Some Two-Point Boundary Value Problems for Systems of Higher Order Linear Functional Differential Equations

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1 Introduction

Consider on the interval $I=[a,b]$ the systems of higher order linear differential equations with argument deviations

$$u_i^{(m_i)}(t)=p_i(t)u_{i+1}(\tau_i(t))+q_i(t) \ \ (i=1,\dots,n), \tag{1.1}$$

where $n\geq 2$, $m_i\geq 2$, $u_{n+1}:=u_1$, $q_i\in L(I;R)$, $p_i\in L_\infty(I;R)$, and $\tau_i:I\to I$ are the measurable functions.

Here we study system (1.1) under the conjugate

$$u_i^{(j_1-1)}(a)=a_{ij_1}, \quad u_i^{(j_2-1)}(b)=b_{ij_2}, \ \ j_1=1,\dots,k_i, \ \ j_2=1,\dots,m_i-k_i, \ \ i=1,\dots,n, \tag{1.2$_1$}$$

and the right-focal

$$u_i^{(j_1-1)}(a)=a_{ij_1}, \quad u_i^{(j_2-1)}(b)=b_{ij_2}, \ \ j_1=1,\dots,k_i, \ \ j_2=k_i+1,\dots,m_i, \ \ i=1,\dots,n, \tag{1.2$_2$}$$

boundary conditions, where $k_i:=[m_i/2]$ is the integer part of the number $m_i/2$.

Throughout the paper we use the following notations.

- $\widetilde{C}^{n-1}(I;R)$ is the set of functions $u:I\to R$ which are absolutely continuous together with their $(n-1)$th derivatives;
- $L(I;R)$ is the Banach space of Lebesgue integrable functions $p:I\to R$ with the norm
$$\|p\|_L=\int\limits_a^b|p(s)|\,ds;$$
- $L_\infty(I;R)$ is the space of essentially bounded measurable functions $p:I\to R$ with the norm
$$\|p\|_\infty=\operatorname{ess\,sup}\big\{|p(t)|:\ t\in I\big\};$$

- $M(I)$ is the set of measurable functions $\tau : I \to I$.

By a solution of problem $(1.1), (1.3_r)$ $r \in \{1, 2\}$ we understand a vector-function $u := (u_i)_{i=1}^n$ where $u_i \in \widetilde{C}^{m_i-1}(I; R)$ $(i = 1, \dots, n)$, which satisfies system (1.1) almost everywhere on I and satisfies conditions (1.3_r).

The aim of our work is to study the question of unique solvability of the above-mentioned problem. We have proved unimprovable sufficient conditions, which describe the relationship between the coefficient p_i and the deviations of the arguments $|t - \tau_i|$ that guarantees the unique solvability of problems $(1.1), (1.2_1)$ and $(1.1), (1.2_2)$. Our study is motivated by some original results for the functional differential equations with argument deviation (see [1–4]).

2 Main results

To formulate the uniqueness theorems of problems $(1.1), (1.2_1)$ and $(1.1), (1.2_2)$, let us introduce the following notation

$$\rho(y, \tau) = \frac{2\pi}{b-a} \bigg(\int\limits_a^b |y(s)|\, |\tau(s) - s|\, ds \bigg)^{1/2}.$$

Theorem 2.1. *Let the functions* $\tau_i \in M(I)$ *and* $p_i \in L_\infty(I; R)$ $(i = 1, \dots, n)$ *be such that the conditions*

$$0 \le \sigma_i p_i(t) \ \ (i = 1, \dots, n) \ \ \text{for} \ \ t \in I, \tag{2.1}$$

where $\sigma_i \in \{-1, 1\}$, *and*

$$\prod_{i=1}^n \Big(2\|p_i\|_\infty + \rho(p_i, \tau_i) \|p_i\|_\infty^{1/2} \Big) < 4^{\sum\limits_{i=1}^n k_i} \Big(\frac{\pi}{2(b-a)} \Big)^{\sum\limits_{i=1}^n m_i} \tag{2.2_1}$$

hold. Then problem $(1.1), (1.2_1)$ *is uniquely solvable.*

Theorem 2.2. *Let the functions* $\tau_i \in M(I)$ *and* $p_i \in L_\infty(I; R)$ $(i = 1, \dots, n)$ *be such that conditions* (2.1), *and*

$$\prod_{i=1}^n \Big(4\|p_i\|_\infty + \rho(p_i, \tau_i) \|p_i\|_\infty^{1/2} \Big) < 4^n \Big(\frac{\pi}{2(b-a)} \Big)^{\sum\limits_{i=1}^n m_i}, \tag{2.2_2}$$

hold. Then problem $(1.1), (1.2_2)$ *is uniquely solvable.*

Corollary 2.1. *Let the functions* $\tau_i \in M(I)$, $p_i \in L_\infty(I; R)$, *and* $\alpha_i \in L(I; R_+)$ $(i = 1, \dots, n)$ *be such that conditions* (2.1),

$$\|p_i\|_\infty |p_i(t)|\, |\tau_i(t) - t| \le \alpha_i(t) \ \ (i = 1, \dots, n) \ \ \text{a.e on } I, \tag{2.3}$$

and

$$\prod_{i=1}^n \Big[\|p_i\|_\infty + \frac{\pi}{b-a} \|\alpha_i\|_L^{1/2} \Big] < 2^{2\sum\limits_{i=1}^n k_i - n} \Big(\frac{\pi}{2(b-a)} \Big)^{\sum\limits_{i=1}^n m_i}$$

hold. Then problem $(1.1), (1.2_1)$ *is uniquely solvable.*

Corollary 2.2. *Let the functions* $\tau_i \in M(I)$, $p_i \in L_\infty(I; R)$, *and* $\alpha_i \in L(I; R_+)$ $(i = 1, \dots, n)$ *be such that conditions* $(2.1), (2.3)$, *and*

$$\prod_{i=1}^n \Big[2\|p_i\|_\infty + \frac{\pi}{b-a} \|\alpha_i\|_L^{1/2} \Big] < 2^n \Big(\frac{\pi}{2(b-a)} \Big)^{\sum\limits_{i=1}^n m_i}$$

hold. Then problem $(1.1), (1.2_2)$ *is uniquely solvable.*

From the last corollaries with $\alpha_i \equiv 0$ it immediately follows

Corollary 2.3. *Let the functions* $\tau_i \in M(I)$, $p_i \in L_\infty(I;R)$ $(i=1,\dots,n)$, *be such that conditions* (2.1),

$$|p_i(t)|\,|\tau_i(t)-t| = 0 \ \ (i=1,\dots,n) \ \ \textit{a.e. on } I, \tag{2.4}$$

and

$$\prod_{i=1}^n \|p_i\|_\infty < 2^{2\sum\limits_{i=1}^n k_i - n} \Big(\frac{\pi}{2(b-a)}\Big)^{\sum\limits_{i=1}^n m_i}$$

hold. Then problem (1.1), (1.2_1) *is uniquely solvable.*

Corollary 2.4. *Let the functions* $\tau_i \in M(I)$, $p_i \in L_\infty(I;R)$ $(i=1,\dots,n)$, *be such that conditions* (2.1), (2.4), *and*

$$\prod_{i=1}^n \|p_i\|_\infty < \Big(\frac{\pi}{2(b-a)}\Big)^{\sum\limits_{i=1}^n m_i}$$

hold. Then problem (1.1), (1.2_2) *is uniquely solvable.*

For the illustration of Theorems 2.1 and 2.2, we consider here the system

$$u_1''(t) = p_1(t)u_2(t) + q_1(t), \quad u_2''(t) = p_2(t)u_1(\tau(t)) + q_2(t), \tag{2.5}$$

with $p_1, p_2 \in L_\infty(I;R)$ and $\tau \in M(I)$, under the boundary conditions (1.2_1) and (1.2_2), and the equation

$$u^{(m)}(t) = p(t)u(\tau(t)) + q(t), \tag{2.6}$$

where $m = 2k+2$ $(k \in N)$ under the boundary conditions

$$u^{(2j-2)}(a) = c_{1j}, \quad u^{(2j-2)}(b) = c_{2j}, \ \ j=1,\dots,k+1, \tag{2.7_1}$$

and

$$u^{(2j-2)}(a) = c_{1j}, \quad u^{(2j-1)}(b) = c_{2j}, \ \ j=1,\dots,k+1. \tag{2.7_2}$$

Then from Theorems 2.1 and 2.2 it immediately follow

Corollary 2.5. *Let* $r \in \{1,2\}$, *and the functions* $p_1, p_2 \in L_\infty(I;R)$, $\tau \in M(I)$, *be such that conditions* (2.1) *with* $n=2$, *and*

$$\|p_1\|_\infty \bigg(r\|p_2\|_\infty + \frac{\pi}{b-a} \bigg(\|p_2\|_\infty \int\limits_a^b |p_2(s)|\,|\tau(s)-s|\,ds \bigg)^{1/2} \bigg) < \frac{1}{2^{r+1}} \Big(\frac{\pi}{b-a}\Big)^4$$

hold. Then problem (2.5), (0.2_r) *is uniquely solvable.*

Corollary 2.6. *Let* $r \in \{1,2\}$, *and the functions* $p \in L_\infty(I;R_+)$, $\tau \in M(I)$ *be such that the conditions*

$$0 \le \sigma p(t) \ \ \textit{for} \ \ t \in I,$$

where $\sigma \in \{-1,1\}$, *and*

$$r\|p\|_\infty + \frac{\pi}{b-a} \bigg(\|p\|_\infty \int\limits_a^b |p(s)|\,|\tau(s)-s|\,ds \bigg)^{1/2} < 2^{(2-r)k+1} \Big(\frac{\pi}{2(b-a)}\Big)^m$$

hold. Then problem (2.6), (0.9_r) *is uniquely solvable.*

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