paper.

Remark 8. The power function $|s|^{p-2}s$ can be replaced in Theorem 2 by any continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying, for some $p \in (1, 1^*), q \in (1, 2), \vartheta > 1$,

$$\lim_{s \rightarrow 0} \frac{f(s)}{s^{q-1}} = 1, \quad \lim_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} = 1, \quad \lim_{s \rightarrow +\infty} \frac{\vartheta \int_0^s f(t) \, dt - f(s)s}{s} = 0.$$

Based on the proof of Theorem 2 given in [26], the interested reader can provide the necessary details for establishing this statement.

Remark 9. A counterpart of Theorem 2 for problem (1) where the Neumann condition is replaced by a homogeneous Dirichlet condition can be established by combining [10, Theorem 2.1] and [27, Theorem 4.4]. In this setting (h_1) – (h_3) can still be assumed, although the condition $\int_{\Omega} w \, dx < 0$ in (h_3) is not necessary anymore. As a result, the situation is substantially less delicate in the Dirichlet case and the corresponding proofs are somehow simpler.

References

- [1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
- [2] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl. (4)* **135** (1983), 293–318 (1984).
- [3] G. Anzellotti, Traces of bounded vector-fields and the divergence theorem. *Tech. rep., Università degli Studi di Trento, Dipartimento di Matematica, September, 1983*.
- [4] G. Anzellotti, The Euler equation for functionals with linear growth. *Trans. Amer. Math. Soc.* **290** (1985), no. 2, 483–501.
- [5] E. Bombieri and E. Giusti, Local estimates for the gradient of non-parametric surfaces of prescribed mean curvature. *Comm. Pure Appl. Math.* **26** (1973), 381–394.
- [6] D. Giachetti, F. Oliva and F. Petitta, Bounded solutions for non-parametric mean curvature problems with nonlinear terms. *J. Geom. Anal.* **34** (2024), no. 9, Paper no. 265, 32 pp.
- [7] M. Giaquinta, G. Modica and J. Souček, Functionals with linear growth in the calculus of variations. I, II. *Comment. Math. Univ. Carolin.* **20** (1979), no. 1, 143–156, 157–172.
- [8] E. Giusti, Superfici cartesiane di area minima. (Italian) *Rend. Sem. Mat. Fis. Milano* **40** (1970), 135–153.
- [9] E. Giusti, Boundary value problems for non-parametric surfaces of prescribed mean curvature. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **3** (1976), no. 3, 501–548.
- [10] P. Habets and P. Omari, Positive solutions of an indefinite prescribed mean curvature problem on a general domain. *Adv. Nonlinear Stud.* **4** (2004), no. 1, 1–13.
- [11] H. Lewy and G. Stampacchia, On existence and smoothness of solutions of some non-coercive variational inequalities. *Arch. Rational Mech. Anal.* **41** (1971), 241–253.
- [12] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.* **12** (1988), no. 11, 1203–1219.
- [13] J. López-Gómez and P. Omari, Global components of positive bounded variation solutions of a one-dimensional indefinite quasilinear Neumann problem. *Adv. Nonlinear Stud.* **19** (2019), no. 3, 437–473.
- [14] J. López-Gómez and P. Omari, Characterizing the formation of singularities in a superlinear indefinite problem related to the mean curvature operator. *J. Differential Equations* **269** (2020), no. 2, 1544–1570.
- [15] J. López-Gómez and P. Omari, Regular versus singular solutions in a quasilinear indefinite problem with an asymptotically linear potential. *Adv. Nonlinear Stud.* **20** (2020), no. 3, 557–578.
- [16] J. López-Gómez and P. Omari, Branches of positive solutions of a superlinear indefinite problem driven by the one-dimensional curvature operator. *Appl. Math. Lett.* **126** (2022), Paper no. 107807, 10 pp.
- [17] J. López-Gómez and P. Omari, Optimal regularity results for the one-dimensional prescribed curvature equation via the strong maximum principle. *J. Math. Anal. Appl.* **518** (2023), no. 2, Paper no. 126719, 22 pp.
- [18] J. López-Gómez and P. Omari, Regular versus singular solutions in quasilinear indefinite problems with sublinear potentials. *J. Differential Equations* **372** (2023), 1–54.

- [19] J. López-Gómez, P. Omari and S. Rivetti, Bifurcation of positive solutions for a one-dimensional indefinite quasilinear Neumann problem. *Nonlinear Anal.* **155** (2017), 1–51.
- [20] J. López-Gómez, P. Omari and S. Rivetti, Positive solutions of a one-dimensional indefinite capillarity-type problem: a variational approach. *J. Differential Equations* **262** (2017), no. 3, 2335–2392.
- [21] M. Marzocchi, Multiple solutions of quasilinear equations involving an area-type term. *J. Math. Anal. Appl.* **196** (1995), no. 3, 1093–1104.
- [22] F. Obersnel and P. Omari, Positive solutions of the Dirichlet problem for the prescribed mean curvature equation. *J. Differential Equations* **249** (2010), no. 7, 1674–1725.
- [23] F. Obersnel and P. Omari, Positive solutions of the prescribed mean curvature equation with non-homogeneous mixed boundary conditions. *J. Differential Equations* **441** (2025), Paper no. 113462, 57 pp.
- [24] F. Obersnel and P. Omari, On the lower and upper solutions method for mean curvature problems with general boundary conditions. *J. Math. Anal. Appl.* **553** (2026), no. 1, Paper no. 129845, 42 pp.
- [25] P. Omari, Infinitely many regular weak solutions for odd symmetric prescribed mean curvature problems. *J. Geom. Anal.* **33** (2023), no. 6, Paper no. 189, 14 pp.
- [26] P. Omari, Pairs of positive solutions of a quasilinear elliptic Neumann problem driven by the mean curvature operator. *Rend. Istit. Mat. Univ. Trieste* **57** (2025), Art no. 7, 30 pp.
- [27] P. Omari and E. Sovrano, Positive solutions of superlinear indefinite prescribed mean curvature problems. *Commun. Contemp. Math.* **23** (2021), no. 3, Paper no. 2050017, 25 pp.
- [28] M. T. O. Pimenta and M. Montenegro, Existence of a BV solution for a mean curvature equation. *J. Differential Equations* **299** (2021), 51–64.
- [29] P. Pucci and J. Serrin, *The Maximum Principle*. Progress in Nonlinear Differential Equations and their Applications, 73. Birkhäuser Verlag, Basel, 2007.