On Continuity in a Parameter of Solutions to Generic Inhomogeneous Boundary-Value Problems in Sobolev Spaces

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An investigation of solutions to systems of ODEs is an important part of numerous problems of contemporary analysis and its applications. Unlike solutions to the Cauchy problem, solutions to boundary-value problems may not exist or may not be unique. Boundary-value problems that depend on a discrete or continuous numerical parameter are less studied than the Cauchy problem. For a long time, the widest class of inhomogeneous boundary conditions for which limit theorems for solutions were established concerned the class of *general* boundary-value problems of the form $Ly = f$, $By = c$, where L is a system of m linear differential equations of order $r \in \mathbb{N}$ with summable coefficients and right-hand sides f, and B is a linear continuous finite-dimensional operator $B : (C^{r-1})^m \to \mathbb{C}^{rm}$. Therefore, the general boundary conditions can contain derivatives of the unknown function only of order $\leq r-1$. For such problems, sufficient conditions for the continuity of solutions in the case $r = 1$ were established in the seminal article of I. T. Kiguradze [2].

In some applied tasks and problems of optimization theory, problems naturally arise with boundary conditions that contain derivatives of an unknown function of integer or fractional orders, which may be greater than the order of the differential equation. These are *generic* boundary-value problems in Sobolev spaces. This new direction in the theory of linear boundary-value problems was developed in a series of works by V. A. Mikhailets and his followers. Observe that for such boundary-value problems the formally conjugate problem and Lagrange formula are not defined. Therefore, their analysis requires the use of new approaches and methods. They have also found application for the investigation of generic boundary-value problems in other spaces of differentiable functions. It is worth noting that even in the case of $n = 0$, $p = 1$ the class of generic boundary-value problems is wider than the class of general boundary-value problems.

We study a wide class of linear inhomogeneous boundary-value problems of the r-th order ODEs whose solutions range over the Sobolev spaces $(W_p^{n+r})^m$, $n \in \mathbb{N} \cup \{0\}$, $\{m, r\} \subset \mathbb{N}$, $1 \leq p \leq \infty$. The boundary conditions for this problem are the most general forms $By = c$, where B is an arbitrary continuous operator from $(W_p^{n+r})^m$ to $\mathbb{C}^{rm}$. These boundary conditions may contain derivatives of the unknown vector-valued function of integer and/or fractional orders $\geq r$. For problems of this class, constructive necessary and sufficient conditions for the continuity of solutions with respect to a parameter from an abstract metric space have been found.

We arbitrarily choose a finite interval $(a, b) \subset \mathbb{R}$ and the following parameters:

$$n \in \mathbb{N} \cup \{0\}, \ \ \{m, r, k\} \subset \mathbb{N}, \ \text{ and } \ 1 \leq p \leq \infty.$$

As usual,

$$W_p^{n+r}([a,b];\mathbb{C}) := \Big\{ y \in C^{n+r-1}([a,b];\mathbb{C}) : \ y^{(n+r-1)} \in AC[a,b], \ y^{(n+r)} \in L_p[a,b] \Big\}$$

is a complex Sobolev space; set $W_p^0 := L_p$. This space is a Banach one with respect to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r} \|y^{(k)}\|_p,$$

with $\|\cdot\|_p$ standing for the norm in the Lebesgue space $L_p([a,b];\mathbb{C})$. We need the Sobolev spaces

$$(W_p^{n+r})^m := W_p^{n+r}([a,b];\mathbb{C}^m) \text{ and } (W_p^{n+r})^{m\times m} := W_p^{n+r}([a,b];\mathbb{C}^{m\times m}).$$

They respectively consist of vector-valued functions and matrix-valued functions whose elements belong to W_p^{n+r}. The norms in these spaces are defined to be the sums of the relevant norms in W_p^{n+r} of all elements of a vector-valued or matrix-valued function. We preserve the same notation $\|\cdot\|_{n+r,p}$ for these norms. It will be clear from the context to which space (scalar or vector-valued or matrix-valued functions) relates the designation of the norm. The same concerns all other Banach spaces used in the sequel. Certainly, the above Sobolev spaces coincide in the $m = 1$ case. If $p < \infty$, they are separable and have a Schauder basis.

Let $\mathcal{M}$ be an arbitrary metric space, a parameter $\mu \in \mathcal{M}$ runs through the entire space $\mathcal{M}$, and an arbitrary point $\mu_0 \in \mathcal{M}$ is fixed. Let us consider a linear boundary-value problem of the form

$$(L(\mu_0)y(\mu_0))(t) := y^{(r)}(t,\mu_0) + \sum_{l=1}^{r} A_{r-l}(t,\mu_0)y^{(r-l)}(t,\mu_0) = f(t,\mu_0), \quad t \in (a,b), \tag{1}$$

$$B(\mu_0)y(\mu_0) = c(\mu_0). \tag{2}$$

We assume that matrix-valued functions $A_{r-l}(\,\cdot\,,\mu_0) \in (W_p^n)^{m\times m}$, a vector-valued function $f(\,\cdot\,,\mu_0) \in (W_p^n)^m$, a vector $c(\mu_0) \in \mathbb{C}^{rm}$, a finite-dimensional linear continuous operator

$$B(\mu_0) : (W_p^{n+r})^m \to \mathbb{C}^{rm} \tag{3}$$

are arbitrarily given, and that a vector-valued function $y(\,\cdot\,,\mu_0) \in (W_p^{n+r})^m$ is unknown.

The boundary condition (3) consists of rm linearly independent scalar condition for system of m differential equations of r-th order, we representing vectors and vector-valued functions as columns.

A solution to the boundary-value problem (1), (2) is understood as the vector-valued function $y(\,\cdot\,) \in (W_p^{n+r})^m$ that satisfies both equation (1) (everywhere if $n \geq 1$, and almost everywhere if $n = 0$) on (a,b) and equality (2). Indeed, if the right-hand side $f(\,\cdot\,,\mu_0)$ of the system passes through the entire space $(W_p^n)^m$, then the solution $y(\,\cdot\,,\mu_0)$ of the system passes through the entire space $(W_p^{n+r})^m$. The boundary condition (2) with an arbitrary continuous operator (3) is the most general (generic) for the differential system (1). This condition covers all classical types of boundary conditions, such as initial conditions in the Cauchy problem, various multipoint conditions, integral conditions, conditions used in mixed boundary-value problems, as well as non-classical conditions containing derivatives, usually fractional, and the order of these derivatives may exceed the order of the differential equation.

Let us consider a family of inhomogeneous boundary-value problems of the form (1), (2) for a system of m linear differential equations of order r:

$$L(\mu)y(t,\mu) := y^{(r)}(t,\mu) + \sum_{l=1}^{r} A_{r-l}(t,\mu)y^{(r-l)}(t,\mu) = f(t,\mu), \quad t \in (a,b), \tag{4}$$

$$B(\mu)y(\,\cdot\,,\mu) = c(\mu). \tag{5}$$

Here, for each $\mu \in \mathcal{M}$, the unknown vector-valued function $y(\,\cdot\,,\mu) \in (W_p^{n+r})^m$, the matrix-valued functions $A_{r-l}(\,\cdot\,,\mu) \in (W_p^n)^{m\times m}$, and the family of linear continuous operators $B(\mu) : (W_p^{n+r})^m \to \mathbb{C}^{rm}$ are the same as before.

Let us write the family of inhomogeneous boundary-value problems (4), (5) in the form of a linear operator equation

$$(L(\mu), B(\mu))y(\mu) = (f(\mu), c(\mu)),$$

where $(L(\mu), B(\mu))$ is the family of finite-dimensional linear continuous operators

$$(L(\mu), B(\mu)) : (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^{rm}. \tag{6}$$

According to [3], the family of operators (6) are bounded Fredholm operators with zero index for each $\mu \in \mathcal{M}$.

Definition. The solution to the boundary-value problem (4), (5) depends continuously on the parameter μ at the limit point μ_0 of the space $\mathcal{M}$, if the following two conditions are satisfied:

- ($*$) There exists a positive number ε such that, for any $\mu \in \mathcal{B}(\mu_0, \varepsilon)$ and arbitrary right-hand sides $f(\,\cdot\,;\mu) \in (W_p^n)^m$ and $c(\mu) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\,\cdot\,;\mu)$ from the space $(W_p^{n+r})^m$;
- ($**$) The convergence of the right-hand sides $f(\,\cdot\,;\mu) \to f(\,\cdot\,;\mu_0)$ in $(W_p^n)^m$ and $c(\mu) \to c(\mu_0)$ in $\mathbb{C}^{rm}$ as $\mu \to \mu_0$ implies the convergence of the solutions $y(\,\cdot\,,\mu) \to y(\,\cdot\,,\mu_0)$ in $(W_p^{n+r})^m$ as $\mu \to \mu_0$.

It is easy to verify that if the operators $(L(\mu), B(\mu))$ converge to the operator $(L(\mu_0), B(\mu_0))$ in the uniform operator topology, then the solutions of problems (4), (5) are continuous in the parameter. In this paper, we will find more subtle conditions for the continuity of the solutions to the problem in the parameter, which are not only sufficient, but also necessary in the sense of Definition (see Theorem 1 in abstract form and Theorem 5 in constructive form).

Let us consider the following condition for the fixed point $\mu_0 \in \mathcal{M}$. Henceforth, we will assume that condition (0) is always fulfilled.

Condition (0). A homogeneous boundary-value problem has only a trivial solution

$$L(\mu_0)y(t,\mu_0) = 0, \ \ t \in (a,b), \quad B(\mu_0)y(\,\cdot\,,\mu_0) = 0.$$

We also consider the following two conditions on the left-hand side of this problem.

Limit conditions as $\mu \to \mu_0$:

- (I) $A_{r-l}(\,\cdot\,;\mu) \to A_{r-l}(\,\cdot\,;\mu_0)$ in the space $(W_p^n)^{m\times m}$ for each number $l \in \{1,\dots,r\}$;
- (II) $B(\mu)y \to B(\mu_0)y$ in the space $\mathbb{C}^m$ for every $y \in (W_p^{n+r})^m$.

Let us formulate necessary and sufficient conditions for the continuity of solutions to the boundary-value problem (4), (5) with respect to an abstract parameter.

Theorem 1. *The solution to the boundary-value problem* (4), (5) *depends continuously on the parameter* μ *at* $\mu_0 \in \mathcal{M}$ *if and only if this problem satisfies Condition* (0) *and Limit Conditions* (I) *and* (II).

Corollary. *If Condition* (0) *and Limit Conditions* (I) *and* (II) *are satisfied for each* $\mu \in \mathcal{M}$ *and vector-valued functions* f *and vector* c *are fixed, then the solution to the boundary-value problem* (4), (5) *exists and is unique for arbitrary right-hand sides of the problem and belongs to the space* $C(\mathcal{M}; (W_p^{n+r})^m)$.

It is worth noting that the case of an arbitrary metric space $\mathcal{M}$ allows us to study the cases of continuous and discrete parameters from a unique point of view.

Note that in the case of $r = 1$, $\mathcal{M} = [0, \varepsilon_0]$, $\varepsilon_0 > 0$, $\mu_0 = 0$, Theorem 1 is proved in [1]. In the case of $r = 1$, $\mathcal{M} = I \subset \mathbb{R}$, $\varepsilon_0 > 0$, $\mu_0 = 0$, Theorem 1 is proved in [4].

Let us find conditions under which $(L(\mu), B(\mu))$ converge to the operator $(L(\mu_0), B(\mu_0))$ in the strong and uniform operator topologies in terms of conditions on the coefficients of differential expressions and operators $B(\mu)$.

First, we formulate necessary and sufficient conditions for strong and uniform convergence of the family of operators $L(\mu)$ to the operator $L(\mu_0)$.

Theorem 2. *In the case of* $1 \leq p \leq \infty$*, the following three conditions are equivalent to each other as* $\mu \to \mu_0$*:*

(I) $A_{r-l}(\,\cdot\,, \mu) \to A_{r-l}(\,\cdot\,, \mu_0)$ *in the Banach space* $(W_p^n)^{m \times m}$ *for each* $l \in \{1, \ldots, r\}$*;*

(II) $L(\mu) \to L(\mu_0)$ *in the uniform operator topology;*

(III) $L(\mu) \to L(\mu_0)$ *in the strong operator topology.*

Note that in the case where the metric parameter μ is a natural number, Theorem 2 was proved in the paper [3, Lemma 6.1].

In order to proceed to the study of the convergence of the family of operators $B(\mu)$, we need the following result.

Theorem 3. *Let* $1 \leq p \leq \infty$*,* t_0 *be a fixed point in the interval* $[a, b]$*, the matrix* $(\alpha_s)_{s=1}^{n+1-r} \subset \mathbb{C}^{rm \times rm}$*, and the matrix-valued function* $\Phi(\,\cdot\,) \in L_{p'}([a, b]; \mathbb{C}^{rm \times rm})$*,* $1/p + 1/p' = 1$*.*

(1) *The operator*

$$By = \sum_{s=0}^{n+r-1} \alpha_s y^{(s)}(t_0) + \int_a^b \Phi(t) y^{(n+r)}(t)\, \mathrm{d}t, \quad y(\,\cdot\,) \in (W_p^{n+r})^m \tag{7}$$

acts continuously from the space $(W_p^{n+r})^m$ *to the space* $\mathbb{C}^{rm}$ *and its norm satisfies the inequality*

$$\|B\| \leq \gamma \max_s \{|\alpha_s|\} + \|\Phi\|_{L_{p'}},$$

where $0 < \gamma$ *is a positive constant that does not depend on the choice of the matrices* α_s *and the matrix-valued function* $\Phi(\,\cdot\,)$*.*

(2) *If* $p \neq \infty$*, then every bounded operator* $B : (W_p^{n+r})^m \to \mathbb{C}^{rm}$ *admits a canonical representation of the form* (7) *and it is unique.*

It should be noted that in the case of $p = \infty$, the formula (7) does not give a complete description of the operators B. There are continuous operators B that are defined by integrals over finitely additive measures.

Let us formulate necessary and sufficient conditions for strong and uniform convergence of the family of operators $B(\mu)$.

Let us consider the following asymptotic conditions as $\mu \to \mu_0$.

(a) $\alpha_s(\mu) \to \alpha_s(\mu_0)$ in $\mathbb{C}^{rm \times rm}$ for each number $s \in \{0, \ldots, n + r - 1\}$;

(b) $\|\Phi(\,\cdot\,, \mu)\|_{p'} = O(1)$;

(c) $\int\limits_a^t \Phi(\tau,\mu)\,d\tau \to \int\limits_a^t \Phi(\tau,\mu_0)\,d\tau$ in the space $\mathbb{C}^{rm\times rm}$ for any $t\in(a,b]$;

(d) $\|\Phi(\,\cdot\,,\mu)-\Phi(\,\cdot\,,\mu_0)\|_{p'}\to 0$.

Theorem 4. *Let $1\le p<\infty$. Strong convergence of the operators $B(\mu)$ to operator $B(\mu_0)$ is equivalent to conditions (a), (b), (c) as $\mu\to\mu_0$. Uniform convergence of the operators $B(\mu)$ to operator $B(\mu_0)$ is equivalent to conditions (a), (d) as $\mu\to\mu_0$.*

It is easy to see that condition (d) is stronger than conditions (b) and (c).

The following statement follows from Theorems 1 and 4:

Theorem 5.

1. *The solution to the boundary-value problem* (4), (5) *depends continuously on the parameter μ at the point μ_0 if and only if this problem satisfies the conditions* (I), (a), (b), (c).
2. *The family of operators $(L(\mu),B(\mu))$ uniformly converges to the limit operator $(L(\mu_0),B(\mu_0))$ if and only if the problem* (4), (5) *satisfies the conditions* (I), (a), (d).

Conclusions. We have introduced the new concept of continuity in a parameter of the solutions to the inhomogeneous boundary-value problems with generic boundary conditions in Sobolev spaces. We obtained abstract and contractive forms of necessary and sufficient conditions for continuity in a parameter of solutions to the problem.

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