Periodic Solutions of the Admissibly Perturbed Langford System

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1 Introduction

A wide range of natural and engineered phenomena are modeled using systems of ordinary differential equations. Most of these systems cannot be integrated even by quadratures. This raises the question of studying the solutions of such systems of differential equations based on the systems themselves (i.e., applying the qualitative theory of differential equations). One of the modern and powerful tools in the qualitative theory of differential equations is the Mironenko reflecting function (MRF), introduced by professor V. I. Mironenko (see [8, 9]).

For a system of ordinary differential equations

$$\dot{x} = X(t, x), \quad t \in \mathbb{R}, \quad x \in D \subset \mathbb{R}^n \tag{1.1}$$

with a general solution in Cauchy form $x = \varphi(t; t_0, x_0)$ the MRF is defined by the formula $F(t, x) := \varphi(-t; t, x)$ (see [9, p. 11], and also [8, p. 62]). The theory of MRF allows one to study the qualitative behavior of solutions even of systems that are not integrable in closed form, despite the fact that the MRF is defined (formally) through the general solution of this system. The MRF allows one to solve such problems of the qualitative theory of differential equations as the questions of existence and stability of periodic solutions [9], the existence of solutions to boundary value problems [7], and questions of the global behavior of families of solutions to differential systems [8]. Research by J. Zhou, Z. Zhou, M. S. Belokursky, V. I. Mironenko, V. V. Mironenko, E. V. Musafirov, and others (see [1, 10, 11, 13, 19, 22]) further develops these applications.

Any continuously differentiable function $F(t, x)$, satisfying the condition

$$F(-t, F(t, x)) \equiv F(0, x) \equiv x,$$

is an MRF of the set of systems (see [8]). All systems with the same MRF have the same shift operator (see [5]). Therefore, all 2ω-periodic systems with the same MRF have the same mapping over the period $[-\omega; \omega]$.

Let system (1.1) and the system

$$\dot{y} = Y(t, x), \quad t \in \mathbb{R}, \quad y \in D \subset \mathbb{R}^n \tag{1.2}$$

have the same MRF $F(t, x)$, and let system (1.1) be 2ω-periodic. Then if the solution $\phi(t; -\omega, x)$ of system (1.1) and the solution $\psi(t; -\omega, x)$ of system (1.2) can be extended to a segment $[-\omega, \omega]$, then the mapping over the period $[-\omega, \omega]$ for system (1.1) is $\phi(\omega; -\omega, x) \equiv F(-\omega, x) \equiv \psi(\omega; -\omega, x)$, although system (1.2) may be non-periodic. That is, between 2ω-periodic solutions of system (1.1) and solutions of the two-point problem $y(-\omega) = y(\omega)$ for system (1.2), a one-to-one correspondence can be established (see [8]).

Hence, systems with common MRF exhibit similar qualitative solution properties. Therefore, when studying the qualitative properties of solutions of systems, it is advisable to replace a complicated system with a simpler one.

2 Admissible perturbations

The statement from [10] allows us to find perturbations of differential systems that do not change the MRF (we will call such perturbations *admissible*).

Lemma 2.1. *Let the vector functions $\Delta_i(t,x)$ $(i=\overline{1,m})$ are solutions of the partial differential equation*

$$\frac{\partial\Delta(t,x)}{\partial t}+\frac{\partial\Delta(t,x)}{\partial x}X(t,x)-\frac{\partial X(t,x)}{\partial x}\,\Delta(t,x)=0. \tag{2.1}$$

Then the MRF of the perturbed differential system of the form

$$\dot x=X(t,x)+\sum_{i=1}^{m}\alpha_i(t)\Delta_i(t,x),\quad t\in\mathbb{R},\quad x\in D\subset\mathbb{R}^n$$

coincides with the MRF of system (1.1)*, where $\alpha_i(t)$ are arbitrary continuous scalar odd functions.*

The aim of this work is to find admissible perturbations for well–known Langford system (see [2–4, 6, 12, 14–18, 21] and the references therein) which is modelling turbulence in a liquid:

$$\begin{aligned}\dot x&=(a-1)x-y+xz,\\ \dot y&=x+(a-1)y+yz,\\ \dot z&=az-(x^2+y^2+z^2);\quad a,x,y,z\in\mathbb{R}.\end{aligned} \tag{2.2}$$

System (2.2) has two equilibrium (steady state) points $O_1(0,0,0)$, $O_2(0,0,a)$ and very rich bifurcation behavior.

Using Lemma 2.1, we found admissible perturbations for system (2.2).

Theorem 2.1. *Let $\alpha_i(t)$, $i=\overline{1,4}$ be arbitrary scalar continuous odd functions.*

I. *The MRF of system* (2.2) *coincides with the MRF of the system*

$$\begin{aligned}\dot x&=((a-1+z)x-y)(1+\alpha_1(t))+(a-1+z)x\,\alpha_2(t)+y\alpha_3(t),\\ \dot y&=(x+(a-1+z)y)(1+\alpha_1(t))+(a-1+z)y\,\alpha_2(t)-x\alpha_3(t),\\ \dot z&=(az-x^2-y^2-z^2)(1+\alpha_1(t)+\alpha_2(t)).\end{aligned} \tag{2.3}$$

II. *For $a=2/3$ the MRF of system* (2.2) *coincides with the MRF of the system*

$$\begin{aligned}\dot x&=\Big(\Big(z-\frac13\Big)x-y\Big)(1+\alpha_1(t))+\Big(z-\frac13\Big)x\,\alpha_2(t)\\ &\qquad+y\alpha_3(t)-y(x^2+y^2)\Big(x^2+y^2+2z^2-\frac43\,z\Big)\alpha_4(t),\\ \dot y&=\Big(x+\Big(z-\frac13\Big)y\Big)(1+\alpha_1(t))+\Big(z-\frac13\Big)y\alpha_2(t)-x\alpha_3(t)\\ &\qquad+x(x^2+y^2)\Big(x^2+y^2+2z^2-\frac43\,z\Big)\alpha_4(t),\\ \dot z&=-\Big(x^2+y^2+z^2-\frac23\,z\Big)(1+\alpha_1(t)+\alpha_2(t)).\end{aligned}$$

Proof. The proof follows from Lemma 2.1 by successive verification of identity (2.1) for each vector factor for $\alpha_i(t)$. □

Remark. Typically, the dynamics of processes are modeled on a non-negative time semi-axis, so the requirement that the functions be odd $\alpha_i(t)$ is not essential, since the functions $\alpha_i(t)$ (provided that $\alpha_i(0)=0$) can be extended continuously in an odd way to the negative time semi-axis.

Theorem 2.1 in combination with MRF theory allows one to transfer qualitative properties, such as periodicity and stability, of the unperturbed Langford system to admissibly perturbed systems.

3 Periodic solution

According to [20, Theorem 9], for $1/2 < a < 1$, system (2.2) has a 2π-periodic solution

$$\begin{aligned} x(t) &= -\sqrt{(1-a)(2a-1)}\,\sin t, \\ y(t) &= \sqrt{(1-a)(2a-1)}\,\cos t, \\ z(t) &= 1-a \end{aligned} \tag{3.1}$$

which lies on the invariant cycle $x^2 + y^2 = (1-a)(2a-1)$, $z = 1-a$. Moreover, this solution is asymptotically stable for $a < 2/3$ and unstable for $a > 2/3$. Analogous statements hold for systems (2.3) and (2.4).

Lemma 3.1. *Let $\alpha_i(t)$ $(i = \overline{1,4})$ be scalar continuous functions (not necessarily odd).*

I. *For $1/2 < a < 1$, system (2.3) has a solution*

$$\begin{aligned} x(t) &= -\sqrt{(1-a)(2a-1)}\,\sin\Big(t + \int_0^t \big(\alpha_1(s)-\alpha_3(s)\big)\,\mathrm{d}s\Big), \\ y(t) &= \sqrt{(1-a)(2a-1)}\,\cos\Big(t + \int_0^t \big(\alpha_1(s)-\alpha_3(s)\big)\mathrm{d}s\Big), \\ z(t) &= 1-a \end{aligned} \tag{3.2}$$

on the cycle $x^2 + y^2 = (1-a)(2a-1)$, $z = 1-a$.

II. *System (2.4) has a solution*

$$\begin{aligned} x(t) &= (1-a)\sin\Big(t + \int_0^t \big(\alpha_1(s)-\alpha_3(s)-(a-1)^4\alpha_4(s)\big)\,\mathrm{d}s\Big), \\ y(t) &= (a-1)\cos\Big(t + \int_0^t \big(\alpha_1(s)-\alpha_3(s)-(a-1)^4\alpha_4(s)\big)\,\mathrm{d}s\Big), \\ z(t) &= 1-a \end{aligned} \tag{3.3}$$

on the cycle $x^2 + y^2 = (a-1)^2$, $z = 1-a$.

Proof. Direct substitution of (3.2) into (2.3) and (3.3) into (2.4) confirms both assertions. □

Theorem 3.1. *Assume $\alpha_i(t)$ $(i = \overline{1,4})$ are twice continuously differentiable odd functions, and the right-hand sides of systems (2.3) and (2.4) are 2π-periodic in t.*

I. *For $1/2 < a < 1$, if $\int_{-\pi}^{0} (\alpha_1(s) - \alpha_3(s))\,ds = 2\pi k$ for some $k \in \mathbb{Z}$, then solution (3.2) of system (2.3) is 2π-periodic. It is asymptotically stable for $a < 2/3$ and unstable for $a > 2/3$.*

II. *If $\int_{-\pi}^{0} (\alpha_1(s) - \alpha_3(s) - (a-1)^4\alpha_4(s))\,ds = 2\pi k$ for some $k \in \mathbb{Z}$, then solution (3.3) of system (2.4) is 2π-periodic.*

Proof.

I. Theorem 2.1 implies systems (2.3) and (2.2) share the same MRF. By [20, Theorem 9], for $1/2 < a < 1$, system (2.2) has a 2π-periodic solution (3.1) with stated stability. Lemma 3.1 gives solution (3.2) for (2.3). Let $\overline{\gamma}(t)$ denote (3.1) and $\overline{\chi}(t)$ denote (3.2). The given integral condition implies $\overline{\chi}(-\pi) = \overline{\gamma}(-\pi)$. The conclusion follows from [11, Theorem 5], which ensures the identity of the mapping over the period and stability properties.

II. For $a = 2/3$, system (2.4) has the same MRF as (2.2). The analogous reasoning applies using solution (3.3) and the stated integral condition. The conclusion again follows from [11, Theorem 5]. □

Theorem 3.2. *Let $\alpha_i(t)$ be continuous functions (not necessarily odd).*

I. *If $1/2 < a < 1$, $\alpha_1(t) - \alpha_3(t)$ is 2π-periodic, and $\int\limits_{-2\pi}^{0} (\alpha_1(s) - \alpha_3(s))\,ds = 0$, then solution (3.2) is 2π-periodic (the period is not necessarily minimal).*

II. *If $\alpha_1(t) - \alpha_3(t) - (a-1)^4\alpha_4(t)$ is 2π-periodic and its integral over $[-2\pi, 0]$ vanishes, then solution (3.3) is 2π-periodic (the period is not necessarily minimal).*

Proof.

I. Define $A(t) = \int\limits_{t-2\pi}^{t} (\alpha_1(s) - \alpha_3(s))\,ds$. Periodicity implies $\dot{A}(t) = 0$, so $A(t)$ is constant; this constant equals $\int\limits_{-2\pi}^{0} (\alpha_1(s) - \alpha_3(s))$, which is zero by hypothesis. Thus, the phase increment over any interval of length 2π vanishes, yielding 2π-periodicity of (3.2).

II. Second part follows identically. □

Proposition. *In Theorem 3.2, the integral conditions can be replaced by oddness assumptions:*

I. *If $\alpha_1(t) - \alpha_3(t)$ is 2π-periodic and odd, then $\int\limits_{-2\pi}^{0} (\alpha_1(s) - \alpha_3(s))\,ds = 0$.*

II. $\int\limits_{-2\pi}^{0} \big(\alpha_1(s) - \alpha_3(s) - (a-1)^4\alpha_4(s)\big)\,\mathrm{d}s = 0$ *if $\alpha_1(t) - \alpha_3(t) - (a-1)^4\alpha_4(t)$ is 2π-periodic and odd.*

Proof. Oddness implies $\int\limits_{-l}^{0} f(s)\,ds = -\int\limits_{0}^{l} f(s)\,ds$. For a 2π-periodic odd function, the integrals over $[0, 2\pi]$ and $[-2\pi, 0]$ are negatives of each other, hence both must be zero. □

4 Conclusion

This paper is devoted to the application of the MRF theory to the study of the Langford system, a well-known model for turbulence in fluids. The core of our work lies in constructing a wide class of non-autonomous perturbations that preserve the MRF of the original autonomous system. Such admissible perturbations allow us to generate families of more complex, time-dependent systems that inherit key qualitative properties from the original Langford system. We have explicitly found the form of these admissible perturbations and investigated their impact on the system's dynamics. Specifically, for the perturbed systems, we have:

- identified explicit forms of solutions corresponding to limit cycles;

- established verifiable sufficient conditions under which these solutions are periodic;
- shown that the stability character of these periodic solutions in the perturbed systems coincides with that of the corresponding solutions in the original Langford system.

The significance of these findings is that the extensive research on the bifurcations and stability of the classical Langford system can be directly leveraged to understand the behavior of a much broader class of nonstationary systems. This provides a powerful tool for analyzing complicated, time-varying models by relating them back to a simpler, well-studied autonomous case, thereby offering new insights for modeling real-world processes with inherent non-autonomous features.

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