On Qualitative and Asymptotic Properties of Solutions to the Riccati Equation with Stabilizing Real Roots of the Right-Hand Side

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Abstract

The authors investigate extensibility of solutions and the limits of these solutions at the boundaries of their domains for the Riccati equation with strictly ordered real roots of the right-hand side having different finite limits at $+\infty$ ($-\infty$ or both). In their previous works, the authors have already obtained some results on properties of solutions to such an equation, but using the additional assumption of monotonic stabilization of the roots. The results of this paper are obtained without that assumption, and the earlier theorems were supplemented and generalized.

Consider the equation

$$y' = (y - \alpha_1(x))(y - \alpha_2(x)), \tag{0.1}$$

where $\alpha_{1,2}(\,\cdot\,)$ are continuous real-valued functions.

Let $\mathrm{Dom}(y)$ denote the maximum interval (finite or infinite) on which the partial solution $y(\,\cdot\,)$ to a differential equation can be extended. Further, when we talk about solutions to a differential equation, we mean maximally extended solutions, i.e. solutions $y(\,\cdot\,)$ defined on $\mathrm{Dom}(y)$. We say that the solution $y(\,\cdot\,)$ to a differential equation is defined on the interval Δ if $y(\,\cdot\,)$ is defined at every point of the interval Δ, i.e. $\Delta \subset \mathrm{Dom}(y)$.

1 Considering (0.1) in a neighbourhood of $+\infty$ ($-\infty$)

In this section we consider equation (0.1) for $x \in [a, +\infty)$, $a \in \mathbb{R}$ unless otherwise specified.

Lemma 1.1. *If*

(1) $\alpha_1, \alpha_2 \in C[a, +\infty)$;

(2) $\alpha_1(x) < \alpha_2(x)$ *for* $x \in [a, +\infty)$;

(3) $\alpha := \frac{\alpha_1+\alpha_2}{2} \in C^1[a, +\infty)$;

(4) *there exist* $\lim\limits_{x\to+\infty} \alpha_j(x) \equiv \alpha_j^+ \in \mathbb{R}$, $j = 1, 2$;

(5) $\alpha_1^+ \neq \alpha_2^+$,

then there exists a solution $y(\,\cdot\,)$ to equation (0.1) *such that it is defined on $[x_0, +\infty)$, $x_0 \geq a$ and* $\lim\limits_{x\to+\infty} y(x) = \alpha_1^+$.

Lemma 1.2. *Let statements* (1)–(5) *from Lemma 1.1 be true. Then there exists a unique solution $y_*(\,\cdot\,)$ to equation* (0.1) *such that it is defined on $[b, +\infty)$ for some $b \geq a$ and* $\lim\limits_{x\to+\infty} y_*(x) = \alpha_2^+$.

Theorem 1.1. *Let statements* (1)–(5) *from Lemma 1.1 be true. Then there exists a unique solution $y_*(\,\cdot\,)$ to equation* (0.1) *such that it is defined on $[b, +\infty)$ for some $b \geq a$ and* $\lim\limits_{x\to+\infty} y_*(x) = \alpha_2^+$. *Let $y(\,\cdot\,)$ be a solution to* (0.1) *defined at a point $x_0 \in [a; +\infty)$.*

- *If $x_0 \in \operatorname{Dom}(y_*)$, $y(x_0) < y_*(x_0)$, then $y(\,\cdot\,)$ is defined on $[x_0, +\infty)$ and* $\lim\limits_{x\to+\infty} y(x) = \alpha_1^+$;
- *If $x_0 \in \operatorname{Dom}(y_*)$, $y(x_0) > y_*(x_0)$ or $x_0 \notin \operatorname{Dom}(y_*)$, then there exists $\overline{x} \in \mathbb{R}$ such that $\overline{x} > x_0$, $y(\,\cdot\,)$ is defined on $[x_0, \overline{x})$, and*
$$\lim_{x\to\overline{x}-0} y(x) = +\infty.$$

Proof. The first statement of the theorem is Lemma 1.2. By Lemma 1.1, there exists a solution $y_2(\,\cdot\,)$ defined on $[c, +\infty)$ for some $c > a$ such that $\lim\limits_{x\to+\infty} y_2(x) = \alpha_1^+$. Denote $d := \max(b, c)$.

1st case. Let $x_0 \in \operatorname{Dom}(y_*)$, $y(x_0) < y_*(x_0)$.

- Let $x_0 \geq d$. Then, by virtue of Theorem 3.2.3 [2], $y(\,\cdot\,)$ is defined on $[x_0, +\infty)$ and $\lim\limits_{x\to+\infty} y(x) = \alpha_1^+$.
- Let $x_0 < d$. We have $x_0 \in \operatorname{Dom}(y_*)$, on the other hand $y_*(\,\cdot\,)$ is defined on $[d, +\infty)$. Hence, $y_*(\,\cdot\,)$ is defined on $[x_0, +\infty)$. Thus, $y(x) < y_*(x) \leq \max\limits_{x\geq x_0} y(x)$ for $x \in \operatorname{Dom}(y) \cap [x_0, +\infty)$ due to Picard's theorem. Denote $\overline{x} := \sup \operatorname{Dom}(y)$. If $\overline{x} \in \mathbb{R}$, then, by Lemma 4.1 [2], $\lim\limits_{x\to\overline{x}-0} y(x) = +\infty$. We obtain a contradiction with boundedness of $y(\,\cdot\,)$ on $\operatorname{Dom}(y) \cap [x_0, +\infty)$. Hence, $y(\,\cdot\,)$ is defined on $[x_0, +\infty)$ and $y(x) < y_*(x)$ for $x \geq x_0$. In particular, $y(d) < y_*(d)$. According to Theorem 3.2.3 [2], $\lim\limits_{x\to+\infty} y(x) = \alpha_1^+$.

2nd case. Let $x_0 \in \operatorname{Dom}(y_*)$, $y(x_0) > y_*(x_0)$.

- Let $x_0 \geq d$. Then, by virtue of Theorem 3.2.2 [2], $y_*(\,\cdot\,)$ is a principal solution on $(x_0, +\infty)$. Hence, $y(\,\cdot\,)$ can't be defined on $[x_0, +\infty)$ and $\sup \operatorname{Dom}(y) =: \overline{x} \in \mathbb{R}$. Then, by Lemma 4.1 [2], $\lim\limits_{x\to\overline{x}-0} y(x) = +\infty$.
- Let $x_0 < d$. Assume that $y(\,\cdot\,)$ is defined on $[x_0, +\infty)$. Due to Picard's theorem $y(d) > y_*(d)$. We have that solutions $y_*(\,\cdot\,)$, $y_2(\,\cdot\,)$ are defined on $[d, +\infty)$. Then, by virtue of Theorem 3.2.2 [2], $y_*(\,\cdot\,)$ is a principal solution on $(d, +\infty)$. But $y(\,\cdot\,)$ is defined on $[d, +\infty)$ and $y(d) > y_*(d)$. We have a contradiction. Therefore, our assumption is incorrect, then $\sup \operatorname{Dom}(y) =: \overline{x} \in \mathbb{R}$, and according to Lemma 4.1 [2], $\lim\limits_{x\to\overline{x}-0} y(x) = +\infty$.

3rd case. Let $x_0 \notin \operatorname{Dom}(y_*)$.

The solutions $y_*(\,\cdot\,)$, $y_2(\,\cdot\,)$ are defined on $[d, +\infty)$ and have limits α_2^+ and α_1^+, respectively, as $x \to +\infty$. Hence, by Theorem 3.2.2 [2], $y_*(\,\cdot\,)$ is a principal solution on $(d, +\infty)$. On the other hand, we have $x_0 < \inf \operatorname{Dom}(y_*) =: u < d$, hence, by Lemma 4.1 [2], $\lim\limits_{x\to u+0} y_*(x) = -\infty$. Assume that $y(\cdot)$ is defined on $[x_0, +\infty)$. We know that $y_*(\cdot)$ is a principal solution on $(d, +\infty)$, hence $y(d) < y_*(d)$. We also have

$$\lim_{x \to u+0} y_*(x) = -\infty,$$
$$\lim_{x \to u+0} y(x) = y(u) \in \mathbb{R}.$$

Hence, there exists $\delta \in (0, d-u)$ such that $y(u) - 1 < y(x) < y(u) + 1$ and $y_*(x) < y(u) - 2$ for $x \in (u, u+\delta]$. In particular, $y_*(u+\delta) < y(u+\delta)$. Given that $u + \delta < d$ and $y(d) < y_*(d)$, we obtain that there exists $\xi \in (u+\delta, d)$ such that $y(\xi) = y_*(\xi)$. Hence, $\mathrm{Dom}(y) = \mathrm{Dom}(y_*)$ and $y \equiv y^*$. We obtain a contradiction. Therefore, our assumption is incorrect and $y(\cdot)$ is not defined on $[x_0, +\infty)$, hence, $\sup \mathrm{Dom}(y) =: \overline{x} \in \mathbb{R}$, and according to Lemma 4.1 [2], $\lim_{x \to \overline{x}-0} y(x) = +\infty$. □

Corollary 1.1. *Let statements* (1)–(5) *from Lemma 1.1 be true. If* $y(\cdot)$ *is a solution to* (0.1) *defined on* $[x_0, +\infty)$ *for some* $x_0 \geq a$*, then there exists* $\lim_{x \to +\infty} y(x) = \alpha_1^+$ *or* α_2^+.

Remark 1.1. By the substitution

$$x = x(t) = -t, \quad u(t) = -y(x(t))$$

in equation (0.1) we obtain

$$\frac{du}{dt} = -\frac{d(y(x(t)))}{dt} = -\frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx}\,.$$

Then the equation takes the form

$$\frac{du}{dt} = (u - \overline{\alpha}_1(t))(u - \overline{\alpha}_2(t)), \tag{1.1}$$

where $\overline{\alpha}_1(t) = -\alpha_1(x(t))$, $\overline{\alpha}_2(t) = -\alpha_2(x(t))$.

Given Remark 1.1, we apply Theorem 1.1 to equation (1.1) for $t \in (-a, +\infty)$ and then return to the original variables. We obtain the following

Theorem 1.2. *Consider equation* (0.1) *for* $x \in (-\infty, a]$, $a \in \mathbb{R}$*. If*

(1) α_1, $\alpha_2 \in C(-\infty, a]$;

(2) $\alpha_1(x) < \alpha_2(x)$ *for* $x \in (-\infty, a]$;

(3) $\alpha := \frac{\alpha_1 + \alpha_2}{2} \in C^1(-\infty, a]$;

(4) *there exist* $\lim_{x \to -\infty} \alpha_j(x) \equiv \alpha_j^- \in \mathbb{R}$, $j = 1, 2$;

(5) $\alpha_1^- \neq \alpha_2^-$,

then there exists a unique solution $y^*(\cdot)$ *to equation* (0.1) *such that it is defined on* $(-\infty, b]$ *for some* $b \leq a$ *and* $\lim_{x \to -\infty} y^*(x) = \alpha_1^-$*. Let* $y(\cdot)$ *be a solution to* (0.1) *defined at a point* $x_0 \in (-\infty, a]$.

- *If* $x_0 \in \mathrm{Dom}(y^*)$, $y(x_0) > y^*(x_0)$*, then* $y(\cdot)$ *is defined on* $(-\infty, x_0]$ *and*

$$\lim_{x \to -\infty} y(x) = \alpha_2^-;$$

- *If* $x_0 \in \mathrm{Dom}(y^*)$, $y(x_0) < y^*(x_0)$ *or* $x_0 \notin \mathrm{Dom}(y^*)$, *then there exists* $\underline{x} \in \mathbb{R}$ *such that* $\underline{x} < x_0$, $y(\,\cdot\,)$ *is defined on* $(\underline{x}, x_0]$, *and*
$$\lim_{x \to \underline{x}+0} y(x) = -\infty.$$

Corollary 1.2. *Consider equation* (0.1) *for* $x \in (-\infty, a]$, $a \in \mathbb{R}$. *Let statements* (1)–(5) *from Theorem 1.2 be true. If* $y(\,\cdot\,)$ *is a solution to* (0.1) *defined on* $(-\infty, x_0]$ *for some* $x_0 \leq a$, *then there exists* $\lim\limits_{x \to -\infty} y(x) = \alpha_1^-$ *or* α_2^-.

Remark 1.2. Theorem 1.1 refines and generalizes Theorem 3.3.1 [2], Theorem 1.2 refines and generalizes Theorem 3.3.1′ [2]. Note that Theorems 1.1, 1.2, unlike Theorems 3.3.1 [2] and 3.3.1′ [2], do not require monotonous stabilization of the roots (3.6) [2].

2 Considering (0.1) on $\mathbb{R}$

In this section we consider equation (0.1) for $x \in \mathbb{R}$.

Theorem 2.1. *If*

(1) α_1, $\alpha_2 \in C(\mathbb{R})$;

(2) $\alpha_1(x) < \alpha_2(x)$ *for* $x \in \mathbb{R}$;

(3) $\alpha := \frac{\alpha_1 + \alpha_2}{2} \in C^1(\mathbb{R})$;

(4) *there exist* $\lim\limits_{x \to \pm\infty} \alpha_j(x) \equiv \alpha_j^\pm \in \mathbb{R}$, $j = 1, 2$;

(5) $\alpha_1^+ \neq \alpha_2^+$, $\alpha_1^- \neq \alpha_2^-$,

then every bounded solution $y(\,\cdot\,)$ *to equation* (0.1) *is stabilizing with the limits* $\lim\limits_{x \to \pm\infty} y(x) =: y_\pm$, *and all such solutions are divided into the following four types according to the values of these limits:*

$$\begin{array}{ll} \text{type I:} \quad y_- = \alpha_1^-, \ \ y_+ = \alpha_1^+; & \text{type II:} \quad y_- = \alpha_2^-, \ \ y_+ = \alpha_1^+; \\ \text{type III:} \ y_- = \alpha_2^-, \ \ y_+ = \alpha_2^+; & \text{type IV:} \ y_- = \alpha_1^-, \ \ y_+ = \alpha_2^+. \end{array}$$

Proof. Let $y(\,\cdot\,)$ be a bounded solution to (0.1). By Lemma 4.1 [2], $\mathrm{Dom}(y) = \mathbb{R}$. Let's fix arbitrary $a \in \mathbb{R}$. We have:

- $y(\,\cdot\,)$ is defined on $[a, +\infty)$, hence, by Corollary 1.1, there exists $\lim\limits_{x \to +\infty} y(x) = \alpha_1^+$ or α_2^+.
- $y(\,\cdot\,)$ is defined on $(-\infty, a]$, hence, by Corollary 1.2, there exists $\lim\limits_{x \to -\infty} y(x) = \alpha_1^-$ or α_2^-. ☐

Further in this section, we assume conditions (1)–(5) of Theorem 2.1 to be satisfied.

Theorem 2.2. *There exists a unique solution* $y_*(\,\cdot\,)$ *to equation* (0.1) *such that it is defined in a neighborhood of* $+\infty$ *and* $\lim\limits_{x \to +\infty} y_*(x) = \alpha_2^+$. *There exists a unique solution* $y^*(\,\cdot\,)$ *to equation* (0.1) *such that it is defined in a neighborhood of* $-\infty$ *and* $\lim\limits_{x \to -\infty} y^*(x) = \alpha_1^-$. *If* $x_* := \inf \mathrm{Dom}(y_*)$, $x^* := \sup \mathrm{Dom}(\mathrm{y}^*)$, *then exactly one of the following statements is true for the solutions* $y^*(\,\cdot\,)$ *and* $y_*(\,\cdot\,)$:

(1) *x_*, $x^* \in \mathbb{R}$, $\lim\limits_{x\to x_*} y_*(x) = -\infty$, $\lim\limits_{x\to x^*} y^*(x) = +\infty$, if $x_* < x^*$, then $y_*(x) < y^*(x)$ for $x \in (x_*, x^*)$;*

(2) *$x_* = -\infty$, $x^* = +\infty$, $y_*(x) > y^*(x)$ for $x \in \mathbb{R}$, $y_*(\,\cdot\,)$ is a* type III *solution, $y^*(\,\cdot\,)$ is a* type I *solution;*

(3) *$x_* = -\infty$, $x^* = +\infty$, $y_*(x) = y^*(x)$ for $x \in \mathbb{R}$, $y_*(\,\cdot\,) = y^*(\,\cdot\,)$ is a* type IV *solution.*

Proof. By Theorem 1.1, there exists a unique solution $y_*(\,\cdot\,)$ to equation (0.1) such that it is defined in a neighborhood of $+\infty$ and $\lim\limits_{x\to+\infty} y_*(x) = \alpha_2^+$. By Theorem 1.2, there exists a unique solution $y^*(\,\cdot\,)$ to equation (0.1) such that it is defined in a neighborhood of $-\infty$ and $\lim\limits_{x\to-\infty} y^*(x) = \alpha_1^-$.

1st case. Let $x_* \in \mathbb{R}$.

By Lemma 4.1 [2], $\lim\limits_{x\to x_*+0} y_*(x) = -\infty$.

Assume that $x^* = +\infty$. Therefore, $\mathrm{Dom}(y^*) = \mathbb{R}$. But $\mathrm{Dom}(y_*) = (x_*, +\infty)$, hence $y^*(x) \neq y_*(x)$ for $x > x_*$. So, by Theorem 1.1, $\lim\limits_{x\to+\infty} y^*(x) = \alpha_1^+ < \alpha_2^+$. Then there exists $M > x_*$ such that $y^*(x) < y_*(x)$ for $x \geq M$.

On the other hand, $\lim\limits_{x\to x_*+0} y_*(x) = -\infty$ and $\lim\limits_{x\to x_*+0} y^*(x) = y^*(x_*)$. Therefore, as in the proof of Theorem 1.1, there exists $\delta \in (0, M - x_*)$ such that $y_*(x) < y^*(x)$ for $x \in (x_*, x_* + \delta]$. So, $x_* + \delta < M$, $y_*(x_* + \delta) < y^*(x_* + \delta)$, $y_*(M) > y^*(M)$. Hence, there exists $\xi \in (x_* + \delta, M)$ such that $y_*(\xi) = y^*(\xi)$. We obtain a contradiction. Therefore, $x^* \in \mathbb{R}$ and, by Lemma 4.1 [2], $\lim\limits_{x\to x^*+0} y^*(x) = +\infty$.

If $x_* < x^*$, then there exists $\delta_0 \in (0, x^* - x_*)$ such that $y_*(x) < y^*(x)$ for $x \in (x_*, x_* + \delta_0]$, hence, $y_*(x) < y^*(x)$ for $x \in (x_*, x^*)$.

2nd case. Let $x_* = -\infty$.

We have $\mathrm{Dom}(y_*) = \mathbb{R}$. According to Theorem 1.2, we have two possibilities: 1) $y_* \equiv y^*$ or 2) $y_*(x) > y^*(x)$ for $x \in (-\infty, x^*)$ and $\lim\limits_{x\to-\infty} y_*(x) = \alpha_2^-$. If $y_* \equiv y^*$, then we obtain statement (3) of the theorem being proved. Consider the second sub-case. We have that $y_*(\,\cdot\,)$ is a type III solution. Let's fix arbitrary $x_0 < x^*$. We have $y^*(x_0) < y_*(x_0)$, hence, by Theorem 1.1, $x^* = +\infty$ and $\lim\limits_{x\to+\infty} y^*(x) = \alpha_1^+$. Therefore $y^*(\,\cdot\,)$ is a type I solution. □

Definition 2.1. We call solutions $y_*(\,\cdot\,)$ and $y^*(\,\cdot\,)$ from Theorem 2.2 *the right principal solution* to (0.1) and *the left principal solution* to (0.1), respectively.

Theorem 2.3. *Suppose that statement* (1) *of Theorem 2.2 is true. Let $y(\,\cdot\,)$ be a solution to* (1.1) *defined at a point $x_0 \in \mathbb{R}$.*

(1) *If $x_0 \in \mathrm{Dom}(y_*)$, $y(x_0) < y_*(x_0)$, then*

$$\lim_{x\to+\infty} y(x) = \alpha_1^+$$

and there exists $\underline{x} \in \mathbb{R}$ such that $x_* \leq \underline{x} < x_0$, $\mathrm{Dom}(y) = (\underline{x}, +\infty)$, and

$$\lim_{x\to\underline{x}+0} y(x) = -\infty;$$

(2) *If $x_0 \in \mathrm{Dom}(y^*)$, $y(x_0) > y^*(x_0)$, then*

$$\lim_{x\to-\infty} y(x) = \alpha_2^-$$

and there exists $\overline{x} \in \mathbb{R}$ such that $x_0 < \overline{x} \leq x^*$, $\mathrm{Dom}(y) = (-\infty, \overline{x})$, and

$$\lim_{x\to\overline{x}-0} y(x) = +\infty;$$

(3) *otherwise, i.e. if*

$$(x_0 \notin \operatorname{Dom}(y_*)) \,|\, (x_0 \in \operatorname{Dom}(y_*)) \,\&\, (y(x_0) > y_*(x_0))$$

and

$$(x_0 \notin \operatorname{Dom}(y^*)) \,|\, (x_0 \in \operatorname{Dom}(y^*)) \,\&\, (y(x_0) < y^*(x_0)),$$

then there exist $\underline{x}, \overline{x} \in \mathbb{R}$ *such that* $\operatorname{Dom}(y) = (\underline{x}, \overline{x})$ *and*

$$\lim_{x \to \underline{x}+0} y(x) = -\infty, \quad \lim_{x \to \overline{x}-0} y(x) = +\infty.$$

Proof. The first part of Theorem 2.3's statement (1) follows from Theorem 1.1. Denote $\underline{x} := \inf \operatorname{Dom}(y)$. If $\underline{x} < x_*$, then $y(x) < y_*(x)$ for $x > x_*$. But $\lim\limits_{x \to x_*} y_*(x) = -\infty$, hence, $\lim\limits_{x \to x_*} y(x) = -\infty$ and $\inf \operatorname{Dom}(y) = x_*$. We obtain a contradiction. Therefore, $\underline{x} > x_*$ and Lemma 4.1 [2] finishes the proof of Theorem 2.3's statement (1).

Statement (2) of Theorem 2.3 can be proved similarly to statement (1) of Theorem 2.3 by using Theorem 1.2 and Lemma 4.1 [2].

Let $x_0 \in \operatorname{Dom}(y_*)$, $y(x_0) > y_*(x_0)$. It follows from Theorem 1.1 that there exists $\overline{x} \in \mathbb{R}$ such that $x_0 < \overline{x}$, $y(\,\cdot\,)$ is defined on $[x_0, \overline{x})$, and

$$\lim_{x \to \overline{x}-0} y(x) = +\infty.$$

Let $x_0 \notin \operatorname{Dom}(y_*)$, i.e., $x_0 < x_*$. If $\overline{x} := \sup \operatorname{Dom}(y) = +\infty$, then reasoning as in the 1st case of the Theorem 2.2's proof, we obtain $y(\xi) = y_*(\xi)$ for some $\xi > x_*$. We have a contradiction. Hence, $\overline{x} \in \mathbb{R}$ and, by Lemma 4.1 [2],

$$\lim_{x \to \overline{x}-0} y(x) = +\infty.$$

Let $x_0 \in \operatorname{Dom}(y^*)$, $y(x_0) < y^*(x_0)$. It follows from Theorem 1.2 that there exists $\underline{x} \in \mathbb{R}$ such that $x_0 > \underline{x}$, $y(\,\cdot\,)$ is defined on $(\underline{x}, x_0]$, and

$$\lim_{x \to \underline{x}+0} y(x) = -\infty.$$

Let $x_0 \notin \operatorname{Dom}(y^*)$, i.e., $x_0 > x^*$. If $\underline{x} := \inf \operatorname{Dom}(y) = -\infty$, then $y(\xi) = y^*(\xi)$ for some $\xi < x^*$ and we obtain a contradiction. Hence, $\underline{x} \in \mathbb{R}$ and, by Lemma 4.1 [2],

$$\lim_{x \to \underline{x}+0} y(x) = -\infty.$$

Thus, statement (3) of Theorem 2.3 is proved. □

The next two theorems follow from Theorem 1.1 and Theorem 1.2.

Theorem 2.4. *Suppose that statement* (2) *of Theorem 2.2 is true. Then, for each solution* $y(\,\cdot\,)$ *to equation* (1.1), *we have:*

(1) *if* $y^* < y < y_*$, *then* $y(\,\cdot\,)$ *is a* type II *solution;*

(2) *if* $y > y_*$, *then there exists* $\overline{x} \in \mathbb{R}$ *such that* $y(\,\cdot\,)$ *is defined on* $(-\infty, \overline{x})$ *and* $y_- = \alpha_2^-$, $\lim\limits_{x \to \overline{x}-0} y(x) = +\infty$;

(3) *if* $y < y^*$, *then there exists* $\underline{x} \in \mathbb{R}$ *such that* $y(\,\cdot\,)$ *is defined on the interval* $(\underline{x}, +\infty)$ *and* $y_+ = \alpha_1^+$, $\lim\limits_{x \to \underline{x}+0} y(x) = -\infty$.

Theorem 2.5. *Suppose that statement* (3) *of Theorem 2.2 is true. Then for each solution* $y(\,\cdot\,)$ *to equation* (1.1)*, we have:*

(1) *if* $y > y_*$*, then there exists* $\overline{x} \in \mathbb{R}$ *such that* $y(\,\cdot\,)$ *is defined on* $(-\infty, \overline{x})$ *and* $y_- = \alpha_2^-$*,* $\lim\limits_{x \to \overline{x}-0} y(x) = +\infty$*;*

(2) *if* $y < y_*$*, then there exists such* $\underline{x} \in \mathbb{R}$ *that* $y(\,\cdot\,)$ *is defined on the interval* $(\underline{x}, +\infty)$ *and* $y_+ = \alpha_1^+$*,* $\lim\limits_{x \to \underline{x}+0} y(x) = -\infty$*.*

Theorems 2.1–2.5 have two important Corollaries.

Corollary 2.1. *If equation* (0.1) *has a* type II *solution, then there exist unique solutions* y_I *and* y_{III} *of* types I *and* III*, respectively; moreover, for each solution* $y(\,\cdot\,)$ *to equation* (0.1)*, we have:*

(1) *if* $y_{\mathrm{I}} < y < y_{\mathrm{III}}$*, then* $y(\,\cdot\,)$ *is a* type II *solution;*

(2) *if* $y > y_{\mathrm{III}}$*, then there exists* $\overline{x} \in \mathbb{R}$ *such that* $y(\,\cdot\,)$ *is defined on* $(-\infty, \overline{x})$ *and* $y_- = \alpha_2^-$*,* $\lim\limits_{x \to \overline{x}-0} y(x) = +\infty$*;*

(3) *if* $y < y_{\mathrm{I}}$*, then there exists* $\underline{x} \in \mathbb{R}$ *such that* $y(\,\cdot\,)$ *is defined on* $(\underline{x}, +\infty)$ *and* $y_+ = \alpha_1^+$*,* $\lim\limits_{x \to \underline{x}+0} y(x) = -\infty$*.*

Corollary 2.2. *Exactly one of the following statements is true for equation* (0.1)*:*

(a) *there exists a* type II *solution;*

(b) *there is a single stabilizing solution and it is of* type IV*;*

(c) *there are no stabilizing solutions.*

Remark 2.1. Note that Corollaries 2.1–2.2 are Theorems 3.3.3 [2] and 3.3.7 [2], respectively, with omitted monotone stabilization condition (3.6) [2]. In turn, Theorem 3.3.7 [2] refines Theorem 11 [1] and supplements the results of [3].

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